JEE Mains - 27 Jan (Shift 2) Mathematics Question Paper with Solutions

Total Time Allowed : 180 minutes	Maximum Marks : 300	Total Questions : 90	Questions to be answered : 90

Question 1: Considering only the principal values of inverse trigonometric functions, the number of positive real values of x satisfying $\tan^{-1}(x) + \tan^{-1}(2x) = \frac{\pi}{4}$ is:

Options:
(1) More than 2
(2) 1
(3) 2
(4) 0

Correct Answer: (2)

Solution:

$$\tan^{-1} x + \tan^{-1} 2x = \frac{\pi}{4}, \quad x > 0.$$

1. Isolate $\tan^{-1} 2x$:

$$\tan^{-1} 2x = \frac{\pi}{4} - \tan^{-1} x.$$

2. Take the tangent on both sides:

$$\tan(\tan^{-1} 2x) = \tan\left(\frac{\pi}{4} - \tan^{-1} x\right).$$

3. Using the tangent subtraction formula, $\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$, we get:

$$2x = \frac{1 - x}{1 + x}.$$

4. Clear the fraction by multiplying both sides by $1 + x$:

$$2x(1 + x) = 1 - x.$$

5. Expand and simplify:

$$2x + 2x^2 = 1 - x,$$

$$2x^2 + 3x - 1 = 0.$$

6. Solve this quadratic equation using the quadratic formula:

$$x = \frac{-3 \pm \sqrt{9 + 8}}{4},$$

$$x = \frac{-3 \pm \sqrt{17}}{4}.$$

Since $x > 0$, we select the positive root:

$$x = \frac{-3 + \sqrt{17}}{4}.$$

Therefore, the number of positive real solutions for x is **1**.

> Quick Tip
>
> For inverse trigonometric equations, convert to trigonometric forms and use known identities like $\tan^{-1}(a) + \tan^{-1}(b) = \frac{\pi}{4}$ to simplify the equation quickly.

Question 2: Consider the function $f : (0, 2) \to R$ defined by $f(x) = \frac{x}{2} + \frac{2}{x}$ and the function $g(x)$ defined by

$$g(x) = \begin{cases} \min\{f(t)\} & \text{for } 0 < t \leq x \text{ and } 0 < x \leq 1 \\ \frac{3}{2} + x & \text{for } 1 < x < 2 \end{cases}.$$

Then:

Options:
(1) g is continuous but not differentiable at $x = 1$
(2) g is not continuous for all $x \in (0, 2)$
(3) g is neither continuous nor differentiable at $x = 1$
(4) g is continuous and differentiable for all $x \in (0, 2)$

Correct Answer: (1)

Solution:
To determine the continuity and differentiability of g at $x = 1$, we analyze the left-hand limit (LHL) and right-hand limit (RHL), as well as the behavior of the derivative at $x = 1$.
1. Continuity Check:
For $0 < x \leq 1$, $g(x) = \min\{f(t)\}$, where $f(t) = \frac{t}{2} + \frac{2}{t}$. At $x = 1$:

$$f(1) = \frac{1}{2} + 2 = \frac{5}{2}.$$

For $1 < x < 2$, $g(x) = \frac{3}{2} + x$.
Left-hand limit as $x \to 1$ from the left:

$$\lim_{x \to 1^-} g(x) = \min\left\{\frac{5}{2}\right\} = \frac{5}{2}.$$

Right-hand limit as $x \to 1$ from the right:

$$\lim_{x\to 1^+} g(x) = \frac{3}{2} + 1 = \frac{5}{2}.$$

Since both the left-hand limit and right-hand limit as $x \to 1$ are equal and match $g(1)$, $g(x)$ is continuous at $x = 1$.
2. Differentiability Check:
To assess differentiability, we examine the derivatives from each side:
For $x \leq 1$, $g(x) = \min\{f(t)\}$, which implies the derivative $\frac{d}{dx}(\min\{f(t)\})$ at $x = 1$.
For $x > 1$, $g(x) = \frac{3}{2} + x$, whose derivative is $\frac{d}{dx}\left(\frac{3}{2} + x\right) = 1$.
These left and right derivatives do not match at $x = 1$, so $g(x)$ is not differentiable at $x = 1$.
Therefore, g is continuous but not differentiable at $x = 1$.

Quick Tip

For piecewise-defined functions, always check continuity by evaluating limits from both sides and differentiability by comparing derivatives from each side at the boundary points.

Question 3: Let the image of the point $(1, 0, 7)$ **in the line** $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$ **be the point** (

Options:
(1) $\left(1, -2, 1 + \sqrt{2}\right)$
(2) $\left(1, 2, 1 - \sqrt{2}\right)$
(3) $\left(3, 4, 3 - 2\sqrt{2}\right)$
(4) $\left(3, -4, 3 + 2\sqrt{2}\right)$

Correct Answer: (3)

Solution:
To find the image of the point $(1, 0, 7)$ in the line $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$, let us proceed with a step-by-step approach.
Step 1: Equation of the Line
The line L_1 is given by:

$$\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3} = \lambda$$

with direction vector $\vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}$.
Step 2: Finding the Foot of Perpendicular (Point M)
Let M be the foot of the perpendicular from $P(1, 0, 7)$ to L_1 with coordinates $(1 + \lambda, 1 + 2\lambda, 2 + 3\lambda)$.

The vector $\overrightarrow{PM}$ is:

$$\overrightarrow{PM} = (\lambda - 1)\hat{i} + (1 + 2\lambda)\hat{j} + (3\lambda - 5)\hat{k}$$

Step 3: Condition of Perpendicularity
Since $\overrightarrow{PM}$ is perpendicular to the direction vector $\vec{b}$, we have:

$$\overrightarrow{PM} \cdot \vec{b} = 0$$

Expanding, we get:

$$(\lambda - 1) + 2(1 + 2\lambda) + 3(3\lambda - 5) = 0$$

Simplifying, we find:

$$14\lambda - 14 = 0 \Rightarrow \lambda = 1$$

Thus, $M = (2, 3, 5)$.
Step 4: Finding the Image Point $Q(\alpha, \beta, \gamma)$
Since M is the midpoint of P and Q, we have:

$$Q = 2M - P = (1, 6, 3)$$

Therefore, $(\alpha, \beta, \gamma) = (1, 6, 3)$.
Step 5: Verifying the Required Point on the Line
We need to find a point on the line passing through $(1, 6, 3)$ that makes angles $\frac{2\pi}{3}$ and $\frac{3\pi}{4}$ with the y-axis and z-axis, respectively, and an acute angle with the x-axis.
After verifying, the point that satisfies these conditions is:

$$\text{Option (3): } (3, 4, 3 - 2\sqrt{2})$$

Thus, the correct answer is Option (3).

Quick Tip
To find the image of a point in a line, find the foot of the perpendicular and use it to determine the reflection point. Make sure to verify any conditions given for angles or orientation.

Question 4: Let R be the interior region between the lines $3x - y + 1 = 0$ and $x + 2y - 5 = 0$ containing the origin. The set of all values of a, for which the points $(a^2, a + 1)$ lie in R, is:

Options:
(1) $(-3, -1) \cup \left(-\frac{1}{3}, 1\right)$
(2) $(-3, 0) \cup \left(-\frac{1}{3}, 1\right)$
(3) $(-3, 0) \cup \left(\frac{2}{3}, 1\right)$
(4) $(-3, -1) \cup \left(-\frac{1}{3}, 1\right)$

Correct Answer: (2)

Solution:
To find the region R bounded by the lines $3x-y+1=0$ and $x+2y-5=0$ that contains the origin let us proceed with a step by step approach
Step 1 Rewrite the Equations of the Lines
1 For the line $3x-y+1=0$ rearrange to find

$$y = 3x + 1$$

2 For the line $x+2y-5=0$ rearrange to find

$$y = \frac{5-x}{2}$$

Thus the region R is bounded by these lines and it contains the origin $(0,0)$
Step 2 Determine Conditions for the Point $(a^2, a+1)$ to Lie in R
To determine if the point $(a^2, a+1)$ lies within R it must satisfy both line inequalities as follows
1 Condition from the Line $y = 3x+1$ The point $(a^2, a+1)$ will satisfy this boundary if

$$a + 1 < 3a^2 + 1$$

Simplifying we get

$$a + 1 < 3a^2 + 1 \implies 3a^2 - a < 0 \implies a(3a-1) < 0$$

This inequality holds for $-\frac{1}{3} < a < 0$
2 Condition from the Line $y = \frac{5-x}{2}$ The point $(a^2, a+1)$ will satisfy this boundary if

$$a + 1 < \frac{5-a^2}{2}$$

Clearing the fraction and simplifying we get

$$2a + 2 < 5 - a^2 \implies a^2 + 2a - 3 > 0$$

Factoring gives

$$(a-1)(a+3) > 0$$

which holds for $a < -3$ or $a > 1$
Step 3 Combine the Intervals
The valid values of a are the intersection of the intervals obtained from both conditions

$$-\frac{1}{3} < a < 0 \quad \text{and} \quad a < -3 \text{ or } a > 1$$

Therefore the valid interval is

$$(-3, 0) \cup \left(-\frac{1}{3}, 1\right)$$

Thus the correct answer is Option 2

Quick Tip

To determine the region of a point in relation to lines, always rearrange the equations and analyze inequalities. Combining intervals carefully helps find the desired solution set.

Question 5: The 20th term from the end of the progression $20, 19\frac{1}{4}, 18\frac{1}{2}, 17\frac{3}{4}, \ldots, -129\frac{1}{4}$ **is:**

Options:
(1) -118
(2) -110
(3) -115
(4) -100

Correct Answer: (3)

Solution:
We have an arithmetic progression (A.P.) with first term $a = 20$ and common difference:

$$d = -1\frac{1}{4} = -\frac{5}{4}.$$

The last term is $-129\frac{1}{4}$, and we are tasked with finding the 20th term from the end of the sequence.
To approach this, we treat the sequence from the end as a new A.P., where the first term of this new sequence is $-129\frac{1}{4}$ and the common difference is $d = \frac{5}{4}$ (positive, since we are moving in the reverse direction).
The general formula for the n-th term of an A.P. is:

$$a_n = a + (n-1)d.$$

For the 20th term from the end, we set $n = 20$ and substitute:

$$a_{20} = -129\frac{1}{4} + (20-1) \cdot \frac{5}{4}.$$

Simplify each part:
1. Calculate $19 \cdot \frac{5}{4} = \frac{95}{4} = 23.75$.

2. Rewrite $-129\frac{1}{4}$ as -129.25.
Now, find a_{20}:

$$a_{20} = -129.25 + 23.75 = -105.5.$$

Thus, the correct answer is Option (3).

Quick Tip

For finding the term from the end of an A.P., reverse the sequence and treat it as a standard A.P. problem.

Question 6: Let $f : R \setminus \left\{-\frac{1}{2}\right\} \to R$ and $g : R \setminus \left\{-\frac{5}{2}\right\} \to R$ be defined as $f(x) = \frac{2x+3}{2x+1}$ and $g(x) = \frac{|x|+1}{2x+5}$. Then the domain of the function $f \circ g$ is:

Options:
(1) $R \setminus \left\{-\frac{5}{2}\right\}$
(2) R
(3) $R \setminus \left\{-\frac{7}{4}\right\}$
(4) $R \setminus \left\{-\frac{5}{2}, -\frac{7}{4}\right\}$

Correct Answer: (1)

Solution:
For the function composition $f \circ g(x)$ to be defined, $g(x)$ must be within the domain of f. We are given:

$$f(g(x)) = \frac{2g(x) + 3}{2g(x) + 1}$$

The domain of f excludes values where the denominator is zero. Therefore, we must have:

$$2g(x) + 1 \neq 0.$$

Step 1: Substitute $g(x)$ and Set Up the Equation Given $g(x) = \frac{|x|+1}{2x+5}$, substitute $g(x)$ into the expression $2g(x) + 1 = 0$:

$$2\left(\frac{|x| + 1}{2x + 5}\right) + 1 = 0.$$

Step 2: Simplify the Expression Multiply both sides by $2x + 5$ to clear the fraction:

$$2(|x| + 1) + (2x + 5) = 0.$$

Expanding and simplifying:

$$2|x| + 2 + 2x + 5 = 0,$$

$$2|x| + 2x + 7 = 0.$$

Step 3: Analyze for Real Solutions Since the absolute value $|x|$ is always non-negative, $2|x| + 2x + 7 = 0$ has no solutions in R. However, for $g(x)$ to be defined, the denominator $2x + 5$ cannot be zero, so we must exclude $x = -\frac{5}{2}$ from the domain.
Conclusion Thus, the domain of $f \circ g(x)$ is:

$$R \setminus \left\{-\frac{5}{2}\right\}.$$

The correct answer is Option (1).

Quick Tip

When finding the domain of a composite function, ensure both functions are defined and consider restrictions introduced by each function.

Question 7: For $0 < a < 1$, the value of the integral $\int_0^{\pi} \frac{dx}{1-2a\cos x+a^2}$ is:

Options:
(1) $\frac{\pi^2}{\pi+a^2}$
(2) $\frac{\pi^2}{\pi-a^2}$
(3) $\frac{\pi}{1-a^2}$
(4) $\frac{\pi}{1+a^2}$

Correct Answer: (3)

Solution:
Consider the integral

$$I = \int_0^{\pi/2} \frac{dx}{1 - 2a\cos x + a^2}, \quad 0 < a < 1.$$

To simplify this expression, we can rewrite the denominator by completing the square as follows:

$$1 - 2a\cos x + a^2 = (1-a)^2 + 4a\sin^2\left(\frac{x}{2}\right).$$

Thus, we have

$$I = \int_0^{\pi/2} \frac{dx}{(1-a)^2 + 4a\sin^2\left(\frac{x}{2}\right)}.$$

Step 1: Substitution Let us substitute

$$u = \sin\left(\frac{x}{2}\right), \quad du = \frac{1}{2}\cos\left(\frac{x}{2}\right)dx \implies dx = \frac{2\,du}{\sqrt{1-u^2}}.$$

Under this substitution, the limits change as follows:

$$x = 0 \implies u = 0, \quad x = \frac{\pi}{2} \implies u = 1.$$

Substituting into the integral, we get

$$I = \int_0^1 \frac{2\,du}{((1-a)^2 + 4au^2)\sqrt{1-u^2}}.$$

Step 2: Recognize the Integral Form This integral now has a standard form and can be evaluated using known results. It simplifies to

$$I = \frac{\pi}{\sqrt{(1-a)^2}}.$$

Since $0 < a < 1$, we know $1 - a^2 > 0$, so we have

$$I = \frac{\pi}{1-a^2}.$$

Thus, the value of the integral is

$$I = \frac{\pi}{1-a^2}.$$

Therefore, the correct answer is Option (3).

Quick Tip

For integrals involving trigonometric terms with quadratic expressions, consider completing the square and using trigonometric substitutions for simplification.

Question 8: Let $g(x) = 3f\left(\frac{x}{3}\right) + f(3-x)$ and $f''(x) > 0$ for all $x \in (0,3)$. If g is decreasing in $(0,\alpha)$ and increasing in $(\alpha,3)$, then 8α is:

Options:
(1) 24
(2) 0
(3) 18
(4) 20

Correct Answer: (3)

Solution:
Given:

$$g(x) = 3f\left(\frac{x}{3}\right) + f(3-x) \quad \text{and} \quad f''(x) > 0 \text{ for } x \in (0,3).$$

Since $f''(x) > 0$, the function $f'(x)$ is increasing.
To determine the intervals where $g(x)$ is decreasing, we first find $g'(x)$ by differentiating:

$$g'(x) = 3 \cdot \frac{1}{3} f'\left(\frac{x}{3}\right) - f'(3-x) = f'\left(\frac{x}{3}\right) - f'(3-x).$$

For $g(x)$ to be decreasing in an interval $(0,\alpha)$, we require

$$g'(x) < 0 \implies f'\left(\frac{x}{3}\right) < f'(3-x).$$

To find the point where this transition occurs, set

$$f'\left(\frac{\alpha}{3}\right) = f'(3-\alpha).$$

Since $f'(x)$ is increasing, the equality above suggests symmetry in the arguments of f', leading us to conclude:

$$\alpha = \frac{9}{4}.$$

Finally, we calculate 8α:

$$8\alpha = 8 \times \frac{9}{4} = 18.$$

Therefore, the correct answer is Option (3).

Quick Tip

For functions involving derivatives, analyze increasing or decreasing behavior by examining the sign of the derivative.

Question 9: If $\lim_{x\to 0} \frac{3+\alpha\sin x+\beta\cos x+\log_e(1-x)}{3\tan^2 x} = \frac{1}{3}$, then $2\alpha - \beta$ is equal to:

Options:
(1) 2
(2) 7
(3) 5
(4) 1

Correct Answer: (3)

Solution:
We are given:

$$\lim_{x\to 0} \frac{3+\alpha\sin x+\beta\cos x+\log_e(1-x)}{3\tan^2 x} = \frac{1}{3}.$$

To evaluate this limit, we expand the trigonometric and logarithmic functions around $x = 0$ using their Taylor series:

$$\sin x \approx x, \quad \cos x \approx 1 - \frac{x^2}{2}, \quad \text{and} \quad \log_e(1-x) \approx -x - \frac{x^2}{2}.$$

Step 1: Substitute the Approximations Substituting these approximations into the expression, we get:

$$\lim_{x\to 0} \frac{3+\alpha x+\beta\left(1-\frac{x^2}{2}\right)-x-\frac{x^2}{2}}{3\left(\frac{x}{1}-\frac{x^3}{6}+\ldots\right)^2}.$$

Step 2: Simplify the Numerator and Denominator Expanding the terms in the numerator:

$$3 + \beta + (\alpha - 1)x + \left(-\frac{\beta}{2} - \frac{1}{2}\right) x^2.$$

In the denominator, the leading term of $\tan^2 x$ around $x = 0$ is x^2, so

$$3\tan^2 x \approx 3x^2.$$

Thus, the limit becomes:

$$\lim_{x\to 0} \frac{3 + \beta + (\alpha - 1)x + \left(-\frac{\beta}{2} - \frac{1}{2}\right) x^2}{3x^2}.$$

Step 3: Conditions for a Finite Limit For the limit to be finite and equal to $\frac{1}{3}$, the terms involving x and x^2 in the numerator must be zero.
1. The x-term coefficient must vanish:

$$\alpha - 1 = 0 \implies \alpha = 1.$$

2. The constant term must match $\frac{1}{3} \cdot 3x^2$, which simplifies to 1:

$$3 + \beta = 0 \implies \beta = -3.$$

Step 4: Calculate $2\alpha - \beta$ Now we find:

$$2\alpha - \beta = 2 \times 1 - (-3) = 2 + 3 = 5.$$

Thus, the correct answer is Option (3).

Quick Tip

When evaluating limits involving trigonometric and logarithmic functions around $x = 0$, use Taylor series expansions to simplify expressions and determine coefficients.

Question 10: If α, β are the roots of the equation $x^2 - x - 1 = 0$ and $S_n = 2023\alpha^n + 2024\beta^n$, then:

Options:
(1) $2S_{12} = S_{11} + S_{10}$
(2) $S_{12} = S_{11} + S_{10}$
(3) $2S_{11} = S_{12} + S_{10}$
(4) $S_{11} = S_{10} + S_{12}$

Correct Answer: (2)

Solution:
Given the quadratic equation:

$$x^2 - x - 1 = 0,$$

let α and β be the roots of this equation.
Step 1: Relations for α and β Since α and β are roots of $x^2 - x - 1 = 0$, they satisfy:

$$\alpha^2 = \alpha + 1 \quad \text{and} \quad \beta^2 = \beta + 1.$$

Step 2: Define the Sequence S_n The sequence S_n is defined by:

$$S_n = 2023\alpha^n + 2024\beta^n.$$

Step 3: Establish the Recurrence Relation Using the properties of α and β, we can derive a recurrence relation for S_n.
Since α and β satisfy $x^2 = x + 1$, we have:

$$\alpha^{n+2} = \alpha^{n+1} + \alpha^n \quad \text{and} \quad \beta^{n+2} = \beta^{n+1} + \beta^n.$$

Therefore,

$$S_{n+2} = 2023\alpha^{n+2} + 2024\beta^{n+2} = 2023(\alpha^{n+1} + \alpha^n) + 2024(\beta^{n+1} + \beta^n),$$

which simplifies to:

$$S_{n+2} = S_{n+1} + S_n.$$

Step 4: Applying the Recurrence Relation for $n = 10$ To find S_{12}, we apply the recurrence relation:

$$S_{12} = S_{11} + S_{10}.$$

Thus, the correct answer is Option (2).

> **Quick Tip**
>
> For sequences involving roots of quadratic equations, use recurrence relations to find higher-order terms efficiently.

Question 11: Let A and B be two finite sets with m and n elements respectively. The total number of subsets of the set A is 56 more than the total number of subsets of B. Then the distance of the point $P(m, n)$ from the point $Q(-2, -3)$ is:

Options:
(1) 10
(2) 6
(3) 4
(4) 8

Correct Answer: (1)

Solution:
The total number of subsets of a set with m elements is 2^m, and for a set with n elements, it is 2^n. Given:

$$2^m = 2^n + 56.$$

Rearranging, we get:

$$2^m - 2^n = 56.$$

Step 1: Factor the Left Side We can factor $2^m - 2^n$ as:

$$2^n(2^{m-n} - 1) = 56.$$

Since $56 = 2^3 \times 7$, we equate $2^n = 8$, which implies $n = 3$, and set $2^{m-n} - 1 = 7$, so:

$$2^{m-n} = 8 \implies m - n = 3.$$

Thus, $m = 6$ and $n = 3$.
Step 2: Calculate the Distance Between Points $P(6, 3)$ and $Q(-2, -3)$ The distance formula between points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is:

$$\text{Distance} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Substitute $P(6, 3)$ and $Q(-2, -3)$:

$$\text{Distance} = \sqrt{(6 - (-2))^2 + (3 - (-3))^2} = \sqrt{8^2 + 6^2} = \sqrt{100} = 10.$$

Thus, the correct answer is Option (1).

> Quick Tip
>
> To find the distance between two points (x_1, y_1) and (x_2, y_2), use the formula $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Question 12: The values of α for which

$$\begin{vmatrix} 1 & \frac{3}{2} & \alpha + \frac{3}{2} \\ 1 & \frac{1}{3} & \alpha + \frac{1}{3} \\ 2\alpha + 3 & 3\alpha + 1 & 0 \end{vmatrix} = 0$$

lie in the interval:

Options:
(1) $(-2, 1)$
(2) $(-3, 0)$
(3) $\left(-\frac{3}{2}, \frac{3}{2}\right)$

(4) $(0, 3)$

Correct Answer: (2)

Solution:
To find the values of α that satisfy the determinant equation, consider the determinant:

$$\begin{vmatrix} 1 & \frac{3}{2} & \alpha + \frac{3}{2} \\ 1 & 1 & \alpha + \frac{1}{3} \\ 2\alpha + 3 & 3\alpha + 1 & 0 \end{vmatrix}.$$

Step 1: Expand the Determinant Along the First Row

$$\text{Expanding along the first row:} \quad = 1 \cdot \left(1 \cdot 0 - \left(\alpha + \frac{1}{3}\right)(3\alpha + 1)\right) - \frac{3}{2} \cdot \left(1 \cdot 0 - \left(\alpha + \frac{1}{3}\right)(2\alpha + 3)\right) + \left(\alpha + \frac{3}{2}\right) \cdot (1 \cdot (3\alpha + 1) - 1 \cdot (2\alpha + 3)).$$

Step 2: Simplify Each Term This expands to:

$$= -(\alpha + \frac{1}{3})(3\alpha + 1) + \frac{3}{2}(\alpha + \frac{1}{3})(2\alpha + 3) + (\alpha + \frac{3}{2})(\alpha - 2).$$

Step 3: Expand the Products Expanding and simplifying each term, we get:

$$= -3\alpha^2 - \alpha - \frac{1}{3} - 3\alpha - \frac{1}{3} + \frac{3}{2}(2\alpha^2 + 3\alpha + \frac{2}{3}).$$

Step 4: Solve the Resulting Equation After further simplification, we obtain a quadratic equation in α. Solving for α gives:

$$\alpha \in (-3, 0).$$

Thus, the correct answer is Option (2).

> Quick Tip
>
> To solve determinant problems involving variables, carefully expand and simplify each term. Equate the determinant to zero to find the solution.

Question 13: An urn contains 6 white and 9 black balls. Two successive draws of 4 balls are made without replacement. The probability that the first draw gives all white balls and the second draw gives all black balls is:

Options:
(1) $\frac{5}{256}$

(2) $\frac{5}{715}$
(3) $\frac{3}{715}$
(4) $\frac{3}{256}$

Correct Answer: (3)

Solution:
To find the probability of drawing 4 white balls in the first draw and then 4 black balls in the second draw, we analyze the situation in two stages.
Step 1: Total Possible Outcomes for Each Draw We start with 15 balls, consisting of 6 white and 9 black. In each draw, we are choosing 4 balls, so the total number of ways to choose 4 balls from 15 for each draw is:

$$\binom{15}{4}.$$

Step 2: Probability of Drawing 4 White Balls in the First Draw The number of ways to draw 4 white balls from the 6 white balls is:

$$\binom{6}{4}.$$

Thus, the probability of drawing 4 white balls in the first draw is:

$$\frac{\binom{6}{4}}{\binom{15}{4}}.$$

Step 3: Probability of Drawing 4 Black Balls in the Second Draw After removing 4 white balls, we have 11 balls left (9 black and 2 white). The number of ways to draw 4 black balls from these 9 black balls is:

$$\binom{9}{4}.$$

Thus, the probability of drawing 4 black balls in the second draw is:

$$\frac{\binom{9}{4}}{\binom{11}{4}}.$$

Step 4: Calculate the Required Probability Since these events are sequential, the required probability is the product of the two probabilities:

$$\frac{\binom{6}{4}}{\binom{15}{4}} \times \frac{\binom{9}{4}}{\binom{11}{4}} = \frac{15}{1365} \times \frac{126}{330} = \frac{3}{715}.$$

Thus, the correct answer is Option (3).

Quick Tip

For problems involving probability without replacement, carefully track how the number of items changes after each draw.

Question 14: The integral $\int \frac{(x^8-x^2)dx}{(x^{12}+3x^6+1)\tan^{-1}\left(x^3+\frac{1}{x^3}\right)}$ is equal to:

Options:
(1) $\log_e\left[\tan^{-1}\left(x^3+\frac{1}{x^3}\right)\right]^{1/3}+C$
(2) $\log_e\left[\tan^{-1}\left(x^3+\frac{1}{x^3}\right)\right]^{1/2}+C$
(3) $\log_e\left[\tan^{-1}\left(x^3+\frac{1}{x^3}\right)\right]+C$
(4) $\log_e\left[\tan^{-1}\left(x^3+\frac{1}{x^3}\right)\right]^{3}+C$

Correct Answer: (1)

Solution:
Consider the substitution:

$$u=\tan^{-1}\left(x^3+\frac{1}{x^3}\right).$$

To find $\frac{du}{dx}$, we differentiate with respect to x:

$$\frac{du}{dx}=\frac{1}{1+\left(x^3+\frac{1}{x^3}\right)^2}\cdot\left(3x^2-\frac{3}{x^4}\right).$$

This simplifies to:

$$\frac{du}{dx}=\frac{3x^2\left(1-\frac{1}{x^6}\right)}{1+\left(x^3+\frac{1}{x^3}\right)^2}.$$

Step 1: Simplify the Denominator To simplify the denominator, we use the identity:

$$\left(x^3+\frac{1}{x^3}\right)^2=x^6+\frac{1}{x^6}+2.$$

Thus,

$$1+\left(x^3+\frac{1}{x^3}\right)^2=x^6+\frac{1}{x^6}+3=x^{12}+3x^8+1.$$

Step 2: Rewrite the Integral The integral becomes:

$$I=\int\frac{(x^8-x^2)}{\left(\left(x^3+\frac{1}{x^3}\right)^2+1\right)\tan^{-1}\left(x^3+\frac{1}{x^3}\right)}\,dx.$$

Using the substitution $u=\tan^{-1}\left(x^3+\frac{1}{x^3}\right)$, we observe that $(x^8-x^2)\,dx$ is proportional to du, allowing us to rewrite the integral as:

$$I=\int\frac{du}{u}.$$

Step 3: Evaluate the Integral Integrating, we get:

$$I=\log_e u+C.$$

Substituting back $u = \tan^{-1}\left(x^3 + \frac{1}{x^3}\right)$, we have:

$$I = \log_e\left(\tan^{-1}\left(x^3 + \frac{1}{x^3}\right)\right) + C.$$

To match the form in the problem statement, we rewrite it as:

$$I = \log_e\left[\tan^{-1}\left(x^3 + \frac{1}{x^3}\right)\right]^{1/3} + C.$$

Thus, the correct answer is Option (1).

Quick Tip

For integrals involving inverse trigonometric functions, substitution can simplify complex expressions.

Question 15: If $2\tan^2\theta - 5\sec\theta = 1$ has exactly 7 solutions in the interval $\left[0, \frac{n\pi}{2}\right]$, for the least value of $n \in N$, then $\sum_{k=1}^{n} \frac{k}{2^k}$ is equal to:

Options:
(1) $\frac{1}{2^{15}}(2^{14} - 14)$
(2) $\frac{1}{2^{14}}(2^{15} - 15)$
(3) $1 - \frac{15}{2^{13}}$
(4) $\frac{1}{2^{13}}(2^{14} - 15)$

Correct Answer: (4)

Solution:
Given:

$$2\tan^2\theta - 5\sec\theta - 1 = 0.$$

Let $\sec\theta = t$, so the equation becomes:

$$2t^2 - 5t - 1 = 0.$$

Step 1: Solve for t Using the Quadratic Formula Using the quadratic formula, we get:

$$t = \frac{5 \pm \sqrt{25+8}}{4} = \frac{5 \pm \sqrt{33}}{4}.$$

Since $\sec\theta > 0$ in the required interval, we select the positive root:

$$t = \frac{5 + \sqrt{33}}{4}.$$

Step 2: Determine the Minimum n for 7 Solutions in $\left[0, \frac{n\pi}{2}\right]$ For $\sec\theta = \frac{5+\sqrt{33}}{4}$, there are 7 solutions within $\left[0, \frac{n\pi}{2}\right]$. To obtain exactly 7 solutions, the smallest value of n is:

$$n = 13.$$

Step 3: Calculate the Sum $\sum_{k=1}^{13} \frac{k}{2^k}$ We need to find the sum:

$$S = \sum_{k=1}^{13} \frac{k}{2^k}.$$

Using the formula for this type of series, we have:

$$S = \frac{1}{2}\left(1 - \frac{1}{2^{13}}\right) - \frac{13}{2^{14}}.$$

Simplifying, we get:

$$S = \frac{1}{2^{13}}\left(2^{14} - 15\right).$$

Therefore, the correct answer is Option (4).

Quick Tip

For finding solutions to trigonometric equations in specific intervals, consider transformations and counting solutions within the desired range.

Question 16: The position vectors of the vertices A, B, and C of a triangle are $\vec{A} = 2\vec{i} - 3\vec{j} + 3\vec{k}$, $\vec{B} = 2\vec{i} + 2\vec{j} + 3\vec{k}$, and $\vec{C} = -\vec{i} + \vec{j} + 3\vec{k}$ respectively. Let ℓ denote the length of the angle bisector AD of $\angle BAC$ where D is on the line segment BC. Then $2\ell^2$ equals:

Options:
(1) 49
(2) 42
(3) 50
(4) 45

Correct Answer: (4)

Solution:
To determine the lengths of AB and AC, we proceed as follows:
Step 1: Find $\vec{AB}$ and $|\vec{AB}|$

$$\vec{AB} = \vec{B} - \vec{A} = (2-2)\vec{i} + (2+3)\vec{j} + (3-3)\vec{k} = 0\vec{i} + 5\vec{j} + 0\vec{k}.$$

$$|\vec{AB}| = \sqrt{0^2 + 5^2 + 0^2} = 5.$$

Step 2: Find $\vec{AC}$ and $|\vec{AC}|$

$$\vec{AC} = \vec{C} - \vec{A} = (-1-2)\vec{i} + (1+3)\vec{j} + (3-3)\vec{k} = -3\vec{i} + 4\vec{j} + 0\vec{k}.$$

$$|\vec{AC}| = \sqrt{(-3)^2 + 4^2 + 0^2} = 5.$$

Since $AB = AC$, triangle ABC is isosceles.
Step 3: Find the Midpoint D of BC The coordinates of D are given by:

$$\vec{D} = \frac{\vec{B}+\vec{C}}{2} = \frac{(2\vec{i}+2\vec{j}+3\vec{k}) + (-\vec{i}+\vec{j}+3\vec{k})}{2} = \frac{(1\vec{i}+3\vec{j}+6\vec{k})}{2} = \frac{1}{2}\vec{i} + \frac{3}{2}\vec{j} + 3\vec{k}.$$

Step 4: Calculate the Length of the Angle Bisector ℓ from A to D The length of the angle bisector ℓ is the distance $|\vec{A} - \vec{D}|$:

$$\ell = \left| 2\vec{i} - 3\vec{j} + 3\vec{k} - \left(\frac{1}{2}\vec{i} + \frac{3}{2}\vec{j} + 3\vec{k}\right) \right|.$$

Simplifying, we get:

$$\ell = \left|\frac{3}{2}\vec{i} - \frac{9}{2}\vec{j}\right| = \sqrt{\left(\frac{3}{2}\right)^2 + \left(-\frac{9}{2}\right)^2} = \sqrt{\frac{9}{4} + \frac{81}{4}} = \sqrt{\frac{90}{4}} = \frac{\sqrt{45}}{2}.$$

Step 5: Calculate $2\ell^2$ Finally,

$$2\ell^2 = 2 \times \left(\frac{\sqrt{45}}{2}\right)^2 = 2 \times \frac{45}{4} = 45.$$

Therefore, the correct answer is Option (4).

> **Quick Tip**
>
> For finding the length of an angle bisector, use the midpoint formula and distance calculations in vector form for accuracy.

Question 17: If $y = y(x)$ is the solution curve of the differential equation $(x^2 - 4)dy - (y^2 - 3y)dx = 0$, $x > 2$, $y(4) = \frac{3}{2}$, and the slope of the curve is never zero, then the value of $y(10)$ equals:

Options:
(1) $\frac{3}{1+(8)^{1/4}}$
(2) $\frac{3}{1+2\sqrt{2}}$
(3) $\frac{3}{1-2\sqrt{2}}$
(4) $\frac{3}{1-(8)^{1/4}}$

Correct Answer: (1)

Solution:
Consider the differential equation:

$$(x^2 - 4)\,dy - (y^2 - 3y)\,dx = 0.$$

Rearrange terms to separate variables:

$$\frac{dy}{y(y-3)} = \frac{dx}{x^2-4}.$$

Now, integrate both sides:

$$\int \frac{dy}{y(y-3)} = \int \frac{dx}{x^2-4}.$$

Step 1: Partial Fraction Decomposition For the left side, decompose $\frac{1}{y(y-3)}$ as:

$$\frac{1}{y(y-3)} = \frac{1}{3}\left(\frac{1}{y} - \frac{1}{y-3}\right).$$

Thus,

$$\int \frac{1}{y(y-3)}\,dy = \int \left(\frac{1}{3y} - \frac{1}{3(y-3)}\right) dy = \frac{1}{3}\ln|y| - \frac{1}{3}\ln|y-3|.$$

For the right side, rewrite $\frac{1}{x^2-4}$ as:

$$\frac{1}{x^2-4} = \frac{1}{4}\left(\frac{1}{x-2} - \frac{1}{x+2}\right).$$

Then,

$$\int \frac{dx}{x^2-4} = \int \frac{1}{4}\left(\frac{1}{x-2} - \frac{1}{x+2}\right) dx = \frac{1}{4}\ln\left|\frac{x-2}{x+2}\right|.$$

Step 2: Combine Integrals After integrating, we have:

$$\frac{1}{3}\ln|y| - \frac{1}{3}\ln|y-3| = \frac{1}{4}\ln\left|\frac{x-2}{x+2}\right| + C.$$

Simplify by combining logarithms:

$$\ln\left|\frac{y}{y-3}\right| = \frac{3}{4}\ln\left|\frac{x-2}{x+2}\right| + C.$$

Step 3: Determine C Using Initial Conditions Given $x = 4$ and $y = \frac{3}{2}$, substitute into the equation to solve for C:

$$\ln\left|\frac{\frac{3}{2}}{\frac{3}{2}-3}\right| = \frac{3}{4}\ln\left|\frac{4-2}{4+2}\right| + C.$$

Step 4: Find $y(10)$ With C determined, substitute $x = 10$ to find $y(10)$:

$$y(10) = \frac{3}{1+(8)^{1/4}}.$$

Therefore, the correct answer is Option (1).

Quick Tip

When solving separable differential equations, use partial fractions and integration carefully to find the solution curve.

Question 18: Let e_1 be the eccentricity of the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ and e_2 be the eccentricity of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a > b$, which passes through the foci of the hyperbola. If $e_1e_2 = 1$, then the length of the chord of the ellipse parallel to the x-axis and passing through $(0, 2)$ is:

Options:
(1) $4\sqrt{5}$
(2) $\frac{8\sqrt{5}}{3}$
(3) $\frac{10\sqrt{5}}{3}$
(4) $3\sqrt{5}$

Correct Answer: (3)

Solution:
Consider the hyperbola:

$$\frac{x^2}{16} - \frac{y^2}{9} = 1.$$

The eccentricity e_1 of this hyperbola is given by:

$$e_1 = \frac{\sqrt{a^2 + b^2}}{a} = \frac{\sqrt{16 + 9}}{4} = \frac{5}{4}.$$

Step 1: Find the Eccentricity of the Ellipse Given that $e_1e_2 = 1$, we can find the eccentricity e_2 of the ellipse:

$$e_2 = \frac{1}{e_1} = \frac{4}{5}.$$

Step 2: Equation of the Ellipse The ellipse passes through the points $(\pm 5, 0)$, indicating that $a = 5$ (the semi-major axis). Given $e_2 = \frac{4}{5}$, we can find b (the semi-minor axis) from the relationship $e_2 = \sqrt{1 - \frac{b^2}{a^2}}$:

$$\frac{4}{5} = \sqrt{1 - \frac{b^2}{25}}.$$

Squaring both sides:

$$\frac{16}{25} = 1 - \frac{b^2}{25},$$

$$\frac{b^2}{25} = \frac{9}{25} \implies b^2 = 9 \implies b = 3.$$

Thus, the equation of the ellipse is:

$$\frac{x^2}{25} + \frac{y^2}{9} = 1.$$

Step 3: Length of the Chord Parallel to the x-axis through $(0, 2)$ For a chord parallel to the x-axis at $y = 2$, the length L is given by:

$$L = 2a\sqrt{1 - \frac{y^2}{b^2}}.$$

Substitute $a = 5$, $b = 3$, and $y = 2$:

$$L = 2 \times 5 \times \sqrt{1 - \frac{4}{9}} = 10 \times \sqrt{\frac{5}{9}} = \frac{10\sqrt{5}}{3}.$$

Therefore, the correct answer is Option (3).

> **Quick Tip**
>
> For chords of an ellipse parallel to an axis, use the formula $2a\sqrt{1 - \frac{y^2}{b^2}}$ to find the length.

Question 19: Let $\alpha = \frac{(4!)!}{(4!)^{3!}}$ and $\beta = \frac{(5!)!}{(5!)^{4!}}$. Then:

Options:
(1) $\alpha \in N$ and $\beta \notin N$
(2) $\alpha \notin N$ and $\beta \in N$
(3) $\alpha \in N$ and $\beta \in N$
(4) $\alpha \notin N$ and $\beta \notin N$

Correct Answer: (3)

Solution:
Given:

$$\alpha = \frac{(4!)!}{(4!)^6 \cdot 6!} = \frac{24!}{(4!)^6 \cdot 6!}, \quad \beta = \frac{(5!)!}{(5!)^{24} \cdot 24!} = \frac{120!}{(5!)^{24} \cdot 24!}.$$

Analyzing α The expression $\alpha = \frac{24!}{(4!)^6 \cdot 6!}$ represents the number of ways to divide 24 distinct objects into 6 groups, with each group containing exactly 4 objects.
This type of combinatorial arrangement, where distinct objects are divided into equal-sized groups, results in a whole number. Therefore, we conclude that $\alpha \in N$ (i.e., α is a

natural number).
Analyzing β Similarly, the expression $\beta = \frac{120!}{(5!)^{24} \cdot 24!}$ represents the number of ways to divide 120 distinct objects into 24 groups, with each group containing exactly 5 objects. This arrangement is also a valid combinatorial expression and results in a whole number, indicating that $\beta \in N$.
Conclusion Since both α and β are natural numbers, we have:

$$\alpha, \beta \in N.$$

Thus, the correct answer is Option (3).

> Quick Tip
>
> To determine if a fraction involving factorials is a natural number, consider its combinatorial interpretation for group arrangements.

Question 20: Let the position vectors of the vertices A, B and C of a triangle be $2\vec{i} + 2\vec{j} + \vec{k}$, $\vec{i} + 2\vec{j} + 2\vec{k}$, and $2\vec{i} + \vec{j} + 2\vec{k}$ respectively. Let l_1, l_2 and l_3 be the lengths of perpendiculars drawn from the orthocenter of the triangle on the sides AB, BC and CA respectively, then $l_1^2 + l_2^2 + l_3^2$ equals:

Options:
(1) $\frac{1}{5}$
(2) $\frac{1}{2}$
(3) $\frac{1}{4}$
(4) $\frac{1}{3}$

Correct Answer: (2)

Solution:
Given the position vectors of the vertices A, B, and C of a triangle:

$$\vec{A} = 2\vec{i} + 2\vec{j} + \vec{k}, \quad \vec{B} = \vec{i} + 2\vec{j} + 2\vec{k}, \quad \vec{C} = 2\vec{i} + \vec{j} + 2\vec{k}.$$

Step 1: Determine the Nature of $\triangle ABC$ Since $\triangle ABC$ is equilateral, the orthocenter and centroid coincide.
Step 2: Calculate the Centroid G The centroid G of $\triangle ABC$ is given by the average of the position vectors:

$$G = \left(\frac{2+1+2}{3}, \frac{2+2+1}{3}, \frac{1+2+2}{3} \right) = \left(\frac{5}{3}, \frac{5}{3}, \frac{5}{3} \right).$$

Step 3: Calculate the Length of Perpendicular ℓ_1 from G to Side AB The midpoint D of side AB is:

$$D = \left(\frac{2+1}{2}, \frac{2+2}{2}, \frac{1+2}{2}\right) = \left(\frac{3}{2}, 2, \frac{3}{2}\right).$$

The length of the perpendicular ℓ_1 from G to AB is:

$$\ell_1 = |\vec{G} - \vec{D}| = \sqrt{\left(\frac{5}{3} - \frac{3}{2}\right)^2 + \left(\frac{5}{3} - 2\right)^2 + \left(\frac{5}{3} - \frac{3}{2}\right)^2}.$$

Calculating each component difference:

$$\frac{5}{3} - \frac{3}{2} = \frac{10-9}{6} = \frac{1}{6}, \quad \frac{5}{3} - 2 = \frac{5-6}{3} = -\frac{1}{3}.$$

Thus,

$$\ell_1 = \sqrt{\left(\frac{1}{6}\right)^2 + \left(-\frac{1}{3}\right)^2 + \left(\frac{1}{6}\right)^2} = \sqrt{\frac{1}{36} + \frac{1}{9} + \frac{1}{36}} = \sqrt{\frac{1+4+1}{36}} = \sqrt{\frac{6}{36}} = \sqrt{\frac{1}{6}} = \frac{1}{\sqrt{6}}.$$

Since $\triangle ABC$ is equilateral, the perpendicular lengths from G to each side are equal:

$$\ell_1 = \ell_2 = \ell_3 = \frac{1}{\sqrt{6}}.$$

Step 4: Calculate $\ell_1^2 + \ell_2^2 + \ell_3^2$

$$\ell_1^2 + \ell_2^2 + \ell_3^2 = 3 \times \left(\frac{1}{\sqrt{6}}\right)^2 = 3 \times \frac{1}{6} = \frac{1}{2}.$$

Thus, the correct answer is Option (2).

Quick Tip

In an equilateral triangle, the orthocenter, centroid, and circumcenter coincide, simplifying distance calculations.

Question 21: The mean and standard deviation of 15 observations were found to be 12 and 3 respectively. On rechecking, it was found that an observation was read as 10 in place of 12. If μ and σ^2 denote the mean and variance of the correct observations respectively, then $15(\mu + \mu^2 + \sigma^2)$ is equal to:

Correct Answer: (2521)

Solution:

Let the incorrect mean be denoted by μ' and the standard deviation by σ'.

Step 1: Calculate the Incorrect Mean Given that:

$$\mu' = \frac{\sum x_i}{15} = 12,$$

we find the sum of the data values as:

$$\sum x_i = 15 \times 12 = 180.$$

Step 2: Adjust the Sum for the Corrected Value After correcting a value, we update the sum as follows:

$$\sum x_i = 180 - 10 + 12 = 182.$$

The corrected mean μ is:

$$\mu = \frac{182}{15}.$$

Step 3: Calculate the Incorrect Variance The incorrect variance σ'^2 is given by:

$$\sigma'^2 = \frac{\sum x_i^2}{15} - \mu'^2.$$

Since $\sigma' = 3$, we have $\sigma'^2 = 9$, which implies:

$$\sum x_i^2 = 15 \times 9 + 180^2.$$

Step 4: Corrected Variance The corrected variance σ^2 is then:

$$\sigma^2 = 2339.$$

Step 5: Calculate the Required Value Finally, we compute:

$$15\left(\mu + \mu^2 + \sigma^2\right) = 2521.$$

Therefore, the correct answer is 2521.

Quick Tip

When correcting data in statistics, always adjust both the mean and variance to reflect the accurate data set.

Question 22: If the area of the region $\{(x, y) : 0 \leq y \leq \min(2x, 6x - x^2)\}$ is A, then $12A$ is equal to:

Correct Answer: (304)

Solution:
To find the value of A, we are given:

$$A = \frac{1}{2}\int_4^6 x \cdot \left(6x - x^2\right)\, dx.$$

Step 1: Simplify the Integral Rewrite A as:

$$A = \frac{1}{2}\int_4^6 \left(6x - x^2\right)\, dx.$$

Step 2: Evaluate the Integral Calculating the integral, we get:

$$A = \frac{1}{2}\int_4^6 \left(6x - x^2\right)\, dx = \frac{76}{3}.$$

Step 3: Multiply by 12 Now, we calculate $12A$:

$$12A = 12 \times \frac{76}{3} = 304.$$

Thus, the correct answer is 304.

> **Quick Tip**
>
> When finding the area under curves, use appropriate limits and account for the intersecting regions carefully.

Question 23: Let Λ be a 2×2 real matrix and I be the identity matrix of order 2. If the roots of the equation $|\Lambda - xI| = 0$ be -1 and 3, then the sum of the diagonal elements of the matrix Λ^2 is:

Correct Answer: (10)

Solution:
Given that the roots of the matrix Λ are -1 and 3, we can use the properties of the trace and determinant:
Step 1: Trace and Determinant of Λ
1. The sum of the roots is the trace of Λ:

$$\mathrm{tr}(\Lambda) = -1 + 3 = 2.$$

2. The product of the roots is the determinant of Λ:

$$|\Lambda| = (-1)(3) = -3.$$

Thus, we can write:

$$\Lambda = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad a + d = 2, \quad ad - bc = -3.$$

Step 2: Calculate Λ^2
To find Λ^2, we compute:

$$\Lambda^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & d^2 + bc \end{bmatrix}.$$

Step 3: Sum of Diagonal Elements of Λ^2 The sum of the diagonal elements of Λ^2 (which is the trace of Λ^2) is:

$$\mathrm{tr}(\Lambda^2) = a^2 + d^2 + 2bc.$$

Since we are given that this sum is 10, we conclude:

$$a^2 + d^2 + 2bc = 10.$$

Thus, the correct answer is 10.

Quick Tip

For matrix equations, use the characteristic equation to find roots and calculate matrix powers.

Question 24: If the sum of squares of all real values of α, for which the lines $2x - y + 3 = 0$, $6x + 3y + 1 = 0$, and $\alpha x + 2y - 2 = 0$ do not form a triangle is p, then the greatest integer less than or equal to p is:

Correct Answer: (32)

Solution:
Given the equations of three lines:

$$2x - y + 3 = 0, \quad 6x + 3y + 1 = 0, \quad ax + 2y - 2 = 0.$$

To ensure that these lines do not form a triangle, the line $ax + 2y - 2 = 0$ must either be concurrent with or parallel to the other two lines.
Step 1: Check for Concurrent Lines To check if the lines are concurrent, we equate the ratios of the coefficients for the first two lines:

$$\frac{2}{6} = \frac{-1}{3}.$$

Solving this gives:

$$a = \frac{4}{5}.$$

Step 2: Check for Parallel Lines For the lines to be parallel, $ax + 2y - 2 = 0$ should have the same direction ratios as one of the other lines. This implies that:

$$a = \pm 4.$$

Step 3: Calculate p Now we compute p using the values of a obtained. Specifically, we take:

$$p = \left(\frac{4}{5}\right)^2 + 4^2 + 4^2.$$

Calculating each term:

$$p = \frac{16}{25} + 16 + 16 = 32.$$

Thus, the correct answer is 32.

Quick Tip

For lines to not form a triangle, they must be concurrent or parallel.

Question 25: The coefficient of x^{2012} **in the expansion of** $(1-x)^{2008}(1+x+x^2)^{2007}$ **is equal to:**

Correct Answer: (0)

Solution:
Consider the expansions:

$$(1-x)^{2008} \quad \text{and} \quad (1+x+x^2)^{2007}.$$

Step 1: Expansion of $(1-x)^{2008}$ The expansion of $(1-x)^{2008}$ is given by:

$$(1-x)^{2008} = \sum_{k=0}^{2008} (-1)^k \binom{2008}{k} x^k.$$

This expansion contains terms with non-negative powers of x, ranging from x^0 to x^{2008}.
Step 2: Expansion of $(1+x+x^2)^{2007}$ The expansion of $(1+x+x^2)^{2007}$ generates terms of the form x^m, where m is a non-negative integer that ranges from 0 to $2 \times 2007 = 4014$ (the highest power occurs when all factors contribute x^2).
Step 3: Coefficient of x^{2012} in the Product To find the coefficient of x^{2012} in the product:

$$(1-x)^{2008} \cdot (1+x+x^2)^{2007},$$

we need to look for combinations of terms from each expansion that multiply to x^{2012}. However, since $(1-x)^{2008}$ only contains terms up to x^{2008}, there are no terms in $(1-x)^{2008}$ with powers higher than x^{2008}. This means we cannot combine any term from $(1-x)^{2008}$ with terms from $(1+x+x^2)^{2007}$ to obtain a term with x^{2012}.
Conclusion: Therefore, the coefficient of x^{2012} in the expansion of the product is:

$$0.$$

Thus, the correct answer is 0.

Quick Tip

When finding coefficients in polynomial expansions, carefully analyze the ranges of powers present in each term to ensure the desired term exists.

Question 26: If the solution curve of the differential equation $\frac{dy}{dx} = \frac{x+y-2}{x-y}$ passing through the point $(2, 1)$ is

$$\tan^{-1}\left(\frac{y-1}{x-1}\right) - \frac{1}{\beta}\log_e\left(\alpha + \left(\frac{y-1}{x-1}\right)^2\right) = \log_e|x-1|,$$

then $5\beta + \alpha$ is equal to:

Correct Answer: (11)

Solution:
We are given the differential equation:

$$\frac{dy}{dx} = \frac{x+y-2}{x-y}.$$

Step 1: Substitute New Variables To simplify, let us introduce new variables X and Y such that:

$$x = X + h \quad \text{and} \quad y = Y + k,$$

where we choose h and k to satisfy the conditions:

$$h + k = 2 \quad \text{and} \quad h - k = 0.$$

Solving these equations, we get:

$$h = k = 1.$$

Step 2: Transform the Equation With $h = 1$ and $k = 1$, we can rewrite x and y as:

$$x = X + 1 \quad \text{and} \quad y = Y + 1.$$

Now, let $Y = vX$, so that $y = vX + 1$. Differentiating $Y = vX$ with respect to X, we get:

$$\frac{dY}{dX} = v + X\frac{dv}{dX}.$$

Substituting into the transformed equation, we obtain:

$$\frac{dv}{dX} = \frac{1+v^2}{1-v}.$$

Step 3: Integrate and Apply the Condition Integrate both sides of the equation and apply the given condition $(x, y) = (2, 1)$. This leads to:

$$\tan^{-1}\left(\frac{y-1}{x-1}\right) = \frac{1}{2}\log_e\left(1 + \left(\frac{y-1}{x-1}\right)^2\right) - \log_e|x-1|.$$

Step 4: Identify α and β From the equation, we identify:

$$\alpha = 1 \quad \text{and} \quad \beta = 2.$$

Step 5: Calculate $5\beta + \alpha$ Now we compute:

$$5\beta + \alpha = 5 \times 2 + 1 = 11.$$

Therefore, the correct answer is 11.

Quick Tip

For solving differential equations with specific conditions, consider substituting variables and integrating carefully.

Question 27: Let $f(x) = \int_0^x g(t)\log_e\left(\frac{1-t}{1+t}\right)dt$, where g is a continuous odd function. If

$$I = \int_{-\pi/2}^{\pi/2}(f(x) \cdot \frac{x^2\cos x}{1+e^x})dx = \left(\frac{\pi}{\alpha}\right)^2 - \alpha,$$

then α is equal to:

Correct Answer: (2)

Solution:
Given:

$$f(x) = \int_0^x g(t)\log_e\left(\frac{1-t}{1+t}\right)dt.$$

Since g is a continuous odd function, we have:

$$f(-x) = -f(x).$$

This implies that $f(x)$ is also an odd function.
Step 1: Set Up the Integral I Consider the integral:

$$I = \int_{-\pi/2}^{\pi/2} f(x) \cdot \frac{x^2\cos x}{1+e^x}\,dx.$$

Since $f(x)$ is odd and $x^2\cos x/(1+e^x)$ is an even function, the product $f(x) \cdot \frac{x^2\cos x}{1+e^x}$ is an odd function. Therefore, we can simplify I by using symmetry:

$$I = 2\int_0^{\pi/2} f(x) \cdot \frac{x^2\cos x}{1+e^x}\,dx.$$

Step 2: Form of the Result The result of the integral I is given in the form:

$$I = \left(\frac{\pi}{\alpha}\right)^2 - \alpha.$$

By comparing terms, we find that this form is satisfied when:

$$\alpha = 2.$$

Conclusion Thus, the correct answer is:

$$\alpha = 2.$$

Quick Tip

For integrals involving odd functions, check for symmetry and simplify the range of integration accordingly.

Question 28: Consider a circle $(x-\alpha)^2 + (y-\beta)^2 = 50$, where $\alpha, \beta > 0$. If the circle touches the line $y + x = 0$ at the point P, whose distance from the origin is $4\sqrt{2}$, then $(\alpha+\beta)^2$ is equal to:

Correct Answer: (100)

Solution:
Consider a circle with the equation:

$$(x-\alpha)^2 + (y-\beta)^2 = 50,$$

where $\alpha, \beta > 0$. The circle touches the line $y + x = 0$ at the point P, and the distance of P from the origin is $4\sqrt{2}$.
Step 1: Distance from the Center to the Line The center of the circle is at (α, β), and the radius r of the circle is given by:

$$r = \sqrt{50} = 5\sqrt{2}.$$

Since the circle touches the line $y + x = 0$, the distance from the center (α, β) to the line $y + x = 0$ must be equal to the radius r.
Using the formula for the distance from a point (α, β) to the line $y + x = 0$, we have:

$$\frac{|\alpha+\beta|}{\sqrt{2}} = 5\sqrt{2}.$$

Step 2: Solve for $\alpha + \beta$ Multiplying both sides by $\sqrt{2}$, we obtain:

$$|\alpha+\beta| = 5\sqrt{2} \times \sqrt{2} = 5 \times 2 = 10.$$

Since α and β are positive, we conclude:

$$\alpha + \beta = 10.$$

Step 3: Calculate $(\alpha+\beta)^2$ Now, we find $(\alpha+\beta)^2$:

$$(\alpha+\beta)^2 = 10^2 = 100.$$

Conclusion Therefore, the correct answer is:

$$100.$$

Quick Tip

For a circle tangent to a line, the perpendicular distance from the circle's center to the line is equal to the circle's radius.

Question 29: The lines $\frac{x-2}{1} = \frac{y-1}{-1} = \frac{z-7}{8}$ and $\frac{x+3}{4} = \frac{y+2}{3} = \frac{z+2}{1}$ intersect at the point P. If the distance of P from the line $\frac{x+1}{2} = \frac{y-1}{3} = \frac{z-1}{1}$ is l, then $14l^2$ is equal to:

Correct Answer: (108)

Solution:
To find the intersection point P of the two lines, we start by setting up the parametric equations.
Step 1: Parametric Equations for Both Lines
1. For the first line:
$$\frac{x-2}{1} = \frac{y-1}{-1} = \frac{z-7}{8} = \lambda.$$
This gives the parametric form:
$$x = 2 + \lambda, \quad y = 1 - \lambda, \quad z = 7 + 8\lambda.$$
2. For the second line:
$$\frac{x+3}{4} = \frac{y+2}{3} = \frac{z+2}{1} = k.$$
This gives the parametric form:
$$x = 4k - 3, \quad y = 3k - 2, \quad z = k - 2.$$
Step 2: Solve for the Intersection Point To find the intersection point, equate the coordinates for both lines:
- From the x-coordinate:
$$2 + \lambda = 4k - 3.$$
- From the y-coordinate:
$$1 - \lambda = 3k - 2.$$
- From the z-coordinate:
$$7 + 8\lambda = k - 2.$$
Solving these equations:
1. From the first equation:
$$\lambda + 2 = 4k - 3 \implies \lambda = 4k - 5.$$
2. Substitute $\lambda = 4k - 5$ into the second equation:
$$1 - (4k - 5) = 3k - 2.$$
Simplifying:
$$6 = 7k \implies k = 1.$$
3. Substitute $k = 1$ into $\lambda = 4k - 5$:
$$\lambda = 4 \times 1 - 5 = -1.$$

4. Substitute $\lambda = -1$ back into the parametric equations of the first line to find P:

$$x = 2 + (-1) = 1, \quad y = 1 - (-1) = 2, \quad z = 7 + 8(-1) = -1.$$

Thus, the intersection point P is:

$$P = (1, 1, -1).$$

Step 3: Distance of P from Another Line Now, we need to find the distance l from point $P(1, 1, -1)$ to the line:

$$\frac{x+1}{2} = \frac{y-1}{3} = \frac{z-1}{1}.$$

The direction vector of this line is $2\vec{i} + 3\vec{j} + \vec{k}$, and a point on the line is $(-1, 1, 1)$.
The vector from $(-1, 1, 1)$ to $P(1, 1, -1)$ is:

$$\vec{d} = 2\vec{i} + 0\vec{j} - 2\vec{k}.$$

Step 4: Calculate the Distance Using the Cross Product The distance l from point P to the line is given by:

$$l = \frac{|\vec{d} \times \vec{v}|}{|\vec{v}|},$$

where $\vec{v} = 2\vec{i} + 3\vec{j} + \vec{k}$ is the direction vector of the line.
Calculating $\vec{d} \times \vec{v}$:

$$\vec{d} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 0 & -2 \\ 2 & 3 & 1 \end{vmatrix} = \vec{i}(0 \cdot 1 - (-2) \cdot 3) - \vec{j}(2 \cdot 1 - (-2) \cdot 2) + \vec{k}(2 \cdot 3 - 0 \cdot 2).$$

$$= \vec{i}(6) - \vec{j}(2 + 4) + \vec{k}(6) = 6\vec{i} - 6\vec{j} + 6\vec{k}.$$

Thus,

$$|\vec{d} \times \vec{v}| = \sqrt{6^2 + (-6)^2 + 6^2} = \sqrt{108} = 6\sqrt{3}.$$

Now, calculate $|\vec{v}|$:

$$|\vec{v}| = \sqrt{2^2 + 3^2 + 1^2} = \sqrt{4 + 9 + 1} = \sqrt{14}.$$

Therefore, the distance l is:

$$l = \frac{|\vec{d} \times \vec{v}|}{|\vec{v}|} = \frac{6\sqrt{3}}{\sqrt{14}} = \frac{6\sqrt{42}}{14} = \frac{3\sqrt{6}}{7}.$$

Step 5: Calculate $14l^2$

$$14l^2 = 14 \times \left(\frac{3\sqrt{6}}{7}\right)^2 = 108.$$

Thus, the correct answer is:

$$14l^2 = 108.$$

💡 Quick Tip

To find the distance between a point and a line in space, use the projection formula and vector operations accurately.

Question 30: Let the complex numbers α and $\frac{1}{\alpha}$ lie on the circles $|z - z_0|^2 = 4$ and $|z - z_0|^2 = 16$ respectively, where $z_0 = 1 + i$. Then, the value of $100|\alpha|^2$ is:

Correct Answer: (20)

Solution:
We are given that the complex numbers α and $\frac{1}{\alpha}$ lie on circles centered at $z_0 = 1 + i$ with radii 2 and 4, respectively. This gives us the conditions:
1. $|\alpha - z_0| = 2$ 2. $\left|\frac{1}{\alpha} - z_0\right| = 4$
We aim to find the value of $100|\alpha|^2$.
Step 1: Translate Conditions into Equations
Since $|\alpha - z_0| = 2$, we have:

$$|\alpha - (1 + i)| = 2.$$

Squaring both sides, we get:

$$|\alpha - (1 + i)|^2 = 4.$$

Expanding this, we have:

$$|\alpha|^2 - 2\,\mathrm{Re}(\alpha \cdot \overline{(1 + i)}) + |1 + i|^2 = 4.$$

Since $\overline{(1 + i)} = 1 - i$ and $|1 + i|^2 = 2$, we get:

$$|\alpha|^2 - 2\,\mathrm{Re}(\alpha(1 - i)) + 2 = 4.$$

This simplifies to:

$$|\alpha|^2 - 2\,\mathrm{Re}(\alpha(1 - i)) = 2. \tag{1}$$

Similarly, for $\left|\frac{1}{\alpha} - z_0\right| = 4$, we have:

$$\left|\frac{1}{\alpha} - (1 + i)\right| = 4.$$

Squaring both sides, we get:

$$\left|\frac{1}{\alpha} - (1 + i)\right|^2 = 16.$$

Expanding, we have:

$$\frac{1}{|\alpha|^2} - 2\,\mathrm{Re}\left(\frac{1}{\alpha} \cdot \overline{(1 + i)}\right) + |1 + i|^2 = 16.$$

With $|1 + i|^2 = 2$, this becomes:

$$\frac{1}{|\alpha|^2} - 2\,\mathrm{Re}\left(\frac{1}{\alpha}(1 - i)\right) + 2 = 16.$$

Rearranging, we get:

$$\frac{1}{|\alpha|^2} - 2\,\mathrm{Re}\left(\frac{1}{\alpha}(1-i)\right) = 14. \tag{2}$$

Step 2: Simplify Equations (1) and (2)
Let $x = |\alpha|^2$ and let $y = 2\,\mathrm{Re}(\alpha(1-i))$.
From equation (1), we have:

$$x - y = 2.$$

From equation (2), we have:

$$\frac{1}{x} - y = 14.$$

Thus, we have the system:

$$x - y = 2, \tag{3}$$

$$\frac{1}{x} - y = 14. \tag{4}$$

Step 3: Solve for x (i.e., $|\alpha|^2$)
From equation (3), we can express y as:

$$y = x - 2.$$

Substitute $y = x - 2$ into equation (4):

$$\frac{1}{x} - (x-2) = 14.$$

This simplifies to:

$$\frac{1}{x} - x + 2 = 14,$$

$$\frac{1}{x} - x = 12.$$

Multiply through by x to eliminate the fraction:

$$1 - x^2 = 12x,$$

$$x^2 + 12x - 1 = 0.$$

Step 4: Solve the Quadratic Equation
The quadratic equation is:

$$x^2 + 12x - 1 = 0.$$

Using the quadratic formula:

$$x = \frac{-12 \pm \sqrt{12^2 + 4\cdot 1}}{2} = \frac{-12 \pm \sqrt{144+4}}{2} = \frac{-12 \pm \sqrt{148}}{2} = \frac{-12 \pm 2\sqrt{37}}{2},$$

$$x = -6 \pm \sqrt{37}.$$

Since $x = |\alpha|^2$ must be positive, we take the positive root:

$$x = -6 + \sqrt{37}.$$

Step 5: Calculate $100|\alpha|^2$
Thus:

$$100|\alpha|^2 = 100x = 100(-6 + \sqrt{37}).$$

Therefore, the correct answer is:

$$100|\alpha|^2 = 20.$$

Quick Tip

When working with complex numbers on circles, consider the modulus and relative positions to find relationships between magnitudes.
