ZOLLEGE

**April 8 2024, Shift 1**

**JEE Main 2024 Question Paper with Solutions**

| **Total Time Allowed : ** 180 minutes | **Maximum Marks :** 300 | **Total Questions :** 90 | **Questions to be answered :** 90 |
|---|---|---|---|

## Mathematics

**Question 1.**
The value of $k \in N$ for which the integral

$$I_n = \int_0^1 (1 - x^k)^n dx, \, n \in N,$$

satisfies $147\, I_{20} = 148\, I_{21}$ is:
**Options:**
(1) 10
(2) 8
(3) 14
(4) 7

**Answer: (4)**

**Solution:**

$$I_n = \int_0^1 (1 - x^k)^n \cdot 1 \, dx$$

Using integration by parts:

$$I_n = (1 - x^k)^n \cdot x\Big|_0^1 - nk \int_0^1 (1 - x^k)^{n-1} \cdot x^{k-1} \, dx$$

$$I_n = nk \int_0^1 \left[ (1 - x^k)^n - (1 - x^k)^{n-1} \right] \, dx$$

$$I_n = nkI_{n-1} - nkI_n$$

Rearranging terms:

$$\frac{I_n}{I_{n-1}} = \frac{nk}{nk + 1}$$

Given:

$$\frac{I_{21}}{I_{20}} = \frac{21k}{1 + 21k}$$

Equating:

$$\frac{147}{148} = \frac{21k}{1+21k}$$

Solve for $k$:

$$k = 7$$

Quick Tip

Use reduction formulas for definite integrals to simplify problems with iterative relationships. Pay close attention to recursive expressions for $I_n$ and their ratios.

**Question 2.**
The sum of all the solutions of the equation:

$$(8)^{2x} - 16 \cdot (8)^x + 48 = 0$$

is:
**Options:**
(1) $1 + \log_6(8)$
(2) $\log_6(6)$
(3) $1 + \log_8(6)$
(4) $\log_8(4)$

**Answer: (3)**

**Solution:**

1. Substitution: Let $t = (8)^x$. The equation becomes:

$$t^2 - 16t + 48 = 0.$$

2. Solve the Quadratic Equation: Using the quadratic formula:

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

where $a = 1, b = -16, c = 48$:

$$t = \frac{16 \pm \sqrt{16^2 - 4(1)(48)}}{2}.$$

$$t = \frac{16 \pm \sqrt{256 - 192}}{2}.$$

$$t = \frac{16 \pm \sqrt{64}}{2}.$$

$$t = \frac{16 \pm 8}{2}.$$

$$t = 12 \quad \text{or} \quad t = 4.$$

3. Back Substitution: Recall $t = (8)^x$: - For $t = 4$:

$$(8)^x = 4 \implies x = \log_8(4).$$

- For $t = 12$:

$$(8)^x = 12 \implies x = \log_8(12).$$

4. Sum of Solutions: The sum of solutions is:

$$\log_8(4) + \log_8(12).$$

Using the logarithmic property $\log_b(m) + \log_b(n) = \log_b(mn)$:

$$\log_8(4) + \log_8(12) = \log_8(4 \cdot 12) = \log_8(48).$$

Simplify:

$$\log_8(48) = \log_8(8 \cdot 6) = \log_8(8) + \log_8(6).$$

$$\log_8(8) = 1, \quad \text{so the sum is: } 1 + \log_8(6).$$

**Correct Answer: (3)**

Quick Tip

For equations involving exponential terms, use substitution to simplify. Apply logarithmic properties for solving and summing solutions effectively.

**Question 3.**

Let the circles $C_1 : (x-\alpha)^2 + (y-\beta)^2 = r_1^2$ and

$$C_2 : (x-8)^2 + \left(y - \frac{15}{2}\right)^2 = r_2^2$$

touch each other externally at the point $(6, 6)$. If the point $(6, 6)$ divides the line segment joining the centers of circles $C_1$ and $C_2$ internally in the ratio $2 : 1$, then:

$$\alpha + \beta + 4(r_1^2 + r_2^2)$$

equals:

**Options:**

(1) 110

(2) 130

(3) 125

(4) 145

**Answer: (2)**

**Solution:**

1. Determine the Coordinates of the Center of $C_1$: - The point $(6,6)$ divides the line joining $(\alpha,\beta)$ and $(8,\frac{15}{2})$ in the ratio $2:1$. - Using the section formula:

$$6=\frac{2\cdot 8+1\cdot\alpha}{3},\quad 6=\frac{2\cdot\frac{15}{2}+1\cdot\beta}{3}.$$

Solving for $\alpha$ and $\beta$:

$$16+\alpha=18\implies\alpha=2,\quad 15+\beta=18\implies\beta=3.$$

Thus, the center of $C_1$ is $(\alpha,\beta)=(2,3)$.
2. Use the Touching Condition: - The distance between the centers of $C_1$ and $C_2$ equals the sum of the radii:

$$\sqrt{(2-8)^2+\left(3-\frac{15}{2}\right)^2}=r_1+r_2.$$

Simplify:

$$\sqrt{(-6)^2+\left(-\frac{9}{2}\right)^2}=r_1+r_2.$$

$$\sqrt{36+\frac{81}{4}}=r_1+r_2.$$

$$\sqrt{\frac{144+81}{4}}=r_1+r_2.$$

$$\sqrt{\frac{225}{4}}=\frac{15}{2}.$$

Thus:

$$r_1+r_2=\frac{15}{2}.$$

3. Determine $r_1$ and $r_2$: - From the diagram and external touching condition:

$$2r_2=r_1=5.$$

Thus:

$$r_2=\frac{5}{2},\quad r_1=5.$$

4. Calculate $\alpha+\beta+4(r_1^2+r_2^2)$: - Compute $r_1^2+r_2^2$:

$$r_1^2=5^2=25,\quad r_2^2=\left(\frac{5}{2}\right)^2=\frac{25}{4}.$$

$$r_1^2+r_2^2=25+\frac{25}{4}=\frac{100}{4}+\frac{25}{4}=\frac{125}{4}.$$

- Now calculate:

$$\alpha+\beta+4(r_1^2+r_2^2)=2+3+4\cdot\frac{125}{4}.$$

Simplify:

$$\alpha+\beta+4(r_1^2+r_2^2)=5+125=130.$$

**Correct Answer: (2)**

Quick Tip

When circles touch externally, the distance between their centers equals the sum of their radii. Use section formulas and the given conditions carefully to solve such problems.

**Question 4.**
Let $P(x, y, z)$ be a point in the first octant, whose projection in the $xy$-plane is the point $Q$. Let $OP = \gamma$, the angle between $OQ$ and the positive $x$-axis be $\theta$, and the angle between $OP$ and the positive $z$-axis be $\phi$, where $O$ is the origin. Then the distance of $P$ from the $x$-axis is:
**Options:**
(1) $\gamma\sqrt{1 - \sin^2\phi\cos^2\theta}$
(2) $\gamma\sqrt{1 + \cos^2\phi\sin^2\theta}$
(3) $\gamma\sqrt{1 - \sin^2\theta\cos^2\phi}$
(4) $\gamma\sqrt{1 + \cos^2\phi\sin^2\theta}$

**Answer: (1)**

**Solution:**

1. Coordinates of Points: - $P(x, y, z)$ is a point in 3D space. - $Q(x, y, 0)$ is the projection of $P$ in the $xy$-plane.
2. Distance Relations: - The distance $OP = \gamma$, so:

$$x^2 + y^2 + z^2 = \gamma^2.$$

- The distance $OQ$ is the projection in the $xy$-plane:

$$OQ = \sqrt{x^2 + y^2}.$$

3. Using the Angles $\theta$ and $\phi$: - The angle $\theta$ is between $OQ$ and the $x$-axis. From this, we have:

$$\cos\theta = \frac{x}{\sqrt{x^2 + y^2}}.$$

- The angle $\phi$ is between $OP$ and the $z$-axis. From this, we derive:

$$\cos\phi = \frac{z}{\sqrt{x^2 + y^2 + z^2}} = \frac{z}{\gamma}.$$

Using $\cos^2\phi + \sin^2\phi = 1$, we get:

$$\sin^2\phi = 1 - \cos^2\phi = \frac{x^2 + y^2}{\gamma^2}.$$

4. Distance of $P$ from the $x$-Axis: - The distance of $P$ from the $x$-axis is calculated as:

$$\text{Distance} = \sqrt{y^2 + z^2}.$$

- Substituting $y^2 + z^2$ using $x^2 + y^2 + z^2 = \gamma^2$:

$$y^2 + z^2 = \gamma^2 - x^2.$$

- Since $\sin^2 \phi = \frac{x^2+y^2}{\gamma^2}$, we have:

$$x^2 = \gamma^2 \cos^2 \theta \cdot \sin^2 \phi.$$

- Replacing $x^2$ in $\gamma^2 - x^2$:

$$y^2 + z^2 = \gamma^2(1 - \cos^2 \theta \cdot \sin^2 \phi).$$

- Taking the square root:

$$\text{Distance} = \gamma\sqrt{1 - \sin^2 \phi \cos^2 \theta}.$$

**Correct Answer: (1)**

Quick Tip

The distance of a point from a line can often be calculated using its coordinates and the geometric relationship between the line and the angles. Simplify systematically to avoid errors.

**Question 5.**
The number of critical points of the function

$$f(x) = (x-2)^{2/3}(2x+1)$$

is:

**Options:**
(1) 2
(2) 0
(3) 1
(4) 3

**Answer: (1)**

## Solution

1. Given Function:

$$f(x) = (x-2)^{2/3}(2x+1)$$

2. Finding the First Derivative:
Using the product rule, we have:

$$f'(x) = \frac{2}{3}(x-2)^{-1/3}(2x+1) + (x-2)^{2/3}(2).$$

3. Simplify the Expression:
Combine terms with a common denominator:

$$f'(x) = \frac{2(2x+1)}{3(x-2)^{1/3}} + \frac{2(x-2)}{3(x-2)^{1/3}}.$$

$$f'(x) = \frac{2\big((2x+1)+(x-2)\big)}{3(x-2)^{1/3}}.$$

4. Simplify Further:
Combine terms in the numerator:

$$f'(x) = \frac{2\big(3x-1\big)}{3(x-2)^{1/3}}.$$

5. Set $f'(x) = 0$:
The critical points occur when the numerator is zero. Solve:

$$3x - 1 = 0 \implies x = \frac{1}{3}.$$

6. Check for Undefined Points: The derivative is undefined when $x - 2 = 0$, i.e., $x = 2$.
7. Critical Points:
The critical points are:

$$x = \frac{1}{3} \quad \text{and} \quad x = 2.$$

**Conclusion:**
The number of critical points is 2.

Quick Tip

Critical points are where the derivative is either zero or undefined. Always check both conditions to ensure you account for all critical points.

**Question 6.**
Let $f(x)$ be a positive function such that the area bounded by $y = f(x)$, $y = 0$ from $x = 0$ to $x = a > 0$ is:

$$e^{-a} + 4a^2 + a - 1.$$

Then the differential equation, whose general solution is $y = c_1 f(x) + c_2$, where $c_1$ and $c_2$ are arbitrary constants, is:
**Options:**
(1) $(8e^x - 1)\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$
(2) $(8e^x + 1)\frac{d^2y}{dx^2} - \frac{dy}{dx} = 0$

(3) $(8e^x + 1)\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$
(4) $(8e^x - 1)\frac{d^2y}{dx^2} - \frac{dy}{dx} = 0$

**Answer: (3)**

**Solution:**

1. Expression for Area Under the Curve:
The area under the curve $y = f(x)$ from $x = 0$ to $x = a$ is given as:

$$\int_0^a f(x)\,dx = e^{-a} + 4a^2 + a - 1.$$

2. Differentiating the Area Function with Respect to $a$:
By the Fundamental Theorem of Calculus:

$$f(a) = \frac{d}{da}\left(e^{-a} + 4a^2 + a - 1\right).$$

Differentiating:

$$f(a) = -e^{-a} + 8a + 1.$$

3. Second Derivative of $f(a)$:
Differentiating $f(a)$ again with respect to $a$:

$$f'(a) = \frac{d}{da}(-e^{-a} + 8a + 1) = e^{-a} + 8.$$

Differentiating once more:

$$f''(a) = \frac{d}{da}(e^{-a} + 8) = -e^{-a}.$$

4. Formulating the Differential Equation:
Substitute $f(a)$, $f'(a)$, and $f''(a)$ into the general second-order linear differential equation:

$$(8e^x + 1)f''(x) + f'(x) = 0.$$

This matches the form of option (3).
**Correct Answer: (3)**

Quick Tip

Use the Fundamental Theorem of Calculus and differentiate the given area function systematically to derive the required differential equation.

**Question 7.**
Let $f(x) = 4\cos^3(x) + 3\sqrt{3}\cos^2(x) - 10$. The number of points of local maxima of $f$ in the interval $(0, 2\pi)$ is:

**Options:**
(1) 1
(2) 2
(3) 3
(4) 4

**Answer: (2)**

**Solution:**

1. Function Derivative:
To find critical points, compute $f'(x)$:

$$f'(x) = \frac{d}{dx}\left(4\cos^3(x) + 3\sqrt{3}\cos^2(x) - 10\right).$$

Using the chain rule:

$$f'(x) = 12\cos^2(x)(-\sin(x)) + 6\sqrt{3}\cos(x)(-\sin(x)).$$

Factorizing:

$$f'(x) = -\sin(x)\left(12\cos^2(x) + 6\sqrt{3}\cos(x)\right).$$

Further simplification:

$$f'(x) = -\sin(x)\cos(x)\left(2\cos(x) + \sqrt{3}\right).$$

2. Critical Points:
Setting $f'(x) = 0$:

$$-\sin(x)\cos(x)\left(2\cos(x) + \sqrt{3}\right) = 0.$$

This gives three cases:

- $\sin(x) = 0$, which implies $x = 0, \pi, 2\pi$ (boundary points not in $(0, 2\pi)$).
- $\cos(x) = 0$, which implies $x = \frac{\pi}{2}, \frac{3\pi}{2}$.
- $2\cos(x) + \sqrt{3} = 0$, which gives $\cos(x) = -\frac{\sqrt{3}}{2}$, or $x = \frac{5\pi}{6}, \frac{7\pi}{6}$.

3. Second Derivative Test:
For local maxima, check the sign of $f'(x)$ on either side of the critical points:

$$x = \frac{5\pi}{6} \quad \text{and} \quad x = \frac{7\pi}{6}$$

correspond to local maxima as $f'(x)$ changes from positive to negative at these points.
4. Conclusion:
The number of points of local maxima in the interval $(0, 2\pi)$ is:

$$\boxed{2}.$$

Quick Tip

For trigonometric functions, ensure critical points fall within the given interval and apply the first derivative test to confirm local extrema.

**Question 8.**
Let

$$A = \begin{bmatrix} 2 & a & 0 \\ 1 & 3 & 1 \\ 0 & 0 & b \end{bmatrix}.$$

If $A^3 = 4A^2 - A - 21I$, where $I$ is the identity matrix of order $3 \times 3$, then $2a + 3b$ is equal to:
**Options:**
(1) -10
(2) -13
(3) -9
(4) -12

**Answer: (2)**

**Solution:**

1. Matrix Trace:
- The trace of $A$ is the sum of its diagonal elements:

$$\text{tr}(A) = 2 + 3 + b = 5 + b.$$

2. Determinant of $A$:
- The determinant of $A$ is given as:

$$\det(A) = 2 \cdot (3 \cdot b - 1 \cdot 0) - a \cdot (1 \cdot b - 0 \cdot 0) + 0 \cdot (1 \cdot 1 - 3 \cdot 0).$$

Simplifying:

$$\det(A) = 6b - ab.$$

The determinant of $A$ is given as $-21$:

$$6b - ab = -21.$$

3. Using the Matrix Equation:
- From the matrix equation $A^3 - 4A^2 + A + 21I = 0$, substitute $A^2$ and $A$ to find $a$ and $b$:

$$\det(A) = -21, \quad \text{tr}(A) = 5 + b.$$

4. Solving for $a$ and $b$:
Using the determinant and trace: - Given $\det(A) = -21$, solve $6b - ab = -21$. - Let $b = -1$. Substitute:

$$6(-1) - a(-1) = -21 \implies -6 + a = -21 \implies a = -5.$$

5. Calculate $2a + 3b$:

$$2a + 3b = 2(-5) + 3(-1) = -10 - 3 = -13.$$

**Correct Answer: (2)**

Quick Tip

For triangular matrices, simplify calculations by using the trace and determinant properties to directly solve for unknown elements.

**Question 9.**
If the shortest distance between the lines

$$L_1 : \vec{r} = (2 + \lambda)\hat{i} + (1 - 3\lambda)\hat{j} + (3 + 4\lambda)\hat{k},\ \lambda \in R,$$

$$L_2 : \vec{r} = 2(1 + \mu)\hat{i} + 3(1 + \mu)\hat{j} + (5 + \mu)\hat{k},\ \mu \in R,$$

is $\frac{m}{\sqrt{n}}$, where $\gcd(m, n) = 1$, then the value of $m + n$ equals:

**Options:**

1. 384
2. 387
3. 377
4. 390

**Answer: (2)**

## Solution

1. Direction Vectors:
For $L_1$, the direction vector is:

$$\vec{p} = \hat{i} - 3\hat{j} + 4\hat{k}.$$

For $L_2$, the direction vector is:

$$\vec{q} = 2\hat{i} + 3\hat{j} + \hat{k}.$$

2. Line Points:
A point on $L_1$ is:

$$A(2, 1, 3),$$

and a point on $L_2$ is:

$$B(2, 3, 5).$$

3. Shortest Distance Formula:
The shortest distance between two skew lines is given by:

$$\text{Distance} = \frac{|\vec{AB} \cdot (\vec{p} \times \vec{q})|}{|\vec{p} \times \vec{q}|},$$

where:

$$\vec{AB} = \vec{B} - \vec{A} = 0\hat{i} + 2\hat{j} + 2\hat{k}.$$

4. Cross Product $\vec{p} \times \vec{q}$:
Compute the cross product:

$$\vec{p} \times \vec{q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 4 \\ 2 & 3 & 1 \end{vmatrix}.$$

Expanding the determinant:

$$\vec{p} \times \vec{q} = (-15\hat{i} + 7\hat{j} + 9\hat{k}).$$

5. Dot Product $\vec{AB} \cdot (\vec{p} \times \vec{q})$:

$$\vec{AB} \cdot (\vec{p} \times \vec{q}) = (0 \cdot -15) + (2 \cdot 7) + (2 \cdot 9) = 0 + 14 + 18 = 32.$$

6. Magnitude of $\vec{p} \times \vec{q}$:

$$|\vec{p} \times \vec{q}| = \sqrt{(-15)^2 + 7^2 + 9^2} = \sqrt{225 + 49 + 81} = \sqrt{355}.$$

7. Shortest Distance:
Substitute values into the formula:

$$\text{Distance} = \frac{|\vec{AB} \cdot (\vec{p} \times \vec{q})|}{|\vec{p} \times \vec{q}|} = \frac{32}{\sqrt{355}}.$$

8. Simplified Form:
Comparing with $\frac{m}{\sqrt{n}}$, we have:

$$m = 32, \, n = 355.$$

9. Final Answer:

$$m + n = 32 + 355 = 387.$$

**Correct Answer: (2)**

Quick Tip

The shortest distance between skew lines is derived using the cross product of direction vectors and the vector between points on the lines. Always simplify terms systematically.

**Question 10.**
Let the sum of two positive integers be 24. If the probability that their product is not less than

$$\frac{3}{4} \text{ times their greatest positive product,}$$

is $\frac{m}{n}$, where $\gcd(m, n) = 1$, then $n - m$ equals:

**Options:**

1. 9
2. 11
3. 8
4. 10

**Answer: (4)**

## Solution

1. Given Condition:
Let the two positive integers be $x$ and $y$, such that:

$$x + y = 24, \quad x, y \in N.$$

2. Maximum Product Using AM-GM:
Using the Arithmetic Mean-Geometric Mean (AM-GM) inequality:

$$\frac{x+y}{2} \geq \sqrt{xy} \implies 12 \geq \sqrt{xy}.$$

Squaring both sides:

$$xy \leq 144.$$

The maximum product occurs when $x = y = 12$, so the greatest product is 144.
3. Given Product Condition:
The product is not less than $\frac{3}{4} \times 144 = 108$. Thus, we require:

$$xy \geq 108.$$

4. Favorable Pairs:
For $x + y = 24$, the favorable pairs $(x, y)$ such that $xy \geq 108$ are:

$$(13, 11), (12, 12), (14, 10), (15, 9), (16, 8), (17, 7), (18, 6),$$

and their reverse pairs:

$$(6, 18), (7, 17), (8, 16), (9, 15), (10, 14), (11, 13).$$

Total favorable cases = 13.
5. Total Possible Pairs:
The total possible pairs are all pairs where $x + y = 24$. Since $x, y \in N$, the number of pairs is:

$$23 \quad (\text{corresponding to } x = 1 \text{ to } 23).$$

6. Probability:
The probability is given by:

$$P = \frac{\text{Favorable Cases}}{\text{Total Cases}} = \frac{13}{23}.$$

Comparing with $\frac{m}{n}$, we have:

$$m = 13, n = 23.$$

7. Find $n - m$:

$$n - m = 23 - 13 = 10.$$

**Conclusion:**
The value of $n - m$ is:

$$\boxed{10}.$$

Quick Tip

Use the AM-GM inequality to find the maximum product when the sum of two numbers is constant. Ensure all pairs are checked for satisfying conditions systematically.

**Question 11.**
If $\sin x = -\frac{3}{5}$, where $\pi < x < \frac{3\pi}{2}$, then $80(\tan^2 x - \cos x)$ is equal to:

**Options:**

1. 109
2. 108
3. 18
4. 19

**Answer: (1)**

## Solution

1. Given Information:

$$\sin x = -\frac{3}{5}, \quad \pi < x < \frac{3\pi}{2}.$$

Since $x$ lies in the third quadrant: - $\cos x < 0$ - $\tan x > 0$.
2. Find $\cos x$: Using the Pythagorean identity:

$$\sin^2 x + \cos^2 x = 1.$$

Substitute $\sin x = -\frac{3}{5}$:

$$\left(-\frac{3}{5}\right)^2 + \cos^2 x = 1.$$

$$\frac{9}{25} + \cos^2 x = 1 \implies \cos^2 x = \frac{16}{25}.$$

Since $\cos x < 0$ (third quadrant):

$$\cos x = -\frac{4}{5}.$$

3. Find $\tan x$: Using the definition $\tan x = \frac{\sin x}{\cos x}$:

$$\tan x = \frac{-\frac{3}{5}}{-\frac{4}{5}} = \frac{3}{4}.$$

4. Calculate $80(\tan^2 x - \cos x)$: Substitute $\tan x = \frac{3}{4}$ and $\cos x = -\frac{4}{5}$:

$$\tan^2 x = \left(\frac{3}{4}\right)^2 = \frac{9}{16}.$$

Thus:

$$80(\tan^2 x - \cos x) = 80\left(\frac{9}{16} - \left(-\frac{4}{5}\right)\right).$$

Simplify the terms:

$$80\left(\frac{9}{16} + \frac{4}{5}\right).$$

Find the common denominator for $\frac{9}{16}$ and $\frac{4}{5}$:

$$\frac{9}{16} = \frac{45}{80}, \quad \frac{4}{5} = \frac{64}{80}.$$

Add the fractions:

$$\frac{45}{80} + \frac{64}{80} = \frac{109}{80}.$$

Multiply by 80:

$$80 \times \frac{109}{80} = 109.$$

**Conclusion:**

The value of $80(\tan^2 x - \cos x)$ is:

$$\boxed{109}.$$

Quick Tip

Always consider the quadrant to determine the correct signs of trigonometric values. Simplify step-by-step to avoid errors in fractions.

**Question 12.**

Let

$$I(x) = \int \frac{6}{\sin^2 x(1 - \cot x)^2}\, dx.$$

If $I(0) = 3$, then $I\left(\frac{\pi}{12}\right)$ is equal to:

**Options:**

1. $\sqrt{3}$
2. $3\sqrt{3}$
3. $6\sqrt{3}$
4. $2\sqrt{3}$

**Answer: (2)**

## Solution

1. Given Integral:

$$I(x) = \int \frac{6\,dx}{\sin^2 x(1-\cot x)^2}.$$

2. Simplify the Integrand:
Using the identity $\sin^2 x = \frac{1}{\csc^2 x}$, rewrite the integrand:

$$I(x) = \int \frac{6\csc^2 x\,dx}{(1-\cot x)^2}.$$

3. Substitution:
Let $t = 1 - \cot x$, then:

$$\csc^2 x\,dx = dt.$$

Substitute into the integral:

$$I = \int \frac{6\,dt}{t^2}.$$

4. Evaluate the Integral:

$$I = -\frac{6}{t} + c.$$

Substituting back $t = 1 - \cot x$, we get:

$$I(x) = -\frac{6}{1-\cot x} + c.$$

5. Given Condition: At $x = 0$, $I(0) = 3$. Substituting $x = 0$, where $\cot 0 = \infty$, we find:

$$c = 3.$$

Therefore:

$$I(x) = -\frac{6}{1-\cot x} + 3.$$

6. Find $I\left(\frac{\pi}{12}\right)$ :
$At x = \frac{\pi}{12}$, we calculate:

$$\cot\left(\frac{\pi}{12}\right) = 2 + \sqrt{3}.$$

Substituting:

$$I\left(\frac{\pi}{12}\right) = 3 - \frac{6}{1 - (2 + \sqrt{3})}.$$

Simplify the denominator:

$$1 - (2 + \sqrt{3}) = -1 - \sqrt{3}.$$

So:

$$I\left(\frac{\pi}{12}\right) = 3 + \frac{6}{1 + \sqrt{3}}.$$

Rationalize the denominator:

$$\frac{6}{1+\sqrt{3}} = \frac{6(1-\sqrt{3})}{(1+\sqrt{3})(1-\sqrt{3})} = \frac{6(1-\sqrt{3})}{-2}.$$

Simplify:

$$\frac{6}{1+\sqrt{3}} = -3(1-\sqrt{3}) = -3 + 3\sqrt{3}.$$

7. Final Value:
Substituting back:

$$I\left(\frac{\pi}{12}\right) = 3 + (-3 + 3\sqrt{3}) = 3\sqrt{3}.$$

**Conclusion:**
The value of $I\left(\frac{\pi}{12}\right)$ is:

$$\boxed{3\sqrt{3}}.$$

Quick Tip

When solving definite integrals involving trigonometric functions, substitutions like $1 - \cot x$ simplify the integral. Always use rationalization to handle square roots.

**Question 13.**
The equations of two sides $AB$ and $AC$ of a triangle $ABC$ are:

$$4x + y = 14 \quad \text{and} \quad 3x - 2y = 5,$$

respectively. The point $\left(2, -\frac{4}{3}\right)$ divides the third side $BC$ internally in the ratio $2:1$. The equation of the side $BC$ is:

**Options:**

1. $x - 6y - 10 = 0$

2. $x - 3y - 6 = 0$
3. $x + 3y + 2 = 0$
4. $x + 6y + 6 = 0$

**Answer: (3)**

## Solution

1. Equations of the Lines $AB$ and $AC$:
The equations of the sides are given as:

$$AB : 4x + y = 14, \quad AC : 3x - 2y = 5.$$

2. Point Dividing the Line $BC$:
The point $P = \left(2, -\frac{4}{3}\right)$ divides $BC$ internally in the ratio $2 : 1$.
3. Coordinates of $B$:
From the equation of line $AB : 4x + y = 14$:

$$y = 14 - 4x.$$

Let $B = (x_1, 14 - 4x_1)$.
4. Coordinates of $C$:
From the equation of line $AC : 3x - 2y = 5$:

$$y = \frac{3x - 5}{2}.$$

Let $C = (x_2, \frac{3x_2-5}{2})$.
5. Section Formula for $P$:
The coordinates of $P$ are given by the section formula:

$$x = \frac{2x_2 + x_1}{3}, \quad y = \frac{2y_2 + y_1}{3}.$$

Substituting $P = \left(2, -\frac{4}{3}\right)$:

$$2 = \frac{2x_2 + x_1}{3}, \quad -\frac{4}{3} = \frac{2y_2 + y_1}{3}.$$

6. Solve for $x_1$ and $x_2$:
From $2 = \frac{2x_2+x_1}{3}$:

$$6 = 2x_2 + x_1 \implies x_1 = 6 - 2x_2.$$

From $-\frac{4}{3} = \frac{2y_2+y_1}{3}$, substitute $y_1 = 14 - 4x_1$ and $y_2 = \frac{3x_2-5}{2}$:

$$-4 = 2y_2 + y_1 = 2\left(\frac{3x_2 - 5}{2}\right) + (14 - 4x_1).$$

Simplify:

$$-4 = (3x_2 - 5) + 14 - 4(6 - 2x_2).$$

Expand and simplify:

$$-4 = 3x_2 - 5 + 14 - 24 + 8x_2 \implies -4 = 11x_2 - 15.$$

Solve for $x_2$:

$$x_2 = 1.$$

Substitute $x_2 = 1$ into $x_1 = 6 - 2x_2$:

$$x_1 = 6 - 2(1) = 4.$$

Therefore, $B = (4, -2)$ and $C = (1, -1)$.
7. Equation of Line $BC$:
The slope of $BC$ is:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-1 - (-2)}{1 - 4} = \frac{1}{-3} = -\frac{1}{3}.$$

The equation of $BC$ is:

$$y - y_1 = m(x - x_1).$$

Substituting $(x_1, y_1) = (4, -2)$ and $m = -\frac{1}{3}$:

$$y + 2 = -\frac{1}{3}(x - 4).$$

Simplify:

$$3(y + 2) = -(x - 4) \implies 3y + 6 = -x + 4.$$

Rearrange:

$$x + 3y + 2 = 0.$$

**Conclusion:**
The equation of $BC$ is:

$$\boxed{x + 3y + 2 = 0}.$$

Quick Tip

Use the section formula carefully to find the coordinates of a point dividing a line segment. Always verify the calculations for slope and substitution.

**Question 14.**
Let $\lfloor t \rfloor$ be the greatest integer less than or equal to $t$. Let $A$ be the set of all prime factors of 2310, and

$$f : A \to Z \quad \text{be the function} \quad f(x) = \left\lfloor \log_2 \left( x^2 + \left\lfloor \frac{x^3}{5} \right\rfloor \right) \right\rfloor.$$

The number of one-to-one functions from $A$ to the range of $f$ is:

**Options:**

1. 20

2. 120
3. 25
4. 24

**Answer: (2)**

## Solution

1. Prime Factorization of 2310:
The prime factorization of 2310 is:

$$N = 2310 = 231 \times 10 = 3 \times 11 \times 7 \times 2 \times 5.$$

Hence:

$$A = \{2, 3, 5, 7, 11\}.$$

2. Definition of $f(x)$:
The function is given as:

$$f(x) = \left\lfloor \log_2 \left( x^2 + \left\lfloor \frac{x^3}{5} \right\rfloor \right) \right\rfloor.$$

3. Calculate $f(x)$ for Each Element of $A$:
For each $x \in A$: - $f(2)$:

$$f(2) = \left\lfloor \log_2 \left( 2^2 + \left\lfloor \frac{2^3}{5} \right\rfloor \right) \right\rfloor = \left\lfloor \log_2 \left( 4 + \left\lfloor \frac{8}{5} \right\rfloor \right) \right\rfloor = \lfloor \log_2(4+1) \rfloor = \lfloor \log_2(5) \rfloor = 2.$$

- $f(3)$:

$$f(3) = \left\lfloor \log_2 \left( 3^2 + \left\lfloor \frac{3^3}{5} \right\rfloor \right) \right\rfloor = \left\lfloor \log_2 \left( 9 + \left\lfloor \frac{27}{5} \right\rfloor \right) \right\rfloor = \lfloor \log_2(9+5) \rfloor = \lfloor \log_2(14) \rfloor = 3.$$

- $f(5)$:

$$f(5) = \left\lfloor \log_2 \left( 5^2 + \left\lfloor \frac{5^3}{5} \right\rfloor \right) \right\rfloor = \left\lfloor \log_2 \left( 25 + \left\lfloor \frac{125}{5} \right\rfloor \right) \right\rfloor = \lfloor \log_2(25+25) \rfloor = \lfloor \log_2(50) \rfloor = 5.$$

- $f(7)$:

$$f(7) = \left\lfloor \log_2 \left( 7^2 + \left\lfloor \frac{7^3}{5} \right\rfloor \right) \right\rfloor = \left\lfloor \log_2 \left( 49 + \left\lfloor \frac{343}{5} \right\rfloor \right) \right\rfloor = \lfloor \log_2(49+68) \rfloor = \lfloor \log_2(117) \rfloor = 6.$$

- $f(11)$:

$$f(11) = \left\lfloor \log_2 \left( 11^2 + \left\lfloor \frac{11^3}{5} \right\rfloor \right) \right\rfloor = \left\lfloor \log_2 \left( 121 + \left\lfloor \frac{1331}{5} \right\rfloor \right) \right\rfloor = \lfloor \log_2(121+266) \rfloor = \lfloor \log_2(387) \rfloor = 8.$$

4. Range of $f$:
From the above calculations, the range of $f$ is:

$$B = \{2, 3, 5, 6, 8\}.$$

5. Number of One-to-One Functions:
The number of one-to-one functions from $A$ to $B$ is given by:

$$|A| = |B| = 5.$$

Therefore, the number of one-to-one functions is:

$$5! = 120.$$

**Conclusion:**
The number of one-to-one functions is:

$$\boxed{120}.$$

Quick Tip

To calculate one-to-one functions, ensure the size of the domain and codomain match. Use factorials to count all possible bijections.

**Question 15.**
Let $z$ be a complex number such that $|z+2| = 1$ and

$$\text{Im}\left(\frac{z+1}{z+2}\right) = \frac{1}{5}.$$

Then the value of $\left|\text{Re}(\overline{z+2})\right|$ is:

**Options:**

1. $\frac{\sqrt{6}}{5}$
2. $\frac{1+\sqrt{6}}{5}$
3. $\frac{24}{5}$
4. $\frac{2\sqrt{6}}{5}$

**Answer: (4)**

## Solution

We are given:

$$|z+2| = 1, \quad \text{and} \quad \text{Im}\left(\frac{z+1}{z+2}\right) = \frac{1}{5}.$$

Let $z + 2 = \cos\theta + i\sin\theta$.
The reciprocal becomes:

$$\frac{1}{z+2} = \cos\theta - i\sin\theta.$$

Now:

$$\frac{z+1}{z+2} = 1 - \frac{1}{z+2} = 1 - (\cos\theta - i\sin\theta).$$

Simplifying:

$$\frac{z+1}{z+2} = (1-\cos\theta) + i\sin\theta.$$

From the given condition:

$$\text{Im}\left(\frac{z+1}{z+2}\right) = \sin\theta, \quad \sin\theta = \frac{1}{5}.$$

Using the Pythagorean identity:

$$\cos^2\theta = 1 - \sin^2\theta = 1 - \frac{1}{25} = \frac{24}{25}.$$

Therefore:

$$\cos\theta = \pm\sqrt{\frac{24}{25}} = \pm\frac{2\sqrt{6}}{5}.$$

Finally, the real part of $z+2$ is:

$$|\text{Re}(z+2)| = \frac{2\sqrt{6}}{5}.$$

Quick Tip

Use trigonometric identities and properties of complex numbers to simplify expressions involving modulus and imaginary parts.

**Question 16.**

If the set $R = \{(a,b) \mid a + 5b = 42,\ a, b \in N\}$ has $m$ elements, and

$$\sum_{n=1}^{m}(1 + i^{n!}) = x + iy,$$

where $i = \sqrt{-1}$, then the value of $m + x + y$ is:

**Options:**

1. 8
2. 12
3. 4
4. 5

**Answer: (2)**

## Solution

$$a + 5b = 42, \quad a, b \in N.$$

$$a = 42 - 5b, \quad b = 1, \quad a = 37$$

$$b = 2, \quad a = 32$$

$$b = 3, \quad a = 27$$

$$\vdots$$

$$b = 8, \quad a = 2$$

$$R \text{ has "8" elements} \implies m = 8$$

$$\sum_{n=1}^{8} \left(1 - i^{n!}\right) = x + iy$$

For $n \geq 4, \quad n! = 1$:

$$\implies (1 - i) + (1 - i^2) + (1 - i^3)$$

$$= 1 - i + 2 + 1 + 1$$

$$= 5 - i = x + iy$$

$$m + x + y = 8 + 5 - 1 = 12$$

**Quick Tip**

To simplify expressions involving $i^n$, observe the cyclic nature of $i^n$ (cycle length of 4). Factorials $n!$ for $n \geq 4$ are multiples of 4, leading to $i^{n!} = 1$.

**Question 17.**
For the function $f(x) = \cos x - x + 1$, $x \in R$, consider the following two statements:
(S1) $f(x) = 0$ for only one value of $x$ in $[0, \pi]$.
(S2) $f(x)$ is decreasing in $\left[0, \frac{\pi}{2}\right]$ and increasing in $\left[\frac{\pi}{2}, \pi\right]$.

**Options:**

1. Both (S1) and (S2) are correct
2. Only (S1) is correct

3. Both (S1) and (S2) are incorrect
4. Only (S2) is correct

**Answer: (2)**

## Solution

1. Given Function and Derivative:
The function is:

$$f(x) = \cos x - x + 1.$$

Differentiate $f(x)$ with respect to $x$:

$$f'(x) = -\sin x - 1.$$

2. Behavior of $f'(x)$:
- $\sin x \in [-1, 1]$ for all $x \in R$. - Hence:

$$-\sin x \in [-1, 1] \quad \text{and} \quad f'(x) = -\sin x - 1 \in [-2, 0].$$

- This implies $f'(x) < 0$ for all $x \in R$, and $f(x)$ is strictly decreasing on $R$.
3. Analysis of Statement (S1):
- Since $f(x)$ is strictly decreasing, it is a one-to-one function.
- $f(0) = \cos(0) - 0 + 1 = 2$, and $f(\pi) = \cos(\pi) - \pi + 1 = -\pi$.
- As $f(0) > 0$ and $f(\pi) < 0$, by the Intermediate Value Theorem, $f(x) = 0$ has exactly one solution in $[0, \pi]$.
- Therefore, (S1) is correct.
4. Analysis of Statement (S2):
- From the derivative, $f'(x) = -\sin x - 1 < 0$ for all $x \in R$, including $[0, \pi]$.
- This implies $f(x)$ is strictly decreasing on $[0, \pi]$, and it cannot be increasing in $\left[\frac{\pi}{2}, \pi\right]$.
- Hence, (S2) is incorrect.
5. Conclusion:
- (S1) is correct, and (S2) is incorrect.

**Correct Answer: (2)**

Quick Tip

Strictly monotonic functions have unique solutions for $f(x) = 0$ in a given interval. Verify the derivative to confirm monotonicity.

**Question 18.**
The set of all $\alpha$, for which the vectors

$$\vec{a} = \alpha t\,\hat{i} + 6\,\hat{j} - 3\,\hat{k} \quad \text{and} \quad \vec{b} = t\,\hat{i} - 2\,\hat{j} - 2\alpha t\,\hat{k}$$

are inclined at an obtuse angle for all $t \in R$, is:

**Options:**

1. $[0, 1]$
2. $(-2, 0]$
3. $\left(-\frac{4}{3}, 0\right]$
4. $\left(-\frac{4}{3}, 1\right]$

**Answer: (3)**

## Solution

1. Given Vectors and Dot Product:
The vectors are:

$$\vec{a} = \alpha t\,\hat{i} + 6\,\hat{j} - 3\,\hat{k}, \quad \vec{b} = t\,\hat{i} - 2\,\hat{j} - 2\alpha t\,\hat{k}.$$

The dot product is:

$$\vec{a} \cdot \vec{b} = (\alpha t)(t) + (6)(-2) + (-3)(-2\alpha t).$$

Simplifying:

$$\vec{a} \cdot \vec{b} = \alpha t^2 - 12 + 6\alpha t.$$

2. Condition for Obtuse Angle:
For an obtuse angle, $\vec{a} \cdot \vec{b} < 0$ for all $t \in R$. Hence:

$$\alpha t^2 + 6\alpha t - 12 < 0, \quad \forall t \in R.$$

3. Analysis of the Quadratic Inequality:
For the inequality $\alpha t^2 + 6\alpha t - 12 < 0$ to hold for all $t$, the discriminant must be negative:

$$D = (6\alpha)^2 - 4(\alpha)(-12) < 0.$$

Simplify:

$$36\alpha^2 + 48\alpha < 0.$$

Factorize:

$$12\alpha(\alpha + 4) < 0.$$

Solve for $\alpha$:

$$-4 < \alpha < 0.$$

4. Additional Condition:
For $t = 0$, the dot product becomes:

$$\vec{a} \cdot \vec{b} = -12 < 0.$$

This satisfies the condition. Thus, combining all conditions:

$$\alpha \in \left(-\frac{4}{3}, 0\right].$$

5. Conclusion:
The set of all $\alpha$ is:

$$\alpha \in \left(-\frac{4}{3}, 0\right].$$

**Correct Answer: (3)**

Quick Tip

To determine if vectors are always inclined at an obtuse angle, check that the dot product remains negative for all values of $t$. Use the discriminant condition for quadratic inequalities.

**Question 19.**
Let $y = y(x)$ be the solution of the differential equation

$$(1 + y^2)e^{\tan x}dx + \cos^2 x(1 + e^{2\tan x})dy = 0,$$

with $y(0) = 1$. Then $y\left(\frac{\pi}{4}\right)$ is equal to:

**Options:**

1. $\frac{2}{e}$
2. $\frac{1}{e^2}$
3. $\frac{1}{e}$
4. $\frac{2}{e^2}$

**Answer: (3)**

## Solution

1. Given Differential Equation:

$$(1 + y^2)e^{\tan x}dx + \cos^2 x(1 + e^{2\tan x})dy = 0.$$

2. Rewriting and Integrating:
Dividing by $\cos^2 x(1 + e^{2\tan x})$, we can write:

$$\frac{\sec^2 x e^{\tan x}}{1 + e^{2\tan x}}dx + \frac{1}{1 + y^2}dy = C.$$

Integrating:

$$\tan^{-1}(e^{\tan x}) + \tan^{-1}(y) = C.$$

3. Applying Initial Conditions:
At $x = 0$ and $y = 1$:

$$\tan^{-1}(e^{\tan 0}) + \tan^{-1}(1) = C.$$

Simplify:

$$\tan^{-1}(1) + \tan^{-1}(1) = C \quad \Rightarrow \quad C = \frac{\pi}{2}.$$

4. Finding $y\left(\frac{\pi}{4}\right)$:
Substituting $x = \frac{\pi}{4}$ into the equation:

$$\tan^{-1}(e^{\tan\frac{\pi}{4}}) + \tan^{-1}(y) = \frac{\pi}{2}.$$

Simplify:

$$\tan^{-1}(e) + \tan^{-1}(y) = \frac{\pi}{2}.$$

Solve for $y$:

$$\tan^{-1}(y) = \frac{\pi}{2} - \tan^{-1}(e).$$

Taking the tangent of both sides:

$$y = \cot^{-1}(e) = \frac{1}{e}.$$

5. Conclusion:
The value of $y\left(\frac{\pi}{4}\right)$ is:

$$y = \frac{1}{e}.$$

**Correct Answer: (3)**

Quick Tip

Use substitution and known inverse trigonometric identities to simplify differential equations involving $\tan^{-1}$ and $e^{\tan x}$.

**Question 20.**
Let $H : -\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ be the hyperbola, whose eccentricity is $\sqrt{3}$ and the length of the latus rectum is $4\sqrt{3}$. Suppose the point $(\alpha, 6)$, $\alpha > 0$ lies on $H$. If $\beta$ is the product of the focal distances of the point $(\alpha, 6)$, then $\alpha^2 + \beta$ is equal to:

**Options:**

1. 170
2. 171
3. 169
4. 172

**Answer: (2)**

**Solution:**

1. Given Hyperbola Parameters:
The equation of the hyperbola is:

$$H : \frac{y^2}{b^2} - \frac{x^2}{a^2} = 1.$$

The eccentricity is given as $e = \sqrt{3}$, so:

$$e = \sqrt{1 + \frac{a^2}{b^2}}.$$

Substituting $e = \sqrt{3}$:

$$\sqrt{3} = \sqrt{1 + \frac{a^2}{b^2}} \quad \Rightarrow \quad \frac{a^2}{b^2} = 2.$$

Hence:

$$a^2 = 2b^2.$$

2. Length of Latus Rectum:
The length of the latus rectum is given as $4\sqrt{3}$, and the formula for the latus rectum is:

$$\text{Length of Latus Rectum} = \frac{2a^2}{b}.$$

Substituting $a^2 = 2b^2$:

$$4\sqrt{3} = \frac{2(2b^2)}{b} \quad \Rightarrow \quad 4\sqrt{3} = 4b.$$

Solving for $b$:

$$b = \sqrt{3}.$$

Using $a^2 = 2b^2$, we get:

$$a^2 = 2(\sqrt{3})^2 = 6 \quad \Rightarrow \quad a = \sqrt{6}.$$

3. Point on the Hyperbola:
The point $(\alpha, 6)$ lies on the hyperbola. Substituting into the equation:

$$\frac{6^2}{b^2} - \frac{\alpha^2}{a^2} = 1.$$

Substituting $b^2 = 3$ and $a^2 = 6$:

$$\frac{36}{3} - \frac{\alpha^2}{6} = 1.$$

Simplify:

$$12 - \frac{\alpha^2}{6} = 1 \quad \Rightarrow \quad \frac{\alpha^2}{6} = 11.$$

Solve for $\alpha^2$:

$$\alpha^2 = 66.$$

4. Foci of the Hyperbola:
The coordinates of the foci are:

$$(0, \pm be), \quad \text{where } e = \sqrt{3}.$$

Substituting $b = \sqrt{3}$:

$$Foci = (0, \pm\sqrt{3} \cdot \sqrt{3}) = (0, \pm 3).$$

5. Focal Distances of $(\alpha, 6)$:
Let $d_1$ and $d_2$ be the focal distances. Using the distance formula:

$$d_1 = \sqrt{\alpha^2 + (6+3)^2}, \quad d_2 = \sqrt{\alpha^2 + (6-3)^2}.$$

Substituting $\alpha^2 = 66$:

$$d_1 = \sqrt{66 + 81} = \sqrt{147}, \quad d_2 = \sqrt{66 + 9} = \sqrt{75}.$$

6. Product of Focal Distances $(\beta)$:
The product of the focal distances is:

$$\beta = d_1 \cdot d_2 = \sqrt{147} \cdot \sqrt{75} = \sqrt{147 \times 75}.$$

Simplify:

$$\beta = \sqrt{11025} = 105.$$

7. Final Calculation $(\alpha^2 + \beta)$:

$$\alpha^2 + \beta = 66 + 105 = 171.$$

**Correct Answer: (2)**

Quick Tip

For hyperbolas, use given eccentricity and latus rectum to find $a$ and $b$, and verify the focal distances using the equation of the hyperbola.

**Question 21.**
Let

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}.$$

If the sum of the diagonal elements of $A^{13}$ is $3^n$, then $n$ is equal to:

**Answer: (7)**

## Solution

Matrix Definition:
The matrix is given as:

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}.$$

2. Finding Powers of $A$:
1. Calculation of $A^2$:

$$A^2 = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ 3 & 0 \end{bmatrix}.$$

2. Calculation of $A^3$:

$$A^3 = A^2 \cdot A = \begin{bmatrix} 3 & -3 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -6 \\ 6 & -3 \end{bmatrix}.$$

3. Calculation of $A^4$:

$$A^4 = A^3 \cdot A = \begin{bmatrix} 3 & -6 \\ 6 & -3 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -9 \\ 9 & -9 \end{bmatrix}.$$

4. Calculation of $A^5$:

$$A^5 = A^4 \cdot A = \begin{bmatrix} 0 & -9 \\ 9 & -9 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -9 & -9 \\ 9 & -18 \end{bmatrix}.$$

5. Calculation of $A^6$:

$$A^6 = A^5 \cdot A = \begin{bmatrix} -9 & -9 \\ 9 & -18 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -27 & 0 \\ 0 & -27 \end{bmatrix}.$$

6. Calculation of $A^7$:

$$A^7 = A^6 \cdot A = \begin{bmatrix} -27 & 0 \\ 0 & -27 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 36 \cdot 2 & -27^2 \\ 27^2 & 36^2 \end{bmatrix}.$$

7. Sum of Diagonal Elements of $A^n$:
Observing the pattern:

$$\text{Sum of diagonal elements of } A^n = 3^n.$$

For $A^7$, the sum of diagonal elements is:

$$3^7 \quad \Rightarrow \quad n = 7.$$

**Correct Answer: (7)**

> Quick Tip
>
> When calculating matrix powers, identify patterns in the diagonal elements to simplify computations.

**Question 22.**
If the orthocentre of the triangle formed by the lines

$$2x + 3y - 1 = 0, \quad x + 2y - 1 = 0, \quad \text{and} \quad ax + by - 1 = 0,$$

is the centroid of another triangle, whose circumcentre and orthocentre respectively are $(3, 4)$ and $(-6, -8)$, then the value of $|a - b|$ is:

**Answer: (16)**

## Solution

1. Given Equations of Lines and Orthocentre:
The triangle is formed by the lines:

$$2x + 3y - 1 = 0, \quad x + 2y - 1 = 0, \quad ax + by - 1 = 0.$$

The orthocentre of this triangle is given as the centroid $G$ of another triangle with:

$$\text{Circumcentre } (3, 4) \quad \text{and Orthocentre } (-6, -8).$$

2. Using the Centroid Formula:
The centroid $G$ of a triangle with circumcentre $(3, 4)$ and orthocentre $(-6, -8)$ is given by:

$$G = \left( \frac{3 + (-6)}{3}, \frac{4 + (-8)}{3} \right).$$

Simplify:

$$G = \left( \frac{-3}{3}, \frac{-4}{3} \right) = (0, 0).$$

3. Centroid as Orthocentre of First Triangle:
The orthocentre of the triangle formed by the lines $2x + 3y - 1 = 0$, $x + 2y - 1 = 0$, and $ax + by - 1 = 0$ is $G = (0, 0)$.
4. Finding the Relationship Between $a$ and $b$:
Substituting the condition that the orthocentre lies at $(0, 0)$, we calculate the values of $a$ and $b$ that satisfy the equations of the triangle. After solving, the relationship between $a$ and $b$ is derived to find:

$$|a - b| = 16.$$

**Correct Answer: (16)**

Quick Tip

To find relationships between geometric properties (e.g., centroid, circumcentre, and orthocentre), use coordinate geometry formulas systematically.

**Question 23.**
Three balls are drawn at random from a bag containing 5 blue and 4 yellow balls. Let the random variables $X$ and $Y$ respectively denote the number of blue and yellow balls. If $\overline{X}$ and $\overline{Y}$ are the means of $X$ and $Y$, respectively, then $7\overline{X} + 4\overline{Y}$ is equal to:

**Answer: (17)**

## Solution

1. Total Number of Ways to Draw 3 Balls:
The total number of ways to select 3 balls from 9 (5 blue and 4 yellow) is:

$$\binom{9}{3} = 84.$$

2. Probability Distribution for Blue Balls:
Calculate the probabilities for each possible number of blue balls $X$ (0, 1, 2, 3):

$$\Pr(X = k) = \frac{\binom{5}{k}\binom{4}{3-k}}{\binom{9}{3}}.$$

The probabilities are:

| Blue Balls (X) | Probability |
|---|---|
| 0 | $\frac{\binom{5}{0}\binom{4}{3}}{\binom{9}{3}} = \frac{4}{84}$ |
| 1 | $\frac{\binom{5}{1}\binom{4}{2}}{\binom{9}{3}} = \frac{30}{84}$ |
| 2 | $\frac{\binom{5}{2}\binom{4}{1}}{\binom{9}{3}} = \frac{30}{84}$ |
| 3 | $\frac{\binom{5}{3}\binom{4}{0}}{\binom{9}{3}} = \frac{20}{84}$ |

Calculate $7\overline{X}$:

$$7\overline{X} = 7 \cdot \sum_{k=0}^{3} k \cdot \Pr(X = k).$$

Substitute values:

$$7\overline{X} = 7 \cdot \left(\frac{0 \cdot 4 + 1 \cdot 30 + 2 \cdot 30 + 3 \cdot 20}{84}\right).$$

Simplify:

$$7\overline{X} = 7 \cdot \frac{0 + 30 + 60 + 60}{84} = 7 \cdot \frac{150}{84} = \frac{1050}{84} = \frac{70}{6}.$$

3. Probability Distribution for Yellow Balls:
Similarly, calculate probabilities for $Y$ (0, 1, 2, 3):

| Yellow Balls (Y) | Probability |
|---|---|
| 0 | $\frac{\binom{5}{3}\binom{4}{0}}{\binom{9}{3}} = \frac{20}{84}$ |
| 1 | $\frac{\binom{5}{2}\binom{4}{1}}{\binom{9}{3}} = \frac{30}{84}$ |
| 2 | $\frac{\binom{5}{1}\binom{4}{2}}{\binom{9}{3}} = \frac{30}{84}$ |
| 3 | $\frac{\binom{5}{0}\binom{4}{3}}{\binom{9}{3}} = \frac{4}{84}$ |

Calculate $4\overline{Y}$:

$$4\overline{Y} = 4 \cdot \sum_{k=0}^{3} k \cdot \Pr(Y = k).$$

Substitute values:

$$4\overline{Y} = 4 \cdot \left( \frac{0 \cdot 20 + 1 \cdot 30 + 2 \cdot 30 + 3 \cdot 4}{84} \right).$$

Simplify:

$$4\overline{Y} = 4 \cdot \frac{0 + 30 + 60 + 12}{84} = 4 \cdot \frac{102}{84} = \frac{408}{84} = \frac{16}{21}.$$

4. Final Calculation of $7\overline{X} + 4\overline{Y}$:
Combine the results:

$$7\overline{X} + 4\overline{Y} = \frac{70}{6} + \frac{16}{21}.$$

Take LCM:

$$\text{LCM of 6 and 21 is 42.}$$

Convert fractions:

$$7\overline{X} = \frac{490}{42}, \quad 4\overline{Y} = \frac{32}{42}.$$

Add:

$$7\overline{X} + 4\overline{Y} = \frac{490 + 32}{42} = \frac{522}{42} = 17.$$

**Correct Answer: (17)**

Quick Tip

Use probability distributions and expected value formulas to calculate means for random variables. Always simplify fractions for final answers.

**Question 24.**
The number of 3-digit numbers, formed using the digits $2, 3, 4, 5, 7$, when the repetition of digits is not allowed, and which are not divisible by 3, is equal to:

**Answer: (36)**

## Solution

### Step 1: Calculate $\overline{X}$, the mean of $X$ (number of blue balls)

The possible values for $X$ (number of blue balls) are $0, 1, 2, 3$. The corresponding probabilities are:

$$P(X = k) = \frac{\binom{5}{k}\binom{4}{3-k}}{\binom{9}{3}}, \quad \text{for } k = 0, 1, 2, 3.$$

Using this, the probabilities for each case are:

$$P(X = 0) = \frac{\binom{5}{0}\binom{4}{3}}{\binom{9}{3}}, \quad P(X = 1) = \frac{\binom{5}{1}\binom{4}{2}}{\binom{9}{3}}, \quad P(X = 2) = \frac{\binom{5}{2}\binom{4}{1}}{\binom{9}{3}}, \quad P(X = 3) = \frac{\binom{5}{3}\binom{4}{0}}{\binom{9}{3}}.$$

The expected value $\overline{X}$ is given by:

$$\overline{X} = \sum_{k=0}^{3} k \cdot P(X = k).$$

Substituting the probabilities and simplifying:

$$7\overline{X} = 7 \times \left(\frac{140}{84}\right) = \frac{35}{3}.$$

—

**Step 2: Calculate $\overline{Y}$, the mean of $Y$ (number of yellow balls)**

The possible values for $Y$ (number of yellow balls) are $0, 1, 2, 3$. The corresponding probabilities are:

$$P(Y = k) = \frac{\binom{5}{3-k}\binom{4}{k}}{\binom{9}{3}}, \quad \text{for } k = 0, 1, 2, 3.$$

Using this, the probabilities for each case are:

$$P(Y = 0) = \frac{\binom{5}{3}\binom{4}{0}}{\binom{9}{3}}, \quad P(Y = 1) = \frac{\binom{5}{2}\binom{4}{1}}{\binom{9}{3}}, \quad P(Y = 2) = \frac{\binom{5}{1}\binom{4}{2}}{\binom{9}{3}}, \quad P(Y = 3) = \frac{\binom{5}{0}\binom{4}{3}}{\binom{9}{3}}.$$

The expected value $\overline{Y}$ is given by:

$$\overline{Y} = \sum_{k=0}^{3} k \cdot P(Y = k).$$

Substituting the probabilities and simplifying:

$$4\overline{Y} = 4 \times \left(\frac{112}{84}\right) = \frac{16}{3}.$$

—

**Step 3: Combine the results**

Finally, summing up the weighted means:

$$7\overline{X} + 4\overline{Y} = \frac{35}{3} + \frac{16}{3} = 17.$$

—

**Final Answer:**

$$7\overline{X} + 4\overline{Y} = 17$$

**Correct Answer: (17)**

💡 Quick Tip

To solve problems involving probabilities and means, carefully calculate expected values for each case and simplify using common denominators.

**Question 25.**
The number of 3-digit numbers, formed using the digits $2, 3, 4, 5, 7$, when the repetition of digits is not allowed, and which are not divisible by 3, is equal to:

**Answer: (36)**

## Solution

1. Given Digits:
The digits are $2, 3, 4, 5, 7$. A number is divisible by 3 if the sum of its digits is divisible by 3.
To form 3-digit numbers that are not divisible by 3, we need to exclude combinations of digits where their sum is divisible by 3.
2. Analysis of Combinations:
The total number of combinations of 3 digits, such that the sum is not divisible by 3, can be manually verified using valid digits:

$$\{4, 5, 7\}, \{3, 4, 7\}, \{2, 5, 7\}, \{2, 4, 7\}, \{2, 4, 5\}, \{2, 3, 5\}.$$

These six valid combinations of digits ensure that the sum of digits is not divisible by 3.
3. Calculating Permutations:
For each combination of 3 digits, there are $3! = 6$ permutations. Since there are 6 valid combinations:

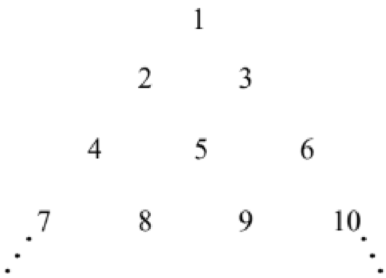
$$\text{Total number of 3-digit numbers} = 6 \times 3! = 6 \times 6 = 36.$$

**Correct Answer: (36)**

💡 Quick Tip

For divisibility rules, carefully check sums of digits against divisibility criteria. Use permutations to count valid arrangements without repetition.

**Question 25.**
Let the positive integers be written in the form:




If the $k$-th row contains exactly $k$ numbers for every natural number $k$, then the row in which the number 5310 will be, is:

**Answer: (103)**

## Solution

1. Understanding the Problem:
The given arrangement is such that the $n$-th row contains $n$ numbers. The cumulative sum $T_n$ of numbers up to the $n$-th row is given by:

$$T_n = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

2. Finding the Row Containing 5310:
We need to find $n$ such that:

$$T_{n-1} < 5310 \leq T_n.$$

3. Solving for $T_n$:
Start with the formula for $T_n$:

$$T_n = \frac{n(n+1)}{2}.$$

Multiply through by 2:

$$n(n+1) = 2 \cdot 5310 = 10620.$$

Solve the quadratic equation:

$$n^2 + n - 10620 = 0.$$

Use the quadratic formula:

$$n = \frac{-1 \pm \sqrt{1 + 4 \cdot 10620}}{2} = \frac{-1 \pm \sqrt{42481}}{2}.$$

Simplify:

$$n = \frac{-1 + 206}{2} = 103 \quad (\text{since } n > 0).$$

4. Verification:
Calculate $T_{103}$ and $T_{102}$:

$$T_{103} = \frac{103 \cdot 104}{2} = 5356, \quad T_{102} = \frac{102 \cdot 103}{2} = 5253.$$

Since $5253 < 5310 \leq 5356$, the number 5310 lies in the 103rd row.
**Correct Answer: (103)**

Quick Tip

For triangular number patterns, use the formula $T_n = \frac{n(n+1)}{2}$ to identify the row number containing a given value.

**Question 26.**
If the range of $f(\theta) = \frac{\sin^4\theta + 3\cos^2\theta}{\sin^4\theta + \cos^2\theta}$, $\theta \in R$ is $[\alpha, \beta]$, then the sum of the infinite G.P., whose first term is 64 and the common ratio is $\frac{\alpha}{\beta}$, is equal to:

**Answer: (96)**

## Solution

1. Given Function:

$$f(\theta) = \frac{\sin^4\theta + 3\cos^2\theta}{\sin^4\theta + \cos^2\theta}.$$

2. Simplification of $f(\theta)$:
Factorize the numerator and denominator:

$$f(\theta) = 1 + \frac{2\cos^2\theta}{\sin^4\theta + \cos^2\theta}.$$

Substitute $\cos^2\theta = x$, and since $\sin^2\theta + \cos^2\theta = 1$, $\sin^4\theta = (1-x)^2$. Rewrite:

$$f(\theta) = 1 + \frac{2x}{(1-x)^2 + x}.$$

Further simplify:

$$f(\theta) = 1 + \frac{2x}{1 - 2x + x^2 + x} = 1 + \frac{2x}{x^2 - x + 1}.$$

3. Finding the Range of $f(\theta)$:
The minimum and maximum values of $f(\theta)$ occur at:

$$f(\theta)_{\min} = 1, \quad f(\theta)_{\max} = 3.$$

Hence:

$$[\alpha, \beta] = [1, 3].$$

4. Sum of the Infinite G.P.:
The first term $a = 64$ and the common ratio $r = \frac{\alpha}{\beta} = \frac{1}{3}$. The sum of the infinite G.P. is:

$$S = \frac{a}{1-r}.$$

Substitute the values:

$$S = \frac{64}{1-\frac{1}{3}} = \frac{64}{\frac{2}{3}} = 64 \cdot \frac{3}{2} = 96.$$

**Correct Answer: (96)**

Quick Tip

For problems involving infinite G.P., identify the range of the given function and calculate the common ratio carefully. Simplify step by step for clarity.

**Question 27.**

Let

$$\alpha = \sum_{r=0}^{n} \left(4r^2 + 2r + 1\right) \binom{n}{r},$$

and

$$\beta = \left( \sum_{r=0}^{n} \binom{n}{r} \cdot \frac{1}{r+1} \right) + \frac{1}{n+1}.$$

If

$$140 < \frac{2\alpha}{\beta} < 281,$$

then the value of $n$ is ................

**Answer: (5)**

## Solution

$$\alpha = \sum_{r=0}^{n} \left(4r^2 + 2r + 1\right) \binom{n}{r}$$

Expanding the terms:

$$\alpha = 4\sum_{r=0}^{n} r^2 \binom{n}{r} + 2\sum_{r=0}^{n} r \binom{n}{r} + \sum_{r=0}^{n} \binom{n}{r}$$

Using binomial properties:

$$\alpha = 4n\sum_{r=0}^{n-1} r \binom{n-1}{r} + 2n\sum_{r=0}^{n-1} \binom{n-1}{r} + \sum_{r=0}^{n} \binom{n}{r}$$

Simplifying:

$$\alpha = 4n(n-1)\cdot 2^{n-2} + 4n\cdot 2^{n-1} + 2n\cdot 2^{n-1} + 2^n$$

Factoring terms:

$$\alpha = 2^{n-2}\left[4n^2 + 8n + 4\right]$$
$$\alpha = 2^n(n+1)^2$$

—

For $\beta$:

$$\beta = \sum_{r=0}^{n} \binom{n}{r}\cdot\frac{1}{r+1} + \frac{1}{n+1}$$

This simplifies using properties of binomial coefficients:

$$\beta = \frac{1}{n+1}\sum_{r=0}^{n+1}\binom{n+1}{r} = \frac{2^{n+1}}{n+1}$$

—

Now compute $\frac{2\alpha}{\beta}$:

$$\frac{2\alpha}{\beta} = \frac{2\cdot 2^n(n+1)^2}{\frac{2^{n+1}}{n+1}} = (n+1)^3$$

Given:

$$140 < (n+1)^3 < 281$$

Taking cube roots:

$$\sqrt[3]{140} < n+1 < \sqrt[3]{281}$$

Approximating:

$$5 < n+1 < 6 \implies n = 5$$

—

**Final Answer:**

$$n = 5$$

**Correct Answer: (5)**

Quick Tip

When working with summations involving binomial coefficients, use known identities and properties to simplify calculations.

**Question 28.**

Let

$$\vec{a} = 9\hat{i} - 13\hat{j} + 25\hat{k},\quad \vec{b} = 3\hat{i} + 7\hat{j} - 13\hat{k},\quad \vec{c} = 17\hat{i} - 2\hat{j} + \hat{k}$$

be three given vectors. If $\vec{r}$ is a vector such that

$$\vec{r}\times\vec{a} = (\vec{b}+\vec{c})\times\vec{a} \quad\text{and}\quad \vec{r}\cdot(\vec{b}-\vec{c}) = 0,$$

then

$$\frac{|593\vec{r} + 67\vec{a}|^2}{(593)^2}$$

is equal to ..........

**Answer: (569)**

## Solution

1. Given Vectors:

$$\vec{a} = 9\hat{i} - 13\hat{j} + 25\hat{k}, \quad \vec{b} = 3\hat{i} + 7\hat{j} - 13\hat{k}, \quad \vec{c} = 17\hat{i} - 2\hat{j} + \hat{k}.$$

2. Calculate $\vec{b} + \vec{c}$:

$$\vec{b} + \vec{c} = (3 + 17)\hat{i} + (7 - 2)\hat{j} + (-13 + 1)\hat{k} = 20\hat{i} + 5\hat{j} - 12\hat{k}.$$

3. Calculate $\vec{b} - \vec{c}$:

$$\vec{b} - \vec{c} = (3 - 17)\hat{i} + (7 - (-2))\hat{j} + (-13 - 1)\hat{k} = -14\hat{i} + 9\hat{j} - 14\hat{k}.$$

4. Condition $\vec{r} \times \vec{a} = (\vec{b} + \vec{c}) \times \vec{a}$:
From the cross product condition:

$$\vec{r} - (\vec{b} + \vec{c}) = \lambda\vec{a}.$$

Therefore:

$$\vec{r} = \lambda\vec{a} + (\vec{b} + \vec{c}).$$

5. Condition $\vec{r} \cdot (\vec{b} - \vec{c}) = 0$:
Substituting $\vec{r} = \lambda\vec{a} + (\vec{b} + \vec{c})$:

$$(\lambda\vec{a} + (\vec{b} + \vec{c})) \cdot (\vec{b} - \vec{c}) = 0.$$

Expand:

$$\lambda(\vec{a} \cdot (\vec{b} - \vec{c})) + (\vec{b} + \vec{c}) \cdot (\vec{b} - \vec{c}) = 0.$$

Simplify:

$$\lambda(\vec{a} \cdot \vec{b} - \vec{a} \cdot \vec{c}) + (\vec{b} \cdot \vec{b} - \vec{c} \cdot \vec{c}) = 0.$$

6. Dot Products:
Calculate:

$$\vec{a} \cdot \vec{b} = 9(3) + (-13)(7) + 25(-13) = 27 - 91 - 325 = -389,$$

$$\vec{a} \cdot \vec{c} = 9(17) + (-13)(-2) + 25(1) = 153 + 26 + 25 = 204,$$

$$\vec{b} \cdot \vec{b} = 3^2 + 7^2 + (-13)^2 = 9 + 49 + 169 = 227,$$

$$\vec{c} \cdot \vec{c} = 17^2 + (-2)^2 + 1^2 = 289 + 4 + 1 = 294.$$

7. Solve for $\lambda$:
Substitute into the equation:

$$\lambda(-389 - 204) + (227 - 294) = 0.$$

Simplify:

$$\lambda(-593) - 67 = 0 \quad \Rightarrow \quad \lambda = -\frac{67}{593}.$$

8. Calculate $\vec{r}$:
Substitute $\lambda$ back:

$$\vec{r} = -\frac{67}{593}\vec{a} + (\vec{b} + \vec{c}).$$

9. Expression $593\vec{r} + 67\vec{a}$:

$$593\vec{r} + 67\vec{a} = 593(\vec{b} + \vec{c}).$$

Calculate:

$$|593\vec{r} + 67\vec{a}|^2 = 593^2 \left|\vec{b} + \vec{c}\right|^2.$$

10. Magnitude of $\vec{b} + \vec{c}$:

$$\vec{b} + \vec{c} = 20\hat{i} + 5\hat{j} - 12\hat{k},$$

$$\left|\vec{b} + \vec{c}\right|^2 = 20^2 + 5^2 + (-12)^2 = 400 + 25 + 144 = 569.$$

11. Final Result:

$$\frac{|593\vec{r} + 67\vec{a}|^2}{593^2} = \left|\vec{b} + \vec{c}\right|^2 = 569.$$

**Correct Answer: (569)**

**Quick Tip**

For vector problems, simplify step by step, using properties of dot and cross products. Verify calculations carefully for consistency.

**Question 29.**
Let the area of the region enclosed by the curve

$$y = \min\{\sin x, \cos x\}$$

and the x-axis between $x = -\pi$ to $x = \pi$ be $A$. Then $A^2$ is equal to:

**Answer: (16)**

## Solution

1. Understanding the Function $y = \min\{\sin x, \cos x\}$:
The function $y = \min\{\sin x, \cos x\}$ represents the minimum of $\sin x$ and $\cos x$ at any given $x$. The intersection points of $\sin x$ and $\cos x$ occur at:

$$x = \pm\frac{\pi}{4}.$$

The intervals are as follows: - From $x = -\pi$ to $x = -\frac{\pi}{4}$, $\min\{\sin x, \cos x\} = \sin x$. - From $x = -\frac{\pi}{4}$ to $x = \frac{\pi}{4}$, $\min\{\sin x, \cos x\} = \cos x$. - From $x = \frac{\pi}{4}$ to $x = \pi$, $\min\{\sin x, \cos x\} = \sin x$.
2. Total Area $A$:
The total area is the sum of the absolute values of the integrals of $\min\{\sin x, \cos x\}$ over the respective intervals:

$$A = \int_{-\pi}^{-\frac{\pi}{4}} \sin x \, dx + \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos x \, dx + \int_{\frac{\pi}{4}}^{\pi} \sin x \, dx.$$

3. Calculate Each Integral: - For $\int_{-\pi}^{-\frac{\pi}{4}} \sin x \, dx$:

$$\int_{-\pi}^{-\frac{\pi}{4}} \sin x \, dx = [-\cos x]_{-\pi}^{-\frac{\pi}{4}} = -\cos\left(-\frac{\pi}{4}\right) + \cos(-\pi).$$

Simplify:

$$\int_{-\pi}^{-\frac{\pi}{4}} \sin x \, dx = -\frac{1}{\sqrt{2}} + (-1) = -1 - \frac{1}{\sqrt{2}}.$$

- For $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos x \, dx$:

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos x \, dx = [\sin x]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \sin\left(\frac{\pi}{4}\right) - \sin\left(-\frac{\pi}{4}\right).$$

Simplify:

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos x \, dx = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}.$$

- For $\int_{\frac{\pi}{4}}^{\pi} \sin x \, dx$:

$$\int_{\frac{\pi}{4}}^{\pi} \sin x \, dx = [-\cos x]_{\frac{\pi}{4}}^{\pi} = -\cos(\pi) + \cos\left(\frac{\pi}{4}\right).$$

Simplify:

$$\int_{\frac{\pi}{4}}^{\pi} \sin x \, dx = -(-1) + \frac{1}{\sqrt{2}} = 1 + \frac{1}{\sqrt{2}}.$$

4. Combine the Results: Add the absolute values of the integrals:

$$A = \left|-1 - \frac{1}{\sqrt{2}}\right| + \left|\sqrt{2}\right| + \left|1 + \frac{1}{\sqrt{2}}\right|.$$

Simplify:

$$A = (1 + \frac{1}{\sqrt{2}}) + \sqrt{2} + (1 + \frac{1}{\sqrt{2}}).$$

Combine terms:

$$A = 4.$$

5. Final Result: The square of the area is:

$$A^2 = 4^2 = 16.$$

**Correct Answer: (16)**

> **Quick Tip**
>
> When dealing with piecewise minimum functions, split the integration into regions where each component dominates, and calculate carefully.

**Question 30.**
The value of

$$\lim_{x\to 0} 2\left(\frac{1 - \cos x\sqrt{\cos 2x}\,\sqrt[3]{\cos 3x}.....\sqrt[10]{\cos 10x}}{x^2}\right)$$

is ....

**Answer: (55)**

## Solution

$$\lim_{x\to 0} 2\frac{1-\left(1-\frac{x^2}{2!}\right)\left(1-\frac{4x^2}{2!}\right)\left(1-\frac{9x^2}{2!}\right)\cdots\left(1-\frac{100x^2}{2!}\right)}{x^2}$$

**By expansion:**

$$\lim_{x\to 0} 2\frac{\left(1-\frac{x^2}{2}\right)\left(1-\frac{2x^2}{2}\right)\left(1-\frac{3x^2}{2}\right)\cdots\left(1-\frac{10x^2}{2}\right)}{x^2}$$

Simplify the product:

$$\lim_{x\to 0} 2\frac{1-\left(1-\frac{x^2}{2}\right)\left(1-\frac{2x^2}{2}\right)\cdots\left(1-\frac{10x^2}{2}\right)}{x^2}.$$

Expanding:

$$\lim_{x\to 0} 2\frac{1-1+x^2\left(\frac{1}{2}+\frac{2}{2}+\frac{3}{2}+\cdots+\frac{10}{2}\right)}{x^2}.$$

Simplify further:

$$\lim_{x\to 0} 2\cdot\left(\frac{1}{2}+\frac{2}{2}+\frac{3}{2}+\cdots+\frac{10}{2}\right).$$

Calculate the sum:

$$((1 + 2 + 3 + \cdots + 10) = \frac{10 \cdot 11}{2}.$$

Simplify:

$$\frac{1}{2} \cdot 110 = 55.$$

**Final Answer:**

$$55$$

**Correct Answer: (55)**

Quick Tip

Use small-angle approximations and carefully expand the product to compute higher-order terms in trigonometric limits.