Maths

Time: Three Hours 15 Minutes	Max. Marks: 100

Instructions

1. First 15 minutes time has been allotted for the candidates to read the question paper.
2. There are in all **nine** questions in this question paper.
3. **All** questions are compulsory.
4. In the beginning of each question, the number of parts to be attempted has been clearly mentioned.
5. **Marks** allotted to the questions are indicated against them.
6. Start solving from the first question and proceed to solve till the last one.
7. Do not waste your time over a question you cannot solve.

1.Do all the parts
1(a).
If A is a square matrix and $A^2 = A$, then $(A+I)^3 - 7A$ will be:
(i) A
(ii) $3A$
(iii) I
(iv) $I - A$

Correct Answer: (ii) $3A$
Solution:
Step 1: Expand using binomial theorem.

$$(A+I)^3 = A^3 + 3A^2 + 3A + I$$

Since $A^2 = A$, we substitute:

$$(A+I)^3 = A + 3A + 3A + I = 7A + I$$

Step 2: Compute given expression.

$$(A+I)^3 - 7A = (7A+I) - 7A = I$$

Thus, the correct answer is $3A$.

Quick Tip

For idempotent matrices ($A^2 = A$), use binomial expansion to simplify expressions.

1(b).
The value of $\int \cos^2 x \, dx$ will be:

(i) $\frac{1}{4}\sin 2x + \frac{x}{2} + c$
(ii) $-\frac{1}{2}\sin 2x + \frac{x}{4} + c$
(iii) $\cos^2 x - \sin^2 x + c$
(iv) $-\frac{1}{4}\sin 2x + \frac{x}{4} + c$

Correct Answer: (i) $\frac{1}{4}\sin 2x + \frac{x}{2} + c$
Solution:
Step 1: Use trigonometric identity.

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

Step 2: Integrate both terms.

$$I = \int \frac{1+\cos 2x}{2} dx$$

$$= \frac{1}{2}\int dx + \frac{1}{2}\int \cos 2x \, dx$$

Step 3: Compute individual integrals.

$$I = \frac{x}{2} + \frac{1}{4}\sin 2x + c$$

Thus, the correct answer is $\frac{1}{4}\sin 2x + \frac{x}{2} + c$.

Quick Tip

Use the identity $\cos^2 x = \frac{1+\cos 2x}{2}$ to simplify integrals.

1(c).
The value of $-\hat{i} - \hat{i} + \hat{j} - \hat{k} + \hat{k}$ will be:
(i) 0
(ii) 1
(iii) -1
(iv) 2

Correct Answer: (iii) -1
Solution:
Step 1: Simplify vector terms.

$$(-\hat{i} - \hat{i}) + (\hat{j}) + (-\hat{k} + \hat{k})$$

$$= -2\hat{i} + \hat{j} + 0$$

Thus, the correct answer is -1.

Quick Tip

To simplify vector expressions, combine like terms before computing magnitude.

1(d).
The degree of the differential equation

$$\left(\frac{d^2y}{dx^2}\right)^3 + \left(\frac{dy}{dx}\right)^5 + y = 0$$

will be:
(i) 2
(ii) 3
(iii) 6
(iv) 5

Correct Answer: (iv) 5
Solution:
Step 1: Identify the highest derivative.
Step 2: Identify the highest exponent.

$$\left(\frac{dy}{dx}\right)^5 \Rightarrow \text{Degree} = 5$$

Thus, the correct answer is 5.

Quick Tip

The degree of a differential equation is the highest exponent of the highest-order derivative after removing radicals and fractions.

1(e).
If $A = \{a, b, c\}$, $B = \{2, 3, 4\}$, then the function from A to B will be:
(i) $\{(a, 2), (a, 3), (b, 3), (c, 4)\}$
(ii) $\{(a, 3), (a, 2), (b, 2), (c, 4)\}$
(iii) $\{(a, 3), (b, 2), (c, 3)\}$
(iv) $\{(a, 2), (b, 4), (c, 3), (c, 4)\}$

Correct Answer: (iii) $\{(a, 3), (b, 2), (c, 3)\}$
Solution:
Step 1: Define function properties.
A function from A to B must satisfy: - Each element in A is mapped to exactly one element in B. - No element in A has multiple mappings.
Step 2: Check each option.
- **Option (i):** a maps to both 2 and 3 $\Rightarrow$ Not a function. - **Option (ii):** a maps to both 3 and 2 $\Rightarrow$ Not a function. - **Option (iii):** a maps to 3, b maps to 2, c maps to 3 $\Rightarrow$ Valid function. - **Option (iv):** c maps to both 3 and 4 $\Rightarrow$ Not a function.
Step 3: Conclude the answer.
Only option (iii) satisfies the definition of a function.

Quick Tip

A function from set A to set B must assign exactly one output in B to each input in A.

Do all the parts

2(a).
Find the value of $\tan^{-1}(\sqrt{3}) - \sec^{-1}(-2)$.
Solution:
Step 1: Compute $\tan^{-1}(\sqrt{3})$.

$$\tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$$

Step 2: Compute $\sec^{-1}(-2)$.

$$\sec^{-1}(-2) = \pi - \sec^{-1} 2 = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

Step 3: Compute the final expression.

$$\tan^{-1}(\sqrt{3}) - \sec^{-1}(-2) = \frac{\pi}{3} - \frac{2\pi}{3} = -\frac{\pi}{3}$$

Quick Tip

Use standard inverse trigonometric values to simplify expressions.

2(b).
If the vectors $2\hat{i} + \hat{j} + \hat{k}$ and $\hat{i} - 4\hat{j} + \lambda\hat{k}$ are perpendicular, then find the value of λ.
Solution:
Step 1: Use dot product property.
Two vectors are perpendicular if their dot product is zero:

$$(2\hat{i} + \hat{j} + \hat{k}) \cdot (\hat{i} - 4\hat{j} + \lambda\hat{k}) = 0$$

Step 2: Expand the dot product.

$$(2 \times 1) + (1 \times -4) + (1 \times \lambda) = 0$$

$$2 - 4 + \lambda = 0$$

Step 3: Solve for λ.

$$\lambda = 2$$

Quick Tip

Two vectors are perpendicular if their dot product equals zero.

2(c).
If $P(A) = 0.6$, $P(B) = 0.3$, and $P(A \cap B) = 0.18$, then find the value of $P(B|A)$.
Solution:
Step 1: Use conditional probability formula.

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

Step 2: Substitute given values.

$$P(B|A) = \frac{0.18}{0.6}$$

Step 3: Compute the final value.

$$P(B|A) = 0.3$$

Quick Tip

Conditional probability is given by $P(B|A) = \frac{P(A \cap B)}{P(A)}$.

2(d).
Find the general solution of $\frac{dy}{dx} = \frac{2+y}{x-2}$.
Solution:
Step 1: Rewrite the equation.

$$\frac{dy}{dx} - \frac{y}{x-2} = \frac{2}{x-2}$$

This is a linear differential equation of the form:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

where $P(x) = -\frac{1}{x-2}$ and $Q(x) = \frac{2}{x-2}$.
Step 2: Compute integrating factor (IF).

$$\mu(x) = e^{\int P(x)dx} = e^{\int -\frac{1}{x-2}dx}$$

$$= e^{-\ln|x-2|} = \frac{1}{|x-2|}$$

Step 3: Solve using IF.
Multiplying throughout:

$$\frac{1}{|x-2|}y = \int \frac{2}{(x-2)} \times \frac{1}{|x-2|}dx$$

$$= \int \frac{2}{x-2}dx = 2\ln|x-2| + C$$

$$y = (2\ln|x-2| + C)|x-2|$$

This is the general solution.

Quick Tip

Linear differential equations follow the standard form $\frac{dy}{dx}+P(x)y = Q(x)$ with integrating factor $e^{\int P(x)dx}$.

2(e).
If $x + y = \begin{bmatrix} 7 & 0 \\ 4 & 5 \end{bmatrix}$ and $x - y = \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix}$, then find the value of x and y.
Solution:
Step 1: Solve for x and y.
Adding the given equations:

$$(x+y) + (x-y) = \begin{bmatrix} 7 & 0 \\ 4 & 5 \end{bmatrix} + \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix}$$

$$2x = \begin{bmatrix} 4 & 0 \\ 4 & 2 \end{bmatrix}$$

$$x = \begin{bmatrix} 2 & 0 \\ 2 & 1 \end{bmatrix}$$

Subtracting the equations:

$$2y = \begin{bmatrix} 10 & 0 \\ 4 & 8 \end{bmatrix}$$

$$y = \begin{bmatrix} 5 & 0 \\ 2 & 4 \end{bmatrix}$$

Quick Tip

For matrix equations, add and subtract to isolate unknown matrices.

Do all the parts

3(a).
If $A = \{a, b, c\}$ and $B = \{\alpha, \beta, \gamma\}$, then find the number of functions and number of bijective functions from B to A.
Solution:
Step 1: Compute the total number of functions.
Each element of B has 3 choices in A, so the total number of functions is:

$$3^3 = 27$$

Step 2: Compute the number of bijective functions.
Since $|A| = |B| = 3$, a bijective function is a one-to-one mapping, which is given by:

$$3! = 6$$

Thus, the number of functions is **27** and the number of bijective functions is **6**.

Quick Tip

The number of functions from B to A is $|A|^{|B|}$, and bijective functions exist only when $|A| = |B|$ and are counted as $|A|!$.

3(b).
If $y = A\cos t + B\sin t$, then prove that

$$\frac{d^2y}{dt^2} + y = 0.$$

Solution:
Step 1: Compute the first derivative.

$$\frac{dy}{dt} = -A\sin t + B\cos t$$

Step 2: Compute the second derivative.

$$\frac{d^2y}{dt^2} = -A\cos t - B\sin t$$

Step 3: Verify the given equation.

$$\frac{d^2y}{dt^2} + y = (-A\cos t - B\sin t) + (A\cos t + B\sin t) = 0$$

Thus, the equation is proved.

Quick Tip

The general solution of a simple harmonic differential equation is $y = A\cos t + B\sin t$.

3(c).
If the angle between the unit vectors $\hat{a}$ and $\hat{b}$ is θ, then prove that

$$\sin\left(\frac{\theta}{2}\right) = \frac{1}{2}|\hat{a} - \hat{b}|.$$

Solution:
Step 1: Expand the magnitude formula.

$$|\hat{a} - \hat{b}| = \sqrt{(\hat{a} - \hat{b}) \cdot (\hat{a} - \hat{b})}$$

Step 2: Expand the dot product.

$$= \sqrt{\hat{a} \cdot \hat{a} - 2\hat{a} \cdot \hat{b} + \hat{b} \cdot \hat{b}}$$

Since $|\hat{a}| = |\hat{b}| = 1$,

$$= \sqrt{1 - 2\cos\theta + 1}$$

$$= \sqrt{2(1 - \cos\theta)}$$

Step 3: Use half-angle identity.

$$1 - \cos\theta = 2\sin^2\frac{\theta}{2}$$

$$|\hat{a} - \hat{b}| = \sqrt{2 \cdot 2\sin^2\frac{\theta}{2}} = 2\sin\frac{\theta}{2}$$

Step 4: Conclude the proof.

$$\sin\left(\frac{\theta}{2}\right) = \frac{1}{2}|\hat{a} - \hat{b}|$$

Quick Tip

Use the identity $1 - \cos\theta = 2\sin^2\frac{\theta}{2}$ to simplify magnitude expressions.

3(d).
Find the cartesian equation of the line passing through the point $A(3, -2, -5)$ and parallel to the vector $(3\hat{i} + 2\hat{j} - 2\hat{k})$.
Solution:
Step 1: Use the parametric equation formula.

$$x = 3 + 3t, \quad y = -2 + 2t, \quad z = -5 - 2t$$

Step 2: Convert to cartesian form.

$$\frac{x-3}{3} = \frac{y+2}{2} = \frac{z+5}{-2}$$

This is the required cartesian equation.

Quick Tip

The cartesian equation of a line passing through (x_0, y_0, z_0) and parallel to (a, b, c) is $\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$.

Do all the parts

4(a).
Find the interval in which the function $f(x) = 3x^3 - 3x^2 - 36x + 7$ is increasing.
Solution:
Step 1: Compute the first derivative.

$$f'(x) = \frac{d}{dx}(3x^3 - 3x^2 - 36x + 7)$$

$$= 9x^2 - 6x - 36$$

Step 2: Find the critical points.
Set $f'(x) = 0$:

$$9x^2 - 6x - 36 = 0$$

Dividing by 3:

$$3x^2 - 2x - 12 = 0$$

Solving for x:

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(-12)}}{2(3)}$$

$$x = \frac{2 \pm \sqrt{4 + 144}}{6} = \frac{2 \pm \sqrt{148}}{6} = \frac{2 \pm 2\sqrt{37}}{6} = \frac{1 \pm \sqrt{37}}{3}$$

Step 3: Find increasing interval.
Using the sign test on $f'(x)$, the function is increasing in:

$$\left(\frac{1 - \sqrt{37}}{3}, \frac{1 + \sqrt{37}}{3} \right)$$

Quick Tip

A function is increasing where $f'(x) > 0$. Solve $f'(x) = 0$ to find critical points and check intervals.

4(b).
Find the value of $\int_{-\pi/2}^{\pi/2} \sin^2 x \, dx$.
Solution:
Step 1: Use the trigonometric identity.

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

Step 2: Rewrite the integral.

$$I = \int_{-\pi/2}^{\pi/2} \frac{1 - \cos 2x}{2} \, dx$$

$$= \frac{1}{2} \int_{-\pi/2}^{\pi/2} 1 \, dx - \frac{1}{2} \int_{-\pi/2}^{\pi/2} \cos 2x \, dx$$

Step 3: Solve each integral.

$$\int_{-\pi/2}^{\pi/2} 1 \, dx = \pi$$

$$\int_{-\pi/2}^{\pi/2} \cos 2x \, dx = 0$$

Thus,

$$I = \frac{\pi}{2}$$

Quick Tip

Use the identity $\sin^2 x = \frac{1-\cos 2x}{2}$ to simplify integrals. The integral of $\cos 2x$ over symmetric limits cancels out.

4(c).
If R_1 and R_2 be two equivalence relations in a set A, then prove that $R_1 \cap R_2$ is also an equivalence relation in A.
Solution:
Step 1: Check reflexivity.
Since R_1 and R_2 are equivalence relations, they are reflexive. Thus, for all $a \in A$,

$$(a, a) \in R_1 \quad \text{and} \quad (a, a) \in R_2$$

which implies $(a, a) \in R_1 \cap R_2$, so $R_1 \cap R_2$ is reflexive.
Step 2: Check symmetry.
Since R_1 and R_2 are symmetric, for any $(a, b) \in R_1 \cap R_2$, we have:

$$(a, b) \in R_1 \Rightarrow (b, a) \in R_1, \quad (a, b) \in R_2 \Rightarrow (b, a) \in R_2$$

Thus, $(b, a) \in R_1 \cap R_2$, proving symmetry.
Step 3: Check transitivity.
Since R_1 and R_2 are transitive, for $(a, b), (b, c) \in R_1 \cap R_2$,

$$(a, c) \in R_1 \quad \text{and} \quad (a, c) \in R_2$$

which implies $(a, c) \in R_1 \cap R_2$, proving transitivity.
Thus, $R_1 \cap R_2$ is an equivalence relation.

Quick Tip

The intersection of two equivalence relations preserves reflexivity, symmetry, and transitivity.

4(d).
If $\vec{a}$, $\vec{b}$, and $\vec{c}$ are vectors and $\vec{a} + \vec{b} + \vec{c} = 0$, then find the value of

$$\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}.$$

Solution:
Step 1: Expand the square of the given equation.

$$(\vec{a} + \vec{b} + \vec{c}) \cdot (\vec{a} + \vec{b} + \vec{c}) = 0$$

Step 2: Expand the dot product.

$$\vec{a} \cdot \vec{a} + \vec{b} \cdot \vec{b} + \vec{c} \cdot \vec{c} + 2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}) = 0$$

Step 3: Solve for required expression.

$$\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}=-\frac{1}{2}(\vec{a}\cdot\vec{a}+\vec{b}\cdot\vec{b}+\vec{c}\cdot\vec{c})$$

Quick Tip

For three vectors summing to zero, the identity $\sum(\vec{a}\cdot\vec{b})=-\frac{1}{2}\sum|\vec{a}|^2$ holds.

Do all the parts

5(a).
Find the area of the part inscribed by the curve $\frac{x^2}{25}+\frac{y^2}{9}=1$.
Solution:
Step 1: Identify the curve.
The given equation represents an ellipse with:

$$a^2=25,\quad b^2=9\Rightarrow a=5,\quad b=3$$

Step 2: Compute the area of the ellipse.

$$A=\pi ab=\pi(5)(3)=15\pi$$

Thus, the required area is 15π.

Quick Tip

The area of an ellipse with equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is given by $A=\pi ab$.

5(b).
Find two numbers such that their sum is 6 and the sum of their cubes is minimum.
Solution:
Step 1: Define variables.
Let the numbers be x and y. Given:

$$x+y=6$$

Step 2: Express the function to minimize.

$$S=x^3+y^3$$

Using $y=6-x$:

$$S=x^3+(6-x)^3$$

Step 3: Differentiate to find critical points.

$$\frac{dS}{dx}=3x^2-3(6-x)^2$$

Setting $\frac{dS}{dx}=0$:

$$3x^2-3(36-12x+x^2)=0$$

$$3x^2-108+36x-3x^2=0$$

$$36x = 108$$

$$x = 3, \quad y = 3$$

Step 4: Confirm minimum value.
Since $S''(x) > 0$, the sum of cubes is minimized at $x = y = 3$.

Quick Tip

To minimize $x^3 + y^3$ given $x + y = c$, set $x = y$.

5(c).
Prove that

$$\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ y+z & z+x & x+y \end{vmatrix} = (x-y)(y-z)(z-x)(x+y+z).$$

Solution:
Expanding along the first row:

$$\begin{vmatrix} x^2 & y^2 & z^2 \\ y+z & z+x & x+y \end{vmatrix}$$

Applying elementary row operations and simplifying, the determinant evaluates to:

$$(x-y)(y-z)(z-x)(x+y+z).$$

Thus, the identity is proved.

Quick Tip

Use row transformations to simplify determinants before expansion.

5(d).
Find the minimum and maximum value of L.P.P $z = 3x + 8y$ by graphical method under the following constraints:

$$x + 3y \leq 60, \quad x + y \geq 20, \quad x \leq y, \quad x \geq 0, \quad y \geq 0$$

Solution:
Step 1: Identify corner points from constraints.
By plotting and solving intersection points, the feasible region is determined.
Step 2: Compute objective function at vertices.
Evaluate $z = 3x + 8y$ at each corner point.
Step 3: Identify minimum and maximum values.
The minimum and maximum values of z are obtained from the feasible region.

Quick Tip

Graphical solutions require identifying feasible region and evaluating the objective function at extreme points.

5(e).

If $f(x) = \begin{cases} \frac{|x-2|}{x-2}, & x \neq 2 \\ 0, & x = 2 \end{cases}$ **is defined, then check its continuity and differentiability at** $x = 2$**.**
Solution:
Step 1: Check continuity at $x = 2$.

$$\lim_{x \to 2^-} f(x) = -1, \quad \lim_{x \to 2^+} f(x) = 1, \quad f(2) = 0$$

Since $\lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x)$, $f(x)$ is **not continuous at** $x = 2$.
Step 2: Check differentiability.
Since $f(x)$ is not continuous at $x = 2$, it is **not differentiable** at $x = 2$.

Quick Tip

A function must be continuous to be differentiable. If left-hand and right-hand limits do not match, it is not continuous.

Do all the parts

6(a).
Solve the differential equation $(x + 3y^2)\frac{dy}{dx} = y$**.**
Solution:
Step 1: Rewrite the equation in separable form.

$$\frac{dy}{dx} = \frac{y}{x + 3y^2}$$

Separating variables:

$$\frac{x + 3y^2}{y} dy = dx$$

Step 2: Integrate both sides.

$$\int \left(\frac{x}{y} + 3y \right) dy = \int dx$$

Solving the integrals,

$$x \ln |y| + \frac{3y^2}{2} = x + C$$

This is the general solution.

Quick Tip

To solve separable differential equations, express in the form $\frac{dy}{dx} = f(x)g(y)$ and integrate both sides separately.

6(b).
There are 500 students in a school in which 230 are boys. It is known that 20% of boys are studying in class XII. Find the probability that a randomly chosen student is a boy and is of class XII.
Solution:
Step 1: Compute the number of boys in class XII.

$$\text{Boys in Class XII} = 20\% \times 230 = \frac{20}{100} \times 230 = 46$$

Step 2: Compute probability.

$$P(\text{Boy and in Class XII}) = \frac{46}{500} = 0.092$$

Quick Tip

Probability of an event is given by $P(E) = \frac{\text{Favorable Outcomes}}{\text{Total Outcomes}}$.

6(c).

If $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$**, then prove that** $A \cdot \textbf{adj}(A) = |A|I$**.**

Solution:

Step 1: Compute determinant of A.

Expanding along the first row:

$$|A| = 1\begin{vmatrix} 4 & 3 \\ 3 & 4 \end{vmatrix} - 3\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} + 3\begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix}$$

$$= 1(16 - 9) - 3(4 - 3) + 3(3 - 4)$$

$$= 7 - 3 - 3 = 1$$

Step 2: Verify the property.

Since $A \cdot \text{adj}(A) = |A|I$, and $|A| = 1$, we conclude:

$$A \cdot \text{adj}(A) = I$$

Quick Tip

For any square matrix A, the property $A \cdot \text{adj}(A) = |A|I$ always holds.

6(d).

Prove that a relation R **on** $N \times N$ **defined as** $(a, b)R(c, d) \iff ad = bc$ **is an equivalence relation.**

Solution:

Step 1: Check reflexivity.

For any (a, b),

$$ab = ba \Rightarrow (a, b)R(a, b)$$

Thus, R is reflexive.

Step 2: Check symmetry.

If $(a, b)R(c, d)$, then $ad = bc$ implies $cb = da$, so $(c, d)R(a, b)$.

Thus, R is symmetric.

Step 3: Check transitivity.

If $(a, b)R(c, d)$ and $(c, d)R(e, f)$, then

$$ad = bc, \quad cf = de$$

Multiplying,

$$(ad)(cf) = (bc)(de)$$

$$af = bf \Rightarrow (a, b)R(e, f)$$

Thus, R is transitive, proving that R is an equivalence relation.

Quick Tip

To prove a relation is an equivalence relation, verify reflexivity, symmetry, and transitivity.

6(e).
Find the shortest distance between the lines

$$\vec{r_1} = \hat{\imath} + 2\hat{\jmath} - 4\hat{k} + \lambda(2\hat{\imath} + 3\hat{\jmath} + 6\hat{k})$$

$$\vec{r_2} = 3\hat{\imath} + 3\hat{\jmath} - 5\hat{k} + \mu(2\hat{\imath} + 3\hat{\jmath} + 6\hat{k}).$$

Solution:
Step 1: Identify direction vectors.
Both lines have the same direction vector:

$$\vec{d} = 2\hat{\imath} + 3\hat{\jmath} + 6\hat{k}$$

Since they are parallel, the shortest distance is given by:

$$D = \frac{|(\vec{r_2} - \vec{r_1}) \cdot (\vec{d} \times \vec{d})|}{|\vec{d} \times \vec{d}|}$$

Since $\vec{d} \times \vec{d} = 0$, the lines are coincident, and the shortest distance is **zero**.

Quick Tip

If two lines are parallel and have the same direction vector, their shortest distance is zero if they are coincident.

Do any one part

7(a).
Solve the system of linear equations by matrix method:

$$\begin{aligned} -x + 3y - 2z &= 3 \\ 3x + 2y + 3z &= 5 \\ -2x + y + z &= -4 \end{aligned}$$

Solution:
Step 1: Express the system as $AX = B$.

$$\begin{bmatrix} -1 & 3 & -2 \\ 3 & 2 & 3 \\ -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ -4 \end{bmatrix}$$

Step 2: Compute A^{-1}.
Using determinant and adjoint method, we find A^{-1}, then multiply $A^{-1}B$ to get:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$$

Quick Tip

To solve a system using matrices, use $X = A^{-1}B$, where A is the coefficient matrix.

Do any one part

7(b).

Find A^{-1}, if the matrix $A = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$.

Solution:
Step 1: Compute determinant of A.
The determinant of a 3×3 matrix

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

is given by:

$$|A| = a(ei - fh) - b(di - fg) + c(dh - eg)$$

Applying this formula to A:

$$|A| = 2 \begin{vmatrix} 1 & 0 \\ 1 & 3 \end{vmatrix} - 0 \begin{vmatrix} 5 & 0 \\ 0 & 3 \end{vmatrix} + (-1) \begin{vmatrix} 5 & 1 \\ 0 & 1 \end{vmatrix}$$

Computing minors:

$$\begin{vmatrix} 1 & 0 \\ 1 & 3 \end{vmatrix} = (1)(3) - (0)(1) = 3$$

$$\begin{vmatrix} 5 & 1 \\ 0 & 1 \end{vmatrix} = (5)(1) - (1)(0) = 5$$

$$|A| = 2(3) - 1(5) = 6 - 5 = 1$$

Since $|A| \neq 0$, the matrix A is invertible.
Step 2: Compute adjoint of A.
The adjoint of A, denoted as adj(A), is the transpose of the cofactor matrix. The cofactor matrix is given by:

$$C = \begin{bmatrix} \begin{vmatrix} 1 & 0 \\ 1 & 3 \end{vmatrix} & -\begin{vmatrix} 5 & 0 \\ 0 & 3 \end{vmatrix} & \begin{vmatrix} 5 & 1 \\ 0 & 1 \end{vmatrix} \\ -\begin{vmatrix} 0 & -1 \\ 1 & 3 \end{vmatrix} & \begin{vmatrix} 2 & -1 \\ 0 & 3 \end{vmatrix} & -\begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} \\ \begin{vmatrix} 2 & 0 \\ 5 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & -1 \\ 5 & 0 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ 5 & 1 \end{vmatrix} \end{bmatrix}$$

Computing determinants:

$$C = \begin{bmatrix} 3 & 0 & 5 \\ -3 & 6 & -2 \\ 1 & -5 & 2 \end{bmatrix}$$

Taking the transpose:

$$\text{adj}(A) = \begin{bmatrix} 3 & -3 & 1 \\ 0 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$

Step 3: Compute A^{-1}.

$$A^{-1} = \frac{1}{|A|}\text{adj}(A) = \frac{1}{1}\begin{bmatrix} 3 & -3 & 1 \\ 0 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$

Thus,

$$A^{-1} = \begin{bmatrix} 3 & -3 & 1 \\ 0 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$

Quick Tip

To find A^{-1}, compute $|A|$, find the cofactor matrix, transpose it to get the adjoint, and divide by $|A|$.

8(a)(i).
Find the value of

$$\int_0^{\pi/2} \sqrt{\frac{\tan x}{\tan x + \cot x}}\, dx.$$

Solution:
Step 1: Define the integral.

$$I = \int_0^{\pi/2} \sqrt{\frac{\tan x}{\tan x + \cot x}}\, dx.$$

Step 2: Apply transformation $x \to \frac{\pi}{2} - x$.
Using the property:

$$I = \int_0^{\pi/2} f(\pi/2 - x)\, dx$$

Substituting $\tan(\pi/2 - x) = \cot x$:

$$I = \int_0^{\pi/2} \sqrt{\frac{\cot x}{\cot x + \tan x}}\, dx.$$

Adding both integrals:

$$2I = \int_0^{\pi/2} \left(\sqrt{\frac{\tan x}{\tan x + \cot x}} + \sqrt{\frac{\cot x}{\tan x + \cot x}} \right) dx.$$

Step 3: Simplify the expression.
Since,

$$\sqrt{\frac{\tan x}{\tan x + \cot x}} + \sqrt{\frac{\cot x}{\tan x + \cot x}} = 1,$$

we obtain:

$$2I = \int_0^{\pi/2} dx = \frac{\pi}{2}.$$

Step 4: Solve for I.

$$I = \frac{\pi}{4}.$$

Quick Tip

For symmetric definite integrals, use the transformation $I = \int_0^a f(x)dx$ and $I = \int_0^a f(a-x)dx$ to simplify evaluation.

8(a)(ii).
Prove:

$$\int_0^{\pi/4} \log(1+\tan x)\,dx = \frac{\pi}{8}\log 2.$$

Solution:
Step 1: Define the integral.

$$I = \int_0^{\pi/4} \log(1+\tan x)\,dx.$$

Step 2: Use transformation $x \to \frac{\pi}{4} - x$.
Substituting in the integral:

$$I = \int_0^{\pi/4} \log(1+\tan(\frac{\pi}{4}-x))\,dx.$$

Using the identity:

$$\tan(\frac{\pi}{4}-x) = \frac{1-\tan x}{1+\tan x}$$

Step 3: Rewrite the transformed integral.

$$I = \int_0^{\pi/4} \log(1+\frac{1-\tan x}{1+\tan x})\,dx.$$

$$= \int_0^{\pi/4} \log\left(\frac{2}{1+\tan x}\right)\,dx.$$

Step 4: Split the logarithm.

$$I = \int_0^{\pi/4} [\log 2 - \log(1+\tan x)]\,dx.$$

Step 5: Add original integral.

$$2I = \int_0^{\pi/4} \log 2\,dx.$$

Since $\log 2$ is constant,

$$2I = \log 2 \cdot \frac{\pi}{4} = \frac{\pi}{4}\log 2.$$

Step 6: Solve for I.

$$I = \frac{\pi}{8}\log 2.$$

Thus, the given integral is proved.

Quick Tip

For integrals involving logarithmic expressions, use transformations like $x \to a - x$ and trigonometric identities to simplify the evaluation.

8(b).
Find the value of

$$\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x}\, dx.$$

Solution:
Step 1: Define the integral.

$$I = \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x}\, dx.$$

Step 2: Use transformation $x \to \pi - x$.
Using the property:

$$I = \int_0^{\pi} f(\pi - x)\, dx.$$

Substituting,

$$f(\pi - x) = \frac{(\pi - x)\sin(\pi - x)}{1 + \cos^2(\pi - x)}.$$

Since $\sin(\pi - x) = \sin x$ and $\cos(\pi - x) = -\cos x$,

$$I = \int_0^{\pi} \frac{(\pi - x)\sin x}{1 + \cos^2 x}\, dx.$$

Step 3: Add the transformed integral.

$$2I = \int_0^{\pi} \frac{x \sin x + (\pi - x)\sin x}{1 + \cos^2 x}\, dx.$$

$$= \int_0^{\pi} \frac{\pi \sin x}{1 + \cos^2 x}\, dx.$$

Step 4: Factor out π.

$$2I = \pi \int_0^{\pi} \frac{\sin x}{1 + \cos^2 x}\, dx.$$

Using substitution $t = \cos x$,

$$dt = -\sin x\, dx.$$

Thus,

$$\int_0^{\pi} \frac{\sin x}{1 + \cos^2 x} dx = \int_1^{-1} \frac{-dt}{1 + t^2}.$$

Since the integral is symmetric,

$$\int_{-1}^{1} \frac{dt}{1 + t^2} = \tan^{-1} 1 - \tan^{-1}(-1) = \frac{\pi}{2}.$$

Step 5: Solve for I.

$$2I = \pi \cdot \frac{\pi}{2} = \frac{\pi^2}{2}.$$

$$I = \frac{\pi}{2}.$$

Thus,

$$\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x}\, dx = \frac{\pi}{2}.$$

Quick Tip

For definite integrals with symmetric limits, use transformations $x \to a - x$ and trigonometric substitutions for simplifications.

Do any one part

9(a).
Solve the differential equation $(1 + y^2)dx = (\tan^{-1} y - x)dy$**.**
Solution:
Step 1: Rewrite the equation in differential form.

$$\frac{dx}{dy} = \frac{\tan^{-1} y - x}{1 + y^2}$$

This is a linear differential equation of the form:

$$\frac{dx}{dy} + P(y)x = Q(y)$$

where

$$P(y) = \frac{1}{1 + y^2}, \quad Q(y) = \frac{\tan^{-1} y}{1 + y^2}$$

Step 2: Find the integrating factor (IF).

$$\mu(y) = e^{\int P(y)dy} = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1} y}$$

Step 3: Solve for x.
Multiplying the equation by $\mu(y)$:

$$e^{\tan^{-1} y} x = \int e^{\tan^{-1} y} \cdot \frac{\tan^{-1} y}{1 + y^2} dy$$

Using integration techniques,

$$x = Ce^{-\tan^{-1} y} + \tan^{-1} y$$

This is the general solution.

Quick Tip

To solve linear differential equations of the form $\frac{dx}{dy} + P(y)x = Q(y)$, use the integrating factor $\mu(y) = e^{\int P(y)dy}$.

9(b)(i).
If $\cos y = x \cos(a + y)$ **and** $\cos a \neq \pm 1$**, then prove that**

$$\frac{dy}{dx} = \frac{\cos^2(a + y)}{\sin a}.$$

Solution:
Step 1: Differentiate both sides with respect to x.

$$\frac{d}{dx}(\cos y) = \frac{d}{dx}(x \cos(a + y))$$

Using the chain rule:

$$-\sin y\frac{dy}{dx} = \cos(a+y) + x(-\sin(a+y)\frac{dy}{dx})$$

Step 2: Solve for $\frac{dy}{dx}$.

$$-\sin y\frac{dy}{dx} + x\sin(a+y)\frac{dy}{dx} = \cos(a+y)$$

Rewriting:

$$\frac{dy}{dx}(\sin(a+y)x - \sin y) = \cos(a+y)$$

Using the identity $\sin y = \sin(a+y)\sin a$,

$$\frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$$

Thus, the required result is proved.

Quick Tip

Use chain rule and trigonometric identities to differentiate composite trigonometric functions.

9(b)(ii).
If $y = (\sin x)^{\tan x}$, then find $\frac{dy}{dx}$.
Solution:
Step 1: Take logarithm on both sides.

$$\ln y = \tan x \ln(\sin x)$$

Step 2: Differentiate both sides.
Using the product rule:

$$\frac{1}{y}\frac{dy}{dx} = \sec^2 x \ln(\sin x) + \tan x\frac{\cos x}{\sin x}$$

$$\frac{dy}{dx} = y\left(\sec^2 x \ln(\sin x) + \frac{\tan x}{\sin x}\cos x\right)$$

Step 3: Substitute $y = (\sin x)^{\tan x}$.

$$\frac{dy}{dx} = (\sin x)^{\tan x}\left(\sec^2 x \ln(\sin x) + \frac{\tan x}{\sin x}\cos x\right)$$

Thus, the derivative is obtained.

Quick Tip

For functions in the form $y = f(x)^{g(x)}$, take logarithm and then differentiate using implicit differentiation.
