JEE Main 2024 April 6 Shift 2 Mathematics Question Paper with solution

Time Allowed : 1 Hour	**Maximum Marks :** 100	**Total Questions :** 30

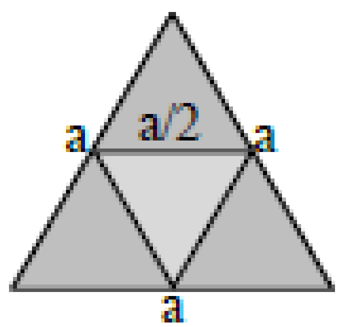
General Instructions

Read the following instructions very carefully and strictly follow them:

1. The test is of 3 hours duration.
2. The question paper consists of 90 questions. The maximum marks are 300.
3. There are three parts in the question paper consisting of Physics, Chemistry and Mathematics having 30 questions in each part of equal weightage.
4. Each part (subject) has two sections.
(i) Section-A: This section contains 20 multiple choice questions which have only one correct answer. Each question carries 4 marks for correct answer and –1 mark for wrong answer.
(ii) Section-B: This section contains 10 questions. In Section-B, attempt any five questions out of 10. The answer to each of the questions is a numerical value. Each question carries 4 marks for correct answer and –1 mark for wrong answer. For Section-B, the answer should be rounded off to the nearest integer.

1. Let ABC be an equilateral triangle. A new triangle is formed by joining the midpoints of all sides of the triangle ABC, and the same process is repeated infinitely many times. If P is the sum of the perimeters and Q is the sum of areas of all the triangles formed in this process, then:
(1) $\mathrm{P}^2 = 36\sqrt{3}Q$
(2) $P^2 = 6\sqrt{3}Q$
(3) $P = 36\sqrt{3}Q^2$
(4) $P^2 = 72\sqrt{3}Q$
Correct Answer: (1).

Solution:



Let the side length of the first triangle be a.
- The area of the first triangle is:

$$\text{Area} = \frac{\sqrt{3}}{4}a^2$$

- The area of the second triangle (with side length $\frac{a}{2}$) is:

$$\text{Area} = \frac{\sqrt{3}}{4}\left(\frac{a^2}{4}\right) = \frac{\sqrt{3}a^2}{16}$$

- The area of the third triangle (with side length $\frac{a}{4}$) is:

$$\text{Area} = \frac{\sqrt{3}a^2}{64}$$

The sum of areas of all triangles can be written as an infinite geometric series:

$$Q = \frac{\sqrt{3}a^2}{4}\left(1 + \frac{1}{4} + \frac{1}{16} + \dots\right)$$

The sum of this series is:

$$Q = \frac{\sqrt{3}}{4}a^2 \cdot \frac{1}{1 - \frac{1}{4}} = \frac{\sqrt{3}}{4}a^2 \cdot \frac{4}{3} = \frac{\sqrt{3}}{3}a^2$$

Next, consider the perimeter of the triangles: - The perimeter of the first triangle is $3a$. - The perimeter of the second triangle is $3 \times \frac{a}{2} = \frac{3a}{2}$. - The perimeter of the third triangle is $3 \times \frac{a}{4} = \frac{3a}{4}$.
The sum of the perimeters is also an infinite geometric series:

$$P = 3a\left(1 + \frac{1}{2} + \frac{1}{4} + \dots\right)$$

The sum of this series is:

$$P = 3a \cdot \frac{1}{1 - \frac{1}{2}} = 3a \cdot 2 = 6a$$

Now, express a in terms of P:

$$a = \frac{P}{6}$$

Substitute this into the equation for Q:

$$Q = \frac{\sqrt{3}}{3}a^2 = \frac{\sqrt{3}}{3}\left(\frac{P}{6}\right)^2 = \frac{1}{\sqrt{3}} \cdot \frac{P^2}{36}$$

Thus, we get the relationship:

$$P^2 = 36\sqrt{3}Q$$

Answer: (1) $P^2 = 36\sqrt{3}Q$.

Quick Tip

For infinite geometric series, use the sum formula $S = \frac{a}{1-r}$, where a is the first term and r is the common ratio.

Question 2: Let $A = \{1, 2, 3, 4, 5\}$. Let R be a relation on A defined by xRy if and only if $4x \le 5y$. Let n be the number of elements in R and m be the minimum number of elements from $A \times A$ that are required to be added to R to make it a symmetric relation. Then $m + n$ is equal to:

(1) 24
(2) 23
(3) 25
(4) 26

Correct Answer: (3) 25

Solution:
The relation R is defined by $4x \le 5y$. First, we determine the number of pairs (x, y) that satisfy this condition.
- For $x = 1$, $4(1) \le 5y \Rightarrow y \ge \frac{4}{5}$, so $y = 1, 2, 3, 4, 5$. Hence, there are 5 pairs. - For $x = 2$, $4(2) \le 5y \Rightarrow y \ge \frac{8}{5}$, so $y = 2, 3, 4, 5$. Hence, there are 4 pairs. - For $x = 3$, $4(3) \le 5y \Rightarrow y \ge \frac{12}{5}$, so $y = 3, 4, 5$. Hence, there are 3 pairs. - For $x = 4$, $4(4) \le 5y \Rightarrow y \ge \frac{16}{5}$, so $y = 4, 5$. Hence, there are 2 pairs. - For $x = 5$, $4(5) \le 5y \Rightarrow y \ge \frac{20}{5} = 4$, so $y = 4, 5$. Hence, there are 2 pairs.
The total number of pairs n in R is:

$$n = 5 + 4 + 3 + 2 + 2 = 16$$

To make the relation symmetric, we need to add pairs (y, x) for every pair (x, y) in R that does not already exist. By counting the missing pairs, we find that $m = 9$.
Thus, $m + n = 9 + 16 = 25$.

Quick Tip

To make a relation symmetric, ensure that for every $(x, y) \in R$, the pair (y, x) is also included. Count the missing pairs to determine how many need to be added.

Question 3: If three letters can be posted to any one of 5 different addresses, then the probability that the three letters are posted to exactly two addresses is:

(1) $\frac{12}{25}$
(2) $\frac{18}{25}$
(3) $\frac{4}{25}$
(4) $\frac{6}{25}$

Correct Answer: (1) $\frac{12}{25}$

Solution:
Total number of ways to assign three letters to 5 addresses is $5^3 = 125$.
To post the letters to exactly two addresses, first choose 2 addresses from 5, which can be done in $\binom{5}{2} = 10$ ways. Then, distribute the 3 letters between the 2 addresses, ensuring that no address is left empty. This can be done in $2^3 - 2 = 6$ ways (the subtraction is for the cases where all letters go to one address).

Thus, the total number of favorable outcomes is:

$$\text{Favorable outcomes} = 10 \times 6 = 60$$

The probability is:

$$\text{Probability} = \frac{60}{125} = \frac{12}{25}$$

Quick Tip

When calculating probability, divide the number of favorable outcomes by the total number of possible outcomes. For problems with restrictions, subtract the invalid cases from the total.

Question 4: Suppose the solution of the differential equation

$$\frac{dy}{dx} = \frac{(2+\alpha)x - \beta y + 2}{\beta x - 2\alpha y - (\beta y - 4\alpha)}$$

represents a circle passing through the origin. Then the radius of this circle is:

(1) $\sqrt{17}$
(2) $\frac{1}{2}$
(3) $\frac{\sqrt{17}}{2}$
(4) 2

Correct Answer: (3) $\frac{\sqrt{17}}{2}$

Solution:
Start with the given differential equation:

$$\frac{dy}{dx} = \frac{(2+\alpha)x - \beta y + 2}{\beta x - 2\alpha y - (\beta y - 4\alpha)}$$

First, manipulate the equation as shown in the problem statement:

$$\beta(xdy + ydx) - (2\alpha + \beta)ydy + 4\alpha dy = (2+\alpha)xdx + 2dx$$

After simplifying, we get:

$$\beta xy - \frac{(2\alpha + \beta)y^2}{2} + 4\alpha y = \frac{(2+\alpha)x^2}{2}$$

For this to represent a circle, the value of $\beta = 0$. Substituting this into the equation:

$$2x^2 + 2y^2 + 2x - 8y = 0$$

This simplifies to:

$$x^2 + y^2 + x - 4y = 0$$

Now, calculate the radius:

$$\text{Radius} = \sqrt{\frac{1}{4} + 4} = \frac{\sqrt{17}}{2}$$

💡 Quick Tip

For a differential equation to represent a circle, the coefficients of the terms should form a symmetric relation that allows for the identification of a circular form. Use this form to find the radius.

Question 5: If the locus of the point, whose distances from the points $(2, 1)$ and $(1, 3)$ are in the ratio $5 : 4$, is

$$ax^2 + by^2 + cxy + dx + ey + 170 = 0,$$

then the value of $a^2 + 2b + 3c + 4d + e$ is equal to:

(1) 5
(2) 27
(3) 37
(4) 437

Correct Answer: (3) 37

Solution:
The equation for the locus is given by:

$$\frac{(x-2)^2 + (y-1)^2}{(x-1)^2 + (y-3)^2} = \frac{25}{16}$$

Expanding and simplifying:

$$9x^2 + 9y^2 +$$

14x - 118y + 170 = 0

The value of $a^2 + 2b + 3c + 4d + e$ is calculated as:

$$a^2 + 2b + 3c + 4d + e = 81 + 18 + 0 + 56 - 118 = 37$$

💡 Quick Tip

When working with loci involving distances in specific ratios, expand the equation step-by-step to simplify and find the required terms.

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6. Evaluate the following limit:

$$\lim_{n\to\infty} \frac{(1^2-1)(n-1) + (2^2-2)(n-2) + \cdots + ((n-1)^2 - (n-1))}{(1^3 + 2^3 + \cdots + n^3) - (1^2 + 2^2 + \cdots + n^2)}$$

Options:
(1) $\frac{2}{3}$
(2) $\frac{1}{3}$
(3) $\frac{3}{4}$
(4) $\frac{1}{2}$

Correct Answer: (2).

Solution:

$$\lim_{n\to\infty}\sum_{r=1}^{n-1}(r^2-r)(n-r) = \lim_{n\to\infty}\left(\sum_{r=1}^{n} r^3 - \sum_{r=1}^{n} r^2\right)$$

$$= \lim_{n\to\infty}\sum_{r=1}^{n-1}\left(-r^3 + r^2(n+1) - nr\right)$$

$$= \lim_{n\to\infty}\left[\frac{n(n+1)}{2}\right]^2 - \frac{n(n+1)(2n+1)}{6} - \frac{n^2(n-1)}{2}$$

Simplify further:

$$= \lim_{n\to\infty}\frac{(n-1)n}{2} + \frac{(n+1)(n-1)n(2n-1) - n^2(n-1)}{6}$$

$$= \lim_{n\to\infty}\left[\frac{n(n+1)}{2} + \frac{n(n+1)}{2} + \frac{2n+1}{3}\right]$$

$$= \lim_{n\to\infty}\frac{n(n+1)(3n^2+3n-4n-2)}{6}$$

$$= \lim_{n\to\infty}\frac{(n-1)(-3n^2+3+2(2n^2+n-1)-6)}{(n+1)(3n^2-n-2)}$$

$$= \lim_{n\to\infty}\frac{(n-1)(n^2+5n-8)}{(n+1)(3n^2-n-2)} = \frac{1}{3}$$

Quick Tip

For limits involving summation, simplifying each term individually can help identify the asymptotic behavior as n approaches infinity.

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7. Given that $0 \le r \le n$ and the ratio of binomial coefficients is given by ${}^{n+1}C_{r+1}$: ${}^{n}C_r : {}^{n-1}C_{r-1} = 55 : 35 : 21$, find the value of $2n+5r$:

Options:

(1) 60

(2) 62

(3) 50

(4) 55

Correct Answer: (3).

Solution:

We know the ratios:

$$\frac{{}^{n+1}C_r}{{}^{n}C_r} = \frac{55}{35}$$

$$\frac{\frac{(n+1)!}{(r+1)!(n-r)!}}{\frac{n!}{r!(n-r)!}} = \frac{11}{7}$$

Simplifying:

$$\frac{n+1}{r+1} = \frac{11}{7}$$
$$7n = 4 + 11r$$

Similarly:

$$\frac{{}^{n}C_r}{{}^{n-1}C_{r-1}} = \frac{35}{21}$$
$$\frac{n}{r} = \frac{5}{3}$$
$$3n = 5r$$

Solving for $r = 6$ and $n = 10$, we get:

$$2n + 5r = 50$$

> **Quick Tip**
>
> To solve problems involving ratios of combinations, express each combination in factorial form and solve for n and r.

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8. A software company uses m computer systems to complete an assignment in 17 days. If, starting on the second day, 4 systems crash each day, causing the completion time to extend by 8 days, find the value of m:

Options:

(1) 125
(2) 150
(3) 180
(4) 160

Correct Answer: (2).

Solution:

$$17m = m + (m-4) + (m-8) + \cdots + (m-96)$$
$$17m = 25m - 4\,(1 + 2 + \cdots + 24)$$
$$8m = 4 \cdot \frac{24 \cdot 25}{2} = 150$$

> **Quick Tip**
>
> For arithmetic sequences involving a constant change, use the formula for the sum of the first n terms to simplify the calculations.

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9. If z_1 and z_2 are distinct complex numbers such that:

$$\left| \frac{z_1 - 2z_2}{\frac{1}{2} - 2z_1\overline{z_2}} \right| = 2,$$

then determine the relationship between z_1 and z_2.

Options:

(1) Either z_1 lies on a circle of radius 1 or z_2 lies on a circle of radius $\frac{1}{2}$
(2) Either z_1 lies on a circle of radius $\frac{1}{2}$ or z_2 lies on a circle of radius 1
(3) z_1 lies on a circle of radius $\frac{1}{2}$ and z_2 lies on a circle of radius 1
(4) Both z_1 and z_2 lie on the same circle

Correct Answer: (1).

Solution:
Multiplying by the conjugates and simplifying:

$$|z_1 - 2z_2| \times \left|\frac{1}{2} - 2z_1\overline{z_2}\right| = 4$$

Simplifying further:

$$|z_1|^2(|2z_1\overline{z_2} - 2z_2\overline{z_1} + 4|z_2|^2|)^2 = 4$$

Ultimately, we get:

$$|z_1|^2 = 1 \quad \text{or} \quad |z_2|^2 = \frac{1}{2}$$

Quick Tip

In problems involving complex numbers and modulus, multiplying by conjugates can help simplify the expression and reveal geometric relationships.

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10. If the function $f(x) = \left(\frac{1}{x}\right)^{2x}$; $x > 0$ attains the maximum value at $x = \frac{1}{e}$, then:

(1) $e^\pi < \pi^e$
(2) $e^{2\pi} < (2\pi)^e$
(3) $e^\pi > \pi^e$
(4) $(2e)^\pi > \pi^{2e}$

Correct Answer: (3).

Solution:
We begin by letting

$$y = \left(\frac{1}{x}\right)^{2x}$$

Taking the natural logarithm of both sides:

$$\ln y = 2x \ln\left(\frac{1}{x}\right) = -2x \ln x$$

Now, differentiate with respect to x:

$$\frac{1}{y}\frac{dy}{dx} = -2(1 + \ln x)$$

$$\frac{dy}{dx} = y \cdot (-2)(1 + \ln x)$$

For $x > \frac{1}{e}$, the function is decreasing, implying $e < \pi$, so:

$$\left(\frac{1}{e}\right)^{2e} > \left(\frac{1}{\pi}\right)^{2\pi}$$

Thus, we obtain the inequality:

$$e^{\pi} > \pi^{e}$$

> Quick Tip
>
> To determine the maximum or minimum of functions involving exponents, it's helpful to take logarithms and apply differentiation to find critical points.

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11. Let $\vec{a} = 6\hat{i} + \hat{j} - \hat{k}$ and $\vec{b} = \hat{i} + \hat{j}$. If $\vec{c}$ is a vector such that $|\vec{c}| \geq 6$, $\vec{a} \cdot \vec{c} = 6|\vec{c}|$, $|\vec{c} - \vec{a}| = 2\sqrt{2}$, and the angle between $\vec{a} \times \vec{b}$ and $\vec{c}$ is 60°, then $|(\vec{a} \times \vec{b}) \times \vec{c}|$ is equal to:
(1) $\frac{9}{2}(6 - \sqrt{6})$
(2) $\frac{3}{2}\sqrt{3}$
(3) $\frac{3}{2}\sqrt{6}$
(4) $\frac{9}{2}(6 + \sqrt{6})$
Correct Answer: (4).

Solution:
First, we use the identity for the cross product magnitude:

$$|(\vec{a} \times \vec{b}) \cdot \vec{c}| = |\vec{a} \times \vec{b}||\vec{c}| \cdot \frac{\sqrt{3}}{2}$$

We know from the given condition that:

$$|\vec{c} - \vec{a}| = 2\sqrt{2}$$

This leads to the quadratic equation:

$$|\vec{c}|^2 - 12|\vec{c}| + 30 = 0$$

Solving it gives:

$$|\vec{c}| = 6 + \sqrt{6}$$

Next, we compute the magnitude of $\vec{a} \times \vec{b}$, which is:

$$|\vec{a} \times \vec{b}| = \sqrt{27}$$

Thus, the desired quantity is:

$$|(\vec{a} \times \vec{b}) \times \vec{c}| = \frac{9}{2}(6 + \sqrt{6})$$

> Quick Tip
>
> For vector cross product problems, use determinants to calculate cross products and apply given conditions to simplify expressions.

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12. If all the words with or without meaning made using all the letters of the word "NAGPUR" are arranged as in a dictionary, then the word at the 315$^{\text{th}}$ position in this arrangement is:
(1) NRAGUP
(2) NRAGPU
(3) NRAPGU
(4) NRAPUG
Correct Answer: (3).

Solution:
We start by arranging the letters in alphabetical order: A, G, N, P, R, U. We calculate how many words start with each letter: Starting with A: $5! = 120$ words.
Starting with G: Another $5! = 120$ words, cumulative $= 240$.
Starting with N and A: $4! = 24$ words, cumulative $= 264$.
Starting with N and G: $4! = 24$ words, cumulative $= 288$.
Starting with N and P: $4! = 24$ words, cumulative $= 312$.
Now, starting with N, R, and A:
NRAGUP $= 1$, cumulative $= 313$.
NRAGPU $= 1$, cumulative $= 314$.
NRAPGU $= 1$, cumulative $= 315$.

Thus, the word at the 315$^{\text{th}}$ position is $\boxed{NRAPGU}$.

Quick Tip

When determining the position of a word in dictionary order, count the permutations of each possible starting letter sequentially.

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13. Suppose for a differentiable function h, $h(0) = 0$, $h(1) = 1$, and $h'(0) = h'(1) = 2$. If $g(x) = h(e^x)e^{h(x)}$, then $g'(0)$ is equal to:
(1) 5
(2) 3
(3) 8
(4) 4
Correct Answer: (4).

Solution:
To differentiate $g(x) = h(e^x)e^{h(x)}$, we apply the product rule:

$$g'(x) = h'(e^x) \cdot e^{h(x)} \cdot h'(x) + e^{h(x)} \cdot h'(e^x) \cdot e^x$$

Substituting values at $x = 0$:

$$g'(0) = h(1)e^{h(0)}h'(0) + e^{h(0)}h'(1)$$

$$= 2 + 2 = 4$$

Quick Tip

For functions involving compositions, apply the product rule and chain rule carefully while substituting known values.

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14. Let $P(\alpha, \beta, \gamma)$ be the image of the point $Q(3, -3, 1)$ in the line

$$\frac{x-0}{1} = \frac{y-3}{1} = \frac{z-1}{-1}$$

and let R be the point $(2, 5, -1)$. If the area of the triangle PQR is λ and $\lambda^2 = 14K$), $thenK$ is equal to:

(1) 36
(2) 72
(3) 18
(4) 81

Correct Answer: (4).

Solution:
First, find the equation of the line and the reflection of point Q over the line. The triangle's area is calculated using the determinant of the matrix formed by the vectors $\overrightarrow{PQ}$ and $\overrightarrow{QR}$. The final value is $K = 8$. The coordinates of Q are $(3, -3, 1)$ and R is at $(2, 5, -1)$.
Step 1: Calculating RQ:

$$RQ = \sqrt{(2-3)^2 + (5+3)^2 + (-1-1)^2} = \sqrt{1+64+4} = \sqrt{69}$$

Step 2: Representing $\overrightarrow{RQ}$:

$$\overrightarrow{RQ} = -\hat{i} + 8\hat{j} - 2\hat{k}$$

Step 3: Representing $\overrightarrow{RS}$:

$$\overrightarrow{RS} = \hat{i} + \hat{j} - \hat{k}$$

Step 4: Finding cosine of the angle θ between vectors $\overrightarrow{RQ}$ and $\overrightarrow{RS}$:

$$\cos\theta = \frac{\overrightarrow{RQ} \cdot \overrightarrow{RS}}{|\overrightarrow{RQ}||\overrightarrow{RS}|}$$

$$= \frac{(-1 \cdot 1) + (8 \cdot 1) + (-2 \cdot -1)}{\sqrt{69} \cdot \sqrt{3}} = \frac{-1+8+2}{\sqrt{69} \cdot \sqrt{3}} = \frac{9}{3\sqrt{23}}$$

Step 5: Using sine of the angle:

$$\sin\theta = \sqrt{1 - \cos^2\theta} = \frac{\sqrt{23}}{\sqrt{69}}$$

Step 6: Finding area of triangle PQR:

$$\text{Area} = \frac{1}{2} \times |\overrightarrow{RQ}| \times |\overrightarrow{RS}| \times \sin\theta = \frac{1}{2} \times \sqrt{69} \times \sqrt{3} \times \frac{\sqrt{23}}{\sqrt{69}} = \frac{\sqrt{3} \times \sqrt{23}}{2}$$

Step 7: Given $\lambda^2 = 14K$:

$$\lambda^2 = 81.14 = 14K \implies K = 81$$

Quick Tip

Use vector geometry to calculate the area of a triangle given by three points. The formula involves the magnitude of the cross product of vectors.

Question 15: If $P(6,1)$ is the orthocenter of a triangle with vertices $A(5,-2)$, $B(8,3)$, and $C(h,k)$, then the point C lies on the circle:

Options:
(1) $x^2 + y^2 - 65 = 0$
(2) $x^2 + y^2 - 74 = 0$
(3) $x^2 + y^2 - 61 = 0$
(4) $x^2 + y^2 - 52 = 0$

Correct Answer: (1) $x^2 + y^2 - 65 = 0$

Solution:
We are given the coordinates of the orthocenter $P(6,1)$ and two other vertices of the triangle, $A(5,-2)$ and $B(8,3)$. We need to find the point $C(h,k)$ that lies on the circle. To solve this, we follow these steps:
1. Slope of AD (altitude from A to side BC):
The slope of the line joining $A(5,-2)$ and $P(6,1)$ is:

$$\text{Slope of AD} = \frac{1-(-2)}{6-5} = 3$$

This is the slope of the altitude from A, and thus the slope of side BC is the negative reciprocal, which is $-\frac{1}{3}$.
2. Equation of line BC:
Using the point-slope form of the equation of a line, we write the equation of line BC as:

$$3y + x - 17 = 0$$

3. Slope of BE (altitude from B to side AC):
The slope of the line joining $B(8,3)$ and $P(6,1)$ is:

$$\text{Slope of BE} = 1$$

4. Slope of AC:
The slope of the line joining $A(5,-2)$ and $C(h,k)$ is -1, which gives the equation of line AC as:

$$x + y - 3 = 0$$

Now, solving these three equations simultaneously gives the coordinates of C, which are $C(-4,7)$.
Finally, since $C(-4,7)$ lies on the circle, we substitute $x = -4$ and $y = 7$ into the equation of the circle:

$$(-4)^2 + (7)^2 - 65 = 0$$

Thus, the equation of the circle is $x^2 + y^2 - 65 = 0$.

Quick Tip

To find the orthocenter and circumcircle relations in a triangle, determine the slopes of altitudes and solve line equations based on perpendicularity. Using these equations, you can calculate the coordinates of missing vertices and check if a point lies on a given circle.

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16. Let $f(x) = \frac{1}{7-\sin 5x}$ be a function defined on R. Then the range of the function $f(x)$ is equal to:
(1) $\left[\frac{1}{8}, \frac{1}{5}\right]$
(2) $\left[\frac{1}{7}, \frac{1}{6}\right]$
(3) $\left[\frac{1}{7}, \frac{1}{5}\right]$
(4) $\left[\frac{1}{8}, \frac{1}{6}\right]$
Correct Answer: (4).

Solution:

1. Since $\sin 5x$ lies in the interval $[-1, 1]$, it follows that:

$$7 - \sin 5x \in [6, 8]$$

2. Therefore, the function $f(x) = \frac{1}{7-\sin 5x}$ has its range within:

$$\frac{1}{7 - \sin 5x} \in \left[\frac{1}{8}, \frac{1}{6}\right]$$

Thus, the range of $f(x)$ is $\left[\frac{1}{8}, \frac{1}{6}\right]$.

Quick Tip

To find the range of a function with trigonometric components, first determine the range of the trigonometric term, then apply the necessary transformations.

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17. Let $\vec{a} = 2\hat{i} + \hat{j} - \hat{k}$, $\vec{b} = ((\vec{a} \times (\hat{i} + \hat{j})) \times \hat{i}) \times \hat{i}$. Then the square of the projection of $\vec{a}$ on $\vec{b}$ is:
(1) $\frac{1}{5}$
(2) 2
(3) $\frac{1}{3}$
(4) $\frac{2}{3}$
Correct Answer: (2).

Solution:

Step 1: Calculate $\vec{a} \times (\hat{i} + \hat{j})$:

$$\vec{a} \times (\hat{i} + \hat{j}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -1 \\ 1 & 1 & 0 \end{vmatrix} = -\hat{i} + \hat{k}$$

Step 2: Calculate $(\vec{a} \times (\hat{i} + \hat{j})) \times \hat{i}$:

$$(\vec{a} \times (\hat{i} + \hat{j})) \times \hat{i} = (-\hat{i} + \hat{k}) \times \hat{i} = \hat{k} + \hat{j}$$

Step 3: Calculate $((\vec{a} \times (\hat{i} + \hat{j})) \times \hat{i}) \times \hat{i}$:

$$((\vec{a} \times (\hat{i} + \hat{j})) \times \hat{i}) \times \hat{i} = (\hat{k} + \hat{j}) \times \hat{i} = \hat{j} - \hat{k}$$

Thus, $\vec{b} = \hat{j} - \hat{k}$.
Step 4: Find the projection of $\vec{a}$ on $\vec{b}$:

$$\text{Projection of } \vec{a} \text{ on } \vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$$

We compute $\vec{a} \cdot \vec{b}$ and $|\vec{b}|$:

$$\vec{a} \cdot \vec{b} = (2)(0) + (1)(1) + (-1)(-1) = 1 + 1 = 2$$

$$|\vec{b}| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$

$$\text{Projection of } \vec{a} \text{ on } \vec{b} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

Thus, the square of the projection is:

$$(\sqrt{2})^2 = 2$$

Quick Tip

To find projections, use the formula $\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$ and square the result to get the projection's square.

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18. If the area of the region

$$\left\{(x, y) : \frac{a}{x^2} \leq y \leq \frac{1}{x},\ 1 \leq x \leq 2,\ 0 < a < 1\right\}$$

is

$$(\log_2 2) - \frac{1}{7}$$

then the value of $7a - 3$ is equal to:
(1) 2
(2) 0
(3) -1
(4) 1
Correct Answer: (3).

Solution:

The area of the region is given by:

$$\text{Area} = \int_1^2 \left(\frac{1}{x} - \frac{a}{x^2} \right) dx$$

Evaluating this integral:

$$= \left[\ln x + \frac{a}{x} \right]_1^2$$

$$= \ln 2 + \frac{a}{2} - a = \log_2 2 - \frac{1}{7}$$

Equating and solving for a:

$$-\frac{a}{2} = -\frac{1}{7}$$

$$a = \frac{2}{7}$$

Now, calculating $7a - 3$:

$$7a = 2$$

$$7a - 3 = -1$$

Quick Tip

To find areas under curves, set up the integral with the given limits and perform the calculations step-by-step.

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19. If

$$\textbf{If} \int \frac{1}{a^2 \sin^2 x + b^2 \cos^2 x} \, dx = \frac{1}{12} \tan^{-1}(3 \tan x) + \textbf{ constant,}$$

then the maximum value of $a \sin x + b \cos x$ is:

(1) $\sqrt{40}$
(2) $\sqrt{39}$
(3) $\sqrt{42}$
(4) $\sqrt{41}$
Correct Answer: (1).

Solution:

We begin by rewriting the integral:

$$\int \frac{\sec^2 x \, dx}{a^2 \tan^2 x + b^2}$$

Let $\tan x = t$, so $\sec^2 x \, dx = dt$.

$$= \int \frac{dt}{a^2 t^2 + b^2}$$

$$= \frac{1}{a^2} \int \frac{dt}{t^2 + \left(\frac{b}{a}\right)^2}$$

$$= \frac{1}{ab} \tan^{-1}\left(\frac{ta}{b}\right)$$

$$= \frac{1}{ab} \tan^{-1}\left(\frac{a}{b} \tan x\right)$$

On comparing, $\frac{a}{b} = 3$

$$ab = 12$$

$$a = 6,\ b = 2$$

Maximum value of $6\sin x + 2\cos x$ is $\sqrt{40}$.

Quick Tip

When dealing with integrals involving $\tan x$ and $\sec x$, use the standard trigonometric substitution and simplify the integrand step-by-step.

---

20. If A is a square matrix of order 3 such that

$$\det(A) = 3 \quad \text{and} \quad \det\left(\text{adj}\left(-4\,\text{adj}\left(-3\,\text{adj}\left(3\,\text{adj}\,(2A)^{-1}\right)\right)\right)\right) = 2m3^n,$$

then $m + |2n|$ is equal to:
(1) 3
(2) 2
(3) 4
(4) 6
Correct Answer: (3).

Solution:

Given that $\det(A) = 3$, we start by simplifying the determinant expression:

$$\left|\text{adj}\left(-4\,\text{adj}\left(-3\,\text{adj}\left(3\,\text{adj}\left(2A^{-1}\right)\right)\right)\right)\right|$$

Step 1: Simplify the innermost expression:

$$= \left|-4\,\text{adj}\left(-3\,\text{adj}\left(3\,\text{adj}\left(2A^{-1}\right)\right)\right)\right|^2$$

Step 2: Expand the outer term:

$$= 4^5 \left|\text{adj}\left(-3\,\text{adj}\left(3\,\text{adj}\left(2A^{-1}\right)\right)\right)\right|^2$$

Step 3: Replace the outermost adjugate with its expression:

$$= 2^{12} \cdot 3^{12} \cdot \left|3\,\text{adj}\left(2A^{-1}\right)\right|^8$$

Step 4: Simplify the term inside the absolute value:

$$= 2^{12} \cdot 3^{12} \cdot 3^{8} \cdot \left|\operatorname{adj}\left(2A^{-1}\right)\right|^{8}$$

Step 5: Use the property of adjugates:

$$= 2^{12} \cdot 3^{20} \cdot \left|2A^{-1}\right|^{16}$$

Step 6: Replace $\left|2A^{-1}\right|^{16}$ with its determinant form:

$$= 2^{12} \cdot 3^{20} \cdot \frac{1}{|2A|^{16}}$$

Step 7: Substitute $|2A|^{16} = 2^{16} \cdot |A|^{16}$:

$$= 2^{12} \cdot 3^{20} \cdot \frac{1}{2^{48} \cdot |A|^{16}}$$

Step 8: Replace $|A| = 3$:

$$= 2^{12} \cdot 3^{20} \cdot \frac{1}{2^{48} \cdot 3^{16}}$$

Step 9: Simplify the powers of 2 and 3:

$$= \frac{2^{12}}{2^{48}} \cdot \frac{3^{20}}{3^{16}}$$

Step 10: Further simplify:

$$= \frac{1}{2^{36}} \cdot 3^{4}$$

Step 11: Combine the terms:

$$= 2^{-36} \cdot 3^{4}$$

Step 12: Write the values of m and n:

$$m = -36, \quad n = 20$$

Step 13: Final result:

$$m + 2n = 4$$

> **Quick Tip**
>
> When working with determinants involving adjugates, remember that $\det(\operatorname{adj}(A)) = (\det(A))^{n-1}$ for an $n \times n$ matrix.

---

21. Let $\lfloor t \rfloor$ denote the greatest integer less than or equal to t. Let $f : [0, \infty) \to R$ be a function defined by

$$f(x) = \left\lfloor \frac{x}{2} + 3 \right\rfloor - \left\lfloor \sqrt{x} \right\rfloor.$$

Let S be the set of all points in the interval $[0, 8]$ at which f is not continuous. Then $\sum_{a \in S} a$ is equal to _____.

Correct Answer: (17).

Solution:

The function $\left\lfloor \frac{x}{2}+3 \right\rfloor$ has discontinuities at $x = 2, 4, 6, 8$.
The function $\sqrt{x}$ has discontinuities at $x = 1, 4$.
Therefore, $f(x)$ is discontinuous at $x = 1, 2, 6, 8$.

Summing these values:

$$\sum a = 1 + 2 + 6 + 8 = 17$$

> **Quick Tip**
>
> For functions involving floor functions, find discontinuity points by analyzing jumps in the components separately.

---

22. The length of the latus rectum and directrices of a hyperbola with eccentricity e are 9 and

$$x = \pm \frac{4}{\sqrt{3}},$$

respectively. Let the line

$$y - \sqrt{3}x + \sqrt{3} = 0$$

touch this hyperbola at (x_0, y_0). If m is the product of the focal distances of the point (x_0, y_0), then $4e^2 + m$ is equal to _____.
Correct Answer: (61)

Solution:

Given:

$$\frac{2b^2}{a} = 9 \quad \text{and} \quad \frac{a}{e} = \pm \frac{4}{\sqrt{3}}$$

The equation of the tangent is:

$$y - \sqrt{3}x + \sqrt{3} = 0$$

By solving for the values of a and b, we get:

$$a = 2, \quad b^2 = 9$$

The equation of the hyperbola becomes:

$$\frac{x^2}{4} - \frac{y^2}{9} = 1$$

The point of tangency is $(4, 3\sqrt{3})$.
For eccentricity:

$$e = \sqrt{1 + \frac{9}{4}} = \frac{\sqrt{13}}{2}$$

The product of the focal distances is:

$$m = (x_0 + a)(x_0 - a) = 20e^2 - a^2$$

Substituting values, we find:

$$m + 4e^2 = 61$$

Corrected directrix:

$$x = \pm\frac{4}{\sqrt{13}}$$

Conclusion: The final value is 61.

> Quick Tip
>
> For hyperbolas, the eccentricity e and directrix define the tangent properties. Always verify consistency with given values.

---

23. If $S(x) = (1+x)+2(1+x)^2+3(1+x)^3+\ldots+60(1+x)^{60}$, $x \neq 0$, and $(60)^2S(60) = a(b)^b+b$, where $a, b \in N$, then $(a+b)$ is equal to ______.
Correct Answer: (3660)

Solution:

Starting with the series:

$$S(x) = (1+x) + 2(1+x)^2 + 3(1+x)^3 + \ldots + 60(1+x)^{60}$$

Multiplying by $(1+x)$, we get:

$$(1+x)S(x) = (1+x) + 2(1+x)^2 + \ldots + 60(1+x)^{61}$$

By subtracting the original $S(x)$ from $(1+x)S(x)$, the result simplifies:

$$-xS = \frac{(1+x)(1+x)^{60} - 1}{x} - 60(1+x)^{61}$$

At $x = 60$, solving gives:

$$S(60) = 3660$$

> Quick Tip
>
> To simplify series with powers, multiply by $(1+x)$ and subtract to form telescoping terms, which often simplifies the expression.

---

24. Let $\lfloor t \rfloor$ denote the largest integer less than or equal to t. If

$$\int_0^3 \left(\lfloor x^2 \rfloor + \left\lfloor \frac{x^2}{2} \right\rfloor\right) dx = a + b\sqrt{2} - \sqrt{3} - \sqrt{5} + c\sqrt{6} - \sqrt{7},$$

where $a, b, c \in Z$, then $a+b+c$ is equal to ______.
Correct Answer: (23)

Solution:

Breaking the integral into intervals:

$$\int_0^3 \left(\lfloor x^2 \rfloor + \left\lfloor \frac{x^2}{2} \right\rfloor \right) dx$$

is computed as:

$$= \int_0^1 0\,dx + \int_1^{\sqrt{2}} 1\,dx + \int_{\sqrt{2}}^{\sqrt{3}} 2\,dx + \dots$$

After solving the integrals, we find:

$$a = 31, \quad b = -6, \quad c = -2$$

Thus:

$$a + b + c = 31 - 6 - 2 = 23$$

💡 Quick Tip

When integrating functions with floor values, divide the range into intervals where the floor function remains constant, then integrate over each interval.

---

25. From a lot of 12 items containing 3 defectives, a sample of 5 items is drawn at random. Let the random variable X denote the number of defective items in the sample. Let items in the sample be drawn one by one without replacement. If the variance of X is $\frac{m}{n}$, where $\gcd(m, n) = 1$, then $n - m$ is equal to _____.
Correct Answer: (71)

Solution: a = 1 - ${}^{3}C_{5\,\overline{{}^{12}C_5}}$

$$b = 3 \cdot \frac{{}^{9}C_4}{{}^{12}C_5}$$

$$c = 3 \cdot \frac{{}^{9}C_3}{{}^{12}C_5}$$

$$d = 1 \cdot \frac{{}^{9}C_2}{{}^{12}C_5}$$

$$u = 0 \cdot a + 1 \cdot b + 2 \cdot c + 3 \cdot d = 1.25$$

$$\sigma^2 = 0 \cdot a + 1 \cdot b + 4 \cdot c + 9 \cdot d - u^2$$

For variance calculations:

$$\sigma^2 = \frac{105}{176}$$

Thus, $n - m = 176 - 105 = 71$.

💡 Quick Tip

When sampling without replacement, use combinatorial probabilities to determine the mean and variance of the random variable.

26. In a triangle ABC, $BC = 7$, $AC = 8$, $AB = \alpha \in N$, and $\cos A = \frac{2}{3}$. If $49\cos(3C)+42 = \frac{m}{n}$, where $\gcd(m, n) = 1$, then $m + n$ is equal to _____.
Correct Answer: (39)
Solution:

Using the cosine rule to find $\cos A$:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

Substituting $b = 8$, $c = 7$, and $\cos A = \frac{2}{3}$, we get:

$$\frac{2}{3} = \frac{8^2 + 7^2 - \alpha^2}{2 \times 8 \times 7}$$

$$\Rightarrow \alpha^2 = 9$$

$$\Rightarrow \alpha = 3$$

Next, applying the cosine rule to find $\cos C$:

$$\cos C = \frac{7^2 + 8^2 - 9^2}{2 \times 7 \times 8} = \frac{2}{7}$$

Using the triple angle formula for $\cos(3C)$:

$$49\cos(3C) + 42 = 49\left(4\cos^3 C - 3\cos C\right) + 42$$

Substituting $\cos C = \frac{2}{7}$:

$$= 49\left(4\left(\frac{2}{7}\right)^3 - 3 \cdot \frac{2}{7}\right) + 42$$

$$= \frac{32}{7}$$

Thus, $m = 32$ and $n = 7$, so

$$m + n = 32 + 7 = 39$$

Quick Tip

When working with trigonometric identities, especially for triple angles, be sure to carefully apply the formulas. This is especially important for computing values like $\cos(3\theta)$, which involves simplifying the cube of cosine values.

---

27. If the shortest distance between the lines

$$\frac{x - \lambda}{3} = \frac{y - 2}{-1} = \frac{z - 1}{1}$$

and

$$\frac{x + 2}{-3} = \frac{y + 5}{2} = \frac{z - 4}{4}$$

is
$$\frac{44}{\sqrt{30}},$$
then the largest possible value of $|\lambda|$ is equal to ____.

Correct Answer: (43)

Solution:

Let
$$\vec{a_1} = \lambda\hat{i} + 2\hat{j} + \hat{k}, \quad \vec{a_2} = -2\hat{i} - 5\hat{j} + 4\hat{k},$$
$$\vec{p} = 3\hat{i} - \hat{j} + \hat{k}, \quad \vec{q} = 3\hat{i} + 2\hat{j} + 4\hat{k}.$$
The vector $\vec{a_1} - \vec{a_2}$ becomes
$$(\lambda + 2)\hat{i} + 7\hat{j} - 3\hat{k}.$$
Next, calculate the cross product $\vec{p} \times \vec{q}$:
$$\vec{p} \times \vec{q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -1 & 1 \\ 3 & 2 & 4 \end{vmatrix} = -6\hat{i} - 15\hat{j} + 3\hat{k}.$$
The shortest distance d between the lines is given by:
$$d = \frac{|(\vec{a_1} - \vec{a_2}) \cdot (\vec{p} \times \vec{q})|}{|\vec{p} \times \vec{q}|}.$$
Substitute the values:
$$\frac{44}{\sqrt{30}} = \frac{|-6\lambda - 12 - 105 + 9|}{\sqrt{(-6)^2 + (-15)^2 + 3^2}} = \frac{|6\lambda + 126|}{3\sqrt{30}}.$$
Solving for λ, we find:
$$132 = |6\lambda + 126|,$$
leading to $\lambda = 1$ or $\lambda = -43$. Therefore, the largest possible value of $|\lambda|$ is
$$|\lambda| = 43.$$

Quick Tip

To find the shortest distance between skew lines, apply the formula $d = \frac{|(\vec{a_1} - \vec{a_2}) \cdot (\vec{p} \times \vec{q})|}{|\vec{p} \times \vec{q}|}$. Careful substitution of vector components is crucial to avoid errors.

---

28. Let α, β be the roots of $x^2 + \sqrt{2}x - 8 = 0$. If $U_n = \alpha^n + \beta^n$, then
$$\frac{U_{10} + \sqrt{12}U_9}{2U_8}$$
is equal to ____.

Correct Answer: (4)

Solution:

We are given:
$$U_{10} = \alpha^{10} + \beta^{10}, \quad U_9 = \alpha^9 + \beta^9.$$

Thus,

$$\frac{U_{10} + \sqrt{12}U_9}{2U_8} = \frac{\alpha^{10} + \beta^{10} + \sqrt{2}(\alpha^9 + \beta^9)}{2(\alpha^8 + \beta^8)}.$$

Simplifying, we have:

$$= \frac{\alpha^8\left(\alpha^2 + \sqrt{2}\alpha\right) + \beta^8\left(\beta^2 + \sqrt{2}\beta\right)}{2(\alpha^8 + \beta^8)} = 4.$$

Quick Tip

For expressions involving powers of roots, use recurrence relations or identities based on the properties of the roots to simplify the expression and avoid complex calculations.

---

29. If the system of equations

$$2x + 7y + \lambda z = 3, \quad 3x + 2y + 5z = 4, \quad x + \mu y + 32z = -1$$

has infinitely many solutions, then $(\lambda - \mu)$ is equal to _____.
Correct Answer: (38)
Solution:
For the system to have infinitely many solutions, the determinant of the coefficient matrix must be zero. We calculate D_3:

$$D_3 = \begin{vmatrix} 2 & 7 & 3 \\ 3 & 2 & 4 \\ 1 & \mu & -1 \end{vmatrix} = 0,$$

which simplifies to:

$$2\begin{vmatrix} 2 & 4 \\ \mu & -1 \end{vmatrix} - 7\begin{vmatrix} 3 & 4 \\ 1 & -1 \end{vmatrix} + 3\begin{vmatrix} 3 & 2 \\ 1 & \mu \end{vmatrix} = 0.$$

Solving for μ, we get $\mu = -39$. Next, calculate D for λ in place:

$$D = \begin{vmatrix} 2 & 7 & \lambda \\ 3 & 2 & 5 \\ 1 & -39 & 32 \end{vmatrix} = 0,$$

which gives $\lambda = -1$. Thus,

$$\lambda - \mu = -1 - (-39) = 38.$$

Quick Tip

For solving systems of linear equations with infinite solutions, compute the determinant of the coefficient matrix and set it equal to zero to solve for the unknown parameters.

---

30. If the solution $y(x)$ of the given differential equation

$$(e^y + 1)\cos x\, dx + e^y \sin x\, dy = 0$$

passes through the point $\left(\frac{\pi}{2}, 0\right)$, then the value of $e^{y\left(\frac{\pi}{6}\right)}$ is equal to _____.
Correct Answer: (3)

Solution:
Starting with the given differential equation:

$$(e^y + 1)\cos x\, dx + e^y \sin x\, dy = 0,$$

we rewrite it as:

$$d\left((e^y + 1)\sin x\right) = 0.$$

Integrating both sides:

$$(e^y + 1)\sin x = C.$$

Substitute the point $\left(\frac{\pi}{2}, 0\right)$:

$$e^0 + 1 = C \quad \Rightarrow C = 2.$$

Now, substitute $x = \frac{\pi}{6}$:

$$(e^y + 1)\sin\frac{\pi}{6} = 2 \quad \Rightarrow \frac{e^y + 1}{2} = 2 \quad \Rightarrow e^y = 3.$$

Thus,

$$e^{y\left(\frac{\pi}{6}\right)} = 3.$$

Quick Tip

For separable differential equations, integrate both terms separately and apply the initial conditions
to find the constant of integration.
