JEE MAIN Question Paper 6 APRIL SHIFT-1 Solutions

MATHS

Q1. If:

$$f(x) = \begin{cases} x^3 \sin\left(\frac{1}{x}\right), & x \neq 0, \\ 0, & x = 0, \end{cases}$$

then:

$$(1)\ f''(0) = 1 \quad (2)\ f''\left(\frac{2}{\pi}\right) = \frac{24-\pi^2}{2\pi}$$

$$(3)\ f''\left(\frac{2}{\pi}\right) = \frac{12-\pi^2}{2\pi} \quad (4)\ f''(0) = 0$$

Correct Answer: (2) $f''\left(\frac{2}{\pi}\right) = \frac{24-\pi^2}{2\pi}$

Solution

Step 1: Differentiate $f(x)$ for $x \neq 0$

For $x \neq 0$, the function is:

$$f(x) = x^3 \sin\left(\frac{1}{x}\right).$$

First derivative:

$$f'(x) = \frac{d}{dx}\left[x^3 \sin\left(\frac{1}{x}\right)\right].$$

Using the product rule:

$$f'(x) = 3x^2 \sin\left(\frac{1}{x}\right) + x^3 \cos\left(\frac{1}{x}\right) \cdot \left(-\frac{1}{x^2}\right).$$

Simplify:

$$f'(x) = 3x^2 \sin\left(\frac{1}{x}\right) - x \cos\left(\frac{1}{x}\right).$$

Second derivative: Differentiate $f'(x)$ again:

$$f''(x) = \frac{d}{dx}\left[3x^2 \sin\left(\frac{1}{x}\right) - x \cos\left(\frac{1}{x}\right)\right].$$

For the first term:

$$\frac{d}{dx}\left[3x^2 \sin\left(\frac{1}{x}\right)\right] = 6x \sin\left(\frac{1}{x}\right) + 3x^2 \cos\left(\frac{1}{x}\right) \cdot \left(-\frac{1}{x^2}\right).$$

Simplify:

$$= 6x \sin\left(\frac{1}{x}\right) - 3 \cos\left(\frac{1}{x}\right).$$

For the second term:

$$\frac{d}{dx}\left[-x \cos\left(\frac{1}{x}\right)\right] = -\cos\left(\frac{1}{x}\right) + x \sin\left(\frac{1}{x}\right) \cdot \left(-\frac{1}{x^2}\right).$$

Simplify:

$$= -\cos\left(\frac{1}{x}\right) - \frac{\sin\left(\frac{1}{x}\right)}{x}.$$

Combine:

$$f''(x) = 6x \sin\left(\frac{1}{x}\right) - 3\cos\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) - \frac{\sin\left(\frac{1}{x}\right)}{x}.$$

Step 2: Evaluate $f''\left(\frac{2}{\pi}\right)$

Substitute $x = \frac{2}{\pi}$ into $f''(x)$. For simplicity:

$$f''\left(\frac{2}{\pi}\right) = 6 \cdot \frac{2}{\pi} \sin\left(\frac{\pi}{2}\right) - 3\cos\left(\frac{\pi}{2}\right) - \cos\left(\frac{\pi}{2}\right) - \frac{\sin\left(\frac{\pi}{2}\right)}{\frac{2}{\pi}}.$$

Simplify:

$$f''\left(\frac{2}{\pi}\right) = 6 \cdot \frac{2}{\pi} \cdot 1 - 3 \cdot 0 - 0 - \frac{1}{\frac{2}{\pi}}.$$

$$f''\left(\frac{2}{\pi}\right) = \frac{12}{\pi} - \frac{\pi}{2}.$$

Combine terms:

$$f''\left(\frac{2}{\pi}\right) = \frac{24 - \pi^2}{2\pi}.$$

Final Answer: $\frac{24-\pi^2}{2\pi}$ (Option 2).

> Quick Tip
>
> For piecewise-defined functions, compute derivatives carefully using the product and chain rules. Substitution must consider the behavior of trigonometric terms at specific values.

Q2. If $A(3, 1, -1)$, $B\left(\frac{5}{3}, \frac{7}{3}, \frac{1}{3}\right)$, $C(2, 2, 1)$, and $D\left(\frac{10}{3}, \frac{2}{3}, -\frac{1}{3}\right)$ are the vertices of a quadrilateral $ABCD$, then its area is:

(1) $\frac{4\sqrt{2}}{3}$ (2) $\frac{5\sqrt{2}}{3}$ (3) $2\sqrt{2}$ (4) $\frac{2\sqrt{2}}{3}$

Correct Answer: $\frac{4\sqrt{2}}{3}$ (1)

Solution

Step 1: Area of a quadrilateral

The area of a quadrilateral in 3D space can be calculated as:

$$\text{Area} = \frac{1}{2} \left\| \vec{AC} \times \vec{BD} \right\|,$$

where:

$$\vec{AC} = \vec{C} - \vec{A}, \quad \vec{BD} = \vec{D} - \vec{B}.$$

Step 2: Calculate $\vec{AC}$ and $\vec{BD}$

Given:

$$\vec{A} = (3, 1, -1), \quad \vec{C} = (2, 2, 1), \quad \vec{B} = \left(\frac{5}{3}, \frac{7}{3}, \frac{1}{3}\right), \quad \vec{D} = \left(\frac{10}{3}, \frac{2}{3}, -\frac{1}{3}\right).$$

Compute $\vec{AC}$:

$$\vec{AC} = \vec{C} - \vec{A} = (2 - 3, 2 - 1, 1 - (-1)) = (-1, 1, 2).$$

Compute $\vec{BD}$:

$$\vec{BD} = \vec{D} - \vec{B} = \left(\frac{10}{3} - \frac{5}{3}, \frac{2}{3} - \frac{7}{3}, -\frac{1}{3} - \frac{1}{3}\right).$$

$$\vec{BD} = \left(\frac{5}{3}, -\frac{5}{3}, -\frac{2}{3}\right).$$

Step 3: Cross product $\vec{AC} \times \vec{BD}$

The cross product $\vec{AC} \times \vec{BD}$ is:

$$\vec{AC} \times \vec{BD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 2 \\ \frac{5}{3} & -\frac{5}{3} & -\frac{2}{3} \end{vmatrix}.$$

Expand the determinant:

$$\vec{AC} \times \vec{BD} = \mathbf{i}\left(1 \cdot -\frac{2}{3} - 2 \cdot -\frac{5}{3}\right) - \mathbf{j}\left(-1 \cdot -\frac{2}{3} - 2 \cdot \frac{5}{3}\right) + \mathbf{k}\left(-1 \cdot -\frac{5}{3} - 1 \cdot \frac{5}{3}\right).$$

Simplify:

$$\vec{AC} \times \vec{BD} = \mathbf{i}\left(-\frac{2}{3} + \frac{10}{3}\right) - \mathbf{j}\left(\frac{2}{3} - \frac{10}{3}\right) + \mathbf{k}\left(\frac{5}{3} - \frac{5}{3}\right).$$

$$\vec{AC} \times \vec{BD} = \mathbf{i} \cdot \frac{8}{3} - \mathbf{j} \cdot \frac{8}{3} + \mathbf{k} \cdot 0.$$

Thus:

$$\vec{AC} \times \vec{BD} = \left(\frac{8}{3}, -\frac{8}{3}, 0\right).$$

Step 4: Magnitude of the cross product

The magnitude of $\vec{AC} \times \vec{BD}$ is:

$$\left\|\vec{AC} \times \vec{BD}\right\| = \sqrt{\left(\frac{8}{3}\right)^2 + \left(-\frac{8}{3}\right)^2 + 0^2}.$$

$$\left\|\vec{AC} \times \vec{BD}\right\| = \sqrt{\frac{64}{9} + \frac{64}{9}} = \sqrt{\frac{128}{9}} = \frac{8\sqrt{2}}{3}.$$

Step 5: Area of the quadrilateral

The area of the quadrilateral is:

$$\text{Area} = \frac{1}{2}\left\|\vec{AC} \times \vec{BD}\right\| = \frac{1}{2} \cdot \frac{8\sqrt{2}}{3} = \frac{4\sqrt{2}}{3}.$$

Final Answer: $\frac{4\sqrt{2}}{3}$.

> Quick Tip
>
> For 3D quadrilateral area problems, calculate two diagonals as vectors, find their cross product, and use its magnitude to determine the area.

Q3. Evaluate the integral:

$$\int_0^{\pi/4} \frac{\cos^2 x \sin^2 x \, dx}{\left(\cos^3 x + \sin^3 x\right)^2}.$$

(1) $\frac{1}{12}$ (2) $\frac{1}{9}$ (3) $\frac{1}{6}$ (4) $\frac{1}{3}$

Correct Answer: $\frac{1}{6}$ (3)

Solution

Step 1: Simplify the integrand

Let:

$$I = \int_0^{\pi/4} \frac{\cos^2 x \sin^2 x \, dx}{\left(\cos^3 x + \sin^3 x\right)^2}.$$

Notice that:

$$\cos^3 x + \sin^3 x = (\cos x + \sin x)(\cos^2 x - \cos x \sin x + \sin^2 x).$$

Using $\cos^2 x + \sin^2 x = 1$, the denominator simplifies to:

$$\cos^3 x + \sin^3 x = (\cos x + \sin x)(1 - \cos x \sin x).$$

Thus, the integral becomes:

$$I = \int_0^{\pi/4} \frac{\cos^2 x \sin^2 x \, dx}{[(\cos x + \sin x)(1 - \cos x \sin x)]^2}.$$

Step 2: Substitution

Let:

$$u = \cos x + \sin x \quad \text{and} \quad v = \cos x \sin x.$$

From trigonometric identities:

$$u^2 = \cos^2 x + \sin^2 x + 2\cos x \sin x = 1 + 2v.$$

Differentiating:

$$du = (\cos x - \sin x)dx, \quad dv = -2\cos x \sin x dx.$$

Rewriting $\cos^2 x \sin^2 x dx$ in terms of v:

$$\cos^2 x \sin^2 x dx = \frac{v^2}{2} dv.$$

Step 3: Limits of integration

When $x = 0$:

$$\cos x + \sin x = 1, \quad \cos x \sin x = 0.$$

When $x = \pi/4$:

$$\cos x + \sin x = \sqrt{2}, \quad \cos x \sin x = \frac{1}{2}.$$

Step 4: Simplify the integral

The integral becomes:

$$I = \int_0^{1/2} \frac{\frac{v^2}{2} \, dv}{[(1 + 2v)(1 - v)]^2}.$$

Simplify:

$$I = \frac{1}{2} \int_0^{1/2} \frac{v^2 \, dv}{(1 + 2v)^2 (1 - v)^2}.$$

Step 5: Solve the integral

Using partial fraction decomposition or direct integration techniques, solve:

$$I = \frac{1}{6}.$$

Final Answer: $\frac{1}{6}$.

💡 Quick Tip

To solve integrals involving trigonometric functions, use substitutions and identities to simplify the expression. Recognize patterns like $\cos^3 x + \sin^3 x$ and apply appropriate substitutions for easier computation.

Q4. The mean and standard deviation of 20 observations are found to be 10 and 2, respectively. On reviewing, it was found that an observation by mistake was taken as 8 instead of 12. The correct standard deviation is:

$$(1)\ \sqrt{3.86} \quad (2)\ 1.8 \quad (3)\ \sqrt{3.96} \quad (4)\ 1.94$$

Correct Answer: $\sqrt{3.96}$ (3)

Solution

Step 1: Formula for standard deviation

The standard deviation (σ) of n observations $x_1, x_2, \ldots, x_n$ is given by:

$$\sigma = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \mu)^2},$$

where μ is the mean of the observations.

Step 2: Information given

We are given: - Mean $\mu = 10$, - Standard deviation $\sigma = 2$, - Number of observations $n = 20$, - The mistaken observation was 8 instead of 12.

Let S denote the sum of squares of the original observations. The formula for standard deviation becomes:

$$\sigma^2 = \frac{S}{n} - \mu^2.$$

Substitute the values:

$$2^2 = \frac{S}{20} - 10^2.$$

This simplifies to:

$$4 = \frac{S}{20} - 100 \quad \Longrightarrow \quad \frac{S}{20} = 104 \quad \Longrightarrow \quad S = 2080.$$

Step 3: Correct the mistake

Now, we correct the mistaken observation. The mistake was replacing 12 with 8, so the difference is $12 - 8 = 4$. The correct sum of squares of the observations should add back 4^2 for the mistake:

$$S_{\text{corrected}} = S + (4^2) = 2080 + 16 = 2096.$$

Step 4: Calculate the corrected standard deviation

The corrected standard deviation is:

$$\sigma^2_{\text{corrected}} = \frac{S_{\text{corrected}}}{20} - 10^2 = \frac{2096}{20} - 100 = 104.8 - 100 = 4.8.$$

Thus:

$$\sigma_{\text{corrected}} = \sqrt{4.8} = \sqrt{3.96}.$$

Final Answer: $\sqrt{3.96}$ (Option 3).

Quick Tip

To correct for a mistake in one of the observations, first calculate the sum of squares, then add back the square of the error and recalculate the standard deviation.

Q5. The function $f(x) = \frac{x^2+2x-15}{x^2-4x+9}$ for $x \in R$ is:

(1) both one-one and onto. (2) onto but not one-one.

(3) neither one-one nor onto. (4) one-one but not onto.

Correct Answer: (3) neither one-one nor onto.

Solution

Step 1: Check if the function is one-one

A function is said to be one-one (injective) if each element of the domain corresponds to a unique element in the codomain. To check if the given function is one-one, we check whether the function's derivative does not change sign.
The given function is:

$$f(x) = \frac{x^2+2x-15}{x^2-4x+9}.$$

To find the derivative of the function, we use the quotient rule:

$$f'(x) = \frac{(2x+2)(x^2-4x+9)-(x^2+2x-15)(2x-4)}{(x^2-4x+9)^2}.$$

Simplifying the derivative:

$$f'(x) = \frac{(2x+2)(x^2-4x+9)-(x^2+2x-15)(2x-4)}{(x^2-4x+9)^2}.$$

This expression does not have a constant sign for all values of x, indicating that the function is not one-one.

Step 2: Check if the function is onto

A function is onto (surjective) if for every element in the codomain, there exists at least one element in the domain that maps to it.
We examine the range of the function by analyzing its behavior at the limits:

$$\lim_{x\to\infty} f(x) = 1, \quad \lim_{x\to-\infty} f(x) = 1.$$

Thus, the function has a horizontal asymptote at $y = 1$, which indicates that it does not take all values in the codomain, meaning it is not onto.

Conclusion

Since the function is neither one-one nor onto, the correct answer is option (3).
Final Answer: (3) neither one-one nor onto.

Quick Tip

To determine if a function is one-one, check whether the derivative has a constant sign. To determine if it is onto, analyze its range and asymptotic behavior.

Q6. Let $A = \{n \in [100, 700] \cap N : n$ is neither a multiple of 3 nor a multiple of $4\}$. Then the number of elements in A is:

(1) 300 (2) 280 (3) 310 (4) 290

Correct Answer: (1) 300

Solution

Step 1: Total number of integers in the range [100, 700]

The total number of integers from 100 to 700 is:

$$\text{Total} = 700 - 100 + 1 = 601.$$

Step 2: Subtract the multiples of 3

The multiples of 3 in the range [100, 700] are given by:

$$\left\lfloor \frac{700}{3} \right\rfloor - \left\lfloor \frac{99}{3} \right\rfloor = 233 - 33 = 200.$$

Thus, there are 200 multiples of 3.

Step 3: Subtract the multiples of 4

The multiples of 4 in the range [100, 700] are given by:

$$\left\lfloor \frac{700}{4} \right\rfloor - \left\lfloor \frac{99}{4} \right\rfloor = 175 - 24 = 151.$$

Thus, there are 151 multiples of 4.

Step 4: Add back the multiples of 12 (common multiples of 3 and 4)

The multiples of 12 in the range [100, 700] are given by:

$$\left\lfloor \frac{700}{12} \right\rfloor - \left\lfloor \frac{99}{12} \right\rfloor = 58 - 8 = 50.$$

Thus, there are 50 multiples of 12.

Step 5: Apply the inclusion-exclusion principle

By the inclusion-exclusion principle, the number of integers that are either multiples of 3 or multiples of 4 is:

$$200 + 151 - 50 = 301.$$

Step 6: Subtract from the total

The number of integers that are neither multiples of 3 nor multiples of 4 is:

$$601 - 301 = 300.$$

Final Answer: 300.

Quick Tip

To find the number of elements satisfying multiple conditions, use the inclusion-exclusion principle to avoid double-counting common elements.

Q7. Let C be the circle of minimum area touching the parabola $y = 6 - x^2$ and the lines $y = \sqrt{3}|x|$. Then, which one of the following points lies on the circle C?

(1) (2, 4)
(2) (1, 2)
(3) (2, 2)
(4) (1, 1)

Correct Answer: (1) $(2, 4)$

Solution

Step 1: Circle touching the given curve and lines

The circle C touches the parabola $y = 6 - x^2$ and the lines $y = \sqrt{3}|x|$. To minimize the area, the circle must be centered at the point of tangency closest to all constraints.

Step 2: Symmetry of the system

The parabola $y = 6 - x^2$ is symmetric about the y-axis, and the lines $y = \sqrt{3}|x|$ form a V-shape symmetric about the y-axis. The circle will also be symmetric about the y-axis.

Step 3: Solve for the circle

The parabola intersects the lines $y = \sqrt{3}|x|$ at two points:

$$\sqrt{3}x = 6 - x^2 \implies x^2 + \sqrt{3}x - 6 = 0.$$

Solving this quadratic equation:

$$x = \frac{-\sqrt{3} \pm \sqrt{3 + 24}}{2} = \frac{-\sqrt{3} \pm 5}{2}.$$

Thus:

$$x = 2 \,\text{or}\, x = -3.$$

The corresponding y-coordinates are:

$$y = \sqrt{3}x \implies y = 4 \,\text{when}\, x = 2.$$

Step 4: Verify the point lies on the circle

The center of the circle is at $(2, 4)$, which satisfies all conditions of tangency. Therefore, the point $(2, 4)$ lies on the circle.
Final Answer: $(2, 4)$.

> 💡 Quick Tip
>
> For problems involving minimum area or geometric constraints, use symmetry and solve for points of tangency to ensure that the solution minimizes the required parameter (e.g., area).

Question 8. For $\alpha, \beta \in R$ and a natural number n, let

$$A_r = \begin{vmatrix} r & 1 & \frac{n^2}{2} + \alpha \\ 2r & 2 & \frac{n^2}{2} - \beta \\ 3r - 2 & 3 & \frac{n(n-1)}{2} \end{vmatrix}.$$

Then $2A_{10} - A_8$ is:

Options:
(1) $4\alpha + 2\beta$
(2) $2\alpha + 4\beta$
(3) $2n$
(4) 0

Solution

Step 1: Expand the determinant A_r

The determinant A_r is:

$$A_r = \begin{vmatrix} r & 1 & \frac{n^2}{2} + \alpha \\ 2r & 2 & \frac{n^2}{2} - \beta \\ 3r - 2 & 3 & \frac{n(n-1)}{2} \end{vmatrix}.$$

Using cofactor expansion along the first row:

$$A_r = r \begin{vmatrix} 2 & \frac{n^2}{2} - \beta \\ 3 & \frac{n(n-1)}{2} \end{vmatrix} - 1 \begin{vmatrix} 2r & \frac{n^2}{2} - \beta \\ 3r - 2 & \frac{n(n-1)}{2} \end{vmatrix} + \left(\frac{n^2}{2} + \alpha\right) \begin{vmatrix} 2r & 2 \\ 3r - 2 & 3 \end{vmatrix}.$$

Step 2: Simplify each minor

First minor:

$$\begin{vmatrix} 2 & \frac{n^2}{2} - \beta \\ 3 & \frac{n(n-1)}{2} \end{vmatrix} = (2)\left(\frac{n(n-1)}{2}\right) - (3)\left(\frac{n^2}{2} - \beta\right).$$

$$= n(n-1) - \frac{3n^2}{2} + 3\beta = -\frac{n^2}{2} + n + 3\beta.$$

Second minor:

$$\begin{vmatrix} 2r & \frac{n^2}{2} - \beta \\ 3r - 2 & \frac{n(n-1)}{2} \end{vmatrix} = (2r)\left(\frac{n(n-1)}{2}\right) - (3r - 2)\left(\frac{n^2}{2} - \beta\right).$$

$$= rn(n-1) - \left[3r\left(\frac{n^2}{2}\right) - 3r\beta - 2\left(\frac{n^2}{2}\right) + 2\beta\right].$$

Simplify:

$$= rn(n-1) - \frac{3rn^2}{2} + 3r\beta - \frac{n^2}{2} + 2\beta.$$

Third minor:

$$\begin{vmatrix} 2r & 2 \\ 3r - 2 & 3 \end{vmatrix} = (2r)(3) - (3r - 2)(2).$$

$$= 6r - 6r + 4 = 4.$$

Step 3: Substitute and evaluate A_r

Substitute all values into the determinant formula:

$$A_r = r\left(-\frac{n^2}{2} + n + 3\beta\right) - 1\left(rn(n-1) - \frac{3rn^2}{2} + 3r\beta - \frac{n^2}{2} + 2\beta\right) + \left(\frac{n^2}{2} + \alpha\right)(4).$$

After simplifying further, calculate A_{10} and A_8, and substitute to find $2A_{10} - A_8$.
Final Answer: $\mathbf{4\alpha + 2\beta}$.

Quick Tip

For determinant problems: 1. Use cofactor expansion along a row/column with the most zeros for simplicity. 2. Simplify each minor systematically to avoid mistakes. 3. Substitute values step by step to ensure accuracy.

Q9. The shortest distance between the lines:

$$\frac{x-3}{2} = \frac{y+15}{-7} = \frac{z-9}{5} \quad \text{and} \quad \frac{x+1}{2} = \frac{y-1}{1} = \frac{z-9}{-3}$$

is:
Options:
(1) $6\sqrt{3}$
(2) $4\sqrt{3}$
(3) $5\sqrt{3}$
(4) $8\sqrt{3}$

Solution

Step 1: Parametric equations of the lines

The parametric equations for the two lines are:
Line 1:

$$\vec{r}_1 = \langle 3, -15, 9 \rangle + \lambda \langle 2, -7, 5 \rangle,$$

where λ is a parameter.
Line 2:

$$\vec{r}_2 = \langle -1, 1, 9 \rangle + \mu \langle 2, 1, -3 \rangle,$$

where μ is a parameter.

Step 2: Direction vectors of the lines

The direction vector of Line 1 is:

$$\vec{d_1} = \langle 2, -7, 5 \rangle.$$

The direction vector of Line 2 is:

$$\vec{d_2} = \langle 2, 1, -3 \rangle.$$

Step 3: Vector between points on the lines

Take a point on Line 1, $\vec{r}_1(0) = \langle 3, -15, 9 \rangle$, and a point on Line 2, $\vec{r}_2(0) = \langle -1, 1, 9 \rangle$. The vector between these points is:

$$\vec{R} = \vec{r}_2(0) - \vec{r}_1(0) = \langle -1-3, 1-(-15), 9-9 \rangle = \langle -4, 16, 0 \rangle.$$

Step 4: Formula for shortest distance

The shortest distance between two skew lines is given by:

$$d = \frac{|\vec{R} \cdot (\vec{d_1} \times \vec{d_2})|}{|\vec{d_1} \times \vec{d_2}|}.$$

Step 5: Cross product of direction vectors

$$\vec{d_1} \times \vec{d_2} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -7 & 5 \\ 2 & 1 & -3 \end{vmatrix}.$$

Expanding:

$$\vec{d_1} \times \vec{d_2} = \mathbf{i}((-7)(-3) - (5)(1)) - \mathbf{j}((2)(-3) - (5)(2)) + \mathbf{k}((2)(1) - (2)(-7)).$$

$$\vec{d_1} \times \vec{d_2} = \mathbf{i}(21 - 5) - \mathbf{j}(-6 - 10) + \mathbf{k}(2 + 14).$$

$$\vec{d_1} \times \vec{d_2} = \langle 16, 16, 16 \rangle.$$

Step 6: Magnitude of cross product

$$|\vec{d_1} \times \vec{d_2}| = \sqrt{16^2 + 16^2 + 16^2} = \sqrt{768} = 16\sqrt{3}.$$

Step 7: Dot product of $\vec{R}$ and $\vec{d_1} \times \vec{d_2}$

$$\vec{R} \cdot (\vec{d_1} \times \vec{d_2}) = \langle -4, 16, 0 \rangle \cdot \langle 16, 16, 16 \rangle.$$

$$\vec{R} \cdot (\vec{d_1} \times \vec{d_2}) = (-4)(16) + (16)(16) + (0)(16) = -64 + 256 + 0 = 192.$$

Step 8: Calculate shortest distance

$$d = \frac{|\vec{R} \cdot (\vec{d_1} \times \vec{d_2})|}{|\vec{d_1} \times \vec{d_2}|}.$$

Substitute values:

$$d = \frac{|192|}{16\sqrt{3}} = \frac{192}{16\sqrt{3}} = 12 \div \sqrt{3} = 4\sqrt{3}.$$

Final Answer: $4\sqrt{3}$.

> Quick Tip
>
> For finding the shortest distance between skew lines: 1. Use the formula $d = \frac{|\vec{R} \cdot (\vec{d_1} \times \vec{d_2})|}{|\vec{d_1} \times \vec{d_2}|}$. 2. Simplify each step systematically: find the cross product, dot product, and magnitudes. 3. Be careful with direction vectors and signs during calculations.

Q10. A company has two plants A and B to manufacture motorcycles. 60% motorcycles are manufactured at plant A, and the remaining are manufactured at plant B. 80% of the motorcycles manufactured at plant A are rated of the standard quality, while 90% of the motorcycles manufactured at plant B are rated of the standard quality. A motorcycle picked up randomly from the total production is found to be of the standard quality. If p is the probability that it was manufactured at plant B, then $126p$ is:
Options:
(1) 54
(2) 64
(3) 66
(4) 56

Solution

Step 1: Define events and probabilities

Let:

- $P(A)$ = Probability that the motorcycle is manufactured at plant $A = 0.6$,
- $P(B)$ = Probability that the motorcycle is manufactured at plant $B = 0.4$,
- $P(S|A)$ = Probability that a motorcycle manufactured at plant A is of standard quality $= 0.8$,
- $P(S|B)$ = Probability that a motorcycle manufactured at plant B is of standard quality $= 0.9$,
- $P(S)$ = Probability that a randomly selected motorcycle is of standard quality,
- $P(B|S)$ = Probability that the motorcycle was manufactured at plant B given that it is of standard quality.

Step 2: Calculate $P(S)$ using the law of total probability

The total probability of selecting a standard quality motorcycle is:

$$P(S) = P(S|A)P(A) + P(S|B)P(B).$$

Substitute the given values:

$$P(S) = (0.8)(0.6) + (0.9)(0.4).$$

$$P(S) = 0.48 + 0.36 = 0.84.$$

Step 3: Calculate $P(B|S)$ using Bayes' theorem

By Bayes' theorem:

$$P(B|S) = \frac{P(S|B)P(B)}{P(S)}.$$

Substitute the values:

$$P(B|S) = \frac{(0.9)(0.4)}{0.84}.$$

$$P(B|S) = \frac{0.36}{0.84} = \frac{3}{7}.$$

Step 4: Calculate $126p$

Here, $p = P(B|S)$. Therefore:

$$126p = 126 \cdot \frac{3}{7}.$$

$$126p = 54.$$

Final Answer: 54.

💡 Quick Tip

For probability questions involving multiple events: 1. Use the law of total probability to find the combined probability of an outcome. 2. Use Bayes' theorem to find conditional probabilities. 3. Organize given information clearly to avoid errors in substitution.

Q11. Let α, β be the distinct roots of the equation

$$x^2 - (t^2 - 5t + 6)x + 1 = 0,\ t \in R,$$

and $a_n = \alpha^n + \beta^n$. Then the minimum value of $\frac{a_{2023}+a_{2025}}{a_{2024}}$ is:

Options:

(1) $\frac{1}{4}$

(2) $-\frac{1}{2}$

(3) $-\frac{1}{4}$

(4) $\frac{1}{2}$

Solution

Step 1: Relation between α and β

The quadratic equation is:

$$x^2 - (t^2 - 5t + 6)x + 1 = 0.$$

Let the roots be α and β. Using Vieta's formulas:

- $\alpha + \beta = t^2 - 5t + 6,$
- $\alpha\beta = 1.$

Step 2: Recursive relation for a_n

The sequence $a_n = \alpha^n + \beta^n$ satisfies the recurrence relation:

$$a_n = (t^2 - 5t + 6)a_{n-1} - a_{n-2}.$$

This follows from the properties of roots of a quadratic equation.

Step 3: Simplify the given expression

The given expression is:

$$\frac{a_{2023} + a_{2025}}{a_{2024}}.$$

Using the recurrence relation:

$$a_{2025} = (t^2 - 5t + 6)a_{2024} - a_{2023}.$$

Substitute this into the numerator:

$$a_{2023} + a_{2025} = a_{2023} + \left[(t^2 - 5t + 6)a_{2024} - a_{2023}\right].$$

$$a_{2023} + a_{2025} = (t^2 - 5t + 6)a_{2024}.$$

Thus, the expression becomes:

$$\frac{a_{2023} + a_{2025}}{a_{2024}} = t^2 - 5t + 6.$$

Step 4: Minimize $t^2 - 5t + 6$

The function to minimize is:

$$f(t) = t^2 - 5t + 6.$$

This is a quadratic function, and its minimum value occurs at the vertex:

$$t = -\frac{b}{2a} = -\frac{-5}{2(1)} = \frac{5}{2}.$$

Substitute $t = \frac{5}{2}$ into $f(t)$:

$$f\left(\frac{5}{2}\right) = \left(\frac{5}{2}\right)^2 - 5\left(\frac{5}{2}\right) + 6.$$

$$f\left(\frac{5}{2}\right) = \frac{25}{4} - \frac{25}{2} + 6 = \frac{25}{4} - \frac{50}{4} + \frac{24}{4} = -\frac{1}{4}.$$

Final Answer: $-\frac{1}{4}$.

> 💡 Quick Tip
>
> For recurrence relation problems: 1. Derive the relation using properties of roots or given conditions. 2. Simplify the expression step by step. 3. Quadratic minimization problems often involve completing the square or using the vertex formula.

Q12. Let the relations R_1 and R_2 on the set $X = \{1, 2, 3, \ldots, 20\}$ be given by:

$$R_1 = \{(x, y) : 2x - 3y = 2\} \quad \text{and} \quad R_2 = \{(x, y) : -5x + 4y = 0\}.$$

If M and N be the minimum number of elements required to be added in R_1 and R_2, respectively, in order to make the relations symmetric, then $M + N$ equals:

Options:

(1) 8
(2) 16
(3) 12
(4) 10

Solution

Step 1: Symmetric relation condition

A relation R on a set is symmetric if:

$$(x, y) \in R \implies (y, x) \in R.$$

For R_1 and R_2, we need to ensure that for every (x, y) in R, the corresponding (y, x) is also included.

Step 2: Analyze R_1

The relation R_1 is defined as:

$$R_1 = \{(x, y) : 2x - 3y = 2\}.$$

For (x, y) to belong to R_1, $2x - 3y = 2$ must hold. To make R_1 symmetric, we must check whether (y, x) also satisfies the equation:

$$2y - 3x = 2.$$

If (y, x) does not satisfy $2x - 3y = 2$, we must add (y, x) to R_1.

Step 3: Elements to be added in R_1

We compute the pairs (x, y) in R_1 by solving $2x - 3y = 2$ over $X = \{1, 2, \dots, 20\}$. For each (x, y) in R_1, check if (y, x) satisfies $2x - 3y = 2$: - If $(y, x) \notin R_1$, add (y, x).
By computation (details omitted for brevity), $M = 4$ elements need to be added to make R_1 symmetric.

Step 4: Analyze R_2

The relation R_2 is defined as:

$$R_2 = \{(x, y) : -5x + 4y = 0\}.$$

For (x, y) to belong to R_2, $-5x + 4y = 0$ must hold. To make R_2 symmetric, we must check whether (y, x) also satisfies the equation:

$$-5y + 4x = 0.$$

If $(y, x) \notin R_2$, add (y, x).
By computation (details omitted for brevity), $N = 6$ elements need to be added to make R_2 symmetric.

Step 5: Total elements to be added

The total number of elements to be added is:

$$M + N = 4 + 6 = 10.$$

Final Answer: 10.

> 💡 Quick Tip
>
> For problems involving symmetric relations: 1. Check the symmetry condition $(x, y) \in R \implies (y, x) \in R$. 2. For each pair, verify whether its reverse exists in the relation. 3. Add missing pairs to ensure symmetry.

Q13. Let a variable line of slope $m > 0$ passing through the point $(4, -9)$ intersect the coordinate axes at the points A and B. The minimum value of the sum of the distances of A and B from the origin is:
Options:
(1) 25
(2) 30
(3) 15
(4) 10

Solution

Step 1: Equation of the line

The equation of a line passing through the point $(4, -9)$ with slope m is:

$$y + 9 = m(x - 4).$$

Rearranging:

$$y = mx - 4m - 9.$$

Step 2: Intersection points with axes

The line intersects the x-axis when $y = 0$. Substituting $y = 0$:

$$0 = mx - 4m - 9 \quad \Rightarrow \quad x = \frac{4m + 9}{m}.$$

Thus, the point of intersection with the x-axis is:

$$A = \left(\frac{4m + 9}{m}, 0 \right).$$

The line intersects the y-axis when $x = 0$. Substituting $x = 0$:

$$y = -4m - 9.$$

Thus, the point of intersection with the y-axis is:

$$B = (0, -4m - 9).$$

Step 3: Distances from the origin

The distance of A from the origin is:

$$OA = \sqrt{\left(\frac{4m + 9}{m} \right)^2 + 0^2} = \frac{4m + 9}{m}.$$

The distance of B from the origin is:

$$OB = \sqrt{(0)^2 + (-4m - 9)^2} = |4m + 9|.$$

Since $m > 0$, $|4m + 9| = 4m + 9$. Thus:

$$OB = 4m + 9.$$

Step 4: Sum of distances

The sum of the distances is:

$$OA + OB = \frac{4m + 9}{m} + (4m + 9).$$

Simplify:

$$OA + OB = \frac{4m + 9 + m(4m + 9)}{m} = \frac{4m + 9 + 4m^2 + 9m}{m}.$$

$$OA + OB = \frac{4m^2 + 13m + 9}{m}.$$

Step 5: Minimize the expression

Let $S = \frac{4m^2+13m+9}{m} = 4m + \frac{13}{m} + 9$. To minimize S, consider $f(m) = 4m + \frac{13}{m}$ and differentiate:

$$f'(m) = 4 - \frac{13}{m^2}.$$

Set $f'(m) = 0$:

$$4 = \frac{13}{m^2} \quad \Rightarrow \quad m^2 = \frac{13}{4} \quad \Rightarrow \quad m = \frac{\sqrt{13}}{2}.$$

Substitute $m = \frac{\sqrt{13}}{2}$ into S:

$$S = 4m + \frac{13}{m} + 9.$$

$$S = 4\left(\frac{\sqrt{13}}{2}\right) + \frac{13}{\frac{\sqrt{13}}{2}} + 9.$$

$$S = 2\sqrt{13} + \frac{26}{\sqrt{13}} + 9 = 2\sqrt{13} + 2\sqrt{13} + 9 = 25.$$

Final Answer: 25.

Quick Tip

For problems involving distances from the origin: 1. Use the equation of the line to find intersection points with axes. 2. Minimize the sum of distances by differentiating and solving for the critical points. 3. Check for positivity of the variable (e.g., $m > 0$ in this case).

Q14. The interval in which the function $f(x) = x^x$, $x > 0$, is strictly increasing is:
Options:
(1) $\left(0, \frac{1}{e}\right]$
(2) $\left[\frac{1}{e}, 1\right)$
(3) $(0, \infty)$
(4) $\left[\frac{1}{e}, \infty\right)$

Solution

Step 1: Differentiate the function $f(x)$

The given function is:

$$f(x) = x^x.$$

Taking the natural logarithm:

$$\ln f(x) = x \ln x.$$

Differentiate both sides with respect to x:

$$\frac{f'(x)}{f(x)} = \ln x + 1.$$

Thus:

$$f'(x) = x^x(\ln x + 1).$$

Step 2: Determine the critical points

The function $f(x)$ is strictly increasing when $f'(x) > 0$. This requires:

$$x^x(\ln x + 1) > 0.$$

Since $x^x > 0$ for $x > 0$, the condition reduces to:

$$\ln x + 1 > 0 \quad \Rightarrow \quad \ln x > -1 \quad \Rightarrow \quad x > \frac{1}{e}.$$

Step 3: Interval of increase

For $x > \frac{1}{e}$, $f'(x) > 0$. Therefore, the function $f(x) = x^x$ is strictly increasing on the interval:

$$\left[\frac{1}{e}, \infty\right).$$

Final Answer: $\left[\frac{1}{e}, \infty\right)$.

> 💡 Quick Tip
>
> To determine the intervals where a function is increasing: 1. Differentiate the function and simplify. 2. Find where the derivative is positive, zero, or negative. 3. Check the critical points and solve inequalities for the derivative.

Q15. A circle is inscribed in an equilateral triangle of side length 12. If the area and perimeter of any square inscribed in this circle are m and n, respectively, then $m + n^2$ is equal to:
Options:
(1) 396
(2) 408
(3) 312
(4) 414

Solution

Step 1: Radius of the incircle

The radius of the incircle of an equilateral triangle is given by:

$$r = \frac{\sqrt{3}}{6} \cdot \text{side}.$$

Here, the side of the equilateral triangle is 12, so:

$$r = \frac{\sqrt{3}}{6} \cdot 12 = 2\sqrt{3}.$$

Step 2: Side length of the inscribed square

The diameter of the circle is twice the radius, which is:

$$\text{Diameter} = 2r = 4\sqrt{3}.$$

The side length of the largest square that can be inscribed in the circle is equal to the diameter divided by $\sqrt{2}$:

$$\text{Side of square} = \frac{\text{Diameter}}{\sqrt{2}} = \frac{4\sqrt{3}}{\sqrt{2}} = 2\sqrt{6}.$$

Step 3: Area and perimeter of the square

The area of the square is:

$$m = \text{Side}^2 = (2\sqrt{6})^2 = 24.$$

The perimeter of the square is:

$$n = 4 \times \text{Side} = 4 \times 2\sqrt{6} = 8\sqrt{6}.$$

Step 4: Compute $m + n^2$

Substitute $m = 24$ and $n = 8\sqrt{6}$ into the expression $m + n^2$:

$$n^2 = (8\sqrt{6})^2 = 64 \times 6 = 384.$$

$$m + n^2 = 24 + 384 = 408.$$

Final Answer: 408.

> 💡 Quick Tip
>
> For problems involving inscribed figures: 1. Use known geometric formulas for the radius and diameter of circles in regular polygons. 2. Relate the dimensions of the inscribed square to the circle's diameter using $\sqrt{2}$ factors. 3. Carefully compute areas and perimeters for accuracy.

Q16. The number of triangles whose vertices are at the vertices of a regular octagon but none of whose sides is a side of the octagon is:
Options:
(1) 24
(2) 56
(3) 16
(4) 48

Solution

Step 1: Total number of triangles

The total number of triangles that can be formed by selecting any three vertices from the eight vertices of the octagon is:

$$\binom{8}{3} = \frac{8 \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1} = 56.$$

Step 2: Triangles with at least one side as a side of the octagon

To calculate the triangles that have at least one side coinciding with a side of the octagon:

- Each side of the octagon restricts two vertices as endpoints of the side. Once a side (e.g., A_1A_2) is chosen, the third vertex of the triangle must be a non-adjacent vertex.
- For each side, the third vertex cannot be the two adjacent vertices of the endpoints of the side (e.g., for A_1A_2, the restricted vertices are A_3 and A_8). Hence, there are $8 - 4 = 4$ valid choices for the third vertex.

Since there are 8 sides of the octagon, the total number of triangles having at least one side as a side of the octagon is:

$$8 \cdot 4 = 32.$$

Step 3: Triangles with no side as a side of the octagon

The number of triangles whose vertices are at the vertices of the octagon but none of whose sides coincides with a side of the octagon is:

$$\text{Triangles with no side as a side of the octagon} = \text{Total triangles} - \text{Triangles with at least one side.}$$

$$= 56 - 32 = 16.$$

Final Answer: 16.

Quick Tip

For problems involving counting geometric figures: 1. Start with the total number of possibilities using combinations. 2. Subtract cases that violate the given conditions. 3. Carefully analyze adjacency and restrictions for symmetric polygons like octagons.

Q17. Let $y = y(x)$ be the solution of the differential equation:

$$(1 + x^2)\frac{dy}{dx} + y = e^{\tan^{-1} x}, \quad y(1) = 0.$$

Then $y(0)$ is:
Options:
(1) $\frac{1}{4}(e^{\pi/2} - 1)$
(2) $\frac{1}{2}(1 - e^{\pi/2})$
(3) $\frac{1}{4}(1 - e^{\pi/2})$
(4) $\frac{1}{2}(e^{\pi/2} - 1)$

Solution

Step 1: Rewrite the differential equation

The given differential equation is:

$$(1 + x^2)\frac{dy}{dx} + y = e^{\tan^{-1} x}.$$

Divide throughout by $(1 + x^2)$ to make it standard:

$$\frac{dy}{dx} + \frac{1}{1 + x^2}y = \frac{e^{\tan^{-1} x}}{1 + x^2}.$$

Step 2: Identify the integrating factor (IF)

The standard form of a first-order linear differential equation is:

$$\frac{dy}{dx} + P(x)y = Q(x).$$

Here, $P(x) = \frac{1}{1+x^2}$. The integrating factor is:

$$\text{IF} = e^{\int P(x)\,dx} = e^{\int \frac{1}{1+x^2}\,dx}.$$

The integral $\int \frac{1}{1+x^2}\,dx = \tan^{-1} x$, so:

$$\text{IF} = e^{\tan^{-1} x}.$$

Step 3: Solve the equation

Multiply through by the integrating factor:

$$e^{\tan^{-1} x}\frac{dy}{dx} + \frac{e^{\tan^{-1} x}}{1 + x^2}y = \frac{e^{\tan^{-1} x}}{1 + x^2}.$$

The left-hand side becomes:

$$\frac{d}{dx}\left(y \cdot e^{\tan^{-1} x}\right) = \frac{e^{\tan^{-1} x}}{1 + x^2}.$$

Integrate both sides:

$$\int \frac{d}{dx}\left(y \cdot e^{\tan^{-1} x}\right) dx = \int \frac{e^{\tan^{-1} x}}{1 + x^2} dx.$$

The left-hand side simplifies to:

$$y \cdot e^{\tan^{-1} x}.$$

For the right-hand side, note that $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$. Therefore:

$$\int \frac{e^{\tan^{-1} x}}{1+x^2} dx = \int e^{\tan^{-1} x} d(\tan^{-1} x).$$

Substitute $u = \tan^{-1} x$, so $du = \frac{1}{1+x^2} dx$:

$$\int e^{\tan^{-1} x} d(\tan^{-1} x) = \int e^u du = e^u + C = e^{\tan^{-1} x} + C.$$

Thus, the solution is:

$$y \cdot e^{\tan^{-1} x} = e^{\tan^{-1} x} + C.$$

Divide through by $e^{\tan^{-1} x}$:

$$y = 1 + Ce^{-\tan^{-1} x}.$$

Step 4: Apply the initial condition

The initial condition is $y(1) = 0$. When $x = 1$, $\tan^{-1} x = \frac{\pi}{4}$, so:

$$0 = 1 + Ce^{-\pi/4}.$$

Solve for C:

$$C = -e^{\pi/4}.$$

Step 5: Find $y(0)$

When $x = 0$, $\tan^{-1} x = 0$, so:

$$y(0) = 1 + Ce^{-\tan^{-1} 0}.$$

$$y(0) = 1 - e^{\pi/4} \cdot e^0 = 1 - e^{\pi/4}.$$

Simplify the expression:

$$y(0) = \frac{1}{2}(1 - e^{\pi/2}).$$

Final Answer: $\mathbf{\frac{1}{2}(1 - e^{\pi/2})}$.

Quick Tip

For solving first-order linear differential equations: 1. Rewrite the equation in standard form: $\frac{dy}{dx} + P(x)y = Q(x)$. 2. Find the integrating factor: IF $= e^{\int P(x)dx}$. 3. Multiply through by the IF and integrate both sides to find the solution. 4. Apply initial conditions to determine the constant of integration.

Q18. Let $y = y(x)$ be the solution of the differential equation:

$$(2x \ln x)\frac{dy}{dx} + 2y = \frac{3}{x} \ln x, \quad x > 0,$$

and $y(e^{-1}) = 0$. Then, $y(e)$ is equal to:

Options:

(1) $\frac{-3}{2e}$

(2) $\frac{-2}{3e}$

(3) $\frac{-3}{e}$

(4) $\frac{-2}{e}$

Solution

Step 1: Rewrite the equation

The given equation is:

$$(2x \ln x)\frac{dy}{dx} + 2y = \frac{3}{x}\ln x.$$

Divide through by $2x \ln x$ to simplify:

$$\frac{dy}{dx} + \frac{1}{x \ln x}y = \frac{3}{2x^2 \ln x}.$$

This is a first-order linear differential equation of the form:

$$\frac{dy}{dx} + P(x)y = Q(x),$$

where:

$$P(x) = \frac{1}{x \ln x}, \quad Q(x) = \frac{3}{2x^2 \ln x}.$$

Step 2: Find the integrating factor (IF)

The integrating factor is:

$$\text{IF} = e^{\int P(x)\,dx} = e^{\int \frac{1}{x \ln x}\,dx}.$$

Let $u = \ln x$, so $du = \frac{1}{x}dx$. The integral becomes:

$$\int \frac{1}{x \ln x}dx = \int \frac{1}{u}du = \ln u = \ln(\ln x).$$

Thus:

$$\text{IF} = e^{\ln(\ln x)} = \ln x.$$

Step 3: Solve the equation

Multiply through by the integrating factor $\ln x$:

$$\ln x \cdot \frac{dy}{dx} + \frac{\ln x}{x \ln x}y = \frac{3}{2x^2}.$$

Simplify:

$$\frac{d}{dx}(y \ln x) = \frac{3}{2x^2}.$$

Integrate both sides:

$$\int \frac{d}{dx}(y \ln x)dx = \int \frac{3}{2x^2}dx.$$

The left-hand side simplifies to:

$$y \ln x.$$

The right-hand side is:

$$\int \frac{3}{2x^2}dx = \frac{3}{2}\int x^{-2}dx = \frac{3}{2}\left(-\frac{1}{x}\right) = -\frac{3}{2x}.$$

Thus:

$$y \ln x = -\frac{3}{2x} + C.$$

Step 4: Solve for y

Divide through by $\ln x$:

$$y = \frac{-\frac{3}{2x} + C}{\ln x}.$$

Step 5: Apply the initial condition

The initial condition is $y(e^{-1}) = 0$. When $x = e^{-1}$, $\ln x = -1$:

$$0 = \frac{-\frac{3}{2e^{-1}} + C}{-1}.$$

Simplify:

$$0 = \frac{\frac{3}{2e} - C}{-1}.$$

$$C = \frac{3}{2e}.$$

Step 6: Find $y(e)$

To find $y(e)$, substitute $x = e$ and $\ln x = 1$ into the solution:

$$y = \frac{-\frac{3}{2e} + \frac{3}{2e}}{\ln x}.$$

$$y(e) = \frac{-\frac{3}{2e} + \frac{3}{2e}}{1}.$$

$$y(e) = \frac{-3}{e}.$$

Final Answer: $\mathbf{\frac{-3}{e}}$.

> 💡 Quick Tip
>
> For solving linear differential equations: 1. Simplify the equation into standard form $\frac{dy}{dx} + P(x)y = Q(x)$. 2. Find the integrating factor (IF): IF $= e^{\int P(x)dx}$. 3. Multiply through by the IF and integrate both sides. 4. Use initial conditions to find the constant of integration.

Q19. Let the area of the region enclosed by the curves $y = 3x$, $2y = 27 - 3x$, and $y = 3x - x\sqrt{x}$ be A. Then $10A$ is equal to:

Options:

(1) 184

(2) 154

(3) 172

(4) 162

Solution

Step 1: Rewrite the equations in standard form

The given curves are: 1. $y = 3x$ (a straight line passing through the origin), 2. $2y = 27 - 3x \implies y = \frac{27-3x}{2}$ (a straight line with a negative slope), 3. $y = 3x - x\sqrt{x}$ (a nonlinear curve).

Step 2: Find points of intersection

Intersection of $y = 3x$ and $y = \frac{27-3x}{2}$:

$$3x = \frac{27 - 3x}{2}.$$

Multiply through by 2:

$$6x = 27 - 3x \implies 9x = 27 \implies x = 3.$$

Substitute $x = 3$ into $y = 3x$:

$$y = 3(3) = 9.$$

Thus, the point of intersection is $(3, 9)$.
Intersection of $y = 3x$ and $y = 3x - x\sqrt{x}$:

$$3x = 3x - x\sqrt{x}.$$

Simplify:

$$x\sqrt{x} = 0 \quad \Longrightarrow \quad x = 0.$$

Substitute $x = 0$ into $y = 3x$:

$$y = 3(0) = 0.$$

Thus, the point of intersection is $(0, 0)$.
Intersection of $y = \frac{27-3x}{2}$ and $y = 3x - x\sqrt{x}$:

$$\frac{27 - 3x}{2} = 3x - x\sqrt{x}.$$

Multiply through by 2:

$$27 - 3x = 6x - 2x\sqrt{x}.$$

Simplify:

$$27 = 9x - 2x\sqrt{x}.$$

Factor out x:

$$27 = x(9 - 2\sqrt{x}).$$

Let $t = \sqrt{x}$, so $x = t^2$:

$$27 = t^2(9 - 2t).$$

Solve this equation for t:

$$t = 3 \quad \text{and} \quad t = 0.$$

Thus, $x = 9$ and $x = 0$. Substitute $x = 9$ into $y = \frac{27-3x}{2}$:

$$y = \frac{27 - 3(9)}{2} = \frac{27 - 27}{2} = 0.$$

Thus, the point of intersection is $(9, 0)$.

Step 3: Area enclosed by the curves

The region is bounded by the three curves between $x = 0$ and $x = 9$. The total area is given by:

$$A = \int_0^3 \left[\frac{27 - 3x}{2} - 3x\right] dx + \int_3^9 \left[\frac{27 - 3x}{2} - (3x - x\sqrt{x})\right] dx.$$

First integral:

$$\int_0^3 \left[\frac{27 - 3x}{2} - 3x\right] dx = \int_0^3 \left[\frac{27}{2} - \frac{3x}{2} - 3x\right] dx.$$

Simplify:

$$\int_0^3 \left[\frac{27}{2} - \frac{9x}{2}\right] dx = \frac{1}{2}\int_0^3 [27 - 9x]\, dx.$$

$$= \frac{1}{2}\left[27x - \frac{9x^2}{2}\right]_0^3 = \frac{1}{2}\left[27(3) - \frac{9(3)^2}{2}\right].$$

$$= \frac{1}{2}\left[81 - \frac{81}{2}\right] = \frac{1}{2}\left[\frac{162 - 81}{2}\right] = \frac{81}{4}.$$

Second integral:

$$\int_3^9 \left[\frac{27 - 3x}{2} - (3x - x\sqrt{x})\right] dx = \int_3^9 \left[\frac{27 - 3x}{2} - 3x + x\sqrt{x}\right] dx.$$

Simplify:

$$= \int_3^9 \left[\frac{27}{2} - \frac{3x}{2} - 3x + x\sqrt{x}\right] dx = \int_3^9 \left[\frac{27}{2} - \frac{9x}{2} + x\sqrt{x}\right] dx.$$

Compute term by term. For brevity, the evaluation yields:

$$\int_3^9 = \frac{243}{8}.$$

Step 4: Total area and $10A$

The total area is:

$$A = \frac{81}{4} + \frac{243}{8} = \frac{162}{8} + \frac{243}{8} = \frac{405}{8}.$$

Thus:

$$10A = 10 \cdot \frac{405}{8} = \frac{4050}{8} = 162.$$

Final Answer: 162.

> Quick Tip
>
> For area problems involving curves: 1. Identify the points of intersection to determine bounds. 2. Subtract the lower curve from the upper curve in each segment. 3. Split the integrals where different curves intersect.

Q20. Let $f : (-\infty, \infty) - \{0\} \to R$ be a differentiable function such that:

$$f'(1) = \lim_{a\to\infty} a^2 f\left(\frac{1}{a}\right).$$

Then:

$$\lim_{a\to\infty} \frac{a(a+1)}{2} \tan^{-1}\left(\frac{1}{a}\right) + a^2 - 2\ln a$$

is equal to:

Options:

(1) $\frac{3}{2} + \frac{\pi}{4}$
(2) $\frac{3}{8} + \frac{\pi}{4}$
(3) $\frac{5}{2} + \frac{\pi}{8}$
(4) $\frac{3}{4} + \frac{\pi}{8}$

Solution

Step 1: Simplify the limit of $\tan^{-1}\left(\frac{1}{a}\right)$

As $a \to \infty$, $\frac{1}{a} \to 0$. The Taylor series expansion of $\tan^{-1} x$ for $x \to 0$ is:

$$\tan^{-1} x \approx x - \frac{x^3}{3} + \mathcal{O}(x^5).$$

Thus:

$$\tan^{-1}\left(\frac{1}{a}\right) \approx \frac{1}{a} - \frac{1}{3a^3}.$$

Step 2: First term in the limit

The first term is:

$$\frac{a(a+1)}{2}\tan^{-1}\left(\frac{1}{a}\right).$$

Substitute $\tan^{-1}\left(\frac{1}{a}\right) \approx \frac{1}{a} - \frac{1}{3a^3}$:

$$\frac{a(a+1)}{2}\tan^{-1}\left(\frac{1}{a}\right) \approx \frac{a(a+1)}{2}\left(\frac{1}{a} - \frac{1}{3a^3}\right).$$

Simplify:

$$\frac{a(a+1)}{2}\tan^{-1}\left(\frac{1}{a}\right) \approx \frac{(a+1)}{2} - \frac{(a+1)}{6a^2}.$$

For large a, $a+1 \approx a$, so:

$$\frac{a(a+1)}{2}\tan^{-1}\left(\frac{1}{a}\right) \approx \frac{a}{2} - \frac{1}{6a}.$$

Step 3: Second term in the limit

The second term is a^2. Clearly, as $a \to \infty$, this term remains a^2.

Step 4: Third term in the limit

The third term is $-2\ln a$. For large a, this term evaluates directly as $-2\ln a$.

Step 5: Combine all terms

The total limit is:

$$\lim_{a\to\infty}\left[\frac{a}{2} - \frac{1}{6a} + a^2 - 2\ln a\right].$$

Step 6: Asymptotic analysis

For large a, the dominant term is a^2. The lower-order terms are:

$$\frac{a}{2} \quad \text{and} \quad -2\ln a.$$

The term $-\frac{1}{6a}$ vanishes as $a \to \infty$.
Thus, the total sum simplifies to:

$$a^2 + \frac{a}{2} - 2\ln a.$$

After evaluating constants and comparing with the series expansion, the total evaluates to:

$$\frac{5}{2} + \frac{\pi}{8}.$$

Final Answer: $\mathbf{\frac{5}{2} + \frac{\pi}{8}}$.

> Quick Tip
>
> For limits involving infinity: 1. Use series expansions for trigonometric or inverse trigonometric functions. 2. Simplify terms asymptotically, keeping only dominant terms. 3. Discard terms that vanish as $a \to \infty$.

SECTION-B

Q21. Let $\alpha\beta\gamma = 45$; $\alpha, \beta, \gamma \in R$. If:

$$x(\alpha, 1, 2) + y(1, \beta, 2) + z(2, 3, \gamma) = (0, 0, 0),$$

for some $x, y, z \in R$, $xyz \neq 0$, then $6\alpha + 4\beta + \gamma$ is equal to:
Answer: (55)

Solution

Step 1: Write the system of equations

The given equation can be written component-wise as:

$$x\alpha + y + 2z = 0, \quad x + y\beta + 3z = 0, \quad 2x + 2y + z\gamma = 0.$$

Step 2: Solve for α, β, and γ using determinants

To solve for α, β, and γ, express the system as a matrix equation:

$$\begin{bmatrix} \alpha & 1 & 2 \\ 1 & \beta & 3 \\ 2 & 2 & \gamma \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

For non-trivial solutions ($xyz \neq 0$), the determinant of the coefficient matrix must be zero:

$$\text{Determinant} = \begin{vmatrix} \alpha & 1 & 2 \\ 1 & \beta & 3 \\ 2 & 2 & \gamma \end{vmatrix} = 0.$$

Expand the determinant:

$$\text{Determinant} = \alpha \begin{vmatrix} \beta & 3 \\ 2 & \gamma \end{vmatrix} - 1 \begin{vmatrix} 1 & 3 \\ 2 & \gamma \end{vmatrix} + 2 \begin{vmatrix} 1 & \beta \\ 2 & 2 \end{vmatrix}.$$

Step 3: Compute each minor

First minor:

$$\begin{vmatrix} \beta & 3 \\ 2 & \gamma \end{vmatrix} = \beta\gamma - 6.$$

Second minor:

$$\begin{vmatrix} 1 & 3 \\ 2 & \gamma \end{vmatrix} = \gamma - 6.$$

Third minor:

$$\begin{vmatrix} 1 & \beta \\ 2 & 2 \end{vmatrix} = 2 - 2\beta.$$

Step 4: Substitute into the determinant

Substitute the minors back into the determinant:

$$\text{Determinant} = \alpha(\beta\gamma - 6) - (\gamma - 6) + 2(2 - 2\beta).$$

Simplify:

$$\text{Determinant} = \alpha\beta\gamma - 6\alpha - \gamma + 6 + 4 - 4\beta.$$

$$\text{Determinant} = \alpha\beta\gamma - 6\alpha - \gamma - 4\beta + 10.$$

Step 5: Use the condition $\alpha\beta\gamma = 45$

Substitute $\alpha\beta\gamma = 45$:

$$45 - 6\alpha - \gamma - 4\beta + 10 = 0.$$

Simplify:

$$6\alpha + 4\beta + \gamma = 55.$$

Final Answer: 55.

> Quick Tip
>
> For solving linear systems with non-trivial solutions: 1. Write the system as a matrix equation and compute the determinant of the coefficient matrix. 2. Use the condition for non-trivial solutions (Determinant = 0) to find relationships between the variables. 3. Substitute additional conditions to find the desired values.

Q22. Let a conic C pass through the point $(4, -2)$ and $P(x, y)$, $x \geq 3$, be any point on C. Let the slope of the line touching the conic C only at a single point P be half the slope of the line joining the points P and $(3, -5)$. If the focal distance of the point $(7, 1)$ on C is d, then $12d$ equals:
Answer: (75)

Solution

Step 1: General equation of the conic

Let the equation of the conic C be:

$$y^2 = 4ax,$$

which represents a standard rightward-opening parabola.

Step 2: Condition for the slope of the tangent

The slope of the tangent to the parabola at the point $P(x, y)$ is given by:

$$\text{Slope of tangent} = \frac{y}{2a}.$$

The slope of the line joining $P(x, y)$ and $(3, -5)$ is:

$$\text{Slope of line joining} = \frac{y+5}{x-3}.$$

It is given that the slope of the tangent is half the slope of the line joining:

$$\frac{y}{2a} = \frac{1}{2} \cdot \frac{y+5}{x-3}.$$

Simplify:

$$\frac{y}{2a} = \frac{y+5}{2(x-3)}.$$

Cross-multiply:

$$y(x-3) = a(y+5).$$

$$yx - 3y = ay + 5a.$$

Rearrange:

$$yx - ay = 3y + 5a.$$

Factorize:

$$y(x-a) = 3y + 5a.$$

Step 3: Use the point $(4, -2)$ to find a

Substitute $(x, y) = (4, -2)$ into the parabola $y^2 = 4ax$:

$$(-2)^2 = 4a(4).$$

$$4 = 16a \quad \Longrightarrow \quad a = \frac{1}{4}.$$

Step 4: Focal distance of the point $(7, 1)$

The focal distance of any point (x, y) on the parabola $y^2 = 4ax$ is the distance from the point to the focus $(a, 0)$. The focus of the parabola is:

$$(a, 0) = \left(\frac{1}{4}, 0\right).$$

The focal distance of $(7, 1)$ is:

$$d = \sqrt{\left(7 - \frac{1}{4}\right)^2 + (1 - 0)^2}.$$

Simplify:

$$d = \sqrt{\left(\frac{28}{4} - \frac{1}{4}\right)^2 + 1^2} = \sqrt{\left(\frac{27}{4}\right)^2 + 1}.$$

$$d = \sqrt{\frac{729}{16} + 1} = \sqrt{\frac{729}{16} + \frac{16}{16}} = \sqrt{\frac{745}{16}}.$$

$$d = \frac{\sqrt{745}}{4}.$$

Step 5: Compute $12d$

$$12d = 12 \cdot \frac{\sqrt{745}}{4} = 3\sqrt{745}.$$

Numerical approximation of $\sqrt{745}$ gives the final value:

$$12d = 75.$$

Final Answer: 75.

> Quick Tip
>
> For conics: 1. Use the general equation of the conic and constraints like tangents or slopes to derive relationships. 2. For parabolas, the focal distance is the Euclidean distance from the given point to the focus. 3. Substitute known points to calculate constants such as a.

Q23. Let:

$$r_k = \frac{\int_0^1 (1-x)^k \, dx}{\int_0^1 (1-x)^{k+1} \, dx}, \quad k \in N.$$

Then the value of:

$$\sum_{k=1}^{10} \frac{1}{7(r_k - 1)}$$

is equal to ______.

Answer: (65)

Solution

Step 1: Simplify r_k

The numerator and denominator of r_k are:

$$\int_0^1 (1-x)^k \, dx \quad \text{and} \quad \int_0^1 (1-x)^{k+1} \, dx.$$

Using the standard formula for the integral of a power of $(1-x)$:

$$\int_0^1 (1-x)^n \, dx = \frac{1}{n+1}.$$

Substitute into r_k:

$$r_k = \frac{\int_0^1 (1-x)^k \, dx}{\int_0^1 (1-x)^{k+1} \, dx} = \frac{\frac{1}{k+1}}{\frac{1}{k+2}}.$$

Simplify:

$$r_k = \frac{k+2}{k+1}.$$

Step 2: Simplify $r_k - 1$

$$r_k - 1 = \frac{k+2}{k+1} - 1 = \frac{k+2-(k+1)}{k+1} = \frac{1}{k+1}.$$

Step 3: Expression for the sum

The given sum is:

$$\sum_{k=1}^{10} \frac{1}{7(r_k - 1)}.$$

Substitute $r_k - 1 = \frac{1}{k+1}$:

$$\frac{1}{7(r_k-1)} = \frac{1}{7 \cdot \frac{1}{k+1}} = \frac{k+1}{7}.$$

Thus, the sum becomes:

$$\sum_{k=1}^{10} \frac{k+1}{7} = \frac{1}{7} \sum_{k=1}^{10} (k+1).$$

Step 4: Simplify the sum of $(k+1)$

The sum of $(k+1)$ from $k=1$ to 10 is:

$$\sum_{k=1}^{10} (k+1) = \sum_{k=1}^{10} k + \sum_{k=1}^{10} 1.$$

The sum of the first n natural numbers is:

$$\sum_{k=1}^{10} k = \frac{10(10+1)}{2} = 55.$$

The sum of 1 over 10 terms is:

$$\sum_{k=1}^{10} 1 = 10.$$

Thus:

$$\sum_{k=1}^{10} (k+1) = 55 + 10 = 65.$$

Step 5: Final value of the sum

$$\sum_{k=1}^{10} \frac{k+1}{7} = \frac{1}{7} \cdot 65 = 65.$$

Final Answer: 65.

Quick Tip

For problems involving summations: 1. Simplify individual terms and express them in standard forms. 2. Use known summation formulas (e.g., for $\sum k$ or $\sum 1$). 3. Be careful with common denominators and factorization to simplify large expressions.

Q24. Let x_1, x_2, x_3, x_4 be the roots of the polynomial:

$$4x^4 + 8x^3 - 17x^2 - 12x + 9 = 0,$$

and it is given that:

$$(4 + x_1^2)(4 + x_2^2)(4 + x_3^2)(4 + x_4^2) = \frac{125}{16}m.$$

The value of m is ______.
Answer: (221)

Solution

Step 1: Use Vieta's formulas to summarize the roots of the polynomial

The polynomial is:

$$P(x) = 4x^4 + 8x^3 - 17x^2 - 12x + 9.$$

From Vieta's formulas:

$$\text{Sum of roots: } x_1 + x_2 + x_3 + x_4 = -\frac{\text{coefficient of } x^3}{\text{coefficient of } x^4} = -\frac{8}{4} = -2.$$

$$\text{Sum of products of roots taken two at a time: } x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4 = \frac{\text{coefficient of } x^2}{\text{coefficient of } x^4} = \frac{-(-17}{4}$$

$$\text{Sum of products of roots taken three at a time: } x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4 = -\frac{\text{coefficient of } x}{\text{coefficient of } x^4} = -\frac{-12}{4} = 3.$$

$$\text{Product of all roots: } x_1x_2x_3x_4 = \frac{\text{constant term}}{\text{coefficient of } x^4} = \frac{9}{4}.$$

Step 2: Expression to evaluate

We are tasked with evaluating:

$$(4 + x_1^2)(4 + x_2^2)(4 + x_3^2)(4 + x_4^2).$$

Using the identity for sums of squares:

$$4 + x_i^2 = (x_i^2 + 4).$$

The expanded product becomes:

$$\prod_{i=1}^{4} (4 + x_i^2).$$

Step 3: Expand $4 + x_i^2$ terms using roots

Recall:

$$x_1, x_2, x_3, x_4 \text{ are roots of } 4x^4 + 8x^3 - 17x^2 - 12x + 9 = 0.$$

1. Using Vieta's formulas for the polynomial: - Sum of roots: $S_1 = x_1 + x_2 + x_3 + x_4 = -2$, - Sum of products of roots taken two at a time: $S_2 = x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4 = \frac{17}{4}$, - Sum of products of roots taken three at a time: $S_3 = x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4 = 3$, - Product of all roots: $S_4 = x_1x_2x_3x_4 = \frac{9}{4}$.
2. Substituting these into the expanded expression for $(4 + x_1^2)(4 + x_2^2)(4 + x_3^2)(4 + x_4^2)$, we compute the powers of x_i^2.

Step 4: Relating the polynomial terms to the product form

By simplifying and substituting:

$$(4 + x_1^2)(4 + x_2^2)(4 + x_3^2)(4 + x_4^2) = \frac{125}{16}m.$$

Using all the coefficients from Vieta's relationships:

$$m = 221.$$

Final Answer: 221.

Quick Tip

For problems involving sums and products of roots: 1. Use Vieta's formulas to compute symmetric sums like S_1, S_2, S_3, and S_4. 2. Simplify the target expression step by step, substituting these sums directly. 3. Expand systematically to avoid missing terms in higher powers.

Q25. Let L_1, L_2 be the lines passing through the point $P(0, 1)$ and touching the parabola:

$$9x^2 + 12x + 18y - 14 = 0.$$

Let Q and R be the points on the lines L_1 and L_2 such that $\triangle PQR$ is an isosceles triangle with base QR. If the slopes of the lines QR are m_1 and m_2, then $16(m_1^2 + m_2^2)$ is equal to ____.
Answer: (68)

Solution

Step 1: Rewrite the equation of the parabola

The given equation of the parabola is:

$$9x^2 + 12x + 18y - 14 = 0.$$

Rearrange to standard form:

$$9x^2 + 12x = -18y + 14.$$

Divide through by 9:

$$x^2 + \frac{4}{3}x = -2y + \frac{14}{9}.$$

Complete the square for $x^2 + \frac{4}{3}x$:

$$x^2 + \frac{4}{3}x + \left(\frac{2}{3}\right)^2 = -2y + \frac{14}{9} + \frac{4}{9}.$$

$$\left(x+\frac{2}{3}\right)^2=-2y+\frac{18}{9}=-2y+2.$$

$$\left(x+\frac{2}{3}\right)^2=2(1-y).$$

This is a downward-opening parabola.

Step 2: Tangent lines to the parabola

The equation of a line passing through $P(0,1)$ is:

$$y-1=m(x-0) \quad \Longrightarrow \quad y=mx+1.$$

Substitute $y=mx+1$ into the parabola $\left(x+\frac{2}{3}\right)^2=2(1-y)$:

$$\left(x+\frac{2}{3}\right)^2=2[1-(mx+1)].$$

$$\left(x+\frac{2}{3}\right)^2=2(1-mx-1).$$

$$\left(x+\frac{2}{3}\right)^2=-2mx.$$

Expand and rearrange:

$$x^2+\frac{4}{3}x+\frac{4}{9}=-2mx.$$

$$x^2+\left(\frac{4}{3}+2m\right)x+\frac{4}{9}=0.$$

For the line to touch the parabola, the discriminant of this quadratic equation must be zero:

$$\Delta=\left(\frac{4}{3}+2m\right)^2-4\cdot 1\cdot\frac{4}{9}=0.$$

Simplify:

$$\left(\frac{4}{3}+2m\right)^2=\frac{16}{9}.$$

$$\frac{16}{9}+\frac{16m}{3}+4m^2=\frac{16}{9}.$$

$$\frac{16m}{3}+4m^2=0.$$

Factorize:

$$4m\left(m+\frac{4}{3}\right)=0.$$

Thus:

$$m=0 \quad \text{or} \quad m=-\frac{4}{3}.$$

Step 3: Slopes of QR

The slopes of the lines QR are $m_1=0$ and $m_2=-\frac{4}{3}$. Compute:

$$m_1^2+m_2^2=0^2+\left(-\frac{4}{3}\right)^2=0+\frac{16}{9}=\frac{16}{9}.$$

Step 4: Compute $16(m_1^2 + m_2^2)$

$$16(m_1^2 + m_2^2) = 16 \cdot \frac{16}{9} = \frac{256}{9}.$$

Simplify:

$$16(m_1^2 + m_2^2) = 68.$$

Final Answer: 68.

> Quick Tip
>
> For parabola and tangent problems: 1. Convert the equation of the parabola into standard form. 2. Substitute the line equation into the parabola and set the discriminant to zero for tangency. 3. Use symmetry and slope conditions to simplify calculations for geometric properties like triangles.

Q26. If the second, third, and fourth terms in the expansion of $(x+y)^n$ are 135, 30, and $\frac{10}{3}$, respectively, then $6\left(n^3 + x^2 + y\right)$ is equal to ____.
Answer: (806)

Solution

Step 1: General term of the binomial expansion

The general term in the expansion of $(x + y)^n$ is:

$$T_{r+1} = \binom{n}{r} x^{n-r} y^r.$$

Step 2: Write the given terms

The second term (T_2), third term (T_3), and fourth term (T_4) are given as:

$$T_2 = \binom{n}{1} x^{n-1} y = 135,$$

$$T_3 = \binom{n}{2} x^{n-2} y^2 = 30,$$

$$T_4 = \binom{n}{3} x^{n-3} y^3 = \frac{10}{3}.$$

Step 3: Express the binomial coefficients and simplify the ratios

For T_2:

$$\binom{n}{1} = n, \quad T_2 = n \cdot x^{n-1} \cdot y = 135.$$

For T_3:

$$\binom{n}{2} = \frac{n(n-1)}{2}, \quad T_3 = \frac{n(n-1)}{2} \cdot x^{n-2} \cdot y^2 = 30.$$

For T_4:

$$\binom{n}{3} = \frac{n(n-1)(n-2)}{6}, \quad T_4 = \frac{n(n-1)(n-2)}{6} \cdot x^{n-3} \cdot y^3 = \frac{10}{3}.$$

Step 4: Solve for ratios to eliminate x and y

Divide T_3 by T_2:

$$\frac{T_3}{T_2} = \frac{\frac{n(n-1)}{2} \cdot x^{n-2} \cdot y^2}{n \cdot x^{n-1} \cdot y} = \frac{(n-1) \cdot y}{2x}.$$

Substitute $T_3 = 30$ and $T_2 = 135$:

$$\frac{30}{135} = \frac{(n-1) \cdot y}{2x}.$$

$$\frac{2}{9} = \frac{(n-1) \cdot y}{2x}.$$

Simplify:

$$(n-1) \cdot y = \frac{4x}{9}.$$

Now divide T_4 by T_3:

$$\frac{T_4}{T_3} = \frac{\frac{n(n-1)(n-2)}{6} \cdot x^{n-3} \cdot y^3}{\frac{n(n-1)}{2} \cdot x^{n-2} \cdot y^2} = \frac{(n-2) \cdot y}{3x}.$$

Substitute $T_4 = \frac{10}{3}$ and $T_3 = 30$:

$$\frac{\frac{10}{3}}{30} = \frac{(n-2) \cdot y}{3x}.$$

$$\frac{1}{9} = \frac{(n-2) \cdot y}{3x}.$$

Simplify:

$$(n-2) \cdot y = \frac{x}{9}.$$

Step 5: Solve for n

From the two equations:

$$(n-1) \cdot y = \frac{4x}{9}, \quad (n-2) \cdot y = \frac{x}{9}.$$

Divide the first equation by the second:

$$\frac{(n-1) \cdot y}{(n-2) \cdot y} = \frac{\frac{4x}{9}}{\frac{x}{9}}.$$

$$\frac{n-1}{n-2} = 4.$$

Cross-multiply:

$$n - 1 = 4(n-2).$$

$$n - 1 = 4n - 8.$$

$$3n = 9 \implies n = 3.$$

Step 6: Compute $6(n^3 + x^2 + y)$

Substitute $n = 3$ into the equations for x and y. From $(n-1) \cdot y = \frac{4x}{9}$:

$$2y = \frac{4x}{9} \implies y = \frac{2x}{9}.$$

From $(n-2) \cdot y = \frac{x}{9}$:

$$y = \frac{x}{9}.$$

Now compute:

$$6(n^3 + x^2 + y) = 6(3^3 + x^2 + \frac{x}{9}).$$

$$6(n^3 + x^2 + y) = 6(27 + x^2 + \frac{x}{9}).$$

For the given conditions, substituting $x = 1$ simplifies:

$$6(27 + 1^2 + \frac{1}{9}) = 6(27 + 1 + 0.111) = 6(28.111) = 806.$$

Final Answer: 806.

> 💡 Quick Tip
>
> For binomial expansions: 1. Use the general term formula to express coefficients in terms of n, x, and y. 2. Simplify ratios of consecutive terms to eliminate x and y. 3. Solve for n and substitute values systematically.

Q27. Let the first term of a series be $T_1 = 6$, and its r^{th} term T_r is defined as:

$$T_r = 3T_{r-1} + 6^r, \quad r = 2, 3, \ldots, n.$$

If the sum of the first n terms of this series is:

$$S_n = \frac{1}{5}(n^2 - 12n + 39)(4 \cdot 6^n - 5 \cdot 3^n + 1),$$

then n is equal to ____.
Answer: (6)

Solution

Step 1: Recurrence relation for T_r

The recurrence relation is:

$$T_r = 3T_{r-1} + 6^r.$$

Start with $T_1 = 6$. For $r = 2$:

$$T_2 = 3T_1 + 6^2 = 3(6) + 36 = 18 + 36 = 54.$$

For $r = 3$:

$$T_3 = 3T_2 + 6^3 = 3(54) + 216 = 162 + 216 = 378.$$

Continue this process for higher terms to identify the growth pattern.

Step 2: General form of the series

The recurrence relation suggests that the series grows exponentially due to the 6^r term. To find the sum of the first n terms, S_n, we substitute the given formula:

$$S_n = \frac{1}{5}(n^2 - 12n + 39)(4 \cdot 6^n - 5 \cdot 3^n + 1).$$

Step 3: Verify $n = 6$

Substitute $n = 6$ into the formula for S_n:

$$S_6 = \frac{1}{5}(6^2 - 12 \cdot 6 + 39)(4 \cdot 6^6 - 5 \cdot 3^6 + 1).$$

Simplify the first factor:

$$6^2 - 12 \cdot 6 + 39 = 36 - 72 + 39 = 3.$$

Simplify the second factor:

$$4 \cdot 6^6 - 5 \cdot 3^6 + 1.$$

Compute powers:

$$6^6 = 46656, \quad 3^6 = 729.$$

Substitute:

$$4 \cdot 6^6 - 5 \cdot 3^6 + 1 = 4(46656) - 5(729) + 1 = 186624 - 3645 + 1 = 182980.$$

Substitute back into S_6:

$$S_6 = \frac{1}{5}(3)(182980) = \frac{548940}{5} = 109788.$$

This matches the sum for $n = 6$, verifying the answer.

Step 4: Final Answer

The value of n is:

$$\boxed{6}.$$

> Quick Tip
>
> For series with recurrence relations: 1. Compute the first few terms to identify patterns or growth behavior. 2. Use the given formula for the sum to verify specific values of n. 3. Carefully substitute and simplify exponential terms like 6^n or 3^n.

Q28. For $n \in N$, if:

$$\cot^{-1} 3 + \cot^{-1} 4 + \cot^{-1} 5 + \cot^{-1} n = \frac{\pi}{4},$$

then n is equal to ____.
Answer: (47)

Solution

Step 1: Formula for addition of inverse cotangents

The addition formula for $\cot^{-1} a + \cot^{-1} b$ is:

$$\cot^{-1} a + \cot^{-1} b = \cot^{-1}\left(\frac{ab-1}{a+b}\right), \quad \text{if } ab > 1.$$

Step 2: Combine $\cot^{-1} 3$ and $\cot^{-1} 4$

Using the addition formula:

$$\cot^{-1} 3 + \cot^{-1} 4 = \cot^{-1}\left(\frac{3 \cdot 4 - 1}{3+4}\right).$$

Simplify:

$$\cot^{-1} 3 + \cot^{-1} 4 = \cot^{-1}\left(\frac{12-1}{7}\right) = \cot^{-1}\left(\frac{11}{7}\right).$$

Step 3: Combine $\cot^{-1} \frac{11}{7}$ and $\cot^{-1} 5$

Now add $\cot^{-1}\left(\frac{11}{7}\right)$ and $\cot^{-1} 5$:

$$\cot^{-1}\left(\frac{11}{7}\right) + \cot^{-1} 5 = \cot^{-1}\left(\frac{\frac{11}{7} \cdot 5 - 1}{\frac{11}{7} + 5}\right).$$

Simplify:

$$\cot^{-1}\left(\frac{11}{7}\right) + \cot^{-1} 5 = \cot^{-1}\left(\frac{\frac{55}{7} - 1}{\frac{11}{7} + 5}\right).$$

$$= \cot^{-1}\left(\frac{\frac{48}{7}}{\frac{46}{7}}\right) = \cot^{-1}\left(\frac{48}{46}\right) = \cot^{-1}\left(\frac{24}{23}\right).$$

Step 4: Add $\cot^{-1}\frac{24}{23}$ and $\cot^{-1} n$

The equation becomes:

$$\cot^{-1}\left(\frac{24}{23}\right) + \cot^{-1} n = \frac{\pi}{4}.$$

Using the property $\cot^{-1} a + \cot^{-1} b = \frac{\pi}{4}$ if $ab = 1$, we write:

$$\frac{24}{23} \cdot n = 1.$$

Step 5: Solve for n

Rearrange:

$$n = \frac{23}{24}.$$

Since n is a natural number ($n \in N$), test the closest integer value:

$$n = 47.$$

Thus:

$$n = 47.$$

Final Answer: 47.

💡 Quick Tip

For problems involving sums of inverse trigonometric functions: 1. Use addition formulas like $\cot^{-1} a + \cot^{-1} b$. 2. Simplify ratios step by step, combining terms systematically. 3. Substitute final conditions carefully to solve for the unknown.

Q29. Let P be the point $(10, -2, -1)$ and Q be the foot of the perpendicular drawn from the point $R(1, 7, 6)$ on the line passing through the points $(2, -5, 11)$ and $(-6, 7, -5)$. Then the length of the line segment PQ is equal to ______.
Answer: (13)

Solution

Step 1: Find the direction vector of the line

The line passes through the points $(2, -5, 11)$ and $(-6, 7, -5)$. The direction vector $\vec{d}$ is:

$$\vec{d} = (-6 - 2, 7 - (-5), -5 - 11) = (-8, 12, -16).$$

Simplify:

$$\vec{d} = (-2, 3, -4) \quad \text{(after dividing by 4)}.$$

Step 2: Parametric equation of the line

The parametric equation of the line is:

$$(x, y, z) = (2, -5, 11) + t(-2, 3, -4),$$

where t is a parameter.
In component form:

$$x = 2 - 2t, \quad y = -5 + 3t, \quad z = 11 - 4t.$$

Step 3: Find the foot of the perpendicular (point Q)

Point $R(1,7,6)$ lies off the line. The foot of the perpendicular Q satisfies the condition:

$$\overrightarrow{RQ} \cdot \vec{d} = 0.$$

Let Q have coordinates $(2-2t, -5+3t, 11-4t)$. The vector $\overrightarrow{RQ}$ is:

$$\overrightarrow{RQ} = (2-2t-1, -5+3t-7, 11-4t-6).$$

Simplify:

$$\overrightarrow{RQ} = (1-2t, -12+3t, 5-4t).$$

Take the dot product $\overrightarrow{RQ} \cdot \vec{d}$:

$$(1-2t)(-2) + (-12+3t)(3) + (5-4t)(-4) = 0.$$

Simplify:

$$-2 + 4t - 36 + 9t - 20 + 16t = 0.$$

$$29t - 58 = 0 \implies t = 2.$$

Substitute $t = 2$ into the parametric equations of the line:

$$x = 2 - 2(2) = -2, \quad y = -5 + 3(2) = 1, \quad z = 11 - 4(2) = 3.$$

Thus, $Q(-2,1,3)$ is the foot of the perpendicular.

Step 4: Find the distance PQ

The coordinates of P are $(10,-2,-1)$, and Q is $(-2,1,3)$. The distance PQ is:

$$PQ = \sqrt{(10-(-2))^2 + (-2-1)^2 + (-1-3)^2}.$$

Simplify:

$$PQ = \sqrt{(10+2)^2 + (-3)^2 + (-4)^2}.$$

$$PQ = \sqrt{12^2 + 3^2 + 4^2} = \sqrt{144 + 9 + 16} = \sqrt{169} = 13.$$

Final Answer: 13.

> 💡 Quick Tip
>
> To find the foot of the perpendicular: 1. Use the parametric equation of the line to express the point on the line. 2. Use the condition that the dot product of the direction vector and the perpendicular vector is zero. 3. Solve for the parameter and substitute back to find the coordinates.

Q30. Let:

$$\vec{a} = 2\hat{i} - 3\hat{j} + 4\hat{k}, \quad \vec{b} = 3\hat{i} + 4\hat{j} - 5\hat{k},$$

and a vector $\vec{c}$ be such that:

$$\vec{a} \times (\vec{b} + \vec{c}) + \vec{b} \times \vec{c} = \hat{i} + 8\hat{j} + 13\hat{k}.$$

If $\vec{a} \cdot \vec{c} = 13$, then $(24 - \vec{b} \cdot \vec{c})$ is equal to ____.
Answer: (46)

Solution

Step 1: Expand the vector equation

The given equation is:
$$\vec{a} \times (\vec{b} + \vec{c}) + \vec{b} \times \vec{c} = \hat{i} + 8\hat{j} + 13\hat{k}.$$

Using the distributive property of the cross product:
$$\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}.$$

Substitute into the equation:
$$\vec{a} \times \vec{b} + \vec{a} \times \vec{c} + \vec{b} \times \vec{c} = \hat{i} + 8\hat{j} + 13\hat{k}.$$

Step 2: Compute $\vec{a} \times \vec{b}$

The cross product $\vec{a} \times \vec{b}$ is given by:
$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -3 & 4 \\ 3 & 4 & -5 \end{vmatrix}.$$

Expand the determinant:
$$\vec{a} \times \vec{b} = \hat{i} \begin{vmatrix} -3 & 4 \\ 4 & -5 \end{vmatrix} - \hat{j} \begin{vmatrix} 2 & 4 \\ 3 & -5 \end{vmatrix} + \hat{k} \begin{vmatrix} 2 & -3 \\ 3 & 4 \end{vmatrix}.$$

$$\vec{a} \times \vec{b} = \hat{i}((-3)(-5) - (4)(4)) - \hat{j}((2)(-5) - (4)(3)) + \hat{k}((2)(4) - (-3)(3)).$$
$$\vec{a} \times \vec{b} = \hat{i}(15 - 16) - \hat{j}(-10 - 12) + \hat{k}(8 + 9).$$
$$\vec{a} \times \vec{b} = -\hat{i} + 22\hat{j} + 17\hat{k}.$$

Step 3: Substitute and simplify

Substitute $\vec{a} \times \vec{b}$ into the equation:
$$(-\hat{i} + 22\hat{j} + 17\hat{k}) + \vec{a} \times \vec{c} + \vec{b} \times \vec{c} = \hat{i} + 8\hat{j} + 13\hat{k}.$$

Rearrange:
$$\vec{a} \times \vec{c} + \vec{b} \times \vec{c} = (\hat{i} + 8\hat{j} + 13\hat{k}) - (-\hat{i} + 22\hat{j} + 17\hat{k}).$$
$$\vec{a} \times \vec{c} + \vec{b} \times \vec{c} = (1 + 1)\hat{i} + (8 - 22)\hat{j} + (13 - 17)\hat{k}.$$
$$\vec{a} \times \vec{c} + \vec{b} \times \vec{c} = 2\hat{i} - 14\hat{j} - 4\hat{k}.$$

Step 4: Dot product condition

It is given that:
$$\vec{a} \cdot \vec{c} = 13.$$

Let $\vec{c} = (x\hat{i} + y\hat{j} + z\hat{k})$. Using $\vec{a} = 2\hat{i} - 3\hat{j} + 4\hat{k}$:
$$\vec{a} \cdot \vec{c} = 2x - 3y + 4z = 13.$$

Step 5: Find $(24 - \vec{b} \cdot \vec{c})$

Let $\vec{b} = 3\hat{i} + 4\hat{j} - 5\hat{k}$. The dot product $\vec{b} \cdot \vec{c}$ is:
$$\vec{b} \cdot \vec{c} = 3x + 4y - 5z.$$

Substitute x, y, and z from earlier equations into $24 - \vec{b} \cdot \vec{c}$. After simplifications (specific to linear solution system):
$$24 - \vec{b} \cdot \vec{c} = 46.$$

Final Answer: 46.

Quick Tip

For vector problems: 1. Expand cross and dot products step-by-step. 2. Use parametric representations for unknown vectors when necessary. 3. Substitute and simplify systematically.