Jee Advanced 2023 Jun 4 paper 2 Question Paper with Solutions

Time Allowed : 3 Hour	**Maximum Marks :** 180	**Total Questions :** 51

General Instructions

Read the following instructions very carefully and strictly follow them:

1. The JEE Advanced, Paper 2, will be structured with a total of 180 marks over 3 hours.
2. It will include 51 questions, with 17 questions each in Physics, Chemistry, and Mathematics.
3. Each subject will be segmented into four sections: Section I: 12 marks.
4. The marking scheme also varies, for example, questions may carry 1 mark, 2 marks, 3 marks or 4 marks.
5. There are negative markings of -1 or -2 and some questions can also read to no negative marking.

1. Let $f : [1, \infty) \to R$ be a differentiable function such that $f(1) = \frac{1}{3}$ and $3\int_1^x f(t)dt = xf(x) - \frac{x^3}{3}$, $x \in [1, \infty)$. Let e denote the base of the natural logarithm. Then the value of $f(e)$ is:

(A) $\frac{e^2+4}{3}$
(B) $\frac{\log_e 4+e}{3}$
(C) $\frac{4e^2}{3}$
(D) $\frac{e^2-4}{3}$

Correct Answer: (C) $\frac{4e^2}{3}$

Solution:

Step 1: Rewrite the integral equation using Newton's Leibniz theorem.

$$3f(x) = xf'(x) + f(x) - x^2 \quad \text{(Differentiating both sides w.r.t. } x\text{)}.$$

Step 2: Simplify and solve for $f(x)$.

$$f'(x) + \left(\frac{1}{x}\right) f(x) = x \quad \text{(Rewriting the equation)}.$$

This is a first-order linear differential equation. The integrating factor (I.F.) is:

$$\text{I.F.} = e^{\int \frac{1}{x}dx} = e^{\log x} = x.$$

Step 3: Solve the differential equation. Multiplying through by the I.F.:

$$x\frac{d}{dx}(f(x)) = x^2 \implies f(x) = x\log x + \frac{c}{x}.$$

Step 4: Apply the initial condition $f(1) = \frac{1}{3}$.

$$f(1) = 1 \cdot \log 1 + \frac{c}{1} = \frac{1}{3} \implies c = \frac{1}{3}.$$

Simplifying:

$$f(e) = \frac{4e^2}{3}.$$

> 💡 Quick Tip
>
> For differential equations, use the integrating factor method for first-order linear equations.

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2. Consider an experiment of tossing a coin repeatedly until the outcomes of two consecutive tosses are the same. If the probability of a random toss resulting in a head is $\frac{1}{3}$, then the probability that the experiment stops with a head is:

(A) $\frac{1}{3}$
(B) $\frac{5}{21}$
(C) $\frac{4}{21}$
(D) $\frac{2}{7}$

Correct Answer: (B) $\frac{5}{21}$

Solution:

Step 1: Define the probabilities for heads and tails.

$$P(H) = \frac{1}{3}, \quad P(T) = \frac{2}{3}.$$

Step 2: Compute the probability that the experiment ends with two consecutive heads.

$$P = P(HH) + \big(P(THH) + P(TTHH) + \dots\big).$$

Expanding:

$$P = \frac{1}{9} + \frac{2}{9} \cdot \frac{1}{9} + \frac{2^2}{9} \cdot \frac{1}{9} + \dots$$

Step 3: Simplify using the geometric series.

$$\frac{\frac{1}{3} \cdot \frac{1}{3}}{1 - \frac{2}{3} \cdot \frac{1}{3}} + \frac{\frac{2}{3} \cdot \frac{1}{3} \cdot \frac{1}{3}}{1 - \frac{2}{3} \cdot \frac{1}{3}} = \frac{5}{21}$$

> 💡 Quick Tip
>
> For problems involving geometric probabilities, identify the repeating pattern and use the geometric series formula.

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3. For any $y \in R$, let $\cot^{-1}(y) \in (0, \pi)$ and $\tan^{-1}(y) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then the sum of all the solutions of the equation

$$\tan^{-1}\left(\frac{6y}{9 - y^2}\right) + \cot^{-1}\left(\frac{9 - y^2}{6y}\right) = \frac{2\pi}{3},$$

for $0 < |y| < 3$, is equal to:
(A) $2\sqrt{3} - 3$
(B) $3 - 2\sqrt{3}$
(C) $4\sqrt{3} - 6$
(D) $6 - 4\sqrt{3}$
Correct Answer: (C) $4\sqrt{3} - 6$
Solution:
Step 1: Simplify using trigonometric identities.

$$\tan^{-1}\left(\frac{6y}{9-y^2}\right) + \cot^{-1}\left(\frac{9-y^2}{6y}\right) = \pi \quad \text{(Sum of complementary angles).}$$

Step 2: Solve for y using given conditions.

$$\tan^{-1}\left(\frac{6y}{9-y^2}\right) = \frac{\pi}{3}.$$

Simplifying:

$$\frac{6y}{9-y^2} = \sqrt{3}.$$

Solve the quadratic:

$$\begin{aligned}
&\tan^{-1}\left(\frac{6y}{9-y^2}\right) + \tan^{-1}\left(\frac{6y}{9-y^2}\right) + \pi = \frac{2\pi}{3} \\
\Rightarrow &\tan^{-1}\left(\frac{6y}{9-y^2}\right) = -\frac{\pi}{6} \\
\Rightarrow &\tan\left[\tan^{-1}\left(\frac{6y}{9-y^2}\right)\right] = -\frac{1}{\sqrt{3}} \\
\Rightarrow &6\sqrt{3}y = -9 + y^2 \Rightarrow y^2 - 6\sqrt{3}y - 9 = 0 \\
\Rightarrow &y = \frac{6\sqrt{3} - \sqrt{144}}{2} = \frac{6\sqrt{3} - 12}{2} = 3\sqrt{3} - 6
\end{aligned}$$

Step 3: Sum of solutions.

$$\text{Sum of solutions} = 3\sqrt{3} - 6 + \sqrt{3} = 4\sqrt{3} - 6.$$

Quick Tip
For trigonometric equations, use complementary angle identities to simplify complex terms.

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4. Let the position vectors of the points P, Q, R and S be $\vec{a} = \hat{i} + 2\hat{j} - 5\hat{k}$, $\vec{b} = 3\hat{i} + 6\hat{j} + 3\hat{k}$, $\vec{c} = \frac{17}{5}\hat{i} + \frac{16}{5}\hat{j} + 7\hat{k}$, and $\vec{d} = 2\hat{i} + \hat{j} + \hat{k}$, respectively. Then which of the following statements is true?
(A) The points P, Q, R and S are NOT coplanar.
(B) $\frac{\vec{b}+2\vec{d}}{3}$ is the position vector of a point which divides PR internally in the ratio $5 : 4$.

(C) $\frac{\vec{b}+2\vec{d}}{3}$ is the position vector of a point which divides PR externally in the ratio 5 : 4.
(D) The square of the magnitude of the vector $\vec{b}\times\vec{d}$ is 95.
Correct Answer: (B) $\frac{\vec{b}+2\vec{d}}{3}$ is the position vector of a point which divides PR internally in the ratio 5 : 4.

Solution:
Step 1: Calculate vectors $\vec{PQ}$, $\vec{PR}$, and $\vec{PS}$.

$$\vec{a} = \hat{\imath} + 2\hat{\jmath} - 5\hat{k}$$

,

$$\vec{b} = 3\hat{\imath} + 6\hat{\jmath} + 3\hat{k},$$

$$\vec{c} = \frac{17}{5}\hat{\imath} + \frac{16}{5}\hat{\jmath} + 7\hat{k}$$

$$\vec{PQ} = \vec{b} - \vec{a} = 2\hat{\imath} + 4\hat{\jmath} + 8\hat{k}.$$
$$\vec{PR} = \vec{c} - \vec{a} = \frac{12}{5}\hat{\imath} + \frac{6}{5}\hat{\jmath} + 12\hat{k}.$$
$$\vec{PS} = \vec{d} - \vec{a} = \hat{\imath} - \hat{\jmath} + 6\hat{k}.$$

Step 2: Check coplanarity. Using the scalar triple product:

$$\begin{vmatrix} 2 & 4 & 8 \\ \frac{12}{5} & \frac{6}{5} & 12 \\ 1 & -1 & 6 \end{vmatrix} = 0 \quad \text{(hence coplanar)}.$$

Option (A) is incorrect.
Step 3: Calculate $\frac{\vec{b}+2\vec{d}}{3}$.

$$\frac{\vec{b}+2\vec{d}}{3} = \frac{(3+4)\hat{\imath} + (6+2)\hat{\jmath} + (3+2)\hat{k}}{3} = \frac{7\hat{\imath} + 8\hat{\jmath} + 5\hat{k}}{3}.$$

This corresponds to dividing PR internally in the ratio 5 : 4. Option (B) is correct.
Step 4: Check $\vec{b}\times\vec{d}$.

$$\vec{b}\times\vec{d} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & 6 & 3 \\ 2 & 1 & 1 \end{vmatrix} = 3\hat{\imath} + 3\hat{\jmath} - 9\hat{k}.$$

$$\|\vec{b}\times\vec{d}\|^2 = 9 + 9 + 81 = 99 \quad \text{(not 95)}.$$

Option (D) is incorrect.

$$\frac{\vec{b}+2\vec{d}}{3} = \frac{7\hat{\imath} + 8\hat{\jmath} + 5\hat{k}}{3} = \frac{21\hat{\imath} + 24\hat{\jmath} + 15\hat{k}}{9}$$
$$\frac{5\vec{c}+4\vec{a}}{9} = \frac{\vec{b}+2\vec{d}}{3}$$

B is correct.

> 💡 Quick Tip
>
> Use the scalar triple product to check coplanarity, and remember vector algebra for cross products and dividing line segments.

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5. Let $M = (a_{ij}), i, j \in \{1, 2, 3\}$, be the 3×3 matrix such that $a_{ij} = 1$ if $j+1$ is divisible by i, otherwise $a_{ij} = 0$. Then which of the following statements is(are) true?
(A) M is invertible.
(B) There exists a nonzero column matrix

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

such that

$$M \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

(C) The set $\{X \in R^3 : MX = 0\} \neq \{0\}$, where $0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.
(D) The matrix $M - 2I$ is invertible, where I is the 3×3 identity matrix.
Correct Answer: (B) There exists a nonzero column matrix

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

such that

$$M \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

, (C) The set $\{X \in R^3 : MX = 0\} \neq \{0\}$, where $0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Solution:
Step 1: Write the matrix M.

$$M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Step 2: Compute $\det(M)$.

$$\det(M) = 1(0 \cdot 0 - 1 \cdot 1) - 1(1 \cdot 0 - 1 \cdot 0) + 1(1 \cdot 1 - 0 \cdot 1) = 0.$$

Since $\det(M) = 0$, M is not invertible. Option (A) is incorrect.

Step 3: Solve for $MX = 0$.

$$M \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 + a_3 \\ a_1 + a_3 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This gives:

$$a_1 + a_2 + a_3 = 0, \quad a_1 + a_3 = 0, \quad a_2 = 0.$$

Hence, $a_1 = -a_3$ and $a_2 = 0$, so $X \neq 0$. Options (B) and (C) are correct.

Step 4: Check $M - 2I$.

$$M - 2I = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}.$$

$$\det(M - 2I) = \begin{aligned} &= (-1)^3 - (-2) + 1 \times 1 \\ &= -3 + 2 + 1 = 0 \end{aligned}$$

Hence, $M - 2I$ is non-invertible. Option (D) is incorrect.

> Quick Tip
>
> For invertibility, calculate the determinant. Use row reduction or matrix algebra for verifying other properties.

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6. Let $f : (0,1) \to R$ be the function defined as $f(x) = [4x]\left(x - \frac{1}{4}\right)^2\left(x - \frac{1}{2}\right)$, where $[x]$ denotes the greatest integer less than or equal to x. Which of the following statements is(are) true?

(A) The function f is discontinuous exactly at one point in $(0,1)$.

(B) There is exactly one point in $(0,1)$ at which the function f is continuous but NOT differentiable.

(C) The function f is NOT differentiable at more than three points in $(0,1)$.

(D) The minimum value of the function f is $-\frac{1}{512}$.

Correct Answer: (A) The function f is discontinuous exactly at one point in $(0,1)$.
, (B) There is exactly one point in $(0,1)$ at which the function f is continuous but NOT differentiable.

Solution:

Step 1: Evaluate the piecewise function.

$$f(x) = \begin{cases} 0, & 0 < x < \frac{1}{4}, \\ 1\left(x - \frac{1}{4}\right)^2\left(x - \frac{1}{2}\right), & \frac{1}{4} \leq x < \frac{1}{2}, \\ 2\left(x - \frac{1}{4}\right)^2\left(x - \frac{1}{2}\right), & \frac{1}{2} \leq x < \frac{3}{4}, \\ 3\left(x - \frac{1}{4}\right)^2\left(x - \frac{1}{2}\right), & \frac{3}{4} \leq x < 1. \end{cases}$$

Step 2: Discontinuity at $x = \frac{3}{4}$. The value of $f(x)$ jumps, so f is discontinuous at $x = \frac{3}{4}$.

Step 3: Non-differentiable points. f(x) = $\begin{cases} 0 & ; 0 < x \leq \frac{1}{4} \\ 2\left(x-\frac{1}{4}\right)\left(x-\frac{1}{2}\right)+\left(x-\frac{1}{4}\right)^2 & ; \frac{1}{4} < x < \frac{1}{2} \\ 4\left(x-\frac{1}{4}\right)\left(x-\frac{1}{2}\right)+2\left(x-\frac{1}{4}\right)^2 & ; \frac{1}{2} < x < \frac{3}{4} \\ 6\left(x-\frac{1}{4}\right)\left(x-\frac{1}{2}\right)+3\left(x-\frac{1}{4}\right)^2 & ; \frac{3}{4} < x < 1 \end{cases}$

$$f(x) \text{ is non-differentiable at } x = \frac{1}{2} \text{ and } \frac{3}{4}$$

$$\text{minimum values of } f(x) \text{ occur at } x = \frac{5}{12} \text{ whose value is } -\frac{1}{432}$$

Quick Tip

Break down piecewise functions and analyze continuity and differentiability carefully.

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7. Let S be the set of all twice differentiable functions f from R to R such that $\frac{d^2f}{dx^2}(x) > 0$ for all $x \in (-1, 1)$. For $f \in S$, let X_f be the number of points $x \in (-1, 1)$ for which $f(x) = x$. Then which of the following statements is(are) true?
(A) There exists a function $f \in S$ such that $X_f = 0$.
(B) For every function $f \in S$, we have $X_f \leq 2$.
(C) There exists a function $f \in S$ such that $X_f = 2$.
(D) There does NOT exist any function $f \in S$ such that $X_f = 1$.
Correct Answer: (A) There exists a function $f \in S$ such that $X_f = 0$.
, (B) For every function $f \in S$, we have $X_f \leq 2$.
, (C) There exists a function $f \in S$ such that $X_f = 2$.

Solution:
Step 1: Analyze the condition $\frac{d^2f}{dx^2} > 0$.
The condition implies that $f(x)$ is concave upward in the interval $(-1, 1)$. The graph of $f(x)$ must be a parabola-like curve opening upward.

Step 2: Analyze intersection with $y = x$.
The function $f(x)$ intersects $y = x$ at points where $f(x) - x = 0$.
The number of solutions depends on the shape of $f(x)$: - $X_f = 0$: Possible when $f(x)$ lies entirely above $y = x$ (e.g., $f(x) = x^2 + 1$). - $X_f = 2$: Possible when $f(x)$ intersects $y = x$ at two points (e.g., $f(x) = 2x^2$).

Step 3: Conclusion. - $X_f \leq 2$ for all $f(x) \in S$, since $f(x)$ is concave upward. - Option (D) is incorrect because a tangent intersection $X_f = 1$ is not possible for concave upward curves.

Quick Tip

For intersection problems, set $f(x) - x = 0$ and analyze the roots based on the function's curvature.

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8. For $x \in R$, let $\tan^{-1}(x) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then the minimum value of the function $f : R \to R$ defined by

$$f(x) = \int_0^{x \tan^{-1} x} \frac{e^{(t-\cos t)}}{1 + t^{2023}}\, dt$$

is:

Correct Answer: 0

Solution:

Step 1: Analyze $f'(x)$. Using the Fundamental Theorem of Calculus:

$$f'(x) = \frac{d}{dx} \int_0^{x \tan^{-1} x} \frac{e^{(-\cos t)}}{1 + t^{2023}}\, dt =$$

$$f(x) = \int_0^{x \tan^{-1}(1-\cos t)} \frac{e^{t}}{1 + t^{2023}}\, dt$$

$$f'(x) = \frac{e^{x \tan^{-1}(1-\cos x)}}{1 + (x \tan^{-1} x)^{2023}} \left[\frac{x}{1+x^2} + \tan^{-1} x - 0 \right]$$

Step 2: Solve $f'(x) = 0$. Since the exponential term and $\frac{1}{1+x^2}$ are positive, $f'(x) = 0$ only if $x = 0$.

Step 3: Calculate $f(0)$. x + $\tan^{-1} x(1 + x^2) = 0, \quad x = 0$

Step 4: Conclusion. The minimum value of $f(x)$ is 0.

Quick Tip

The minimum or maximum value of an integral function occurs at critical points where $f'(x) = 0$.

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9. For $x \in R$, let $y(x)$ be a solution of the differential equation

$$(x^2 - 5)\frac{dy}{dx} - 2xy = -2x(x^2 - 5)^2,$$

such that $y(2) = 7$. Then the maximum value of the function $y(x)$ is:

Correct Answer: 16

Solution:

Step 1: Rewrite the differential equation.

$$\frac{dy}{dx} - \frac{2x}{x^2 - 5} y = -2x(x^2 - 5).$$

This is a first-order linear differential equation.

Step 2: Find the integrating factor (I.F.).

$$\text{I.F.} = e^{\int \frac{-2x}{x^2-5}\, dx} = e^{-\ln(x^2-5)} = \frac{1}{x^2 - 5}.$$

Step 3: Solve for $y(x)$. Multiply through by the I.F.:

$$\frac{1}{x^2 - 5}\frac{dy}{dx} - \frac{2x}{(x^2 - 5)^2} y = -2x.$$

This simplifies to:

$$\frac{d}{dx}\left(\frac{y}{x^2-5}\right) = -2x.$$

Integrate both sides:

$$\frac{y}{x^2-5} = -x^2 + c.$$

Solve for y:

$$y = -x^2(x^2-5) + c(x^2-5).$$

Step 4: Apply initial condition $y(2) = 7$.

$$7 = -(2)^2((2)^2 - 5) + c((2)^2 - 5).$$

Solve for c:

$$7 = -4(-1) + c(-1) \implies c = -3.$$

Step 5: Maximum value of $y(x)$.

$$y = -(x^4 - 2x^2 - 15) = -(x^2-1)^2 - 16.$$

The maximum value is 16.

💡 Quick Tip

For first-order linear differential equations, use the integrating factor method to find the general solution.

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10. Let X be the set of all five-digit numbers formed using $1, 2, 2, 2, 4, 4, 0$. For example, 22240 is in X while 02244 and 44422 are not in X. Suppose that each element of X has an equal chance of being chosen. Let p be the conditional probability that an element chosen at random is a multiple of 20 given that it is a multiple of 5. Then the value of $38p$ is equal to:

Correct Answer: 31

Solution: Step 1:

No. of elements in X which are multiples of 5

$$0 \to \begin{bmatrix} 4 \\ 3 \end{bmatrix} = 4$$

$$0 \to \begin{bmatrix} 12 \\ 4 \end{bmatrix} = 12$$

$$0 \to \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 4$$

$$0 \to \begin{bmatrix} 12 \\ 4 \end{bmatrix} = 12$$
$$0 \to \begin{bmatrix} 22 \\ 6 \end{bmatrix} = 6$$
$$0 \to \begin{bmatrix} 12 \\ 2 \end{bmatrix} = 12$$
$$\text{Total} = 38$$

Among these 38 elements, let us calculate when element is not divisible by 20

$$1 \to \begin{bmatrix} 3 \\ 1 \end{bmatrix} = 1$$
$$1 \to \begin{bmatrix} 12 \\ 3 \end{bmatrix} = 3$$
$$1 \to \begin{bmatrix} 12 \\ 3 \end{bmatrix} = 3$$
$$\text{Total} = 7$$

$$p = \frac{38 - 7}{38} \implies 38p = 31$$

Quick Tip

For combinatorics problems, consider symmetries and constraints in the number construction.

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11. Let $A_1, A_2, A_3, \ldots, A_8$ be the vertices of a regular octagon that lie on a circle of radius 2. Let P be a point on the circle, and let PA_k denote the distance between the points P and A_k, for $k = 1, 2, \ldots, 8$. If P varies over the circle, then the maximum value of the product $PA_1 \cdot PA_2 \cdot \ldots \cdot PA_8$ is:

Correct Answer: 512

Solution:

Step 1: Represent point P on the circle. Let P be $2e^{i\theta}$, and the vertices of the octagon are

$A_k = 2e^{i\frac{2k\pi}{8}}$, for $k = 1, 2, \ldots, 8$.

$$\text{In } \Delta A_i OP, \quad \frac{PA_i}{2} = \frac{\sin\theta}{\sin(90^\circ - \frac{\theta}{2})} = 2\sin\frac{\theta}{2}$$

$$PA_i = 4\sin\left(\frac{\pi}{8} + \frac{\theta}{2}\right) = x_i \quad \text{(say)}$$

$$PA_1 = 4\sin\left(\frac{\pi}{8} + \frac{\theta}{2}\right) = x_1$$
$$PA_2 = 4\sin\left(\frac{\pi}{4} + \frac{\theta}{2}\right) = x_2$$
$$PA_3 = 4\sin\left(\frac{3\pi}{8} + \frac{\theta}{2}\right) = x_3$$
$$PA_4 = 4\sin\left(\frac{\pi}{2} + \frac{\theta}{2}\right) = x_4$$

Similarly

$$PA_5 = 4\sin\left(\frac{\phi}{2}\right) = x_5$$
$$PA_6 = 4\sin\left(\frac{\pi}{8} + \frac{\phi}{2}\right) = x_6$$
$$PA_7 = 4\sin\left(\frac{\pi}{4} + \frac{\phi}{2}\right) = x_7$$
$$PA_8 = 4\sin\left(\frac{3\pi}{8} + \frac{\phi}{2}\right) = x_8$$

$$\text{Let } \prod_{i=1}^{8} PA_i = \prod_{i=1}^{8} x_i = E$$

$$\Rightarrow E = 4^8 \sin\left(\frac{\theta}{2}\right)\sin\left(\frac{\pi}{8} + \frac{\theta}{2}\right)\sin\left(\frac{\pi}{4} + \frac{\theta}{2}\right)\sin\left(\frac{3\pi}{8} + \frac{\theta}{2}\right)$$
$$\times \sin\left(\frac{\phi}{2}\right)\sin\left(\frac{\pi}{8} + \frac{\phi}{2}\right)\sin\left(\frac{\pi}{4} + \frac{\phi}{2}\right)\sin\left(\frac{3\pi}{8} + \frac{\phi}{2}\right)$$

$$
\begin{aligned}
E &= 4^4 \left[\frac{\sin\theta \sin\left(\frac{\pi}{4}+\theta\right) \sin\left(\frac{\pi}{2}+\theta\right) \sin\left(\frac{3\pi}{4}+\theta\right)}{2^4} \right] \\
&= 4^4 \left[\sin\theta \cos\theta \sin\left(\frac{\pi}{4}+\theta\right) \cos\left(\frac{\pi}{4}+\theta\right) \right] \\
&= 4^4 \left[\frac{\sin 2\theta \sin\left(\frac{\pi}{2}+2\theta\right)}{4} \right] \\
&= 4^3 \frac{\sin(4\theta)}{2} = 2^9 \sin 4\theta
\end{aligned}
$$

E is maximum when $\sin 4\theta = 1 \Rightarrow \theta = \dfrac{\pi}{8}$

$$E_{\max} = 2^9 = 512$$

> 💡 Quick Tip
>
> Use symmetry and trigonometric simplifications when working with regular polygons inscribed in circles.

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12. Let $R = \begin{bmatrix} a & 3 & b \\ c & 2 & d \\ 0 & 5 & 0 \end{bmatrix}$**, where** $a, b, c, d \in \{0, 3, 5, 7, 11, 13, 17, 19\}$**. Then the number of invertible matrices in** R **is:**

Correct Answer: 3780

Solution:

Step 1: Total number of matrices. Since a, b, c, d can each take 8 values:

$$\text{Total matrices} = 8^4 = 4096.$$

Step 2: Compute the determinant of R**.**

$$\det(R) = 5(bc - ad).$$

For R to be invertible, $bc - ad \neq 0$.

Step 3: Calculate non-invertible cases.

Case 1: If $a, b, c, d \neq 0$, then $bc - ad = 0$ leads to 91 cases.

Case 2: If $a = 0$ or $b = 0$, $bc = ad = 0$, leading to 225 cases.

Step 4: Calculate invertible matrices.

$$\text{Number of invertible matrices} = 4096 - (91 + 225) = 3780.$$

> 💡 Quick Tip
>
> For matrix invertibility, ensure the determinant is non-zero by analyzing all possible cases.

13. Let C_1 be the circle of radius 1 with center at the origin. Let C_2 be the circle of radius r with center at the point $A = (4, 1)$, where $1 < r < 3$. Two distinct common tangents PQ and ST of C_1 and C_2 are drawn. The tangent PQ touches C_1 at P and C_2 at Q. The tangent ST touches C_1 at S and C_2 at T. Midpoints of the line segments PQ and ST are joined to form a line which meets the x-axis at a point B. If $AB = \sqrt{5}$, then the value of r^2 is:
Correct Answer: 2
Solution:
Step 1: Write the equations of the circles.
$C_1 : x^2 + y^2 = 1.$
$C_2 : (x - 4)^2 + (y - 1)^2 = r^2.$

Step 2: Find the radical axis. The radical axis of C_1 and C_2 is:

$$(x^2 + y^2 - 1) - (x^2 + y^2 - 8x - 2y + (17 - r^2)) = 0.$$

Simplify:

$$8x + 2y - (18 - r^2) = 0.$$

Step 3: Find point B. The line meets the x-axis at:

$$y = 0 \implies x = \frac{18 - r^2}{8}.$$

Step 4: Apply the distance condition. Given $AB = \sqrt{5}$, compute the distance:

$$\sqrt{\left(\frac{18 - r^2}{8} - 4\right)^2 + (0 - 1)^2} = \sqrt{5}.$$

Simplify:

$$\frac{(r^2 + 14)^2}{64} + 1 = 5.$$

Solve for r^2:

$$\frac{r^2 + 14}{8} = 2 \implies r^2 = 2.$$

Quick Tip

For problems involving tangents and circles, use radical axes and distance formulas to derive key relationships.

PARAGRAPH "I"
Consider an obtuse angled triangle ABC in which the difference between the largest and the smallest angle is $\frac{\pi}{2}$ and whose sides are in arithmetic progression. Suppose that the vertices of this triangle lie on a circle of radius 1.
(There are two questions based on PARAGRAPH "I", the question given below is one of them)

14. Let a be the area of the triangle ABC. Then the value of $(64a)^2$ is:
Correct Answer: 1008
Solution:
Step 1: Analyze the problem. The triangle ABC is an obtuse-angled triangle with vertices on a circle of radius $R = 1$. The difference between the largest and smallest angles is $\frac{\pi}{2}$, and the sides are in arithmetic progression.
Step 2: Use trigonometric relations. Let $A > B > C$ be the angles, where $A - C = \frac{\pi}{2}$. Using the property of arithmetic progression:

$$a + c = 2b.$$

Since $A + B + C = \pi$:

$$\frac{\pi}{2} + B + C = \pi \implies B + C = \frac{\pi}{2}.$$

Step 3: Calculate sine values. Using the given conditions:

$$\sin C = \frac{-1+\sqrt{7}}{4}, \quad \sin A = \frac{\sqrt{7}+1}{4}, \quad \sin B = \frac{\sqrt{7}}{4}.$$

Step 4: Compute the area. The area of the triangle is:

$$a = 2R^2 \sin A \sin B \sin C = 2(1)^2 \cdot \frac{\sqrt{7}+1}{4} \cdot \frac{\sqrt{7}}{4} \cdot \frac{-1+\sqrt{7}}{4}.$$

Simplify:

$$a = \frac{2}{64}(6\sqrt{7}) = \frac{3\sqrt{7}}{32}.$$

Thus:

$$(64a)^2 = (6\sqrt{7})^2 = 1008.$$

> Quick Tip
>
> For obtuse triangles in a circle, use sine rules and trigonometric identities to compute the area.

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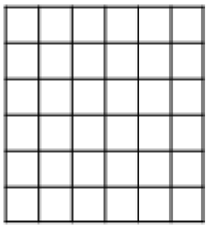
15. Then the inradius of the triangle ABC is:
Correct Answer: 0.25
Solution:
Step 1: Use the formula for the inradius. The inradius r is given by:

$$\begin{aligned} r = \frac{\Delta}{s} &= \frac{1}{2}\frac{n(n+d)\sin\alpha}{\left(\frac{3n}{2}\right)^2} \\ &= \frac{(n+d)\sin\alpha}{3} \\ &= \frac{2\cos\alpha\cdot\sin\alpha}{3} \quad \text{(from (2))} \\ &= \frac{\sin 2\alpha}{3} = \frac{1}{4} \end{aligned}$$

> 💡 Quick Tip
>
> The inradius depends on the area and the semi-perimeter, requiring correct computations of both.

PARAGRAPH "II"
Consider the 6 x 6 square in the figure. Let A1, A2, ..., A49 be the points of intersections (dots in the picture) in some order. We say that A and A are friends if they are adjacent along a row or along a column. Assume that each point A has an equal chance of being chosen.



(There are two questions based on PARAGRAPH "II", the question given below is one of them)
16. Let p_i be the probability that a randomly chosen point has i many friends, $i = 0, 1, 2, 3, 4$. Let X be a random variable such that for $i = 0, 1, 2, 3, 4$, the probability $P(X = i) = p_i$. Then the value of $7E(X)$ is:

Correct Answer: 24
Solution:
Step 1: Construct the probability distribution. From the problem:

$$P(0) = 0, \quad P(1) = 0, \quad P(2) = \frac{4}{49}, \quad P(3) = \frac{20}{49}, \quad P(4) = \frac{25}{49}.$$

Step 2: Compute $E(X)$.

$$E(X) = \sum_{i=0}^{4} i \cdot P(i) = 0 \cdot 0 + 0 \cdot 0 + 2 \cdot \frac{4}{49} + 3 \cdot \frac{20}{49} + 4 \cdot \frac{25}{49}.$$

Simplify:

$$E(X) = \frac{8 + 60 + 100}{49} = \frac{168}{49} = \frac{24}{7}.$$

Step 3: Calculate $7E(X)$.

$$7E(X) = 7 \cdot \frac{24}{7} = 24.$$

> 💡 Quick Tip
>
> For expected value problems, multiply probabilities by outcomes and sum them up systematically.

---

17. Two distinct points are chosen randomly out of the points $A_1, A_2, \ldots, A_{49}$. Let p be the probability that they are friends. Then the value of $7p$ is:

Correct Answer: 0.5

Solution:

Step 1: Compute total pairs. The total number of ways to choose two points is:

$$^{49}C_2$$

Step 2: Compute favorable pairs. Consecutive points are friends. There are $2 \cdot 6 \cdot 7 = 84$ such pairs.

Step 3: Compute probability p.

$$p = \frac{84}{^{49}C_2}.$$

Step 4: Compute $7p$. Simplify:

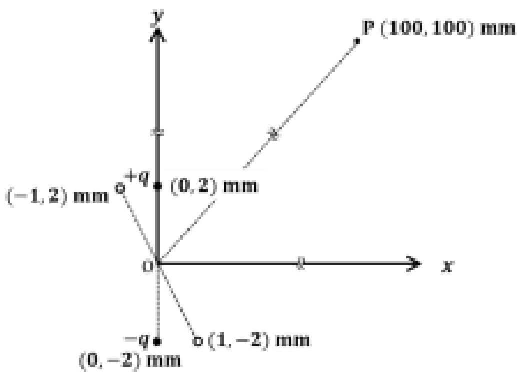
$$7p = 0.5.$$

Quick Tip

For combinatorics problems, carefully count total and favorable outcomes to calculate probabilities.

Physics

1. An electric dipole is formed by two charges $+q$ and $-q$ located in the xy-plane at $(0, 2)$ mm and $(0, -2)$ mm, respectively, as shown in the figure. The electric potential at point $P(100, 100)$ mm due to the dipole is V_0. The charges $+q$ and $-q$ are then moved to the points $(-1, 2)$ mm and $(1, -2)$ mm, respectively. What is the value of the electric potential at P due to the new dipole?



(A) $\frac{V_0}{4}$

(B) $\frac{V_0}{2}$

(C) $\frac{V_0}{\sqrt{2}}$

(D) $\frac{3V_0}{4}$

Correct Answer: (B) $\frac{V_0}{2}$

Solution:
Step 1: Analyze the given dipole configuration.
The electric potential at P due to the initial dipole is given as:

$$V_0 = \frac{kp_0}{r^2}\cos 45^\circ,$$

where p_0 is the dipole moment.
Step 2: Consider the new dipole moment. After the charges are moved, the new dipole moment is:

$$\text{the Component } P_0' \text{ along xaxis} = -\frac{P_0}{2}\hat{i}$$
$$\text{the Component } P_0' \text{ along yaxis} = P_0\hat{j}$$
$$\text{Potential due to } P_0\hat{j}$$
$$\Rightarrow \frac{kP_0\cos 45^\circ}{r^2} = V_0$$
$$\text{Potential due to } -\frac{P_0}{2}\hat{i}$$

The new potential at P is:

$$\frac{k\left(\frac{P_0}{2}\right)\cos 135^\circ}{r^2} = -\frac{V_0}{2}$$

Thus, the new potential is $V_0 - \frac{V_0}{2} = \frac{V_0}{2}$

Quick Tip

For electric dipoles, the potential at a point is proportional to the dipole moment and the cosine of the angle.

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2. Young's modulus of elasticity Y is expressed in terms of three derived quantities, namely, the gravitational constant G, Planck's constant h, and the speed of light c, as $Y = c^\alpha h^\beta G^\gamma$. Which of the following is the correct option?
(A) $\alpha = 7, \beta = -1, \gamma = -2$
(B) $\alpha = -7, \beta = -1, \gamma = -2$
(C) $\alpha = 7, \beta = -1, \gamma = 2$
(D) $\alpha = -7, \beta = 1, \gamma = -2$
Correct Answer: (A) $\alpha = 7, \beta = -1, \gamma = -2$

Solution:
Step 1: Dimensional analysis. The dimensional formula of Y is:

$$[ML^{-1}T^{-2}] = [c^\alpha h^\beta G^\gamma].$$

Step 2: Write dimensional formulas.

$$[c] = [LT^{-1}], \quad [h] = [ML^2T^{-1}], \quad [G] = [M^{-1}L^3T^{-2}].$$

Step 3: Equate powers of M, L, T. Equate the powers of M, L, T:

$$\begin{aligned} \beta - \gamma &= 1 \quad \cdots(i) \\ \alpha + 2\beta + 3\gamma &= -1 \quad \cdots(ii) \\ -\alpha - \beta - 2\gamma &= -2 \\ \alpha + \beta + 2\gamma &= 2 \quad \cdots(iii) \end{aligned}$$

Step 4: Solve the equations. From equations (i), (ii), and (iii):

$$\alpha = 7,\ \beta = -1,\ \gamma = -2.$$

Quick Tip

Dimensional analysis is a powerful method to determine relationships between physical quantities.

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3. A particle of mass m is moving in the xy-plane such that its velocity at a point (x, y) is given as $\vec{v} = \alpha(y\hat{x} + 2x\hat{y})$, where α is a non-zero constant. What is the force $\vec{F}$ acting on the particle?

(A) $\vec{F} = 2m\alpha^2(x\hat{x} + y\hat{y}))$
(B) $\vec{F} = m\alpha^2(y\hat{x} + 2xy)$
(C) $\vec{F} = 2m\alpha^2(y\hat{x} + x\hat{y})$
(D) $\vec{F} = m\alpha^2(x\hat{x} + 2y\hat{y})$
Correct Answer: (A)$\vec{F} = 2m\alpha^2(x\hat{x} + y\hat{y}))$

Solution:
Step 1: Calculate acceleration.

$$\vec{V} = \alpha(y\hat{x} + 2x\hat{y})$$

$$\begin{aligned} v_x &= \alpha y \\ v_y &= 2\alpha x \\ \frac{dv_x}{dt} &= \alpha\frac{dy}{dt} = 2\alpha^2 x \\ \frac{dv_y}{dt} &= 2\alpha v_x = 2\alpha^2 y \\ \vec{F} &= m\vec{a} = 2m\alpha^2(\hat{x}x + \hat{y}y) \end{aligned}$$

Quick Tip

For velocity-dependent motion, differentiate with respect to time to find the acceleration.

---

4. An ideal gas is in thermodynamic equilibrium. The number of degrees of freedom of a molecule of the gas is n. The internal energy of one mole of the gas is U_n and

the speed of sound in the gas is v_n. At a fixed temperature and pressure, which of the following is the correct option?
(A) $v_3 < v_6$ and $U_3 > U_6$
(B) $v_5 > v_3$ and $U_3 > U_5$
(C) $v_5 > v_7$ and $U_5 < U_7$
(D) $v_6 < v_7$ and $U_6 < U_7$
Correct Answer: (C) $v_5 > v_7$ and $U_5 < U_7$

Solution:
Step 1: Analyze internal energy. The internal energy is:

$$U_n = \frac{n}{2}RT \quad \text{(proportional to } n\text{)}.$$

Step 2: Analyze speed of sound. The speed of sound is:

$$v_n = \sqrt{\frac{\gamma RT}{M}},$$

where $\gamma = 1 + \frac{2}{n}$. As n increases, γ decreases, and hence v_n decreases.
Step 3: Conclusion. For $n = 5, 7$, we have:

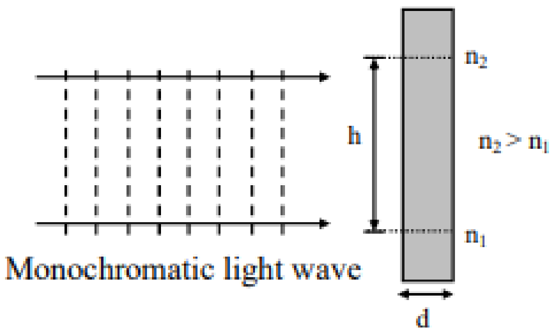
$$v_5 > v_7, \quad U_5 < U_7.$$

Quick Tip

Use degrees of freedom to calculate internal energy and speed of sound for ideal gases.

---

5. A monochromatic light wave is incident normally on a glass slab of thickness d, as shown in the figure. The refractive index of the slab increases linearly from n_1 to n_2 over the height h. Which of the following statements is/are true about the light wave emerging out of the slab?



(A) It will deflect up by an angle $\tan^{-1}\left[\frac{(n_2^2-n_1^2)d}{2h}\right]$
(B) It will deflect up by an angle $\tan^{-1}\left[\frac{(n_2-n_1)d}{h}\right]$
(C) It will not deflect.
(D) The deflection angle depends only on $n_2 - n_1$ and not on the individual values of n_1 and n_2.

Correct Answer: (B) It will deflect up by an angle $\tan^{-1}\left[\frac{(n_2-n_1)d}{h}\right]$
, (D) The deflection angle depends only on n_2-n_1 and not on the individual values of n_1 and n_2.

Solution:
Step 1: Analyze the refractive index gradient. The refractive index increases linearly from n_1 to n_2 over a height h. This causes the wavefront to tilt as the speed of light decreases with increasing refractive index.
Step 2: Calculate the tilt of the wavefront. At time t, the wavefront PQ is given by:

$$\frac{n_2 d}{c} = \frac{n_1 d}{c} + \frac{x}{c}.$$

Simplifying for x:

$$x = d(n_2 - n_1).$$

Step 3: Determine the angle of deflection. The angle of deflection θ is:

$$\tan\theta = \frac{x}{h} = \frac{(n_2 - n_1)d}{h}.$$

Thus:

$$\theta = \tan^{-1}\left[\frac{(n_2 - n_1)d}{h}\right].$$

Step 4: Validate the dependency on $n_2 - n_1$. The deflection angle depends only on the difference $n_2 - n_1$, as seen in the expression for θ, and not on the individual values of n_1 or n_2.

> Quick Tip
>
> In problems involving gradient refractive indices, the deflection angle can be found using basic trigonometric relations between wavefront displacement and the slab's geometry.

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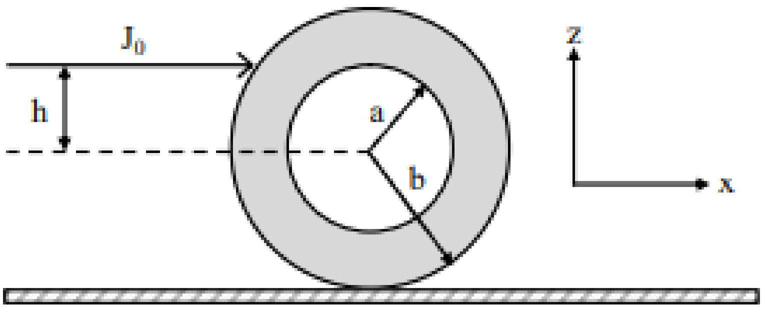
6. An annular disk of mass M, inner radius a, and outer radius b is placed on a horizontal surface with a coefficient of friction μ, as shown in the figure. At some time, an impulse $J_0\hat{x}$ is applied at a height h above the center of the disk. If $h = h_m$, then the disk rolls without slipping along the x-axis. Which of the following statement(s) is/are correct?



(A) For $\mu \neq 0$ and $a \to 0$, $h_m = b/2$
(B) For $\mu \neq 0$ and $a \to b$, $h_m = b$

(C) For $h = h_m$, the initial angular velocity does not depend on the inner radius a.
(D) For $\mu = 0$ and $h = 0$, the wheel always slides without rolling.

Correct Answer: (A) For $\mu \neq 0$ and $a \to 0$, $h_m = b/2$

(B) For $\mu \neq 0$ and $a \to b$, $h_m = b$

(C) For $h = h_m$, the initial angular velocity does not depend on the inner radius a.

(D) For $\mu = 0$ and $h = 0$, the wheel always slides without rolling.

Solution:
Step 1: Angular impulse-momentum relation. The angular impulse-momentum relation gives:

$$Jh = I_{cm}\omega,$$

where J is the impulse and I_{cm} is the moment of inertia about the center of mass.
Step 2: Rolling condition. For rolling without slipping, the linear velocity v_{cm} and angular velocity ω are related by:

$$v_{cm} = b\omega.$$

Step 3: Calculate h_m. The critical height h_m is found by equating:

$$h_m = \frac{I_{cm}}{Mb}.$$

For different cases: 1. For $a \to 0$, $h_m = b/2$. 2. For $a \to b$, $h_m = b$.
Step 4: Additional observations.
1. For $h = 0$ and $\mu = 0$, rolling does not occur due to the absence of torque.
2. When $h = h_m$, rolling without slipping occurs.

Quick Tip

In rolling motion, the relationship between linear velocity, angular velocity, and radius is key to determining rolling conditions.

---

7. The electric field associated with an electromagnetic wave propagating in a dielectric medium is given by

$$\vec{E} = 30(2\hat{x} + \hat{y}) \sin\left[2\pi\left(5 \times 10^{14}t - \frac{10^7 z}{3}\right)\right] \text{ V m}^{-1}.$$

Which of the following option(s) is/are correct?
Given: The speed of light in vacuum, c = 3 × 10$^8 ms^{-1}$

(A) $B_x = -2 \times 10^{-7} \sin\left[2\pi\left(5 \times 10^{14}t - \frac{10^7 z}{3}\right)\right]$ Wb m^{-2}
(B) $B_y = 2 \times 10^{-7} \sin\left[2\pi\left(5 \times 10^{14}t - \frac{10^7 z}{3}\right)\right]$ Wb m^{-2}

(C) The wave is polarized in the xy-plane with a polarization angle of $30°$ with respect to the x-axis.
(D) The refractive index of the medium is 2.

Correct Answer: (A) $B_x = -2 \times 10^{-7} \sin \left[2\pi \left(5 \times 10^{14} t - \frac{10^7 z}{3}\right)\right]$ Wb m^{-2}
, (D) The refractive index of the medium is 2.

Solution:
Step 1: Calculate the speed of the wave in the dielectric medium. The wave number k is:

$$k = \frac{2\pi}{\lambda} = \frac{10^7}{3}.$$

The speed of the wave is:

$$v = \frac{\omega}{k} = \frac{2\pi \cdot 5 \times 10^{14}}{\frac{10^7}{3}} = 1.5 \times 10^8 \, \text{m/s}.$$

Step 2: Determine the refractive index. The refractive index n is given by:

$$n = \frac{c}{v} = \frac{3 \times 10^8}{1.5 \times 10^8} = 2.$$

Step 3: Magnetic field components. The amplitude of the magnetic field B is related to E by:

$$B = \frac{E}{v}.$$

Thus:

$$B = \frac{30}{1.5 \times 10^8} = 2 \times 10^{-7} \, \text{Wb m}^{-2}.$$

Step 4: Magnetic field expressions. The components of $\vec{B}$ are:

$$B_x = -2 \times 10^{-7} \sin \left[2\pi \left(5 \times 10^{14} t - \frac{10^7 z}{3}\right)\right],$$

$$B_y = 4 \times 10^{-7} \sin \left[2\pi \left(5 \times 10^{14} t - \frac{10^7 z}{3}\right)\right].$$

Quick Tip

In EM waves, the electric and magnetic fields are perpendicular to each other and the direction of propagation, and their magnitudes are related by the wave speed.

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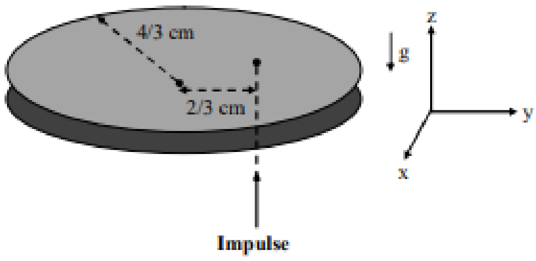
8. A thin circular coin of mass 5 gm and radius $4/3$ cm is initially in a horizontal xy-plane. The coin is tossed vertically up ($+z$-direction) by applying an impulse $\sqrt{\pi/2} \times 10^{-2}$ N-s at a distance $2/3$ cm from its center. The coin spins about its diameter and moves along the $+z$-direction. By the time the coin reaches back to its initial position, it completes n rotations. The value of n is _____.



Correct Answer: $n = 30$
Solution:
Step 1: Angular impulse and linear momentum. The impulse J is given as:

$$J = Mv_{cm}.$$

For angular motion:

$$J \cdot \frac{R}{2} = I\omega,$$

where $I = \frac{MR^2}{4}$ is the moment of inertia about the center of mass.
Step 2: Calculate v_{cm} and ω. From the above equations:

$$v_{cm} = \frac{J}{M}, \quad \omega = \frac{2J}{MR}.$$

Step 3: Time to return to the initial position. The time for the coin to return is:

$$T = \frac{2v_{cm}}{g} = \frac{2J}{gM}.$$

Step 4: Total angle rotated. The angle rotated in time T is:

$$\theta = \omega T = \frac{2J^2}{M^2 gR}.$$

Substituting $J = \sqrt{\frac{\pi}{2}} \times 10^{-2}$, $M = 5\,\text{gm}$, and $R = \frac{4}{3}\,\text{cm}$:

$$\theta = 60\pi.$$

Step 5: Number of rotations. The number of rotations is:

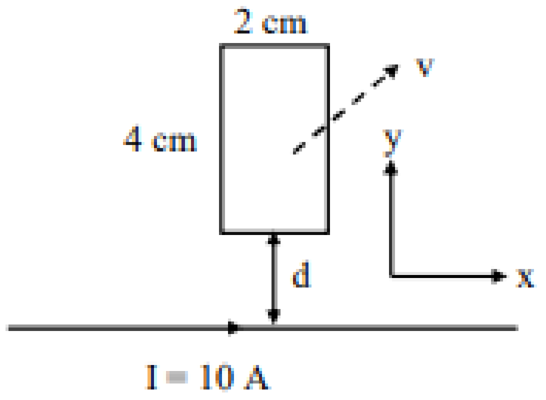
$$n = \frac{\theta}{2\pi} = \frac{60\pi}{2\pi} = 30.$$

Quick Tip
In problems involving rotational motion, angular momentum conservation is key to relating linear impulse and rotational variables.

---

9. A rectangular conducting loop of length $4\,\text{cm}$ and width $2\,\text{cm}$ is in the xy-plane, as shown in the figure. It is being moved away from a thin and long conducting wire along the direction $\frac{\sqrt{3}}{2}\hat{x} + \frac{1}{2}\hat{y}$ with a constant speed v. The wire is carrying a

steady current $I = 10\,\mathbf{A}$ in the positive x-direction. A current of $10\,\mu\mathbf{A}$ flows through the loop when it is at a distance $d = 4\,\mathbf{cm}$ from the wire. If the resistance of the loop is $0.1\,\Omega$, then the value of v is _____.
[Given: The permeability of free space $\mu_0 = 4\pi \times 10^{-7}$ N A^{-2}]



Correct Answer: $v = 4\,\text{m/s}$
Solution:
Step 1: Induced emf in the loop. The induced emf is:

$$\epsilon = \frac{\mu_0 I}{2\pi d} \cdot \frac{v}{2} \cdot b - \frac{\mu_0 I}{2\pi(d+\ell)} \cdot \frac{v}{2} \cdot b$$

Substituting $b = 2\,\text{cm}, \ell = 4\,\text{cm}, d = 4\,\text{cm}$:

$$\Rightarrow \epsilon = \frac{\mu_0 I_0 v b}{4\pi}\left(\frac{1}{d} - \frac{1}{d+\ell}\right)$$

$$\Rightarrow I_0 = \frac{\epsilon}{R} = \frac{\mu_0 I_0 v b \ell}{4\pi d(d+\ell)R}$$

Step 2: Calculate current in the loop. The current is:

$$I_0 = \frac{\mathcal{E}}{R}.$$

Given $I_0 = 10\,\mu\text{A}$, solve for v:

$$v = 4\,\text{m/s}.$$

Quick Tip

In electromagnetic induction, the induced emf depends on the rate of change of magnetic flux through the loop.

Q.10 A string of length 1 m and mass $2 \times 10^{-5}\,\text{kg}$ is under tension T. When the string vibrates, two successive harmonics are found to occur at frequencies $750\,\text{Hz}$ and $1000\,\text{Hz}$. The value of tension T is _____ Newton.
Correct Answer: 5N

Solution:
The frequency of vibration of a string is given by:

$$f = \frac{nv}{2\ell},$$

where: - n is the harmonic number, - v is the speed of the wave, - $\ell = 1\,\text{m}$ is the length of the string.
For two successive harmonics:

$$f_n = \frac{nv}{2\ell} \quad \text{and} \quad f_{n+1} = \frac{(n+1)v}{2\ell}.$$

The difference in frequencies is:

$$f_{n+1} - f_n = \frac{v}{2\ell}.$$

Given:

$$f_{n+1} - f_n = 1000 - 750 = 250\,\text{Hz}.$$

Substitute values:

$$\frac{v}{2 \cdot 1} = 250 \quad \Rightarrow \quad v = 500\,\text{m/s}.$$

The speed v is related to tension T and linear mass density μ by:

$$v = \sqrt{\frac{T}{\mu}},$$

where $\mu = \frac{\text{mass}}{\text{length}} = \frac{2\times10^{-5}}{1} = 2 \times 10^{-5}\,\text{kg/m}$.
Substitute $v = 500$:

$$500 = \sqrt{\frac{T}{2 \times 10^{-5}}}.$$

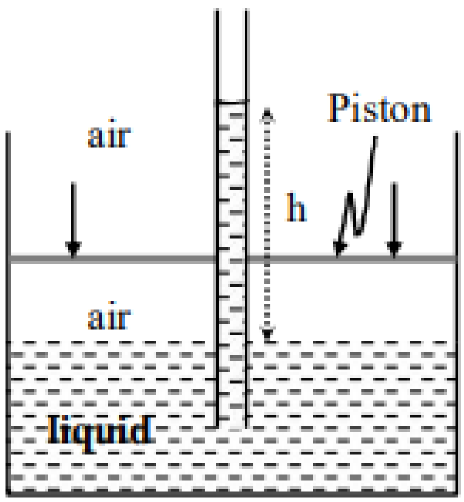
Square both sides:

$$250000 = \frac{T}{2 \times 10^{-5}} \quad \Rightarrow \quad T = 250000 \times 2 \times 10^{-5} = 5\,\text{N}.$$

Quick Tip

The difference between successive harmonics is constant for a vibrating string and is determined by the wave speed and string length.

Q.11 An incompressible liquid is kept in a container having a weightless piston with a hole. A capillary tube of inner radius $0.1\,\text{mm}$ is dipped vertically into the liquid through the airtight piston hole, as shown in the figure. The air in the container is isothermally compressed from its original volume V_0 to $\frac{100}{101}V_0$ with the movable piston. Considering air as an ideal gas, the height (h) of the liquid column in the capillary above the liquid level in cm is _____.
[Given: Surface tension of the liquid is 0.075 N m^{-1}, atmospheric pressure is 10^5 N m^{-2}, acceleration due to gravity (g) is 10 m s^{-2}, density of the liquid is 10^3 kg m^{-3} and contact angle of capillary surface with the liquid is zero]



Correct Answer: 25 cm.
Solution:
Using Boyle's law for isothermal compression:

$$P_0 V_0 = P_A \left(\frac{100}{101} V_0 \right),$$

where P_0 is the initial pressure and P_A is the pressure after compression.
Simplify:

$$P_A = P_0 \frac{101}{100}.$$

The pressure P_A is transmitted to the liquid:

$$P_D = P_A = \frac{101}{100} P_0.$$

The pressure at the capillary is given by:

$$P_B = P_C + \frac{2T}{r},$$

where: - $T = 0.075\,\mathrm{N/m}$ is the surface tension, - $r = 0.1\,\mathrm{mm} = 0.0001\,\mathrm{m}$ is the radius of the capillary.
Thus:

$$P_C = P_B - \frac{2T}{r}.$$

The pressure difference between P_D and P_C causes the liquid column to rise:

$$P_D = P_C + \rho g h.$$

Substitute:

$$\frac{101}{100} P_0 = \left(P_0 - \frac{2T}{r} \right) + \rho g h.$$

Rearrange for h:

$$\frac{101}{100} P_0 - P_0 + \frac{2T}{r} = \rho g h.$$

Simplify:

$$h = \frac{\frac{P_0}{100} + \frac{2T}{r}}{\rho g}.$$

Substitute values:

$$P_0 = 10^5\,\text{Pa},\ T = 0.075\,\text{N/m},\ r = 0.0001\,\text{m},\ \rho = 10^3\,\text{kg/m}^3,\ g = 10\,\text{m/s}^2.$$

Calculate:

$$h = \frac{\frac{10^5}{100} + \frac{2\cdot 0.075}{0.0001}}{10^3 \cdot 10}.$$

Simplify:

$$h = \frac{1000 + 1500}{10000} = \frac{2500}{10000} = 0.25\,\text{m}.$$

Convert to cm:

$$h = 25\,\text{cm}.$$

> Quick Tip
>
> The height of the liquid column in a capillary depends on surface tension, pressure difference, and the radius of the capillary tube.

Q.12 In a radioactive decay process, the activity is defined as $A = -\frac{dN}{dt}$, where $N(t)$ is the number of radioactive nuclei at time t. Two radioactive sources, S_1 and S_2, have the same activity at time $t = 0$. At a later time, the activities of S_1 and S_2 are A_1 and A_2, respectively. When S_1 and S_2 have just completed their 3rd and 7th half-lives, respectively, the ratio $\frac{A_1}{A_2}$ is ______.

Correct Answer: 16.

Solution:

The activity of a radioactive source is proportional to the number of radioactive nuclei present:

$$A(t) = A_0 \left(\frac{1}{2}\right)^n,$$

where A_0 is the initial activity, and n is the number of half-lives.

For S_1 after 3 half-lives:

$$A_1 = A_0 \left(\frac{1}{2}\right)^3 = \frac{A_0}{8}.$$

For S_2 after 7 half-lives:

$$A_2 = A_0 \left(\frac{1}{2}\right)^7 = \frac{A_0}{128}.$$

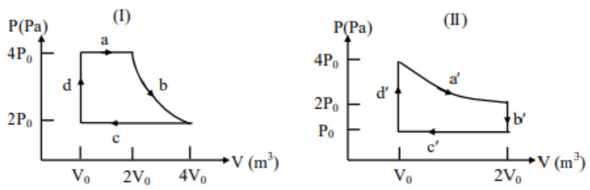
The ratio $\frac{A_1}{A_2}$ is:

$$\frac{A_1}{A_2} = \frac{\frac{A_0}{8}}{\frac{A_0}{128}} = \frac{128}{8} = 16.$$

Quick Tip

In radioactive decay, the activity decreases exponentially with time. Each half-life reduces the activity to half its previous value.

Q.13 One mole of an ideal gas undergoes two different cyclic processes I and II, as shown in the $P-V$ diagrams below. In cycle I, processes a, b, c, d are isobaric, isothermal, isobaric, and isochoric, respectively. In cycle II, processes a', b', c', d' are isothermal, isochoric, isobaric, and isochoric, respectively. The total work done during cycle I is W_I and that during cycle II is W_{II}. The ratio W_I/W_{II} is ______.



Correct Answer: 2.

Solution:

In cycle I:

$$W_I = W_{\text{isobaric}}(a) + W_{\text{isothermal}}(b) + W_{\text{isobaric}}(c) + W_{\text{isochoric}}(d).$$

For processes:

$$W_I = 4P_0V_0 + 8P_0V_0 \ln 2 - 6P_0V_0.$$

Simplify:

$$W_I = 4P_0V_0 + 8P_0V_0 \ln 2 - 6P_0V_0 = 8P_0V_0 \ln 2 - 2P_0V_0.$$

In cycle II:

$$W_{II} = W_{\text{isothermal}}(a') + W_{\text{isochoric}}(b') + W_{\text{isobaric}}(c') + W_{\text{isochoric}}(d').$$

For processes:

$$W_{II} = 4P_0V_0(\ln 2 - 1).$$

Simplify the ratio:

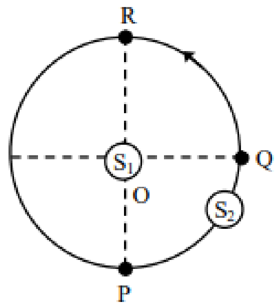
$$\frac{W_I}{W_{II}} = \frac{8P_0V_0 \ln 2 - 2P_0V_0}{4P_0V_0(\ln 2 - 1)} = 2.$$

Quick Tip

For cyclic processes, work done is equal to the area enclosed by the $P - V$ curve. Isothermal processes depend on ln terms for work calculations.

S_1 and S_2 are two identical sound sources of frequency 656 Hz. The source S_1 is located at O, and S_2 moves anticlockwise with a uniform speed $4\sqrt{2}$ m/s on a circular path around O, as shown in the figure. There are three points P, Q, and R

on this path such that P and R are diametrically opposite, while Q is equidistant from them. A sound detector is placed at point P. The source S_1 can move along the direction OP.



[Given: The speed of sound in air is 324 m s^{-1}]

Q14. When only S_2 is emitting sound and it is at Q, the frequency of sound measured by the detector in Hz is:

Correct Answer: 648 Hz

Solution: Step 1: Calculate apparent frequency. The apparent frequency is given by:

$$f = \left(\frac{v}{v + v_s \cos 45°} \right) f_0$$

where $v = 324\,\text{m/s}$, $v_s = 4\sqrt{2}\,\text{m/s}$, and $f_0 = 656\,\text{Hz}$.
Substituting the values:

$$f = \left(\frac{324}{324 + 4} \right) 656$$

$$f = \left(\frac{324}{328} \right) 656 = 648\,\text{Hz}.$$

Quick Tip

For Doppler effect problems, identify the source and observer's relative velocities and use the appropriate frequency shift formula.

Q15. Consider both sources emitting sound. When S_2 is at R and S_1 approaches the detector with a speed of $4\,\text{m/s}$, the beat frequency measured by the detector is:

Correct Answer: 8.2 Hz

Solution: Step 1: Calculate f_1 for S_1. The apparent frequency is given by:

$$f_1 = \left(\frac{v}{v - v_s} \right) f_0,$$

where $v = 324\,\text{m/s}$, $v_s = 4\,\text{m/s}$, and $f_0 = 656\,\text{Hz}$.
Substituting the values:

$$f_1 = \left(\frac{324}{324 - 4} \right) 656 = \frac{324}{320} \times 656 = 664.2\,\text{Hz}.$$

Step 2: Calculate f_2 for S_2. Since S_2 is stationary relative to the detector:

$$f_2 = f_0 = 656\,\text{Hz}.$$

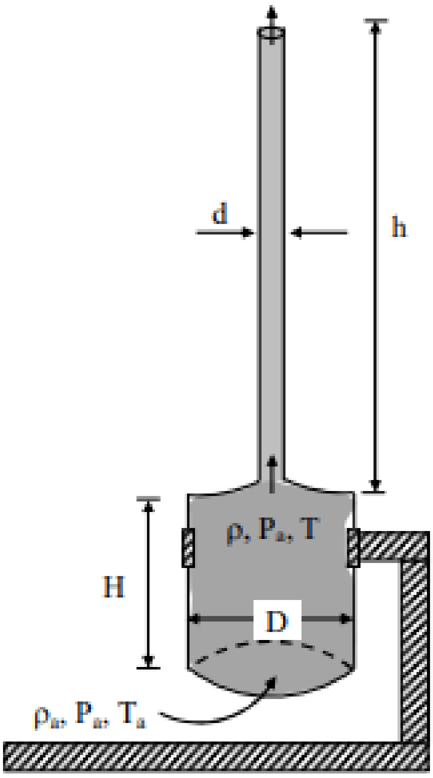
Step 3: Calculate the beat frequency. The beat frequency is given by:

$$f_b = |f_1 - f_2| = |664.2 - 656| = 8.2\,\text{Hz}.$$

💡 Quick Tip

The beat frequency is the absolute difference between the frequencies of two overlapping sound waves.

Paragraph:
A cylindrical furnace has height (H) and diameter (D) both 1 m. It is maintained at a temperature of 360 K. The air gets heated inside the furnace at constant pressure P, and its temperature becomes $T = 360\,\text{K}$. The hot air with density ρ rises up a vertical chimney of diameter $d = 0.1\,\text{m}$ and height $h = 9\,\text{m}$ above the furnace and exits the chimney (see the figure). As a result, atmospheric air of density $\rho_a = 1.2\,\text{kg}\,\text{m}^{-3}$, pressure P_a, and temperature $T_a = 300\,\text{K}$ enters the furnace. Assume air as an ideal gas, neglect the variations in P and T inside the chimney and the furnace. Also, ignore the viscous effects.
[Given: The acceleration due to gravity $g = 10$ m s^{-2} and $\pi = 3.14$]



Q16. Considering the airflow to be streamline, the steady mass flow rate of air exiting the chimney is:
Correct Answer: 49.61 gm/s

Solution: Step 1: Calculate air density. From the ideal gas law:

$$\rho = \rho_0 \frac{T_a}{T} = 1.2 \times \frac{300}{360} = 1\,\mathrm{kg/m}^3.$$

Step 2: Apply Bernoulli's equation. At points 1 and 2:

$$P_a + 0 + 0 = P_a - \rho g h + 0 + \frac{1}{2}\rho v^2,$$

where $h = 9\,\mathrm{m}$. Solving for v:

$$v = \sqrt{2(\rho_a - \rho)gh/\rho} = \sqrt{40}\,\mathrm{m/s}.$$

Step 3: Calculate mass flow rate. The mass flow rate is given by:

$$Q = \rho A v = \pi\frac{d^2}{4} \times v = \pi\frac{(0.1)^2}{4} \times \sqrt{40} \times 1000 = 49.61\,\mathrm{gm/s}.$$

Quick Tip
For streamline flow, apply Bernoulli's principle and calculate density changes using the ideal gas law.

Q17. When the chimney is closed using a cap at the top, a pressure difference ΔP develops between the top and the bottom surfaces of the cap. If the changes in temperature and density of the hot air, due to the stoppage of airflow, are negligible, then ΔP is:

Correct Answer: 20 $\mathrm{N/m}^2$

Solution: Step 1: Calculate pressure difference. At the bottom surface:

$$P_{\text{inside}} = P_a - \rho g h.$$

At the top surface:

$$P_{\text{outside}} = P_a - \rho_a g h.$$

The pressure difference is:

$$\Delta P = P_{\text{inside}} - P_{\text{outside}} = (\rho_a - \rho)gh.$$

Substituting values:

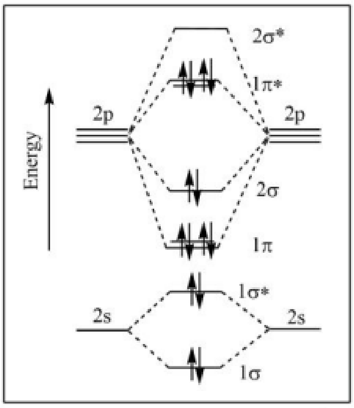
$$\Delta P = (1.2 - 1) \times 10 \times 10 = 20\,\mathrm{N/m}^2.$$

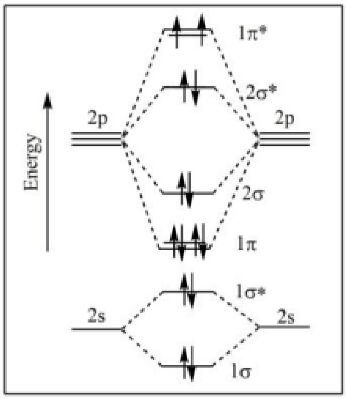
Quick Tip
For pressure difference problems, consider the density variations and height difference using hydrostatic principles.

Chemistry

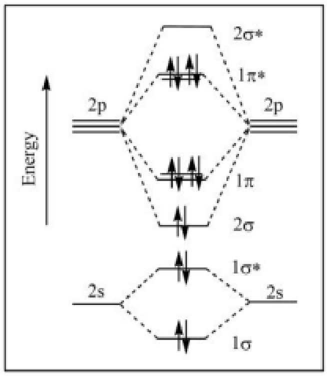
1. The correct molecular orbital diagram for F_2 molecule in the ground state is

(A)

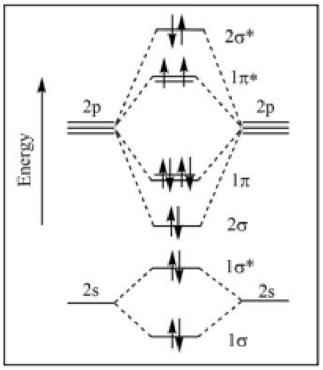




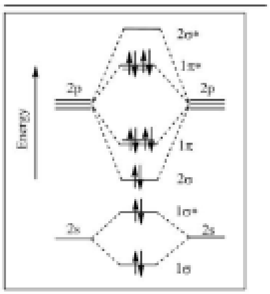
(C)



(D)



(C)



Correct Answer:
Solution:
Atomic Orbitals:

- Each fluorine atom has 9 electrons, with the electronic configuration $1s^2$ $2s^2$ 2p.
- The valence electrons reside in the 2s and 2p orbitals.

Molecular Orbital Formation:

- When two fluorine atoms combine to form F, their atomic orbitals overlap to form molecular orbitals.
- The 2s orbitals overlap to form 2s and *2s molecular orbitals.
- The 2p orbitals overlap in two ways:
  - Head-on overlap of 2pz orbitals along the internuclear axis forms 2pz and *2pz orbitals.
  - Side-on overlap of 2px and 2py orbitals forms two sets of 2px, 2py and *2px, *2py orbitals.

Filling Molecular Orbitals:

- The 14 valence electrons (7 from each fluorine atom) fill the molecular orbitals according to the Aufbau principle (filling orbitals from lowest to highest energy) and Hund's rule (filling degenerate orbitals singly before pairing).

Correct Molecular Orbital Diagram
Based on the above analysis, the correct molecular orbital diagram for F in the ground state is Option C.
Key Features of the Correct Diagram:

- The 2s and *2s orbitals are filled.
- The 2pz orbital is filled.
- The 2px and 2py orbitals are each filled with one electron (due to Hund's rule).
- The *2px and *2py orbitals each have one unpaired electron.

Energy	$\sigma^*_{2p_z}$	$\pi^*_{2p_x} = \pi^*_{2p_y}$	σ_{2p_z}
$\uparrow\downarrow$	$\uparrow$	$\uparrow$	$\uparrow\downarrow$
$\pi_{2p_x} = \pi_{2p_y}$	σ^*_{2s}	σ_{2s}	
$\uparrow\downarrow$	$\uparrow\downarrow$	$\uparrow\downarrow$	

Quick Tip

Explanation for Incorrect Options:
Option A: Incorrectly shows the 2pz orbital lower in energy than the 2px and 2py orbitals.
Option B: Incorrectly shows the 2pz orbital lower in energy than the 2px and *2py orbitals.
Option D: Incorrectly shows the 2px and 2py orbitals filled with two electrons each, violating Hund's rule.

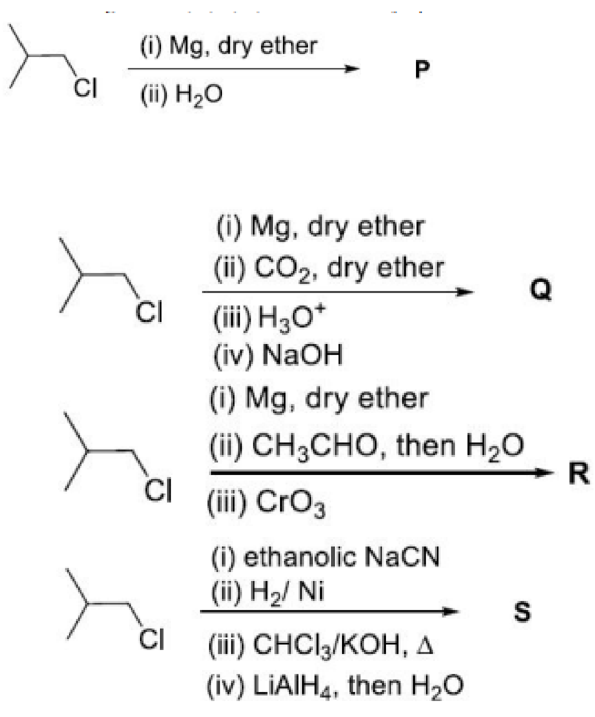
Q2. Consider the following statements related to colloids:
(I) Lyophobic colloids are not formed by simple mixing of dispersed phase and dispersion medium.
(II) For emulsions, both the dispersed phase and the dispersion medium are liquid.
(III) Micelles are produced by dissolving a surfactant in any solvent at any temperature.
(IV) Tyndall effect can be observed from a colloidal solution with dispersed phase having the same refractive index as that of the dispersion medium.

The option with the correct set of statements is:
(A) (I) and (II)
(B) (II) and (III)
(C) (III) and (IV)
(D) (II) and (IV)

Correct Answer: (A) (I) and (II)
Solution: Step 1: Analyze each statement.
(I) Lyophobic colloids are indeed not formed by simple mixing. They require stabilizing agents.
(II) In emulsions, both the dispersed phase and the dispersion medium are liquids. This is correct.
(III) Micelles form at the critical micelle concentration (CMC) and specific temperatures, like the Kraft temperature. This makes the statement incorrect.
(IV) Tyndall effect requires different refractive indices of the dispersed phase and medium, so this statement is incorrect.

> Quick Tip
>
> Colloids exhibit unique properties such as the Tyndall effect, which depends on differences in refractive indices. Lyophobic colloids require stabilizers for formation.

Q3. In the following reactions, $P, Q, R,$ and S are the major products. The correct statement about $P, Q, R,$ and S is:



(A) P is a primary alcohol with four carbons.
(B) Q undergoes Kolbe's electrolysis to give an eight-carbon product.
(C) R has six carbons and it undergoes Cannizzaro reaction.
(D) S is a primary amine with six carbons.

Correct Answer: (B) Q undergoes Kolbe's electrolysis to give an eight-carbon product.
Solution: Step 1: Analyze each reaction step.
P: Reaction with Mg/dry ether followed by H_2O forms a primary alcohol with four carbons.

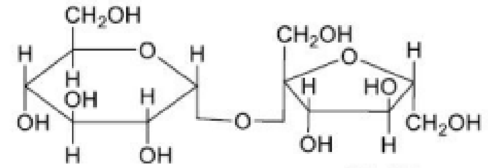
Q: Reaction sequence results in a carboxylic acid that can undergo Kolbe's electrolysis to give an eight-carbon product.
R: Reaction sequence forms an aldehyde that does not undergo Cannizzaro reaction.
S: Reaction sequence forms a primary amine with six carbons.

Quick Tip

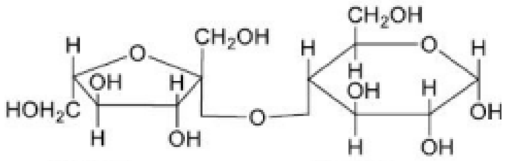
Understand the mechanism of Grignard reagents and the conditions required for reactions like Kolbe's electrolysis and Cannizzaro reactions.

Q4. A disaccharide X cannot be oxidized by bromine water. The acid hydrolysis of X leads to a laevorotatory solution. The disaccharide X is:

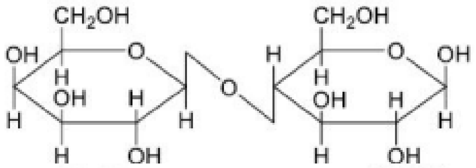
(A)



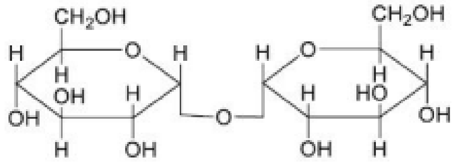
(B)



(C)

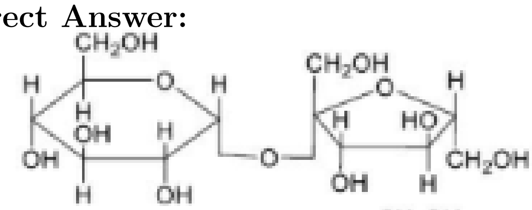


(D)



Correct Answer:

(A)



Solution: Step 1: Analyze the properties of the disaccharides. Sucrose is a non-reducing sugar and cannot be oxidized by bromine water. Upon hydrolysis, it gives glucose and fructose, with the solution becoming laevorotatory due to fructose's higher optical rotation.

Quick Tip

Non-reducing sugars like sucrose do not have free aldehyde or ketone groups, making them unreactive with bromine water.

Q5. The complex(es) which can exhibit the type of isomerism shown by $[Pt(NH_3)_2Br_2]$ is(are):
(A) $[Pt(en)(SCN)_2]$
(B) $[Zn(NH_3)_2Cl_2]$
(C) $[Pt(NH_3)_2Cl_4]$
(D) $[Cr(en)_2(H_2O)(SO_4)]^+$

Correct Answer: (C) $[Pt(NH_3)_2Cl_4]$, (D) $[Cr(en)_2(H_2O)(SO_4)]^+$

Solution: Step 1: Analyze each complex.
$[Pt(NH_3)_2Br_2]$ shows cis-trans isomerism as it is a square planar complex.
(A) $[Pt(en)(SCN)_2]$ does not show isomerism.
(B) $[Zn(NH_3)_2Cl_2]$ is tetrahedral, so it does not exhibit geometrical isomerism.
(C) $[Pt(NH_3)_2Cl_4]$ is octahedral and shows geometrical isomerism.
(D) $[Cr(en)_2(H_2O)(SO_4)]^+$ is also octahedral and exhibits geometrical isomerism.

Quick Tip

For geometrical isomerism, the arrangement of ligands in octahedral or square planar complexes is key. Tetrahedral complexes do not exhibit this type of isomerism.

Q6. Atoms of metals x, y, and z form face-centred cubic (fcc) unit cell of edge length L_x, body-centred cubic (bcc) unit cell of edge length L_y, and simple cubic unit cell of edge length L_z, respectively. If $r_z = \frac{\sqrt{3}}{2}r_y$; $r_y = \frac{8}{\sqrt{3}}$; $M_z = \frac{3}{2}M_y$ and $M_z = 3M_x$, then the correct statement(s) is(are): Given: Mx, My, and Mz are molar masses of metals x, y, and z, respectively. r_x, r_y, and r_z are atomic radii of metals x, y, and z, respectively
(A) Packing efficiency of unit cell of $x >$ Packing efficiency of unit cell of $y >$ Packing efficiency of unit cell of z.
(B) $L_y > L_z$.
(C) $L_x > L_y$.
(D) Density of $x >$ Density of y.

Correct Answer: (A) Packing efficiency of unit cell of $x >$ Packing efficiency of unit cell of $y >$ Packing efficiency of unit cell of z.
(B) $L_y > L_z$.
, (D) Density of $x >$ Density of y.

Solution: Step 1: Analyze edge lengths and radii. For metal x (FCC):

$$L_x = 2\sqrt{2}r_x.$$

For metal y (BCC):

$$L_y = \frac{4r_y}{\sqrt{3}}.$$

For metal z (SC):

$$L_z = 2r_z.$$

Step 2: Packing efficiencies.

(i) For FCC (Z = 4) metal 'x', $4r_x = \sqrt{2}L_x$

$$\text{P.E.} = \frac{Z \times \frac{4}{3}\pi(r_x)^3}{a_x^3} = \frac{4 \times \frac{4}{3}\pi(r_x)^3}{(L_x)^3} = \frac{4 \times \frac{4}{3}\pi(r_x)^3}{\left(\frac{4r_x}{\sqrt{2}}\right)^3} = 0.24\pi$$

(ii) For BCC (Z = 2) metal 'y', $4r_y = \sqrt{3}L_y$

$$\text{P.E.} = \frac{Z \times \frac{4}{3}\pi(r_y)^3}{a_y^3} = \frac{2 \times \frac{4}{3}\pi(r_y)^3}{(L_y)^3} = \frac{2 \times \frac{4}{3}\pi(r_y)^3}{\left(\frac{4r_y}{\sqrt{3}}\right)^3} = 0.22\pi$$

(iii) For SC (Z = 1) metal 'z', $2r_z = L_z$

$$\text{P.E.} = \frac{Z \times \frac{4}{3}\pi(r_z)^3}{a_z^3} = \frac{1 \times \frac{4}{3}\pi(r_z)^3}{(L_z)^3} = \frac{1 \times \frac{4}{3}\pi(r_z)^3}{(2r_z)^3} = \frac{\pi}{6} = 0.17\pi$$

Thus, $P.E_{FCC} > P.E_{BCC} > P.E_{SC}$.

Step 3: Density comparison.

$$\begin{aligned} 4r_x &= \sqrt{2}L_x, \quad 4r_y = \sqrt{3}L_y, \quad 2r_z = L_z \\ L_x &= \frac{4r_x}{\sqrt{2}}, \quad L_y = \frac{4r_y}{\sqrt{3}} \\ \frac{L_y}{L_x} &= \frac{4r_y/\sqrt{3}}{4r_x/\sqrt{2}} = \frac{2r_y}{\sqrt{3}r_x} = \frac{2r_y}{\sqrt{3}.\frac{\sqrt{3}}{2}r_y} = \frac{4}{3} \end{aligned}$$

Therefore, $L_y > L_x$

For option (C)

$$\begin{aligned} 4r_x &= \sqrt{2}L_x, \quad 4r_y = \sqrt{3}L_y \\ L_x &= \frac{4r_x}{\sqrt{2}}, \quad L_y = \frac{4r_y}{\sqrt{3}} \\ \frac{L_y}{L_x} &= \frac{4r_y/\sqrt{3}}{4r_x/\sqrt{2}} = \frac{\sqrt{2}r_y}{\sqrt{3}r_x} = \frac{\sqrt{2} \times 8/\sqrt{3}r_x}{\sqrt{3}r_x} = \frac{8\sqrt{2}}{3} \end{aligned}$$

Therefore, $L_x < L_y$ (incorrect)

For option (D)

$$d_x = \frac{4 \times M_x}{\left(\frac{4r_x}{\sqrt{2}}\right)^3 \times N_A}$$

$$d_y = \frac{4 \times M_y}{\left(\frac{4r_y}{\sqrt{3}}\right)^3 \times N_A}$$

$$r_y = \frac{8}{\sqrt{3}} r_x, \quad \frac{M_x}{M_y} = \frac{1}{2}$$

$$\frac{d_x}{d_y} = \frac{512}{2\sqrt{2}} = \frac{256}{\sqrt{2}}$$

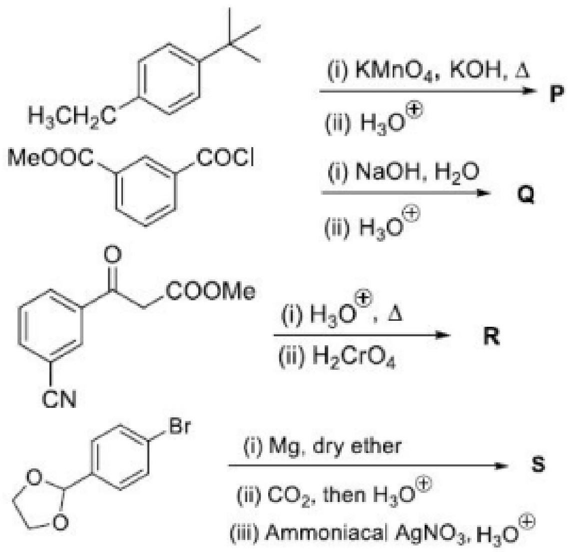
Therefore, $d_x > d_y$ (correct)

$$\rho_x > \rho_y > \rho_z.$$

Quick Tip

Packing efficiency decreases from FCC to BCC to SC. Use relations between edge length and radius to solve such problems.

Q7. In the following reactions, $P, Q, R,$ and S are the major products. The correct statement(s) about $P, Q, R,$ and S is(are):



(A) P and Q are monomers of polymers dacron and glyptal, respectively.
(B) $P, Q,$ and R are dicarboxylic acids.
(C) Compounds Q and R are the same.
(D) R does not undergo aldol condensation and S does not undergo Cannizzaro reaction.

Correct Answer: (C) Compounds Q and R are the same.

(D) R does not undergo aldol condensation and S does not undergo Cannizzaro reaction.

Solution: Step 1: Analyze reactions for $P, Q, R, \text{and}\, S$.
P: Oxidation of the alkyl group leads to a dicarboxylic acid.
Q: Hydrolysis of the acyl chloride leads to the same dicarboxylic acid as R.
R: Product does not contain an alpha-hydrogen, so it cannot undergo aldol condensation.
S: Product cannot undergo Cannizzaro reaction as it is not a non-enolizable aldehyde.

Quick Tip

Use the structural features of the products to analyze their reactivity towards condensation and Cannizzaro reactions.

Q8. H_2S (5 moles) reacts completely with acidified aqueous potassium permanganate solution. In this reaction, the number of moles of water produced is x, and the number of moles of electrons involved is y. The value of $x + y$ is:
Correct Answer: 18
Solution: The reaction is:

$$8H^+ + \mathrm{MnO_4^-} + 5e^- \rightarrow \mathrm{Mn}^{2+} + 4H_2O \ \times 2.$$

$$H_2S \rightarrow S + 2H^+ + 2e^- \ \times 5.$$

Balancing gives:

$$2\mathrm{MnO_4^-} + 16H^+ + 5H_2S \rightarrow 2\mathrm{Mn}^{2+} + 5S + 10H^+ + 8H_2O.$$

Here, $x = 8$ (moles of water) and $y = 10$ (moles of electrons). Thus, $x + y = 18$.

Quick Tip

Balance redox reactions by ensuring conservation of both mass and charge.

Q9. Among $[I_3]^+$, $[\mathrm{SiO_4}]^{4-}$, $\mathrm{SO_2Cl_2}$, $\mathrm{XeF_2}$, $\mathrm{SF_4}$, $\mathrm{ClF_3}$, $\mathrm{Ni(CO)_4}$, $\mathrm{XeO_2F_2}$, $[\mathrm{PtCl_4}]^{2-}$, $\mathrm{XeF_4}$, and $\mathrm{SOCl_2}$, the total number of species having sp^3-hybridised central atom is:

Correct Answer: 5
Solution: The species with sp^3-hybridised central atoms are:
$[\mathrm{SiO_4}]^{4-}$: Tetrahedral geometry, sp^3.
$\mathrm{SOCl_2}$: Tetrahedral geometry, sp^3.
$\mathrm{Ni(CO)_4}$: Tetrahedral geometry, sp^3.
$\mathrm{XeO_2F_2}$: Distorted tetrahedral geometry, sp^3.
$\mathrm{SF_4}$: See-saw geometry, sp^3.

Thus, the total is 5.

💡 Quick Tip

Determine the hybridization by considering the steric number (sum of bonded atoms and lone pairs).

Q10. Consider the following molecules: $\mathbf{Br_3O_8, F_2O, H_2S_4O_6, H_2S_5O_6,}$ and $\mathbf{C_3O_2}$. Count the number of atoms existing in their zero oxidation state in each molecule. Their sum is:

Correct Answer: 6

Solution: Step 1: Analyze the oxidation states of each molecule.

(i) Br_3O_8: One bromine atom is in the zero oxidation state.

(ii) F_2O: Both fluorine atoms are in the zero oxidation state.

(iii) $H_2S_4O_6$: One sulfur atom is in the zero oxidation state.

(iv) $H_2S_5O_6$: Two sulfur atoms are in the zero oxidation state.

(v) C_3O_2: All carbon atoms are in the zero oxidation state.

Adding them: $1 + 2 + 1 + 2 = 6$.

💡 Quick Tip

Identify zero oxidation states by analyzing the bonding and oxidation state rules for each element.

Q11. For $\mathbf{He^+}$, a transition takes place from the orbit of radius 105.8 pm to the orbit of radius 26.45 pm. The wavelength (in nm) of the emitted photon during the transition is: [Use:

Bohr radius, a = 52.9 pm

Rydberg constant, $R_H = 2.2 \times 10^{-18}$ J

Planck's constant, $h = 6.6 \times 10^{-34}$ J s

Speed of light, $c = 3 \times 10^8$ m s^{-1}]

Correct Answer: 30 nm

Solution: The radius of an orbit is given by:

$$r_n = \frac{52.9 \cdot n^2}{Z}\ \text{pm}.$$

For n_1:

$$105.8 = \frac{52.9 \cdot n_1^2}{2} \quad \Rightarrow \quad n_1^2 = 4 \quad \Rightarrow \quad n_1 = 2.$$

For n_2:

$$26.45 = \frac{52.9 \cdot n_2^2}{2} \quad \Rightarrow \quad n_2^2 = 1 \quad \Rightarrow \quad n_2 = 1.$$

Energy difference:

$$\frac{1}{\lambda} = 109677 \cdot Z^2 \left(\frac{1}{n_1^2} - \frac{1}{n_2^2} \right).$$

Substituting $Z = 2$:

$$\frac{1}{\lambda} = 109677 \cdot 4 \left(\frac{1}{1} - \frac{1}{4}\right) = 3 \cdot 109677.$$

$$\lambda = \frac{10^7}{3 \cdot 109677} = 30.3\,\text{nm} = 30\,\text{nm}.$$

Quick Tip

For hydrogen-like atoms, use the radius formula and the Rydberg equation for energy transitions.

Q12. 50 mL of 0.2 molal urea solution ($\rho = 1.012\,\mathbf{g/mL\,at}\,300\,\mathbf{K}$) is mixed with $250\,\mathbf{mL}$ of a solution containing $0.06\,\mathbf{g}$ urea. Both the solutions were prepared in the same solvent. The osmotic pressure (in Torr) of the resulting solution at $300\,\mathbf{K}$ is: [Use: Molar mass of urea = 60 g mol^{-1}; gas constant, R = 62 L Torr K^{-1} mol^{-1};

Assume, $\Delta H_{\text{mix}} = 0, \Delta V_{\text{mix}} = 0$]

Correct Answer: 682 Torr

Solution: Step 1: Calculate the volume of the solution using its density.

The mass of the solution is given as:

$$\text{Mass of solution} = 1012\,\text{g}$$

The density of the solution is:

$$\text{Density (d)} = 1.012\,\text{g/ml}$$

Using the formula for volume:

$$\text{Volume} = \frac{\text{Mass}}{\text{Density}}$$

Substituting the values:

$$\text{Volume} = \frac{1012\,\text{g}}{1.012\,\text{g/ml}} = 1000.00\,\text{ml}$$

Step 2: Determine the number of moles of solute in 1000 ml of solution.

Since 1000 ml of solution contains:

$$0.2\,\text{moles}$$

The number of moles in 50 ml of solution is calculated as:

$$\text{Moles} = \frac{0.2}{1000} \times 50 = 0.01\,\text{moles}$$

Thus:

$$n_{\text{urea}} = 0.01\,\text{moles}$$

Step 3: Calculate the moles of solute in the second solution.

In the second solution:

$$\text{Mass of solute} = 0.06\,\text{g}$$

Using the molar mass of urea (60 g/mol):

$$n_{\text{urea}} = \frac{0.06}{60} = 0.001\,\text{moles}$$

Step 4: Calculate the final molarity (M) of the mixed solution.
The formula for final molarity is:

$$M = \frac{n_1 + n_2}{V_1 + V_2}$$

Substituting the values:

$$M = \frac{0.01 + 0.001}{\frac{(50+250)}{1000}}$$

Simplify:

$$M = \frac{0.011}{300/1000} = \frac{11}{300}$$

Step 5: Calculate the osmotic pressure (π).
The formula for osmotic pressure is:

$$\pi = CRT$$

Substitute the values:

$$\pi = \frac{11}{300} \times 62 \times 300$$

Simplify:

$$\pi = 682\,\text{torr}$$

Final Answer: The osmotic pressure of the solution is:

$$\pi = 682\,\text{torr}$$

Quick Tip

To calculate osmotic pressure, use $\pi = MRT$, ensuring proper conversion of moles and volumes.

Q13. The reaction of 4-methyloct-1-ene (P) (2.52 g) with HBr in the presence of $(C_6H_5CO)_2O_2$ gives two isomeric bromides in a 9 : 1 ratio, with a combined yield of 50%. Of these, the entire amount of the primary alkyl bromide was reacted with an appropriate amount of diethylamine followed by treatment with aq. K_2CO_3 to give a non-ionic product S in 100% yield. The mass (in mg) of S obtained is
Correct Answer: 1791 mg
Solution: The moles of P:

$$n_P = \frac{2.52}{126} = 0.02\,\text{mol}.$$

Major product A:

$$n_A = 0.02 \times \frac{9}{10} = 0.018\,\text{mol}.$$

Mass of S:

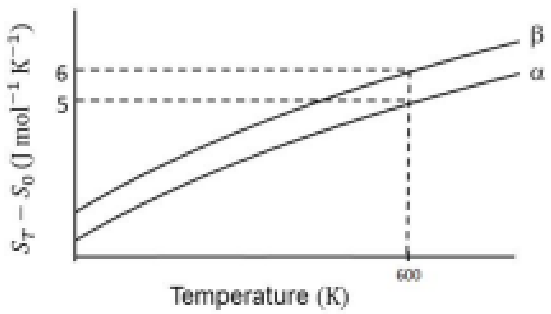
$$\text{Mass} = 0.018 \times (12 \cdot 8 + 2 \cdot 1 + 14) = 1791\,\text{mg}.$$

Quick Tip

Use stoichiometry and mole concept for calculating the yield in such reactions.

"PARAGRAPH I"

The entropy versus temperature plot for phases and at 1 bar pressure is given. S_T and S_0 are entropies of the phases at temperatures T and 0 K, respectively.



The transition temperature for to phase change is 600 K and $C_{p,\alpha} = C_{p,\beta} = 1$ J mol^{-1} K^{-1}. Assume ($C_{p,\beta}$ - $C_{p,\alpha}$) is independent of temperature in the range of 200 to 700 K. $C_{p,\alpha}$ and $C_{p,\beta}$ are heat capacities of and phases, respectively.

Q14. The value of entropy change, $S_\beta - S_\alpha$ (in J mol^{-1} K^{-1}), at 300 K is: [Use: $\ln 2 = 0.69$, Given: $S_\beta - S_\alpha = 0$ at 0 K]
Correct Answer: 0.31
Solution: The entropy is given by:

$$S = S_0 + \int C_p \frac{dT}{T}.$$

For phases α and β:

$$S_\alpha = S_0 + \int (C_p)_\alpha \frac{dT}{T}, S_\beta = S_0 + \int (C_p)_\beta \frac{dT}{T}.$$

The change in entropy is:

$$S_\beta - S_\alpha = \int \left[(C_p)_\beta - (C_p)_\alpha\right] \frac{dT}{T}.$$

Given $(C_p)_\beta - (C_p)_\alpha = 1$:

$$S_\beta - S_\alpha = \ln T + C.$$

At $T_2 = 600\,\text{K}, T_1 = 300\,\text{K}$:

$$(S_\beta - S_\alpha)_{600} - (S_\beta - S_\alpha)_{300} = \ln 600 - \ln 300 = \ln 2 = 0.69.$$

At $T_1 = 300\,\text{K}$:

$$(S_\beta - S_\alpha)_{300} = 1 - 0.69 = 0.31.$$

💡 Quick Tip

To calculate entropy change, use $\Delta S = \int C_p \frac{dT}{T}$, considering given temperature limits.

Q15. The value of enthalpy change, $H_\beta - H_\alpha$ (in J mol^{-1}), at 300 K is:
Correct Answer: $300\,\text{J mol}^{-1}$
Solution: The enthalpy change is:

$$\Delta H = T\Delta S.$$

At $T = 600\,\text{K}$:

$$\Delta H_{600} = 600 \times 1 = 600\,\text{J mol}^{-1}.$$

From Kirchhoff's law:

$$\Delta C_p = \frac{\Delta H_{600} - \Delta H_{300}}{600 - 300}.$$

Given $\Delta C_p = 1$:

$$1 = \frac{600 - \Delta H_{300}}{300}.$$

Solving for ΔH_{300}:

$$\Delta H_{300} = 300\,\text{J mol}^{-1}.$$

💡 Quick Tip

Use Kirchhoff's law to relate heat capacities and enthalpy changes across temperature ranges.

"PARAGRAPH II"

A trinitro compound, 1,3,5-tris-(4-nitrophenyl)benzene, on complete reaction with an excess of Sn/HCl gives a major product, which on treatment with an excess of NaNO/HCl at 0°C provides P as the product. P, upon treatment with excess of HO at room temperature, gives the product Q. Bromination of Q in aqueous medium furnishes the product R. The compound P upon treatment with an excess of phenol under basic conditions gives the product S.

The molar mass difference between compounds Q and R is 474 g mol1 and between compounds P and S is 172.5 g mol1.

Q16. The number of heteroatoms present in one molecule of R is:
Use: Molar mass (in g mol^{-1}): H = 1, C = 12, N = 14, O = 16, Br = 80, Cl = 35.5. Atoms other than C and H are considered as heteroatoms.

Correct Answer: 9
Solution: The compound R contains: - 3 oxygen atoms (O_3), - 3 nitrogen atoms (N_3), - 3 bromine atoms (Br_3).

Total number of heteroatoms:

$$3 + 3 + 3 = 9.$$

Quick Tip

Count heteroatoms explicitly by identifying atoms other than carbon and hydrogen in the molecule's structure.

Q17. The total number of carbon atoms and heteroatoms present in one molecule of S is:

[Use: Molar mass (in g mol^{-1}): H = 1, C = 12, N = 14, O = 16, Br = 80, Cl = 35.5. Atoms other than C and H are considered as heteroatoms.]

Correct Answer: 18

Solution: The compound S contains:
6 carbon atoms (C_6),
3 nitrogen atoms (N_3),
6 oxygen atoms (O_6),
3 bromine atoms (Br_3).
The structure of compound S is given as $C_{24}H_{18}O_3$. Let us count the total number of carbon atoms and heteroatoms.
- Number of carbon atoms:

$$24\,\text{C atoms}.$$

- Number of heteroatoms:

$$3\,\text{Oxygen atoms (O)},\ 3\,\text{Nitrogen atoms (N), and}\,21\,\text{from aromatic rings (excluding hydrogens)}.$$

Total atoms (Carbon + Heteroatoms):

$$24(\text{C}) + 27(\text{O} + \text{N}) = 51.$$

Quick Tip

Sum the total number of carbon atoms and heteroatoms based on molecular structure and stoichiometric data.
