JEE-Main-26-07-2022-Shift-2 (Memory Based)

MATHEMATICS

Question: The interval in which abscissa of point P on $y = x^2$ lies such that its distance from $(x-1)^2 + (y+1)^2 = 1$ is minimum is:

Options:

(a) $0 < x < \frac{1}{4}$

(b) $\frac{1}{4} < x < \frac{1}{2}$

(c) $\frac{1}{2} < x < \frac{3}{4}$

(d) $\frac{3}{4} < x < 1$

Answer: (b)

Solution:

Let $P(x, x^2)$

Distance of P from given circle:

$l = \sqrt{(x-1)^2 + (x^2+1)^2} - 1$

For least value of l, we need to minimize:

$f(x) = (x-1)^2 + (x^2+1)^2$

$f'(x) = 2(x-1) + 4x(x^2+1)$

$= 2\left[2x^3 + 3x - 1\right] = 0$

$\because f'\left(\frac{1}{4}\right)$ is -ve and $f'\left(\frac{1}{2}\right)$ is +ve

So, $f'(x) = 0$ for some $x \in \left(\frac{1}{4}, \frac{1}{2}\right)$

Question: If $z = x + iy, |z| - 2 = 0$ and $|z - i| - |z + 5i| = 0$, then which of the following is TRUE:

Options:

(a) $x^2 + 2y + 4 = 0$

(b) $x^2 - 2y + 4 = 0$

(c) $x + y = 0$

(d) $x^2 - y + 4 = 0$

Answer: (a)

Solution:
As $z = x + iy$

$x^2 + y^2 = 4 \quad ....(1)$

And $y = -2 \quad ....(2)$

So, $x = 0$

Hence, only $x^2 + 2y + 4 = 0$ is true.

Question: $x \sim B(n, p)$, mean = 4, variance $= \frac{4}{3}$, find $P(x \le 2)$.

Answer: $\frac{73}{729}$

Solution:
Given, $np = 4$

$npq = \frac{4}{3}$

$\therefore q = \frac{1}{3}$

$q = \frac{2}{3}$

Thus, $n = 6$

Now, $P(x \le 2) = P(x = 0) + p(x = 1) + P(x = 2)$

$= {}^6C_0 p^0 q^6 + {}^6C_1 p^1 q^5 + {}^6C_2 p^2 q^4$

$= \left(\frac{1}{3}\right)^6 + 6\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^5 + 15\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^4$

Question: Find area between $y = |x^2 - 1|$ & $y = 1$.

Answer: $\frac{4}{3}(\sqrt{2} - 1)$

Solution:

$\int_0^1 -\sqrt{1-y} + \sqrt{1+y}$

$\frac{2}{3}(1-y)^{\frac{3}{2}} + \frac{2}{3}(1+y)^{\frac{3}{2}}\Big|_0^1$

$\frac{2}{3} \times 2^{\frac{3}{2}} - \left(\frac{4}{3}\right)$

$\frac{4\sqrt{2}}{3} - \frac{4}{3} = \frac{4}{3}(\sqrt{2} - 1)$

Question: How 4 digit numbers lying between 1000 & 3000 can be made which are divisible by 4, using digits 1, 2, 3, 4, 5, 6 with no repetition.

Answer: 30.00

Solution:

We will solve the Question in two cases.

Case I: When first digit is 1.

Then last two digits can be 24, 32, 36, 52, 56 and 64.

Number of such numbers $=6\times 3=18$

Case II: When first digit is 2

Then last two digits can be 16, 36, 56 or 64

Number of such numbers $=4\times 3=12$

Total numbers of numbers $=18+12=30$

Question: $\int_0^{20\pi}\left(|\sin x|+|\cos x|\right)^2 dx$

Answer: $20\pi+40$

Solution:

$$\int_0^{20\pi}\left(|\sin x|+|\cos x|\right)^2 dx$$

$$\Rightarrow \int_0^{20\pi}\left(\left(\sin^2 x+\cos^2 x\right)+|\sin 2x|\right)dx$$

$$\Rightarrow \int_0^{20\pi} 1\,dx+\int_0^{20\pi}|\sin 2x|\,dx$$

$$\Rightarrow 20\pi+40\int_0^{\frac{\pi}{2}}\sin 2x\,dx$$

$$\Rightarrow 20\pi+40\left[\frac{-\cos 2x}{2}\right]_0^{\frac{\pi}{2}}$$

$$\Rightarrow 20\pi+20(1+1)$$

$$\Rightarrow 20\pi+40$$

Question: Find equation of common tangent to $y=x^2$ & $y=-(x-2)^2$.

Answer: ()

Solution:

Given, $y=x^2$ & $y=-(x-2)^2$

Tangent for $y=x^2$

$y = mx - \frac{1}{4}m^2$

Tangent for $y = -(x-2)^2$

$y = m(x-2) + \frac{1}{4}m^2$

$y = mx - 2m + \frac{1}{4}m^2$

$-\frac{1}{4}m^2 = -2m + \frac{1}{4}m^2$ (For common tangents)

$2m = \frac{1}{2}m^2$

$m^2 - 4m = 0$

$m = 0, m = 4$

Thus, equation of tangents are $y = 0$ or $y = 4x - 4$

Question: If $\sin^{-1}\left(\frac{x}{\alpha}\right) = \cos^{-1}\left(\frac{x}{\beta}\right)$ then find value of $\sin\left(\frac{2\pi}{\alpha+\beta}\right)$.

Answer: ()

Solution:

Given, $\sin^{-1}\left(\frac{x}{\alpha}\right) = \cos^{-1}\left(\frac{x}{\beta}\right) = k$

$\Rightarrow \alpha = \frac{\sin^{-1} x}{k}, \beta = \frac{\cos^{-1} x}{k}$

$\therefore \sin\left(\frac{2\pi\alpha}{\alpha+\beta}\right) = \sin\left(\frac{2\pi\frac{\sin^{-1} x}{k}}{\frac{\sin^{-1} x + \cos^{-1} x}{k}}\right)$

$\Rightarrow \sin\left(\frac{2\pi\left(\sin^{-1} x\right)}{\frac{\pi}{2}}\right)$

$\Rightarrow \sin\left(4\sin^{-1} x\right)$

$\Rightarrow \sin\left(2\left(\sin^{-1}\left(2x\sqrt{1-x^2}\right)\right)\right)$

$\Rightarrow 2\left(2x\sqrt{1-x^2}\right)\sqrt{1-\left(2x\sqrt{1-x^2}\right)^2}$

$\Rightarrow 4x\sqrt{1-x^2}\sqrt{1-4x^2\left(1-x^2\right)}$

$\Rightarrow 4x\sqrt{1-x^2}\left(2x^2-1\right)$

Question: $\ln 2\times\frac{d}{dx}\left(\frac{\log \operatorname{cosec} x}{\log\cos x}\right)\Big|_{\frac{\pi}{4}}$

Answer: 4.00

Solution:

$$\ln 2\times\frac{d}{dx}\left(\frac{\log \operatorname{cosec} x}{\log\cos x}\right)$$

$$\ln 2\times\frac{d}{dx}\left(-\frac{\log \sin x}{\log\cos x}\right)$$

$$\ln 2\left(\frac{(\log\cos x)\left(-\frac{\cos x}{\sin x}\right)+\log\sin x\left(-\frac{\sin x}{\cos x}\right)}{(\log\cos x)^2}\right)\text{ at } x=\frac{\pi}{4}$$

$$\ln 2\left(\frac{-2\log\frac{1}{\sqrt{2}}}{\left(\log\frac{1}{\sqrt{2}}\right)^2}\right)$$

$$\frac{-2\ln 2}{\log 2^{-\frac{1}{2}}}=\frac{-2}{-\frac{1}{2}}=4$$

Question: If $\beta=\lim_{x\to 0}\frac{\alpha x-\left(e^{3x}-1\right)}{\alpha x\left(e^{3x}-1\right)}$ then $\alpha+\beta=?$

Answer: $\frac{5}{2}$

Solution:

Given, $\beta=\lim_{x\to 0}\frac{\alpha x-\left(e^{3x}-1\right)}{\alpha x\left(e^{3x}-1\right)}$

$$\beta=\lim_{x\to 0}\frac{\alpha x-\left(1+3x+\frac{9x^2}{2}-1\right)}{\alpha x\left(1+3x+\frac{9x^2}{2}-1\right)}$$

$$\beta=\lim_{x\to 0}\frac{x(\alpha-3)-\frac{9}{2}x^2}{\alpha x(3x)}$$

$$\beta=\frac{1}{3\alpha}\lim_{x\to 0}\frac{x(\alpha-3)-\frac{9}{2}x^2}{x^2}$$

$\therefore \alpha = 3,\ \beta = \frac{1}{3\times 3}\times\left(-\frac{9}{2}\right) = \frac{-1}{2}$

$\therefore \alpha + \beta = 3 - \frac{1}{2} = \frac{5}{2}$

Question: Find minimum value of sum of squares of roots of $x^2 + (3-a)x = 2a - 1$

Answer: 6.00

Solution:

Let α, β be the roots of the equation

$x^2 + (3-a)x + 1 - 2a = 0$

Then, $\alpha + \beta = a - 3,\ \alpha\beta = 1 - 2a$

$\therefore \alpha^2 + \beta^2 = (a-3)^2 - 2(1-2a)$

$= a^2 - 2a + 7$

$= (a-1)^2 + 6$

$\therefore$ Minimum value of $\alpha^2 + \beta^2 = 6$

Question: If $\sum_{k=1}^{10} \frac{k}{(k^4 + k^2 + 1)} = \frac{m}{n}$, such that m and n are coprime, then $m + n$ is equal to ___

Answer: $\frac{55}{111}$

Solution:

$$\sum_{k=1}^{10} \frac{k}{(k^4 + k^2 + 1)} = \sum_{k=1}^{10} \frac{k}{(k^2 + k + 1)(k^2 - k + 1)}$$

$$= \sum_{k-1}^{10} \frac{1}{2}\left(\frac{1}{k^2 - k + 1} - \frac{1}{k^2 + k + 1}\right)$$

$$= \frac{1}{2}\left[\left(1 - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{7}\right) + \ldots + \left(\frac{1}{91} - \frac{1}{111}\right)\right]$$

$$= \frac{1}{2}\left(1 - \frac{1}{111}\right)$$

$$\sum_{k=1}^{10} \frac{k}{k^4 + k^2 + 1} = \frac{55}{111}$$

$\therefore m = 55,\ n = 111$

$\therefore m + n = 166$

Question: If $A=\begin{bmatrix}1\\1\\1\end{bmatrix}, B=\begin{vmatrix}9^2 & 10^2 & 11^2\\12^2 & -13^2 & 14^2\\15^2 & 16^2 & -17^2\end{vmatrix}$, then $A'BA$ is equal to:

Answer: 665.00

Solution:

$$A=\begin{bmatrix}1\\1\\1\end{bmatrix}, B=\begin{bmatrix}9^2 & 10^2 & 11^2\\12^2 & -13^2 & 14^2\\15^2 & 16^2 & -17^2\end{bmatrix}$$

$$A'=\begin{bmatrix}1 & 1 & 1\end{bmatrix}$$

$$A'B=\begin{bmatrix}9^2+12^2+15^2 & 10^2-13^2+16^2 & 11^2+14^2-17^2\end{bmatrix}$$

$$A'BA=\begin{bmatrix}9^2+12^2+15^2 & 10^2-13^2+16^2 & 11^2+14^2-17^2\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$A'BA=\begin{bmatrix}9^2+12^2+15^2+10^2-13^2+16^2+11^2+14^2-17^2\end{bmatrix}$$

$$=\begin{bmatrix}665\end{bmatrix}$$

Question: If $ax^2+by^2+2gx+2fy+c=0$ is a circle whose diametric end points are given by $x^2-4x-9=0$ & $y^2+2x-4=0$ then find $a+b-c$.

Answer: 15.00

Solution:

Diametric points be (x_1, y_1) & (x_2, y_2) and equation of circle will be

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$$

$$\Rightarrow x^2-x(x_1+x_2)+x_1x_2+y^2-y(y_1+y_2)+y_1y_2=0$$

$$\Rightarrow x^2-x(4)+(-9)+y^2-y(-2)-4=0$$

$$\Rightarrow x^2+y^2-4x+2y-13=0$$

Composing with $ax^2+by^2+2gx+2fy+c=0$

$$a=1, b=1, c=-13$$

$$a+b-c=1+1+13=15$$

Question: Let $A=\{1,2,3,4,5,6\}, B=\{3,4,6,7,9\}$ and $C=A\cup B$, then number of elements in cartesian product of $C\times B$ is ______.

Answer: 40.00

Solution:

$A=\{1,2,3,4,5,6\}, B=\{3,4,6,7,9\}$

$\therefore C=A\cup B=\{1,2,3,4,5,6,7,9\}$

$\therefore n(C\times B)=8\times 5=40$

Question: $2\sin^2\theta-\cos 2\theta=0$, $2\cos^2\theta+3\sin\theta=0$. If sum of all solutions of θ in $[0,2\pi]$ is $k\pi$, then find k.

Answer: 3.00

Solution:

Given, $2\sin^2\theta-\cos 2\theta=0$

$2\sin^2\theta-1+2\sin^2\theta=0$

$4\sin^2\theta=1$

$\sin^2\theta=\frac{1}{4}$

$\sin\theta=\pm\frac{1}{2}$

$\theta=\frac{\pi}{6},\frac{5\pi}{6},\frac{7\pi}{6},\frac{11\pi}{6}$

$2\cos^2\theta+3\sin\theta=0$

$2-2\sin^2\theta+3\sin\theta=0$

$2\sin^2\theta-3\sin\theta-2=0$

$2\sin^2\theta-4\sin\theta+\sin\theta-2=0$

$(\sin\theta-2)(2\sin\theta+1)=0$

$\sin\theta=-\frac{1}{2}$

$\theta=\frac{7\pi}{6},\frac{11\pi}{6}$

$\therefore$ Sum of all value of common θ

$\frac{7\pi}{6}+\frac{11\pi}{6}=\frac{18\pi}{6}=3\pi$

$\therefore k=3$