

FINAL JEE-MAIN EXAMINATION – MARCH, 2021

(Held On Tuesday 16th March, 2021) TIME : 3 : 00 PM to 6 : 00 PM

MATHEMATICS

SECTION-A

1. The maximum value of

$$f(x) = \begin{vmatrix} \sin^2 x & 1 + \cos^2 x & \cos 2x \\ 1 + \sin^2 x & \cos^2 x & \cos 2x \\ \sin^2 x & \cos^2 x & \sin 2x \end{vmatrix}, x \in \mathbb{R} \text{ is:}$$

- (1) $\sqrt{7}$ (2) $\frac{3}{4}$ (3) $\sqrt{5}$ (4) 5

Official Ans by NTA (3)

- Sol.** $C_1 + C_2 \rightarrow C_1$

$$\begin{vmatrix} 2 & 1 + \cos^2 x & \cos 2x \\ 2 & \cos^2 x & \cos 2x \\ 1 & \cos^2 x & \sin 2x \end{vmatrix}$$

$$R_1 - R_2 \rightarrow R_1$$

$$\begin{vmatrix} 0 & 1 & 0 \\ 2 & \cos^2 x & \cos 2x \\ 1 & \cos^2 x & \sin 2x \end{vmatrix}$$

Open w.r.t. R_1

$$-(2 \sin 2x - \cos 2x)$$

$$\cos 2x - 2 \sin 2x = f(x)$$

$$f(x)|_{\max} = \sqrt{1+4} = \sqrt{5}$$

2. Let A denote the event that a 6-digit integer formed by 0, 1, 2, 3, 4, 5, 6 without repetitions, be divisible by 3. Then probability of event A is equal to :

- (1) $\frac{9}{56}$ (2) $\frac{4}{9}$ (3) $\frac{3}{7}$ (4) $\frac{11}{27}$

Official Ans by NTA (2)

- Sol.** Total cases :

$$6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$

$$n(s) = 6 \cdot 5!$$

Favourable cases :

TEST PAPER WITH SOLUTION

Number divisible by 3 $\equiv$

Sum of digits must be divisible by 3

Case-I

1, 2, 3, 4, 5, 6

Number of ways = $6!$

Case-II

0, 1, 2, 4, 5, 6

Number of ways = $5 \cdot 5!$

Case-III

0, 1, 2, 3, 4, 5

Number of ways = $5 \cdot 5!$

$n(\text{favourable}) = 6! + 2 \cdot 5 \cdot 5!$

$$P = \frac{6! + 2 \cdot 5 \cdot 5!}{6 \cdot 6!} = \frac{4}{9}$$

3. Let $\alpha \in \mathbb{R}$ be such that the function

$$f(x) = \begin{cases} \frac{\cos^{-1}(1-\{x\}^2) \sin^{-1}(1-\{x\})}{\{x\} - \{x\}^3}, & x \neq 0 \\ \alpha, & x = 0 \end{cases}$$

is continuous at $x = 0$, where $\{x\} = x - [x]$, $[x]$

is the greatest integer less than or equal to x .

Then :

$$(1) \alpha = \frac{\pi}{\sqrt{2}} \quad (2) \alpha = 0$$

$$(3) \text{ no such } \alpha \text{ exists} \quad (4) \alpha = \frac{\pi}{4}$$

Official Ans by NTA (3)

Sol.

$$\lim_{x \rightarrow 0^+} f(x) = f(0) = \lim_{x \rightarrow 0^-} f(x)$$

$$\lim_{x \rightarrow 0^+} \frac{\cos^{-1}(1-x^2) \cdot \sin^{-1}(1-x)}{x(1-x)(1+x)}$$

$$\lim_{x \rightarrow 0^+} \frac{\cos^{-1}(1-x^2)}{x \cdot 1 \cdot 1} \cdot \frac{\pi}{2}$$

Let $1 - x^2 = \cos \theta$

$$\frac{\pi}{2} \lim_{x \rightarrow 0^+} \frac{\theta}{\sqrt{1 - \cos \theta}}$$

$$\frac{\pi}{2} \lim_{\theta \rightarrow 0^+} \frac{\theta}{\sqrt{2} \sin \frac{\theta}{2}} = \frac{\pi}{\sqrt{2}}$$

Now, $\lim_{x \rightarrow 0^-} \frac{\cos^{-1}(1 - (1+x)^2) \sin^{-1}(-x)}{(1+x) - (1+x)^3}$

$$\lim_{x \rightarrow 0^-} \frac{\frac{\pi}{2}(-\sin^{-1} x)}{(1+x)(2+x)(-x)}$$

$$\lim_{x \rightarrow 0^-} \frac{\frac{\pi}{2} \cdot \frac{\sin^{-1} x}{x}}{1 \cdot 2} = \frac{\pi}{4}$$

$\Rightarrow \text{RHL} \neq \text{LHL}$

Function can't be continuous

$\Rightarrow$ No value of α exist

4. If (x, y, z) be an arbitrary point lying on a plane P which passes through the point $(42, 0, 0)$, $(0, 42, 0)$ and $(0, 0, 42)$, then the value of expression

$$3 + \frac{x-11}{(y-19)^2(z-12)^2} + \frac{y-19}{(x-11)^2(z-12)^2} + \frac{z-12}{(x-11)^2(y-19)^2} - \frac{x+y+z}{14(x-11)(y-19)(z-12)}$$

(1) 0 (2) 3 (3) 39 (4) -45

Official Ans by NTA (2)

- Sol.** Plane passing through $(42, 0, 0)$, $(0, 42, 0)$, $(0, 0, 42)$

From intercept form, equation of plane is

$$x + y + z = 42$$

$$\Rightarrow (x - 11) + (y - 19) + (z - 12) = 0$$

let $a = x - 11$, $b = y - 19$, $c = z - 12$

$$a + b + c = 0$$

Now, given expression is

$$3 + \frac{a}{b^2 c^2} + \frac{b}{a^2 c^2} + \frac{c}{a^2 b^2} - \frac{42}{14abc}$$

$$3 + \frac{a^3 + b^3 + c^3 - 3abc}{a^2 b^2 c^2}$$

$$\text{If } a + b + c = 0$$

$$\Rightarrow a^3 + b^3 + c^3 = 3abc$$

$$\Rightarrow 3$$

5. Consider the integral

$$I = \int_0^{10} \frac{[x] e^{[x]}}{e^{x-1}} dx,$$

where $[x]$ denotes the greatest integer less than or equal to x . Then the value of I is equal to:

- (1) $9(e - 1)$ (2) $45(e + 1)$
(3) $45(e - 1)$ (4) $9(e + 1)$

Official Ans by NTA (3)

Sol. $I = \int_0^{10} [x] \cdot e^{[x]-x+1} dx$

$$I = \int_0^1 0 dx + \int_1^2 1 \cdot e^{2-x} + \int_2^3 2 \cdot e^{3-x} + \dots + \int_9^{10} 9 \cdot e^{10-x} dx$$

$$\Rightarrow I = \sum_{n=0}^9 \int_n^{n+1} n \cdot e^{n+1-x} dx$$

$$= - \sum_{n=0}^9 n (e^{n+1-x})_n^{n+1}$$

$$= - \sum_{n=0}^9 n \cdot (e^0 - e^1)$$

$$= (e - 1) \sum_{n=0}^9 n$$

$$= (e - 1) \cdot \frac{9 \cdot 10}{2}$$

$$= 45(e - 1)$$

6. Let C be the locus of the mirror image of a point on the parabola $y^2 = 4x$ with respect to the line $y = x$. Then the equation of tangent to C at P(2,1) is :

(1) $x - y = 1$ (2) $2x + y = 5$
(3) $x + 3y = 5$ (4) $x + 2y = 4$

Official Ans by NTA (1)

Sol. Given $y^2 = 4x$

Mirror image on $y = x \Rightarrow C : x^2 = 4y$

$$2x = 4 \cdot \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{x}{2}$$

$$\left. \frac{dy}{dx} \right|_{P(2,1)} = \frac{2}{2} = 1$$

Equation of tangent at (2, 1)

$$\Rightarrow y - 1 = 1(x - 2)$$

$$\Rightarrow x - y = 1$$

7. If $y = y(x)$ is the solution of the differential equation $\frac{dy}{dx} + (\tan x)y = \sin x$, $0 \leq x \leq \frac{\pi}{3}$, with

$y(0) = 0$, then $y\left(\frac{\pi}{4}\right)$ equal to :

(1) $\frac{1}{4} \log_e 2$ (2) $\left(\frac{1}{2\sqrt{2}}\right) \log_e 2$
(3) $\log_e 2$ (4) $\frac{1}{2} \log_e 2$

Official Ans by NTA (2)

Sol. $\frac{dy}{dx} + (\tan x)y = \sin x$; $0 \leq x \leq \frac{\pi}{3}$

$$\text{I.F.} = e^{\int \tan x dx} = e^{\ln \sec x} = \sec x$$

$$y \sec x = \int \tan x dx$$

$$y \sec x = \int \tan x dx$$

$$y \sec x = \ln |\sec x| + C$$

$$x = 0, y = 0 \Rightarrow \therefore c = 0$$

$$y \sec x = \ln |\sec x|$$

$$y = \cos x \cdot \ln |\sec x|$$

$$y|_{x=\frac{\pi}{4}} = \left(\frac{1}{\sqrt{2}}\right) \cdot \ln \sqrt{2}$$

$$y|_{x=\frac{\pi}{4}} = \frac{1}{2\sqrt{2}} \log_e 2$$

8. Let $A = \{2, 3, 4, 5, \dots, 30\}$ and ' $\simeq$ ' be an equivalence relation on $A \times A$, defined by $(a, b) \simeq (c, d)$, if and only if $ad = bc$. Then the number of ordered pairs which satisfy this equivalence relation with ordered pair (4, 3) is equal to :

(1) 5 (2) 6 (3) 8 (4) 7

Official Ans by NTA (4)

Sol. $A = \{2, 3, 4, 5, \dots, 30\}$

$$(a, b) \simeq (c, d) \Rightarrow ad = bc$$

$$(4, 3) \simeq (c, d) \Rightarrow 4d = 3c$$

$$\Rightarrow \frac{4}{3} = \frac{c}{d}$$

$$\frac{c}{d} = \frac{4}{3} \quad \& \quad c, d \in \{2, 3, \dots, 30\}$$

$$(c, d) = \{(4, 3), (8, 6), (12, 9), (16, 12), (20, 15), (24, 18), (28, 21)\}$$

No. of ordered pair = 7

9. Let the lengths of intercepts on x-axis and y-axis made by the circle $x^2 + y^2 + ax + 2ay + c = 0$, ($a < 0$) be $2\sqrt{2}$ and $2\sqrt{5}$, respectively. Then the shortest distance from origin to a tangent to this circle which is perpendicular to the line $x + 2y = 0$, is equal to :

(1) $\sqrt{11}$ (2) $\sqrt{7}$ (3) $\sqrt{6}$ (4) $\sqrt{10}$

Official Ans by NTA (3)

Sol. $x^2 + y^2 + ax + 2ay + c = 0$

$$2\sqrt{g^2 - c} = 2\sqrt{\frac{a^2}{4} - c} = 2\sqrt{2}$$

$$\Rightarrow \frac{a^2}{4} - c = 2 \quad \dots(1)$$

$$2\sqrt{f^2 - c} = 2\sqrt{a^2 - c} = 2\sqrt{5}$$

$$\Rightarrow a^2 - c = 5 \quad \dots(2)$$

(1) & (2)

$$\frac{3a^2}{4} = 3 \Rightarrow a = -2 \quad (a < 0)$$

$$\therefore c = -1$$

$$\text{Circle} \Rightarrow x^2 + y^2 - 2x - 4y - 1 = 0$$

$$\Rightarrow (x-1)^2 + (y-2)^2 = 6$$

$$\text{Given } x + 2y = 0 \Rightarrow m = -\frac{1}{2}$$

$$m_{\text{tangent}} = 2$$

Equation of tangent

$$\Rightarrow (y-2) = 2(x-1) \pm \sqrt{6}\sqrt{1+4}$$

$$\Rightarrow 2x - y \pm \sqrt{30} = 0$$

$$\text{Perpendicular distance from } (0, 0) = \frac{|\pm\sqrt{30}|}{\sqrt{4+1}} = \sqrt{6}$$

10. The least value of $|z|$ where z is complex number which satisfies the inequality

$$\exp\left(\frac{(|z|+3)(|z|-1)}{|z|+1} \log_e 2\right) \geq \log_{\sqrt{2}} |5\sqrt{7} + 9i|,$$

$i = \sqrt{-1}$, is equal to :

- (1) 3 (2) $\sqrt{5}$ (3) 2 (4) 8

Official Ans by NTA (1)

$$\text{Sol. } \exp\left(\frac{(|z|+3)(|z|-1)}{|z|+1} \ln 2\right) \geq \log_{\sqrt{2}} |5\sqrt{7} + 9i|$$

$$\Rightarrow 2^{\frac{(|z|+3)(|z|-1)}{(|z|+1)}} \geq \log_{\sqrt{2}} (16)$$

$$\Rightarrow 2^{\frac{(|z|+3)(|z|-1)}{(|z|+1)}} \geq 2^3$$

$$\Rightarrow \frac{(|z|+3)(|z|-1)}{(|z|+1)} \geq 3$$

$$\Rightarrow (|z|+3)(|z|-1) \geq 3(|z|+1)$$

$$|z|^2 + 2|z| - 3 \geq 3|z| + 3$$

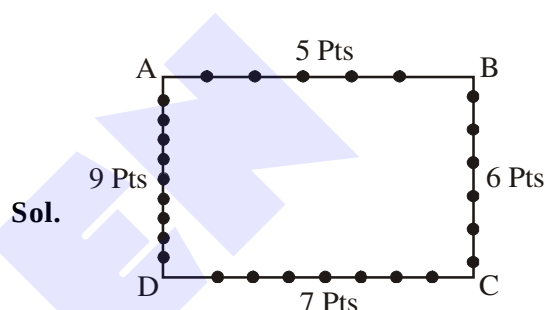
$$\Rightarrow |z|^2 + |z| - 6 \geq 0$$

$$\Rightarrow (|z|-3)(|z|+2) \geq 0 \Rightarrow |z|-3 \geq 0$$

$$\Rightarrow |z| \geq 3 \Rightarrow |z|_{\min} = 3$$

11. Consider a rectangle ABCD having 5, 7, 6, 9 points in the interior of the line segments AB, CD, BC, DA respectively. Let α be the number of triangles having these points from different sides as vertices and β be the number of quadrilaterals having these points from different sides as vertices. Then $(\beta - \alpha)$ is equal to :
- (1) 795 (2) 1173 (3) 1890 (4) 717

Official Ans by NTA (4)



Sol.

α = Number of triangles

$$\alpha = 5 \cdot 6 \cdot 7 + 5 \cdot 7 \cdot 9 + 5 \cdot 6 \cdot 9 + 6 \cdot 7 \cdot 9$$

$$= 210 + 315 + 270 + 378$$

$$= 1173$$

β = Number of Quadrilateral

$$\beta = 5 \cdot 6 \cdot 7 \cdot 9 = 1890$$

$$\beta - \alpha = 1890 - 1173 = 717$$

12. If the point of intersections of the ellipse

$$\frac{x^2}{16} + \frac{y^2}{b^2} = 1 \text{ and the circle } x^2 + y^2 = 4b, b > 4$$

lie on the curve $y^2 = 3x^2$, then b is equal to:

- (1) 12 (2) 5 (3) 6 (4) 10

Official Ans by NTA (1)

$$\text{Sol. } y^2 = 3x^2$$

$$\text{and } x^2 + y^2 = 4b$$

Solve both we get

$$\text{so } x^2 = b$$

$$\frac{x^2}{16} + \frac{3x^2}{b^2} = 1$$

$$\frac{b}{16} + \frac{3}{b} = 1$$

$$b^2 - 16b + 48 = 0$$

$$(b - 12)(b - 4) = 0$$

$$b = 12, b > 4$$

13. Given that the inverse trigonometric functions take principal values only. Then, the number of real values of x which satisfy

$$\sin^{-1}\left(\frac{3x}{5}\right) + \sin^{-1}\left(\frac{4x}{5}\right) = \sin^{-1}x \text{ is equal to:}$$

- (1) 2 (2) 1 (3) 3 (4) 0

Official Ans by NTA (3)

Sol. $\sin^{-1}\frac{3x}{5} + \sin^{-1}\frac{4x}{5} = \sin^{-1}x$

$$\sin^{-1}\left(\frac{3x}{5}\sqrt{1-\frac{16x^2}{25}} + \frac{4x}{5}\sqrt{1-\frac{9x^2}{25}}\right) = \sin^{-1}x$$

$$\frac{3x}{5}\sqrt{1-\frac{16x^2}{25}} + \frac{4x}{5}\sqrt{1-\frac{9x^2}{25}} = x$$

$$x = 0, 3\sqrt{25-16x^2} + 4\sqrt{25-9x^2} = 25$$

$$4\sqrt{25-9x^2} = 25 - 3\sqrt{25-16x^2} \text{ squaring we get}$$

$$16(25-9x^2) = 625 + 9(25-16x^2) - 150\sqrt{25-16x^2}$$

$$400 = 625 + 225 - 150\sqrt{25-16x^2}$$

$$\sqrt{25-16x^2} = 3 \Rightarrow 25 - 16x^2 = 9$$

$$\Rightarrow x^2 = 1$$

Put $x = 0, 1, -1$ in the original equation

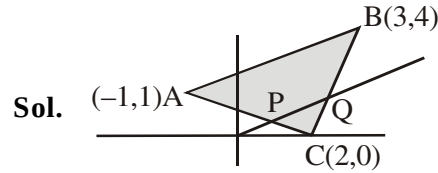
We see that all values satisfy the original equation.

Number of solution = 3

14. Let $A(-1, 1)$, $B(3, 4)$ and $C(2, 0)$ be given three points. A line $y = mx$, $m > 0$, intersects lines AC and BC at point P and Q respectively. Let A_1 and A_2 be the areas of $\triangle ABC$ and $\triangle PQC$ respectively, such that $A_1 = 3A_2$, then the value of m is equal to :

- (1) $\frac{4}{15}$ (2) 1 (3) 2 (4) 3

Official Ans by NTA (2)



$$P \equiv (x_1, mx_1)$$

$$Q \equiv (x_2, mx_2)$$

$$A_1 = \frac{1}{2} \begin{vmatrix} 3 & 4 & 1 \\ 2 & 0 & 1 \\ -1 & 1 & 1 \end{vmatrix} = \frac{13}{2}$$

$$A_2 = \frac{1}{2} \begin{vmatrix} x_1 & mx_1 & 1 \\ x_2 & mx_2 & 1 \\ 2 & 0 & 1 \end{vmatrix}$$

$$A_2 = \frac{1}{2} |2(mx_1 - mx_2)| = m|x_1 - x_2|$$

$$A_1 = 3A_2 \Rightarrow \frac{13}{2} = 3m|x_1 - x_2|$$

$$\Rightarrow |x_1 - x_2| = \frac{16}{6m}$$

$$AC : x + 3y = 2$$

$$BC : y = 4x - 8$$

$$P : x + 3y = 2 \text{ \& } y = mx \Rightarrow x_1 = \frac{2}{1+3m}$$

$$Q : y = 4x - 8 \text{ \& } y = mx \Rightarrow x_2 = \frac{8}{4-m}$$

$$|x_1 - x_2| = \left| \frac{2}{1+3m} - \frac{8}{4-m} \right|$$

$$= \left| \frac{-26m}{(1+3m)(4-m)} \right| = \frac{26m}{(3m+1)|m-4|}$$

$$= \frac{26m}{(3m+1)(4-m)}$$

$$|x_1 - x_2| = \frac{13}{6m}$$

$$\frac{26m}{(3m+1)(4-m)} = \frac{13}{6m}$$

$$\Rightarrow 12m^2 = -(3m+1)(m-4)$$

$$\Rightarrow 12m^2 = -(3m^2 - 11m - 4)$$

$$\Rightarrow 15m^2 - 11m - 4 = 0$$

$$\Rightarrow 15m^2 - 15m + 4m - 4 = 0$$

$$\Rightarrow (15m+4)(m-1) = 0$$

$$\Rightarrow m = 1$$

15. Let f be a real valued function, defined on $\mathbb{R} - \{-1, 1\}$ and given by

$$f(x) = 3 \log_e \left| \frac{x-1}{x+1} \right| - \frac{2}{x-1}.$$

Then in which of the following intervals, function $f(x)$ is increasing?

(1) $(-\infty, -1) \cup \left(\left[\frac{1}{2}, \infty \right) - \{1\} \right)$

(2) $(-\infty, \infty) - \{-1, 1\}$

(3) $\left(-1, \frac{1}{2} \right]$

(4) $\left(-\infty, \frac{1}{2} \right] - \{-1\}$

Official Ans by NTA (1)

Sol. $f(x) = 3 \ln(x-1) - 3 \ln(x+1) - \frac{2}{x-1}$

$$f'(x) = \frac{3}{x-1} - \frac{3}{x+1} + \frac{2}{(x-1)^2}$$

$$f'(x) = \frac{4(2x-1)}{(x-1)^2(x+1)}$$

$$f'(x) \geq 0$$

$$\Rightarrow x \in (-\infty, -1) \cup \left[\frac{1}{2}, 1 \right) \cup (1, \infty)$$

16. Let $f : S \rightarrow S$ where $S = (0, \infty)$ be a twice differentiable function such that $f(x+1) = xf(x)$. If $g : S \rightarrow \mathbb{R}$ be defined as $g(x) = \log_e f(x)$, then the value of $|g''(5) - g''(1)|$ is equal to :

(1) $\frac{205}{144}$ (2) $\frac{197}{144}$ (3) $\frac{187}{144}$ (4) 1

Official Ans by NTA (1)

Sol. $\ln f(x+1) = \ln(xf(x))$

$$\ln f(x+1) = \ln x + \ln f(x)$$

$$\Rightarrow g(x+1) = \ln x + g(x)$$

$$\Rightarrow g(x+1) - g(x) = \ln x$$

$$\Rightarrow g''(x+1) - g''(x) = -\frac{1}{x^2}$$

Put $x = 1, 2, 3, 4$

$$g''(2) - g''(1) = -\frac{1}{1^2} \quad \dots(1)$$

$$g''(3) - g''(2) = -\frac{1}{2^2} \quad \dots(2)$$

$$g''(4) - g''(3) = -\frac{1}{3^2} \quad \dots(3)$$

$$g''(5) - g''(4) = -\frac{1}{4^2} \quad \dots(4)$$

Add all the equation we get

$$g''(5) - g''(1) = -\frac{1}{1^2} - \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{4^2}$$

$$|g''(5) - g''(1)| = \frac{205}{144}$$

17. Let $P(x) = x^2 + bx + c$ be a quadratic polynomial

with real coefficients such that $\int_0^1 P(x) dx = 1$ and

$P(x)$ leaves remainder 5 when it is divided by $(x-2)$. Then the value of $9(b+c)$ is equal to:

(1) 9 (2) 15 (3) 7 (4) 11

Official Ans by NTA (3)

Sol. $\int_0^1 (x^2 + bx + c)dx = 1$

$$\frac{1}{3} + \frac{b}{2} + c = 1 \Rightarrow \frac{b}{2} + c = \frac{2}{3}$$

$$3b + 6c = 4 \quad \dots(1)$$

$$P(2) = 5$$

$$4 + 2b + c = 5$$

$$2b + c = 1 \quad \dots(2)$$

From (1) & (2)

$$b = \frac{2}{9} \quad \& \quad c = \frac{5}{9}$$

$$9(b + c) = 7$$

- 18.** If the foot of the perpendicular from point

$$(4, 3, 8) \text{ on the line } L_1 : \frac{x-a}{l} = \frac{y-2}{3} = \frac{z-b}{4},$$

$l \neq 0$ is $(3, 5, 7)$, then the shortest distance between the line L_1 and line

$$L_2 : \frac{x-2}{3} = \frac{y-4}{4} = \frac{z-5}{5} \text{ is equal to :}$$

$$(1) \frac{1}{2} \quad (2) \frac{1}{\sqrt{6}} \quad (3) \sqrt{\frac{2}{3}} \quad (4) \frac{1}{\sqrt{3}}$$

Official Ans by NTA (2)

- Sol.** $(3, 5, 7)$ satisfy the line L_1

$$\frac{3-a}{l} = \frac{5-2}{3} = \frac{7-b}{4}$$

$$\frac{3-a}{l} = 1 \quad \& \quad \frac{7-b}{4} = 1$$

$$a + l = 3 \quad \dots(1) \quad \& \quad b = 3 \quad \dots(2)$$

$$\vec{v}_1 = \langle 4, 3, 8 \rangle - \langle 3, 5, 7 \rangle$$

$$\vec{v}_1 = \langle 1, -2, 1 \rangle$$

$$\vec{v}_2 = \langle l, 3, 4 \rangle$$

$$\vec{v}_1 \cdot \vec{v}_2 = 0 \Rightarrow l - 6 + 4 = 0 \Rightarrow l = 2$$

$$a + l = 3 \Rightarrow a = 1$$

$$L_1 : \frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$$

$$L_2 : \frac{x-2}{3} = \frac{y-4}{4} = \frac{z-5}{5}$$

$$A = \langle 1, 2, 3 \rangle$$

$$B = \langle 2, 4, 5 \rangle$$

$$\vec{AB} = \langle 1, 2, 2 \rangle$$

$$\vec{p} = 2\hat{i} + 3\hat{j} + 4\hat{k}$$

$$\vec{q} = 3\hat{i} + 4\hat{j} + 5\hat{k}$$

$$\vec{p} \times \vec{q} = -\hat{i} + 2\hat{j} - \hat{k}$$

$$\text{Shortest distance} = \frac{|\vec{AB} \cdot (\vec{p} \times \vec{q})|}{|\vec{p} \times \vec{q}|} = \frac{1}{\sqrt{6}}$$

- 19.** Let C_1 be the curve obtained by the solution of differential equation $2xy \frac{dy}{dx} = y^2 - x^2$, $x > 0$.

Let the curve C_2 be the solution of

$$\frac{2xy}{x^2 - y^2} = \frac{dy}{dx}. \text{ If both the curves pass through}$$

$(1, 1)$, then the area enclosed by the curves C_1 and C_2 is equal to :

$$(1) \pi - 1 \quad (2) \frac{\pi}{2} - 1 \quad (3) \pi + 1 \quad (4) \frac{\pi}{4} + 1$$

Official Ans by NTA (2)

Sol. $\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$, $x \in (0, \infty)$

put $y = vx$

$$x \frac{dv}{dx} + v = \frac{v^2 - 1}{2v}$$

$$\frac{2v}{v^2 + 1} dv = -\frac{dx}{x}$$

Integrate,

$$\ln(v^2 + 1) = -\ln x + C$$

$$\ln\left(\frac{y^2}{x^2} + 1\right) = -\ln x + C$$

put $x = 1, y = 1, C = \ln 2$

$$\ln\left(\frac{y^2}{x^2} + 1\right) = -\ln x + \ln 2$$

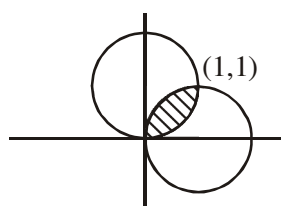
$$\Rightarrow x^2 + y^2 - 2x = 0 \quad (\text{Curve } C_1)$$

Similarly,

$$\frac{dy}{dx} = \frac{2xy}{x^2 - y^2}$$

Put $y = vx$

$$x^2 + y^2 - 2y = 0$$



$$\text{required area} = 2 \int_0^1 (\sqrt{2x - x^2} - x) dx = \frac{\pi}{2} - 1$$

20. Let $\vec{a} = \hat{i} + 2\hat{j} - 3\hat{k}$ and $\vec{b} = 2\hat{i} - 3\hat{j} + 5\hat{k}$. If

$$\vec{r} \times \vec{a} = \vec{b} \times \vec{r}, \quad \vec{r} \cdot (\alpha \hat{i} + 2\hat{j} + \hat{k}) = 3 \quad \text{and}$$

$$\vec{r} \cdot (2\hat{i} + 5\hat{j} - \alpha \hat{k}) = -1, \quad \alpha \in \mathbb{R}, \text{ then the value of}$$

$\alpha + |\vec{r}|^2$ is equal to :

- (1) 9 (2) 15 (3) 13 (4) 11

Official Ans by NTA (2)

Sol. $\vec{r} \times \vec{a} = \vec{b} \times \vec{r} \Rightarrow \vec{r} \times (\vec{a} + \vec{b}) = 0$

$$\vec{r} = \vec{\lambda}(\vec{a} + \vec{b}) \Rightarrow \vec{r} = \vec{\lambda}(\hat{i} + 2\hat{j} - 3\hat{k} + 2\hat{i} - 3\hat{j} + 5\hat{k})$$

$$\vec{r} = \vec{\lambda}(3\hat{i} - \hat{j} + 2\hat{k}) \quad \dots(1)$$

$$\vec{r} \cdot (\alpha \hat{i} + 2\hat{j} + \hat{k}) = 3$$

Put $\vec{r}$ from (1) $\alpha \lambda = 1 \quad \dots(2)$

$$\vec{r} \cdot (2\hat{i} + 5\hat{j} - \alpha \hat{k}) = -1$$

Put $\vec{r}$ from (1) $2\lambda\alpha - \lambda = 1 \quad \dots(3)$

Solve (2) & (3)

$$\alpha = 1, \lambda = 1$$

$$\Rightarrow \vec{r} = 3\hat{i} - \hat{j} + 2\hat{k}$$

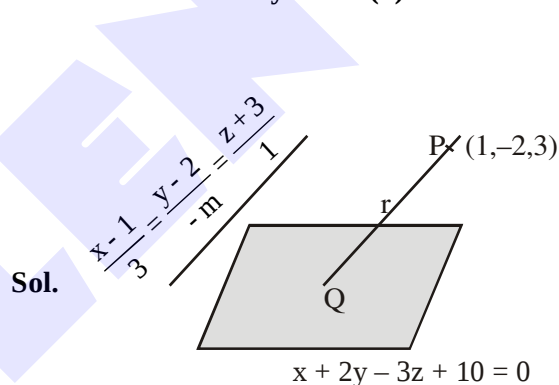
$$|\vec{r}|^2 = 14 \quad \& \quad \alpha = 1$$

$$\alpha + |\vec{r}|^2 = 15$$

SECTION-B

1. If the distance of the point $(1, -2, 3)$ from the plane $x + 2y - 3z + 10 = 0$ measured parallel to the line, $\frac{x-1}{3} = \frac{y-2}{m} = \frac{z+3}{1}$ is $\frac{\sqrt{7}}{2}$, then the value of $|m|$ is equal to _____.

Official Ans by NTA (2)



Sol.

$$\text{DC of line} = \left(\frac{3}{\sqrt{m^2 + 10}}, \frac{-m}{\sqrt{m^2 + 10}}, \frac{1}{\sqrt{m^2 + 10}} \right)$$

$$Q = \left(1 + \frac{3r}{\sqrt{m^2 + 10}}, -2 + \frac{-mr}{\sqrt{m^2 + 10}}, 3 + \frac{r}{\sqrt{m^2 + 10}} \right)$$

$$Q \text{ lies on } x + 2y - 3z + 10 = 0$$

$$1 + \frac{3r}{\sqrt{m^2 + 10}} - 4 - \frac{2mr}{\sqrt{m^2 + 10}} - 9 - \frac{3r}{\sqrt{m^2 + 10}} + 10 = 0$$

$$\Rightarrow \frac{r}{\sqrt{m^2 + 10}} (3 - 2m - 3) = 2$$

$$\Rightarrow \frac{r}{\sqrt{m^2 + 10}} (-2m) = 2$$

$$r^2 m^2 = m^2 + 10$$

$$\frac{7}{2} m^2 = m^2 + 10 \Rightarrow \frac{5}{2} m^2 = 10 \Rightarrow m^2 = 4$$

$$|m| = 2$$

2. Consider the statistics of two sets of observations as follows :

	Size	Mean	Variance
Observation I	10	2	2
Observation II	n	3	1

If the variance of the combined set of these two observations is $\frac{17}{9}$, then the value of n is equal to _____.

Official Ans by NTA (5)

Sol.
$$\sigma^2 = \frac{n_1 \sigma_1^2 + n_2 \sigma_2^2}{n_1 + n_2} + \frac{n_1 n_2}{(n_1 + n_2)^2} (\bar{x}_1 - \bar{x}_2)^2$$

$$n_1 = 10, n_2 = n, \sigma_1^2 = 2, \sigma_2^2 = 1$$

$$\bar{x}_1 = 2, \bar{x}_2 = 3, \sigma^2 = \frac{17}{9}$$

$$\frac{17}{9} = \frac{10 \times 2 + n}{n + 10} + \frac{10n}{(n + 10)^2} (3 - 2)^2$$

$$\Rightarrow \frac{17}{9} = \frac{(n + 20)(n + 10) + 10n}{(n + 10)^2}$$

$$\Rightarrow 17n^2 + 1700 + 340n = 90n + 9(n^2 + 30n + 200)$$

$$\Rightarrow 8n^2 - 20n - 100 = 0$$

$$2n^2 - 5n - 25 = 0$$

$$\Rightarrow (2n + 5)(n - 5) = 0 \Rightarrow n = \frac{-5}{2}, 5$$

↓
(Rejected)

Hence $n = 5$

3. Let $A = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ and $B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ be two 2×1 matrices with real entries such that $A = XB$, where $X = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & -1 \\ 1 & k \end{bmatrix}$, and $k \in \mathbb{R}$. If

$$a_1^2 + a_2^2 = \frac{2}{3} (b_1^2 + b_2^2) \text{ and } (k^2 + 1)b_2^2 \neq -2b_1b_2,$$

then the value of k is _____.

Official Ans by NTA (1)

Sol. $A = XB$

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & -1 \\ 1 & k \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\begin{bmatrix} \sqrt{3} a_1 \\ \sqrt{3} a_2 \end{bmatrix} = \begin{bmatrix} b_1 - b_2 \\ b_1 + kb_2 \end{bmatrix}$$

$$b_1 - b_2 = \sqrt{3} a_1 \quad \dots(1)$$

$$b_1 + kb_2 = \sqrt{3} a_2 \quad \dots(2)$$

Given, $a_1^2 + a_2^2 = \frac{2}{3} (b_1^2 + b_2^2)$

$$(1)^2 + (2)^2$$

$$(b_1 - b_2)^2 + (b_1 + kb_2)^2 = 3(a_1^2 + a_2^2)$$

$$a_1^2 + a_2^2 = \frac{2}{3} b_1^2 + \frac{(1+k^2)}{3} b_2^2 + \frac{2}{3} b_1 b_2 (k-1)$$

Given, $a_1^2 + a_2^2 = \frac{2}{3} b_1^2 + \frac{2}{3} b_2^2$

On comparing we get

$$\frac{k^2 + 1}{3} = \frac{2}{3} \Rightarrow k^2 + 1 = 2$$

$$\Rightarrow k = \pm 1 \quad \dots(3)$$

$$\& \frac{2}{3} (k-1) = 0 \Rightarrow k = 1 \quad \dots(4)$$

From both we get $k = 1$

4. For real numbers α, β, γ and δ , if

$$\int \frac{(x^2 - 1) + \tan^{-1} \left(\frac{x^2 + 1}{x} \right)}{(x^4 + 3x^2 + 1) \tan^{-1} \left(\frac{x^2 + 1}{x} \right)} dx$$

$$= \alpha \log_e \left(\tan^{-1} \left(\frac{x^2+1}{x} \right) \right) + \beta \tan^{-1} \left(\frac{\gamma(x^2-1)}{x} \right) + \delta \tan^{-1} \left(\frac{x^2+1}{x} \right) + C$$

where C is an arbitrary constant, then the value of $10(\alpha + \beta\gamma + \delta)$ is equal to _____.

Official Ans by NTA (6)

Sol.
$$\int \frac{(x^2-1)dx}{(x^4+3x^2+1)\tan^{-1}\left(x+\frac{1}{x}\right)} + \int \frac{dx}{x^4+3x^2+1}$$

$$\int \frac{\left(1-\frac{1}{x^2}\right)dx}{\left(\left(x+\frac{1}{x}\right)^2+1\right)\tan^{-1}\left(x+\frac{1}{x}\right)} + \frac{1}{2} \int \frac{(x^2+1)-(x^2-1)dx}{x^4+3x^2+1}$$

Put $\tan^{-1}\left(x+\frac{1}{x}\right) = t$

$$\int \frac{dt}{t} + \frac{1}{2} \int \frac{\left(1+\frac{1}{x^2}\right)dx}{\left(x-\frac{1}{x}\right)^2+5} - \frac{1}{2} \int \frac{\left(1-\frac{1}{x^2}\right)dx}{\left(x+\frac{1}{x}\right)^2+1}$$

Put $x-\frac{1}{x} = y, x+\frac{1}{x} = z$

$$\log_e t + \frac{1}{2} \int \frac{dy}{y^2+5} - \frac{1}{2} \int \frac{dz}{z^2+1}$$

$$= \log_e \tan^{-1}\left(x+\frac{1}{x}\right) + \frac{1}{2\sqrt{5}} \tan^{-1}\left(\frac{x^2-1}{\sqrt{5}x}\right) - \frac{1}{2} \tan^{-1}\left(\frac{x^2+1}{x}\right) + C$$

$$\alpha = 1, \beta = \frac{1}{2\sqrt{5}}, \gamma = \frac{1}{\sqrt{5}}, \delta = \frac{-1}{2}$$

or

$$\alpha = 1, \beta = \frac{-1}{2\sqrt{5}}, \gamma = \frac{-1}{\sqrt{5}}, \delta = \frac{-1}{2}$$

$$10(\alpha + \beta\gamma + \delta) = 10\left(1 + \frac{1}{10} - \frac{1}{2}\right) = 6$$

5. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ and $g : \mathbb{R} \rightarrow \mathbb{R}$ be defined as

$$f(x) = \begin{cases} x+a, & x < 0 \\ |x-1|, & x \geq 0 \end{cases} \text{ and}$$

$$g(x) = \begin{cases} x+1, & x < 0 \\ (x-1)^2+b, & x \geq 0 \end{cases}$$

where a, b are non-negative real numbers. If $(g \circ f)(x)$ is continuous for all $x \in \mathbb{R}$, then a + b is equal to _____.

Official Ans by NTA (1)

Sol.
$$g[f(x)] = \begin{cases} f(x)+1 & f(x) < 0 \\ (f(x)-1)^2+b & f(x) \geq 0 \end{cases}$$

$$g[f(x)] = \begin{cases} x+a+1 & x+a < 0 \text{ \& } x < 0 \\ |x-1|+1 & |x-1| < 0 \text{ \& } x \geq 0 \\ (x+a-1)^2+b & x+a \geq 0 \text{ \& } x < 0 \\ (|x-1|-1)^2+b & |x-1| \geq 0 \text{ \& } x \geq 0 \end{cases}$$

$$g[f(x)] = \begin{cases} x+a+1 & x \in (-\infty, -a) \text{ \& } x \in (-\infty, 0) \\ |x-1|+1 & x \in \phi \\ (x+a-1)^2+b & x \in [-a, \infty) \text{ \& } x \in (-\infty, 0) \\ (|x-1|-1)^2+b & x \in \mathbb{R} \text{ \& } x \in [0, \infty) \end{cases}$$

$$g[f(x)] = \begin{cases} x+a+1 & x \in (-\infty, -a) \\ (x+a-1)^2+b & x \in [-a, 0) \\ (|x-1|-1)^2+b & x \in [0, \infty) \end{cases}$$

$g(f(x))$ is continuous

at $x = -a$ & at $x = 0$

$1 = b + 1$ & $(a - 1)^2 + b = b$

$b = 0$ & $a = 1$

$\Rightarrow a + b = 1$

6. Let $\frac{1}{16}$, a and b be in G.P. and $\frac{1}{a}, \frac{1}{b}, 6$ be in

A.P., where $a, b > 0$. Then $72(a + b)$ is equal to _____.

Official Ans by NTA (14)

Sol. $a^2 = \frac{b}{16} \Rightarrow \frac{1}{b} = \frac{1}{16a^2}$

$$\frac{2}{b} = \frac{1}{a} + 6$$

$$\frac{1}{8a^2} = \frac{1}{a} + 6$$

$$\frac{1}{a^2} - \frac{8}{a} - 48 = 0$$

$$\frac{1}{a} = 12, -4 \Rightarrow a = \frac{1}{12}, -\frac{1}{4}$$

$$a = \frac{1}{12}, a > 0$$

$$b = 16a^2 = \frac{1}{9}$$

$$\Rightarrow 72(a + b) = 6 + 8 = 14$$

7. In ΔABC , the lengths of sides AC and AB are 12 cm and 5 cm, respectively. If the area of ΔABC is 30 cm^2 and R and r are respectively the radii of circumcircle and incircle of ΔABC , then the value of $2R + r$ (in cm) is equal to _____.

Official Ans by NTA (15)

Sol. $\Delta = \frac{1}{2} \cdot 5 \cdot 12 \cdot \sin A = 30$

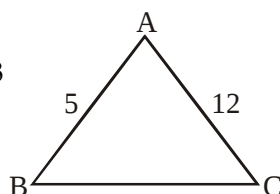
$$\sin A = 1$$

$$A = 90^\circ \Rightarrow BC = 13$$

$$BC = 2R = 13$$

$$r = \frac{\Delta}{S} = \frac{30}{15} = 2$$

$$2R + r = 15$$



8. Let n be a positive integer. Let

$$A = \sum_{k=0}^n (-1)^k n C_k \left[\left(\frac{1}{2}\right)^k + \left(\frac{3}{4}\right)^k + \left(\frac{7}{8}\right)^k + \left(\frac{15}{16}\right)^k + \left(\frac{31}{32}\right)^k \right]$$

If $63A = 1 - \frac{1}{2^{30}}$, then n is equal to _____.

Official Ans by NTA (6)

Sol. $A = \sum_{k=0}^n n C_k \left[\left(-\frac{1}{2}\right)^k + \left(-\frac{3}{4}\right)^k + \left(-\frac{7}{8}\right)^k + \left(-\frac{15}{16}\right)^k + \left(-\frac{31}{32}\right)^k \right]$

$$A = \left(1 - \frac{1}{2}\right)^n + \left(1 - \frac{3}{4}\right)^n + \left(1 - \frac{7}{8}\right)^n + \left(1 - \frac{15}{16}\right)^n + \left(1 - \frac{31}{32}\right)^n$$

$$A = \frac{1}{2^n} + \frac{1}{4^n} + \frac{1}{8^n} + \frac{1}{16^n} + \frac{1}{32^n}$$

$$A = \frac{1}{2^n} \left[\frac{1 - \left(\frac{1}{2}\right)^{5n}}{1 - \frac{1}{2^n}} \right] \Rightarrow A = \frac{\left(1 - \frac{1}{2^{5n}}\right)}{(2^n - 1)}$$

$$(2^n - 1)A = 1 - \frac{1}{2^{5n}}, \text{ Given } 63A = 1 - \frac{1}{2^{30}}$$

Clearly $5n = 30$

$$n = 6$$

9. Let $\vec{c}$ be a vector perpendicular to the vectors

$$\vec{a} = \hat{i} + \hat{j} - \hat{k} \text{ and } \vec{b} = \hat{i} + 2\hat{j} + \hat{k}.$$

If $\vec{c} \cdot (\hat{i} + \hat{j} + 3\hat{k}) = 8$ then the value of $\vec{c} \cdot (\vec{a} \times \vec{b})$ is equal to _____.

Official Ans by NTA (28)

Sol. $\vec{c} = \lambda(\vec{a} \times \vec{b})$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & -1 \\ 1 & 2 & 1 \end{vmatrix}$$

$$(\vec{a} \times \vec{b}) = 3\hat{i} - 2\hat{j} + \hat{k}$$

$$\vec{c} \cdot (\hat{i} + \hat{j} + 3\hat{k}) = \lambda(3\hat{i} - 2\hat{j} + \hat{k}) \cdot (\hat{i} + \hat{j} + 3\hat{k})$$

$$\Rightarrow \lambda(4) = 8 \Rightarrow \lambda = 2$$

$$\vec{c} = 2(\vec{a} \times \vec{b})$$

$$\vec{c} \cdot (\vec{a} \times \vec{b}) = 2 |\vec{a} \times \vec{b}|^2 = 28$$

10. Let

$$S_n(x) = \log_{a^{1/2}} x + \log_{a^{1/3}} x + \log_{a^{1/6}} x$$

$$+ \log_{a^{1/11}} x + \log_{a^{1/18}} x + \log_{a^{1/27}} x + \dots$$

up to n-terms, where $a > 1$. If $S_{24}(x) = 1093$ and $S_{12}(2x) = 265$, then value of a is equal to

Official Ans by NTA (16)

Sol. $S_n(x) = (2+3+6+11+18+27+\dots+n\text{-terms})\log_a x$

$$\text{Let } S_1 = 2 + 3 + 6 + 11 + 18 + 27 + \dots + T_n$$

$$S_1 = 2 + 3 + 6 + \dots + T_n$$

$$T_n = 2 + 1 + 3 + 5 + \dots + n \text{ terms}$$

$$T_n = 2 + (n-1)^2$$

$$S_1 = \Sigma T_n = 2n + \frac{(n-1)n(2n-1)}{6}$$

$$\Rightarrow S_n(x) = \left(2n + \frac{n(n-1)(2n-1)}{6} \right) \log_a x$$

$$S_{24}(x) = 1093 \text{ (Given)}$$

$$\log_a x \left(48 + \frac{23 \cdot 24 \cdot 47}{6} \right) = 1093$$

$$\log_a x = \frac{1}{4} \dots (1)$$

$$S_{12}(2x) = 265$$

$$S_{12}(2x) = 265$$

$$\log_a(2x) \left(24 + \frac{11 \cdot 12 \cdot 23}{6} \right) = 265$$

$$\log_a 2x = \frac{1}{2} \dots (2)$$

$$(2) - (1)$$

$$\log_a 2x - \log_a x = \frac{1}{4}$$

$$\log_a 2 = \frac{1}{4} \Rightarrow a = 16$$