UP Board Class 12 Mathematics - 324(JD) - 2025 Question Paper with Solutions

Time Allowed : 3 Hours	Maximum Marks : 100	Total Questions : 9

General Instructions

Read the following instructions very carefully and strictly follow them:

1. There are in all nine questions in this question paper.
2. All questions are compulsory.
3. In the beginning of each question, the number of parts to be attempted are clearly mentioned.
4. Marks allotted to the questions are indicated against them.
5. Start solving from the first question and proceed to solve till the last one. Do not waste your time over a question you cannot solve.

1. Do all parts.
Select the correct option of each part and write it on your answer-book.

a. If the function $f : \mathbb{R} \to \mathbb{R}$ is defined as $f(x) = 3x$ then f is

(i) one-one and onto
(ii) many-one and onto
(iii) one-one but not onto
(iv) neither one-one nor onto

Correct Answer: (i) one-one and onto

Solution:

Step 1: Understanding the Concepts:
To classify the function $f(x) = 3x$ as one-one (injective) and onto (surjective), we need to examine the following:
- A function is **one-one (injective)** if no two distinct elements in the domain map to the same element in the codomain. Mathematically, $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$. - A function is **onto (surjective)** if for every element in the codomain, there exists an element in the domain that maps to it. In other words, for every $y \in \mathbb{R}$, there exists an $x \in \mathbb{R}$ such that $f(x) = y$.

Step 2: Detailed Explanation:
Checking for Injectivity (One-one):
To verify if $f(x) = 3x$ is injective, we need to check whether distinct inputs always lead to distinct outputs.
Assume $f(x_1) = f(x_2)$ for some $x_1, x_2 \in \mathbb{R}$, i.e.,

$$3x_1 = 3x_2$$

Now, divide both sides of the equation by 3:

$$x_1 = x_2$$

Since $f(x_1) = f(x_2)$ implies that $x_1 = x_2$, we conclude that the function is **one-one**. No two different values of x map to the same value of $f(x)$.

Checking for Surjectivity (Onto):
To check if $f(x) = 3x$ is onto, we need to verify whether for every $y \in \mathbb{R}$ (the codomain), there exists an $x \in \mathbb{R}$ (the domain) such that $f(x) = y$.
Let y be an arbitrary element in the codomain. We need to find an x such that:

$$f(x) = y$$

Substituting the expression for $f(x)$:

$$3x = y$$

Solving for x:

$$x = \frac{y}{3}$$

Since $y \in \mathbb{R}$, it follows that $x = \frac{y}{3}$ is also a real number. Therefore, for every $y \in \mathbb{R}$, there exists an $x \in \mathbb{R}$ (specifically $x = \frac{y}{3}$) such that $f(x) = y$. Thus, the function is **onto**.

Step 3: Conclusion:
Since the function $f(x) = 3x$ satisfies both conditions of being injective (one-one) and surjective (onto), we conclude that it is **both one-one and onto**. Therefore, the correct answer is:

$\boxed{\textbf{(i) one-one and onto}}$.

Quick Tip

Linear functions of the form $f(x) = ax + b$ with $a \neq 0$ are always bijective (one-one and onto) when the domain and codomain are $\mathbb{R}$. Recognizing this pattern can provide a quick answer.

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b. If $X = \{a, b, c\}$ and $Y = \{1, 2, 3\}$ and the mapping f is given by $f(a) = 2, f(b) = 3, f(c) = 1$, then

(i) $f(X) \subset Y$
(ii) $f(X) = Y$
(iii) $f(X) \supset Y$
(iv) All of these

Correct Answer: (iv) All of these

Solution:

Step 1: Understanding the Concept:
We are given a function $f : X \to Y$ where $X = \{a, b, c\}$ is the domain and $Y = \{1, 2, 3\}$ is the codomain. We need to determine the relationship between the range of the function, denoted by $f(X)$, and the codomain Y. The range of the function is the set of all distinct outputs that correspond to the elements in the domain X.

Step 2: Detailed Explanation:
The given mapping is:

$$f(a) = 2, \quad f(b) = 3, \quad f(c) = 1$$

This means the function f maps the element a to 2, the element b to 3, and the element c to 1. Now, we find the range of the function, denoted as $f(X)$, which is the set of all values that f takes:

$$f(X) = \{f(a), f(b), f(c)\} = \{2, 3, 1\}$$

Rearranging the elements in increasing order gives:

$$f(X) = \{1, 2, 3\}$$

Thus, the range of the function $f(X)$ is the set $\{1, 2, 3\}$, which is the same as the codomain Y.
Step 3: Analyzing the Options:
Based on the fact that $f(X) = Y$, let's analyze each option:
- **(i)** $f(X) \subset Y$**:** This means "the range of f is a subset of Y." Since every element of $f(X)$ is also an element of Y, this statement is true. Furthermore, a set is always a subset of itself, so $f(X) \subset Y$ holds.
- **(ii)** $f(X) = Y$**:** This means "the range of f is equal to Y." Since we have already established that $f(X) = Y$, this statement is true.
- **(iii)** $f(X) \supset Y$**:** This means "the range of f is a superset of Y." Since every element of Y is also an element of $f(X)$, this statement is true.

Since all three statements (i), (ii), and (iii) are correct, the correct option is (iv) All of these.

Step 4: Final Answer:
The range of the function is equal to the codomain, which makes all three relationships true. Therefore, the correct option is (iv).

$\boxed{\textbf{(iv) All of these}}$.

Quick Tip
Remember the set theory definitions: - $A \subseteq B$ (A is a subset of B) if every element of A is in B. - $A = B$ if $A \subseteq B$ and $B \subseteq A$. By this definition, if $A = B$, then $A \subseteq B$ and $A \supseteq B$ are both true.

c. If $\int \log x \, dx = x \log x + k(x) + c$ then

(i) $k(x) = \log x$
(ii) $k(x) = -\log x$
(iii) $k(x) = -x$
(iv) $k(x) = -x^2$

Correct Answer: (iii) $k(x) = -x$

Solution:

Step 1: Understanding the Concept:
We are tasked with evaluating the indefinite integral of $\log x$ and determining the function $k(x)$ that satisfies the equation. The integral of a logarithmic function requires using the method of integration by parts.

Step 2: Key Formula or Approach:
To evaluate $\int \log x \, dx$, we apply the integration by parts formula:

$$\int u \, dv = uv - \int v \, du$$

In this case, we need to identify u and dv in the expression $\int \log x \, dx$.

Step 3: Detailed Explanation:
To begin, we rewrite the integral as:

$$\int \log x \cdot 1 \, dx$$

Using the ILATE rule (Inverse, Logarithmic, Algebraic, Trigonometric, Exponential) for choosing u and dv, we let:

$$u = \log x, \quad dv = 1 \, dx$$

Next, we compute du and v:

$$du = \frac{1}{x} \, dx, \quad v = \int 1 \, dx = x$$

Now, we apply the integration by parts formula:

$$\int \log x \, dx = uv - \int v \, du$$

Substituting the values of u, v, and du:

$$\int \log x \, dx = (\log x)(x) - \int x \left(\frac{1}{x}\right) \, dx$$

Simplifying:

$$\int \log x \, dx = x \log x - \int 1 \, dx$$

The integral of 1 with respect to x is simply x, so:

$$\int \log x \, dx = x \log x - x + c$$

We can now compare this result with the given expression:

$$x \log x + k(x) + c$$

By directly comparing the two expressions, we find that:

$$k(x) = -x$$

Step 4: Final Answer:
Thus, the function $k(x)$ is $-x$. Therefore, the correct option is (iii).

$\boxed{\textbf{(iii)}\ k(x) = -x}$.

Quick Tip

The integral of $\log x$ is a standard result that is very useful to memorize for competitive exams: $\int \log x \, dx = x \log x - x + c$. Knowing this can save you the time of performing integration by parts.

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d. The number of arbitrary constants in a general solution of a differential equation of fifth order is

(i) 0
(ii) 4
(iii) 3
(iv) 5

Correct Answer: (iv) 5

Solution:

Step 1: Understanding the Concept:
The order of a differential equation refers to the highest derivative that appears in the equation. For example, in a second-order differential equation, the highest derivative would be the second derivative, and similarly, for higher-order equations, the order is determined by the highest derivative.

The general solution to a differential equation is a solution that contains arbitrary constants, which arise from the integration process. These constants are determined when specific initial or boundary conditions are applied.

Step 2: Detailed Explanation:
A key concept in solving ordinary differential equations (ODEs) is that the general solution to

an n-th order differential equation contains exactly n independent arbitrary constants. This is a direct result of the process of integrating the equation. Each integration introduces one constant of integration.

- For a first-order differential equation, there will be one constant in the solution.
- For a second-order differential equation, there will be two constants in the solution, and so on.

The number of arbitrary constants corresponds to the number of integrations required to solve the equation. In this problem, the given differential equation is of the fifth order. Thus, solving this differential equation would involve performing five integrations, each contributing one arbitrary constant.
Therefore, the general solution of a fifth-order differential equation contains five arbitrary constant.

Step 3: Final Answer:
The number of arbitrary constants is equal to the order of the differential equation, which in this case is 5. Therefore, the correct option is (iv).

$\boxed{\textbf{(iv) 5}}$.

Quick Tip

This is a direct-knowledge question. Remember this simple rule: Number of arbitrary constants in the general solution = Order of the differential equation. A "particular solution" is obtained by finding specific values for these constants and has zero arbitrary constants.

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e. At $t = 2$, the slope of the vector function $\vec{f}(t) = 2\hat{i} + 3\hat{j} + 5t^2\hat{k}$ is

(i) $20\hat{k}$
(ii) $10\hat{k}$
(iii) $5\hat{k}$
(iv) $12\hat{k}$

Correct Answer: (i) $20\hat{k}$

Solution:

Step 1: Understanding the Concept:
The "slope" of a vector function at a particular point refers to the tangent vector at that point. The tangent vector is found by taking the derivative of the vector function with respect to its parameter (in this case, t).

Step 2: Key Formula or Approach:
The tangent vector (or slope) is given by the derivative $\frac{d\vec{f}}{dt}$.

We need to:
1. Differentiate the vector function $\vec{f}(t)$ with respect to t.
2. Evaluate the resulting derivative vector at $t = 2$.

Step 3: Detailed Explanation:
The given vector function is:

$$\vec{f}(t) = 2\hat{i} + 3\hat{j} + 5t^2\hat{k}$$

We differentiate each component of the vector function with respect to t:

$$\frac{d\vec{f}}{dt} = \frac{d}{dt}(2\hat{i}) + \frac{d}{dt}(3\hat{j}) + \frac{d}{dt}(5t^2\hat{k})$$

The derivatives of the constant components are zero:

$$\frac{d\vec{f}}{dt} = 0\hat{i} + 0\hat{j} + (2 \cdot 5t^{2-1})\hat{k}$$

$$\frac{d\vec{f}}{dt} = 10t\hat{k}$$

This is the general expression for the tangent vector. Now, we evaluate it at $t = 2$:

$$\left.\frac{d\vec{f}}{dt}\right|_{t=2} = 10(2)\hat{k} = 20\hat{k}$$

Step 4: Final Answer:
The slope of the vector function at $t = 2$ is $20\hat{k}$. Therefore, the correct option is (i).

Quick Tip

For vector functions $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$, the velocity vector is $\vec{v}(t) = \vec{r}'(t) = x'(t)\hat{i} + y'(t)\hat{j} + z'(t)\hat{k}$, which is tangent to the path of motion. The term "slope" in this context is another way of asking for this tangent/velocity vector.

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2. Do all parts.

a. Prove that $\sin^{-1} x = \tan^{-1}[x/\sqrt{1 - x^2}]$**.**

Solution:

Step 1: Understanding the Concept:
We can prove this identity by converting one inverse trigonometric function to another. A common method is to use a right-angled triangle to represent the relationship given by the first function and then derive the second function from the triangle's dimensions.

Step 2: Key Formula or Approach:
1. Let $\theta = \sin^{-1} x$. This implies $\sin\theta = x$.
2. Construct a right-angled triangle where $\sin\theta = \frac{\text{Opposite}}{\text{Hypotenuse}} = \frac{x}{1}$.
3. Use the Pythagorean theorem to find the length of the adjacent side.
4. From the triangle, find $\tan\theta$ and then express θ in terms of $\tan^{-1}$.

Step 3: Detailed Explanation:
Let $\theta = \sin^{-1} x$. Then, by definition, $\sin\theta = x$. We can write this as $\sin\theta = \frac{x}{1}$.
Consider a right-angled triangle with angle θ. We can set:

- Opposite side $= x$
- Hypotenuse $= 1$

Using the Pythagorean theorem ($\text{Opposite}^2 + \text{Adjacent}^2 = \text{Hypotenuse}^2$):

$$x^2 + (\text{Adjacent})^2 = 1^2$$

$$(\text{Adjacent})^2 = 1 - x^2$$

$$\text{Adjacent} = \sqrt{1 - x^2}$$

Now, we can find the value of $\tan\theta$ from this triangle:

$$\tan\theta = \frac{\text{Opposite}}{\text{Adjacent}} = \frac{x}{\sqrt{1 - x^2}}$$

Taking the inverse tangent of both sides, we get:

$$\theta = \tan^{-1}\left(\frac{x}{\sqrt{1 - x^2}}\right)$$

Since we started with $\theta = \sin^{-1} x$, we have proven that:

$$\sin^{-1} x = \tan^{-1}\left(\frac{x}{\sqrt{1 - x^2}}\right)$$

Step 4: Final Answer:
The identity is proven by equating the two expressions for θ.

Quick Tip

Drawing a simple right-angled triangle is the fastest and most intuitive way to handle conversions between inverse trigonometric functions. This geometric approach helps avoid algebraic mistakes.

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b. Find the unit vector along the vector $\vec{a} = 2\hat{i} + 3\hat{j} + \hat{k}$.

Correct Answer: $\frac{1}{\sqrt{14}}(2\hat{i} + 3\hat{j} + \hat{k})$

Solution:

Step 1: Understanding the Concept:
A unit vector is a vector with a magnitude (or length) of 1. To find the unit vector in the direction of a given vector, we divide the vector by its magnitude.

Step 2: Key Formula or Approach:
The formula for the unit vector $\hat{a}$ in the direction of vector $\vec{a}$ is:

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|}$$

where $|\vec{a}|$ is the magnitude of $\vec{a}$.

Step 3: Detailed Explanation:
The given vector is $\vec{a} = 2\hat{\imath} + 3\hat{\jmath} + \hat{k}$.
First, we calculate the magnitude of $\vec{a}$:

$$|\vec{a}| = \sqrt{(2)^2 + (3)^2 + (1)^2}$$

$$|\vec{a}| = \sqrt{4 + 9 + 1} = \sqrt{14}$$

Now, we divide the vector $\vec{a}$ by its magnitude $|\vec{a}|$ to get the unit vector $\hat{a}$:

$$\hat{a} = \frac{2\hat{\imath} + 3\hat{\jmath} + \hat{k}}{\sqrt{14}}$$

This can also be written as:

$$\hat{a} = \frac{2}{\sqrt{14}}\hat{\imath} + \frac{3}{\sqrt{14}}\hat{\jmath} + \frac{1}{\sqrt{14}}\hat{k}$$

Step 4: Final Answer:
The unit vector along $\vec{a}$ is $\frac{1}{\sqrt{14}}(2\hat{\imath} + 3\hat{\jmath} + \hat{k})$.

> Quick Tip
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> Always double-check your calculation of the magnitude, as it is a common place for errors. The components of the final unit vector are the direction cosines of the original vector.

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c. Find the direction-cosines of the sum of the vectors $\vec{a} = 3\hat{\imath} + 4\hat{\jmath} - 3\hat{k}$ and $\vec{b} = -2\hat{\imath} - 3\hat{\jmath} + \hat{k}$.

Correct Answer: $\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}\right)$

Solution:

Step 1: Understanding the Concept:
Direction cosines of a vector are the cosines of the angles the vector makes with the positive x, y, and z axes. For a vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, the direction cosines are $\frac{x}{|\vec{r}|}$, $\frac{y}{|\vec{r}|}$, and $\frac{z}{|\vec{r}|}$. We first need to find the sum of the given vectors.

Step 2: Key Formula or Approach:
1. Calculate the resultant vector $\vec{s} = \vec{a} + \vec{b}$.
2. Find the magnitude of the resultant vector, $|\vec{s}|$.
3. The direction cosines (l, m, n) are the components of the unit vector in the direction of $\vec{s}$.

Step 3: Detailed Explanation:
Given vectors are:

$$\vec{a} = 3\hat{i} + 4\hat{j} - 3\hat{k}$$
$$\vec{b} = -2\hat{i} - 3\hat{j} + \hat{k}$$

First, find the sum $\vec{s} = \vec{a} + \vec{b}$:

$$\vec{s} = (3-2)\hat{i} + (4-3)\hat{j} + (-3+1)\hat{k}$$
$$\vec{s} = 1\hat{i} + 1\hat{j} - 2\hat{k}$$

Next, find the magnitude of $\vec{s}$:

$$|\vec{s}| = \sqrt{(1)^2 + (1)^2 + (-2)^2}$$
$$|\vec{s}| = \sqrt{1+1+4} = \sqrt{6}$$

Now, calculate the direction cosines:

$$l = \frac{x}{|\vec{s}|} = \frac{1}{\sqrt{6}}$$
$$m = \frac{y}{|\vec{s}|} = \frac{1}{\sqrt{6}}$$
$$n = \frac{z}{|\vec{s}|} = \frac{-2}{\sqrt{6}}$$

Step 4: Final Answer:
The direction-cosines of the sum of the vectors are $\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}\right)$.

> Quick Tip
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> Finding the direction cosines is equivalent to finding the components of the unit vector along the given vector. Both processes involve dividing the vector's components by its magnitude.

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d. If the functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are defined as $f(x) = \cos x$ and $g(x) = 3x^2$ respectively then find gof.

Correct Answer: $(g \circ f)(x) = 3\cos^2 x$

Solution:

Step 1: Understanding the Concept:
The notation 'gof' represents the composition of functions, which is read as "g of f". It means we first apply the function f to x, and then apply the function g to the result of f(x).

Step 2: Key Formula or Approach:
The composition of functions $g \circ f$ is defined as:

$$(g \circ f)(x) = g(f(x))$$

Step 3: Detailed Explanation:
We are given the functions:

$$f(x) = \cos x$$
$$g(x) = 3x^2$$

To find $(g \circ f)(x)$, we substitute $f(x)$ into $g(x)$:

$$(g \circ f)(x) = g(f(x)) = g(\cos x)$$

Now, we apply the function g to the input $\cos x$. The rule for g is to take the input, square it, and multiply by 3.

$$g(\cos x) = 3(\cos x)^2 = 3\cos^2 x$$

Step 4: Final Answer:
The composite function is $(g \circ f)(x) = 3\cos^2 x$.

Quick Tip

Be careful with the order of composition. $g \circ f$ is not the same as $f \circ g$. For example, $(f \circ g)(x) = f(g(x)) = f(3x^2) = \cos(3x^2)$, which is a different function.

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e. If two dice are thrown together then find the probability of getting the sum eight.

Correct Answer: $\frac{5}{36}$

Solution:

Step 1: Understanding the Concept:
This is a problem of classical probability. The probability of an event is the ratio of the number of outcomes favorable to the event to the total number of possible outcomes in the sample space.

Step 2: Key Formula or Approach:

$$\text{Probability} = \frac{\text{Number of favorable outcomes}}{\text{Total number of outcomes}}$$

Step 3: Detailed Explanation:
When two fair six-sided dice are thrown, each die has 6 possible outcomes. The total number of possible outcomes in the sample space is:

$$\text{Total outcomes} = 6 \times 6 = 36$$

We want to find the probability of getting a sum of eight. Let's list all the pairs of outcomes (die 1, die 2) that sum to 8:

$$(2, 6)$$
$$(3, 5)$$
$$(4, 4)$$
$$(5, 3)$$
$$(6, 2)$$

There are 5 such pairs. So, the number of favorable outcomes is 5.
Now, we can calculate the probability:

$$P(\text{sum is } 8) = \frac{5}{36}$$

Step 4: Final Answer:
The probability of getting the sum eight is $\frac{5}{36}$.

> 💡 Quick Tip
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> For problems involving two dice, it can be helpful to visualize or quickly sketch a 6x6 grid of all possible sums. This helps ensure you don't miss any favorable outcomes, especially for more complex conditions.

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3. Do all parts.

a. If the position vectors of the points A and B are $\hat{i} + \hat{j} + \hat{k}$ and $2\hat{i} + 5\hat{j}$ respectively, then find the unit vector along the straight line AB.

Correct Answer: $\frac{1}{\sqrt{18}}(\hat{i} + 4\hat{j} - \hat{k})$ or $\frac{1}{3\sqrt{2}}(\hat{i} + 4\hat{j} - \hat{k})$

Solution:

Step 1: Understanding the Concept:
To find the unit vector along the line AB, we first need to find the vector $\vec{AB}$ by subtracting the position vector of the initial point A from the position vector of the terminal point B. Then, we find the unit vector by dividing $\vec{AB}$ by its magnitude.

Step 2: Key Formula or Approach:
1. Let the position vectors of A and B be $\vec{OA}$ and $\vec{OB}$ respectively.

2. The vector along the line is $\vec{AB} = \vec{OB} - \vec{OA}$.
3. The unit vector is $\widehat{AB} = \frac{\vec{AB}}{|\vec{AB}|}$.

Step 3: Detailed Explanation:
We are given the position vectors:

$$\vec{OA} = \hat{i} + \hat{j} + \hat{k}$$
$$\vec{OB} = 2\hat{i} + 5\hat{j} + 0\hat{k}$$

First, calculate the vector $\vec{AB}$:

$$\vec{AB} = (2\hat{i} + 5\hat{j}) - (\hat{i} + \hat{j} + \hat{k})$$
$$\vec{AB} = (2-1)\hat{i} + (5-1)\hat{j} + (0-1)\hat{k}$$
$$\vec{AB} = 1\hat{i} + 4\hat{j} - 1\hat{k}$$

Next, calculate the magnitude of $\vec{AB}$:

$$|\vec{AB}| = \sqrt{1^2 + 4^2 + (-1)^2}$$
$$|\vec{AB}| = \sqrt{1 + 16 + 1} = \sqrt{18} = 3\sqrt{2}$$

Finally, find the unit vector by dividing $\vec{AB}$ by its magnitude:

$$\widehat{AB} = \frac{\hat{i} + 4\hat{j} - \hat{k}}{\sqrt{18}}$$
$$\widehat{AB} = \frac{1}{\sqrt{18}}\hat{i} + \frac{4}{\sqrt{18}}\hat{j} - \frac{1}{\sqrt{18}}\hat{k}$$

Step 4: Final Answer:
The unit vector along the straight line AB is $\frac{1}{\sqrt{18}}(\hat{i} + 4\hat{j} - \hat{k})$.

> Quick Tip
>
> Remember that the vector from point A to point B is always "Position Vector of B - Position Vector of A". A common mistake is to subtract in the wrong order. Always think "final minus initial".

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2. If the relation R is given by $R = \{(4,5), (1,4), (4,6), (7,6), (3,7)\}$, then find $R^{-1} \circ R^{-1}$.

Correct Answer: $\{(5,1), (6,1), (6,3)\}$

Solution:

Step 1: Understanding the Concept:
First, we need to find the inverse relation, R^{-1}, which is obtained by swapping the first and second elements of each ordered pair in R. Second, we need to find the composition $R^{-1} \circ R^{-1}$.

An ordered pair (a, c) belongs to this composition if there exists an element 'b' such that (a, b) is in R^{-1} and (b, c) is in R^{-1}.

Step 2: Key Formula or Approach:
1. Find R^{-1} from R. 2. For each pair $(a, b) \in R^{-1}$, search for all pairs $(b, c) \in R^{-1}$. 3. The resulting pairs (a, c) form the set $R^{-1} \circ R^{-1}$.

Step 3: Detailed Explanation:
The given relation is:

$$R = \{(4, 5), (1, 4), (4, 6), (7, 6), (3, 7)\}$$

First, find the inverse relation R^{-1} by swapping the elements in each pair:

$$R^{-1} = \{(5, 4), (4, 1), (6, 4), (6, 7), (7, 3)\}$$

Now we find the composition $R^{-1} \circ R^{-1}$ by finding "chains" of length two within R^{-1}.

- We have $(5, 4) \in R^{-1}$ and $(4, 1) \in R^{-1}$. This gives us the pair $(\mathbf{5}, \mathbf{1})$.
- We have $(4, 1) \in R^{-1}$, but there are no pairs in R^{-1} that start with 1. So this chain ends.
- We have $(6, 4) \in R^{-1}$ and $(4, 1) \in R^{-1}$. This gives us the pair $(\mathbf{6}, \mathbf{1})$.
- We have $(6, 7) \in R^{-1}$ and $(7, 3) \in R^{-1}$. This gives us the pair $(\mathbf{6}, \mathbf{3})$.
- We have $(7, 3) \in R^{-1}$, but there are no pairs in R^{-1} that start with 3. So this chain ends.

Combining the resulting pairs, we get the final relation.

Step 4: Final Answer:
The composition $R^{-1} \circ R^{-1}$ is $\{(5, 1), (6, 1), (6, 3)\}$.

> Quick Tip
>
> To find the composition $S \circ T$, think of it as a path. An element (a, c) is in the composition if you can go from 'a' to 'b' using relation T, and then from 'b' to 'c' using relation S. In this problem, both T and S are R^{-1}.

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3. Find the differential equation representing the family of curves $y = 2mx$.

Correct Answer: $x\frac{dy}{dx} - y = 0$

Solution:

Step 1: Understanding the Concept:
To find the differential equation for a family of curves, we need to eliminate the arbitrary constant(s) from the given equation. Since there is one arbitrary constant ('m') in this equation, we will need to differentiate the equation once to get a second equation, and then use the two

equations to eliminate 'm'.

Step 2: Key Formula or Approach:
1. Differentiate the given equation with respect to x. 2. From the original equation, express the constant in terms of x and y. 3. Substitute this expression for the constant into the differentiated equation.

Step 3: Detailed Explanation:
The given equation for the family of curves is:

$$y = 2mx \quad —(1)$$

From equation (1), we can express the term with the constant 'm' as:

$$2m = \frac{y}{x} \quad —(2)$$

Now, differentiate the original equation (1) with respect to x:

$$\frac{d}{dx}(y) = \frac{d}{dx}(2mx)$$

$$\frac{dy}{dx} = 2m \quad —(3)$$

Now we have an expression for $2m$ in terms of the derivative, and another in terms of x and y. We can eliminate $2m$ by substituting equation (2) into equation (3):

$$\frac{dy}{dx} = \frac{y}{x}$$

To write this in a standard form, we can multiply both sides by x:

$$x\frac{dy}{dx} = y$$

$$x\frac{dy}{dx} - y = 0$$

Step 4: Final Answer:
The required differential equation is $x\frac{dy}{dx} - y = 0$.

Quick Tip

The number of arbitrary constants in the equation of a family of curves determines the order of the resulting differential equation. One constant means a first-order differential equation, two constants mean a second-order, and so on.

---

4. If $A = \begin{bmatrix} 3 & \sqrt{3} & 2 \\ 4 & 2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1/4 \\ 0 & 0 \\ 1/2 & 1/8 \end{bmatrix}$, then prove that $|C| = 1$, where $C = (A')'B$.

Solution:

Step 1: Understanding the Concept:
This problem involves matrix operations, including transpose, multiplication, and finding the determinant. The key is to first simplify the expression for matrix C and then perform the necessary calculations to find its determinant.

Step 2: Key Formula or Approach:
1. Simplify the expression for C using the matrix property $(M')' = M$. 2. Perform the matrix multiplication to find the elements of C. 3. Calculate the determinant of the resulting 2x2 matrix C using the formula $|C| = ad - bc$.

Step 3: Detailed Explanation:
We are given the definition of C as:

$$C = (A')'B$$

We use the property that the transpose of a transpose of a matrix is the matrix itself, i.e., $(A')' = A$. So, the expression for C simplifies to:

$$C = AB$$

Now we need to multiply matrix A by matrix B:

$$C = \begin{bmatrix} 3 & \sqrt{3} & 2 \\ 4 & 2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1/4 \\ 0 & 0 \\ 1/2 & 1/8 \end{bmatrix}$$

The resulting matrix C will be a 2x2 matrix. Let's calculate its elements:

- $C_{11} = (3)(0) + (\sqrt{3})(0) + (2)(1/2) = 0 + 0 + 1 = 1$
- $C_{12} = (3)(1/4) + (\sqrt{3})(0) + (2)(1/8) = \frac{3}{4} + 0 + \frac{2}{8} = \frac{3}{4} + \frac{1}{4} = 1$
- $C_{21} = (4)(0) + (2)(0) + (0)(1/2) = 0 + 0 + 0 = 0$
- $C_{22} = (4)(1/4) + (2)(0) + (0)(1/8) = 1 + 0 + 0 = 1$

So, the matrix C is:

$$C = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Finally, we calculate the determinant of C:

$$|C| = (1)(1) - (1)(0) = 1 - 0 = 1$$

Step 4: Final Answer:
The determinant of C is 1. Hence proved.

Quick Tip

Before diving into calculations with matrices, always check if there are any properties you can use to simplify the expression. Recognizing that $(A')' = A$ is the key to making this problem much simpler.

4. Do all parts.

a. Find the vector equation of a straight line passing through the point (5, 2, -4) and parallel to the vector $3\hat{i} + 2\hat{j} - 8\hat{k}$.

Correct Answer: $\vec{r} = (5\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(3\hat{i} + 2\hat{j} - 8\hat{k})$

Solution:

Step 1: Understanding the Concept:
The vector equation of a line describes the position vector $\vec{r}$ of any point on the line. It is determined by a known point on the line and a direction vector that is parallel to the line.

Step 2: Key Formula or Approach:
The vector equation of a line passing through a point with position vector $\vec{a}$ and parallel to a vector $\vec{b}$ is given by:

$$\vec{r} = \vec{a} + \lambda\vec{b}$$

where λ is a scalar parameter.

Step 3: Detailed Explanation:
From the problem statement, we identify the position vector of the given point and the parallel vector.
The line passes through the point (5, 2, -4). The position vector $\vec{a}$ for this point is:

$$\vec{a} = 5\hat{i} + 2\hat{j} - 4\hat{k}$$

The line is parallel to the vector $3\hat{i} + 2\hat{j} - 8\hat{k}$. This is our direction vector $\vec{b}$:

$$\vec{b} = 3\hat{i} + 2\hat{j} - 8\hat{k}$$

Now, we substitute $\vec{a}$ and $\vec{b}$ into the standard formula:

$$\vec{r} = (5\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(3\hat{i} + 2\hat{j} - 8\hat{k})$$

This equation represents all points on the line for different values of λ.

Step 4: Final Answer:
The required vector equation of the straight line is $\vec{r} = (5\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(3\hat{i} + 2\hat{j} - 8\hat{k})$.

Quick Tip

This is a direct application of a fundamental formula. Ensure you can distinguish between the position vector of the point ($\vec{a}$) and the direction vector ($\vec{b}$). The direction vector is always multiplied by the scalar parameter.

b. Without cover a box is formed by 6 m x 16 m rectangular steel sheet on cutting the squares of length x m from its each corner. Then find the maximum volume of the box.

Correct Answer: $\frac{1600}{27}$ m^3

Solution:

Step 1: Understanding the Concept:
This is an optimization problem where we need to find the maximum volume of an open-top box. The volume is expressed as a function of the side 'x' of the square cutouts, and calculus is used to find the maximum value of this function.

Step 2: Key Formula or Approach:
1. Express the dimensions (length, width, height) of the box in terms of x.
2. Formulate the volume function, $V(x)$.
3. Find the derivative $V'(x)$ and set it to zero to find critical points.
4. Use the second derivative test, $V''(x)$, to confirm that the critical point corresponds to a maximum.

Step 3: Detailed Explanation:
The original sheet has dimensions L = 16 m and W = 6 m.
When squares of side x are cut from each corner, the dimensions of the folded box become:

- Height: $h = x$
- Length: $l = 16 - 2x$
- Width: $w = 6 - 2x$

The volume $V(x)$ is given by $l \cdot w \cdot h$:

$$V(x) = (16 - 2x)(6 - 2x)x = (96 - 32x - 12x + 4x^2)x$$

$$V(x) = 4x^3 - 44x^2 + 96x$$

For the dimensions to be physically possible, $x > 0$, $6 - 2x > 0 \Rightarrow x < 3$, and $16 - 2x > 0 \Rightarrow x < 8$. The valid domain for x is $0 < x < 3$.
Find the first derivative to locate critical points:

$$V'(x) = 12x^2 - 88x + 96$$

Set $V'(x) = 0$:

$$12x^2 - 88x + 96 = 0$$

Divide by 4:

$$3x^2 - 22x + 24 = 0$$

Using the quadratic formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$:

$$x = \frac{22 \pm \sqrt{(-22)^2 - 4(3)(24)}}{2(3)} = \frac{22 \pm \sqrt{484 - 288}}{6} = \frac{22 \pm \sqrt{196}}{6} = \frac{22 \pm 14}{6}$$

This gives two possible values: $x = \frac{36}{6} = 6$ and $x = \frac{8}{6} = \frac{4}{3}$.
Within our domain $0 < x < 3$, the only valid critical point is $x = \frac{4}{3}$.
Use the second derivative test to confirm it's a maximum:

$$V''(x) = 24x - 88$$

$$V''\left(\frac{4}{3}\right) = 24\left(\frac{4}{3}\right) - 88 = 32 - 88 = -56$$

Since $V'' < 0$, the volume is maximized at $x = \frac{4}{3}$.
Calculate the maximum volume:

$$V\left(\frac{4}{3}\right) = \left(16 - 2\left(\frac{4}{3}\right)\right)\left(6 - 2\left(\frac{4}{3}\right)\right)\left(\frac{4}{3}\right) = \left(\frac{48-8}{3}\right)\left(\frac{18-8}{3}\right)\left(\frac{4}{3}\right)$$

$$V\left(\frac{4}{3}\right) = \left(\frac{40}{3}\right)\left(\frac{10}{3}\right)\left(\frac{4}{3}\right) = \frac{1600}{27}$$

Step 4: Final Answer:
The maximum volume of the box is $\frac{1600}{27}$ cubic meters.

> **Quick Tip**
>
> Always establish the physical constraints (domain) of the variable in optimization problems. This often helps in discarding extraneous mathematical solutions obtained from setting the derivative to zero.

---

c. Let A and B are independent events and P(A)=0.3 and P(B)=0.4 ; then find P(B/A).

Correct Answer: 0.4

Solution:

Step 1: Understanding the Concept:
This question deals with conditional probability for independent events. Two events are independent if the occurrence of one does not affect the probability of the other. The notation $P(B|A)$ represents the conditional probability of event B occurring given that event A has already occurred.

Step 2: Key Formula or Approach:
The definition of independent events A and B is that $P(A \cap B) = P(A) \cdot P(B)$.
The formula for conditional probability is $P(B|A) = \frac{P(A \cap B)}{P(A)}$.
For independent events, this simplifies directly to $P(B|A) = P(B)$.

Step 3: Detailed Explanation:
Given that events A and B are independent. By the definition of independence, the probability

of B occurring is not affected by whether A has occurred or not. Therefore, the probability of B given A is simply the probability of B.

$$P(B|A) = P(B)$$

We are given that $P(B) = 0.4$. So,

$$P(B|A) = 0.4$$

Alternative Method (using formula): First, calculate $P(A \cap B)$ using the independence rule:

$$P(A \cap B) = P(A) \cdot P(B) = (0.3)(0.4) = 0.12$$

Now, use the conditional probability formula:

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{0.12}{0.3} = \frac{12}{30} = \frac{4}{10} = 0.4$$

Both methods yield the same result.

Step 4: Final Answer:
The value of $P(B|A)$ is 0.4.

> **Quick Tip**
>
> This is a conceptual question. Recognizing that $P(B|A) = P(B)$ for independent events provides the answer instantly without any calculation. This is a very common question type in probability.

---

d. Find the area of a triangle whose vertices are A(2, 2, 2), B(2, 1, 3) and C(3, 2, 1).

Correct Answer: $\frac{\sqrt{3}}{2}$ square units

Solution:

Step 1: Understanding the Concept:
The area of a triangle in three-dimensional space with given vertices can be found using the cross product of two vectors that represent adjacent sides of the triangle. The magnitude of this cross product is equal to the area of the parallelogram formed by these vectors, and the triangle's area is half of that.

Step 2: Key Formula or Approach:
The area of $\triangle ABC$ is given by the formula:

$$\text{Area} = \frac{1}{2}|\vec{AB} \times \vec{AC}|$$

The steps are: 1. Find the vectors for two adjacent sides, e.g., $\vec{AB}$ and $\vec{AC}$. 2. Compute their cross product. 3. Calculate the magnitude of the resulting vector and divide by 2.

Step 3: Detailed Explanation:
The vertices are A(2, 2, 2), B(2, 1, 3), and C(3, 2, 1).
First, determine the vectors for two sides starting from vertex A:

$$\vec{AB} = (\text{Position of B}) - (\text{Position of A}) = (2-2)\hat{i} + (1-2)\hat{j} + (3-2)\hat{k} = 0\hat{i} - 1\hat{j} + 1\hat{k}$$

$$\vec{AC} = (\text{Position of C}) - (\text{Position of A}) = (3-2)\hat{i} + (2-2)\hat{j} + (1-2)\hat{k} = 1\hat{i} + 0\hat{j} - 1\hat{k}$$

Next, calculate the cross product $\vec{AB} \times \vec{AC}$:

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{vmatrix}$$

$$\begin{aligned} &= \hat{i}((-1)(-1) - (1)(0)) - \hat{j}((0)(-1) - (1)(1)) + \hat{k}((0)(0) - (-1)(1)) \\ &= \hat{i}(1-0) - \hat{j}(0-1) + \hat{k}(0+1) \\ &= 1\hat{i} + 1\hat{j} + 1\hat{k} \end{aligned}$$

Now, find the magnitude of this cross product vector:

$$|\vec{AB} \times \vec{AC}| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{1+1+1} = \sqrt{3}$$

Finally, the area of the triangle is half of this magnitude:

$$\text{Area} = \frac{1}{2}|\vec{AB} \times \vec{AC}| = \frac{\sqrt{3}}{2}$$

Step 4: Final Answer:
The area of the triangle is $\frac{\sqrt{3}}{2}$ square units.

> Quick Tip
>
> Remember that the area of the parallelogram formed by vectors $\vec{u}$ and $\vec{v}$ is $|\vec{u} \times \vec{v}|$. The area of the triangle with these adjacent sides is always half of the parallelogram's area. Be careful with sign conventions during the determinant calculation.

---

5. Do all parts.

a. Let function $f : N \to Y$ is defined as $f(x) = 4x + 3$ where $Y = \{y \in N : y = 4x + 3$ for $x \in N\}$. Prove that f is invertible, also find the inverse of the function f.

Correct Answer: The function is invertible and its inverse is $f^{-1}(y) = \frac{y-3}{4}$.

Solution:

Step 1: Understanding the Concept:
A function is invertible if and only if it is bijective, which means it must be both one-one

(injective) and onto (surjective). We need to prove these two properties for the given function f and then find its inverse.

Step 2: Proving One-One (Injectivity):
A function is one-one if different inputs from the domain produce different outputs in the codomain.
Let $x_1, x_2 \in N$ such that $f(x_1) = f(x_2)$.

$$4x_1 + 3 = 4x_2 + 3$$

Subtracting 3 from both sides:

$$4x_1 = 4x_2$$

Dividing by 4:

$$x_1 = x_2$$

Since $f(x_1) = f(x_2)$ implies $x_1 = x_2$, the function f is **one-one**.

Step 3: Proving Onto (Surjectivity):
A function is onto if its range is equal to its codomain.
The domain is N (the set of natural numbers).
The codomain is given as $Y = \{y \in N : y = 4x + 3 \text{ for } x \in N\}$.
The range of f is the set of all possible output values, which is $\{f(x)|x \in N\} = \{4x+3|x \in N\}$.
By the given definition of the codomain Y, the range of f is exactly equal to the codomain Y.
Therefore, the function f is **onto**.

Step 4: Finding the Inverse:
Since f is both one-one and onto, it is a bijective function and hence is invertible.
Let $g : Y \to N$ be the inverse of f. To find the rule for g, we let $y = f(x)$.

$$y = 4x + 3$$

We solve this equation for x in terms of y:

$$y - 3 = 4x$$
$$x = \frac{y-3}{4}$$

So, the inverse function, which we can call f^{-1}, is given by:

$$f^{-1}(y) = \frac{y-3}{4}$$

Step 5: Final Answer:
The function f is one-one and onto, hence it is invertible. The inverse function is $f^{-1} : Y \to N$ defined by $f^{-1}(y) = \frac{y-3}{4}$.

> Quick Tip
>
> When a function's codomain is explicitly defined to be its range, as in this problem, the 'onto' property is satisfied by definition. The main task then becomes proving the 'one-one' property and finding the inverse.

b. Find the minimum value of the objective function $Z = -50x + 20y$ by graphical method under the following constraints :
$2x - y \geq -5$
$3x + y \geq 3$
$2x - 3y \leq 12$
$x \geq 0, y \geq 0$

Correct Answer: -300

Solution:

Step 1: Understanding the Concept:
This is a Linear Programming Problem (LPP). We need to find the minimum value of a linear objective function within a feasible region defined by a set of linear inequalities. The minimum value, if it exists, will occur at one of the corner points (vertices) of the feasible region.

Step 2: Identifying the Feasible Region and Corner Points:
First, we graph the lines corresponding to each constraint:

- $L_1 : 2x - y = -5 \implies y = 2x + 5$
- $L_2 : 3x + y = 3 \implies y = 3 - 3x$
- $L_3 : 2x - 3y = 12 \implies y = \frac{2}{3}x - 4$

The feasible region is the area in the first quadrant ($x \geq 0, y \geq 0$) that satisfies all inequalities:

- $y \leq 2x + 5$ (Below L_1)
- $y \geq 3 - 3x$ (Above L_2)
- $y \geq \frac{2}{3}x - 4$ (Above L_3)

The vertices of the feasible region are the points of intersection of these boundary lines.

- **Point A:** Intersection of $L_2 : 3x + y = 3$ and y-axis ($x = 0$).
  $3(0) + y = 3 \implies y = 3$. So, $\mathbf{A = (0, 3)}$.
- **Point B:** Intersection of $L_2 : 3x + y = 3$ and x-axis ($y = 0$).
  $3x + 0 = 3 \implies x = 1$. So, $\mathbf{B = (1, 0)}$.
- **Point C:** Intersection of $L_3 : 2x - 3y = 12$ and x-axis ($y = 0$).
  $2x - 3(0) = 12 \implies x = 6$. So, $\mathbf{C = (6, 0)}$.

The feasible region is unbounded in the first quadrant.

Step 3: Evaluating the Objective Function at Corner Points:
We evaluate the objective function $Z = -50x + 20y$ at each corner point.

- At A(0, 3): $Z = -50(0) + 20(3) = 60$
- At B(1, 0): $Z = -50(1) + 20(0) = -50$

- At C(6, 0): $Z = -50(6) + 20(0) = -300$

Step 4: Final Answer:
The minimum value of the objective function at the corner points of the feasible region is -300, which occurs at the point (6, 0).

> Quick Tip
>
> For an unbounded feasible region, always test whether a better value for the objective function can be found by moving further into the unbounded area. If so, no optimal solution exists. If not, the optimal solution occurs at a corner point.

---

c. If A and B are two matrices of order n which are invertible, then prove that $(AB)^{-1} = B^{-1}A^{-1}$.

Solution:

Step 1: Understanding the Concept:
The inverse of a square matrix M is a matrix M^{-1} such that their product is the identity matrix I, i.e., $MM^{-1} = M^{-1}M = I$. To prove the given identity, we must show that multiplying AB by $B^{-1}A^{-1}$ results in the identity matrix.

Step 2: Key Approach:
We will use the definition of an inverse. We will show that $(AB)(B^{-1}A^{-1}) = I$.

Step 3: Detailed Proof:
Consider the product of the matrix (AB) and the matrix $(B^{-1}A^{-1})$:

$$(AB)(B^{-1}A^{-1})$$

Using the associative property of matrix multiplication, we can regroup the terms:

$$= A(BB^{-1})A^{-1}$$

Since B is an invertible matrix, by definition, $BB^{-1} = I$, where I is the identity matrix of order n.

$$= A(I)A^{-1}$$

The product of any matrix and the identity matrix is the matrix itself, so $AI = A$.

$$= AA^{-1}$$

Since A is an invertible matrix, by definition, $AA^{-1} = I$.

$$= I$$

Thus, we have shown that $(AB)(B^{-1}A^{-1}) = I$. This proves that $B^{-1}A^{-1}$ is the right inverse of AB. For square matrices, a right inverse is also a left inverse, so $B^{-1}A^{-1}$ is the inverse of AB.

Step 4: Final Answer:
By definition of a matrix inverse and the associative property of multiplication, we have proved that $(AB)^{-1} = B^{-1}A^{-1}$.

Quick Tip

This is often called the "reversal rule" for inverses (or the "socks and shoes" property: to undo putting on socks then shoes, you must first take off the shoes, then the socks). The same reversal rule applies to the transpose of a product of matrices: $(AB)^T = B^T A^T$.

---

d. For the two vectors $\vec{a}$ and $\vec{b}$, prove that $|\vec{a}+\vec{b}| \leq |\vec{a}| + |\vec{b}|$ when $\vec{a} \neq \vec{0}$ and $\vec{b} \neq \vec{0}$.

Solution:

Step 1: Understanding the Concept:
This inequality is known as the triangle inequality for vectors. It states that the magnitude of the sum of two vectors is less than or equal to the sum of their individual magnitudes. We can prove it using the properties of the dot product.

Step 2: Key Approach:
It is easier to work with the squares of the magnitudes to avoid dealing with square roots. We will prove the equivalent inequality $|\vec{a}+\vec{b}|^2 \leq (|\vec{a}| + |\vec{b}|)^2$.

Step 3: Detailed Proof:
We start by considering the square of the magnitude of the sum of the vectors:

$$|\vec{a}+\vec{b}|^2 = (\vec{a}+\vec{b}) \cdot (\vec{a}+\vec{b})$$

Expanding the dot product using its distributive property:

$$= \vec{a}\cdot\vec{a} + \vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{a} + \vec{b}\cdot\vec{b}$$

We know that $\vec{v}\cdot\vec{v} = |\vec{v}|^2$ and the dot product is commutative ($\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$).

$$= |\vec{a}|^2 + 2(\vec{a}\cdot\vec{b}) + |\vec{b}|^2$$

By the definition of the dot product, $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$, where θ is the angle between the vectors. We know that the maximum value of $\cos\theta$ is 1. Therefore, we have the inequality:

$$\vec{a}\cdot\vec{b} \leq |\vec{a}||\vec{b}|$$

Substituting this into our expression:

$$|\vec{a}+\vec{b}|^2 \leq |\vec{a}|^2 + 2|\vec{a}||\vec{b}| + |\vec{b}|^2$$

The right side of the inequality is the expansion of a perfect square:

$$|\vec{a}+\vec{b}|^2 \leq (|\vec{a}| + |\vec{b}|)^2$$

Since magnitude is a non-negative quantity, we can take the square root of both sides without changing the direction of the inequality:

$$|\vec{a}+\vec{b}| \leq |\vec{a}| + |\vec{b}|$$

Step 4: Final Answer:
We have successfully proven the triangle inequality for vectors.

> Quick Tip
>
> The key to proving vector magnitude inequalities is often to square both sides. This allows you to convert the problem into one involving dot products, which have convenient algebraic properties.

---

e. If $I = \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}}$, then prove that $I = \frac{\pi}{12}$.

Solution:

Step 1: Understanding the Concept:
This is a problem involving a definite integral that can be solved elegantly using a specific property of definite integrals, often referred to as the "King's Rule".

Step 2: Key Formula or Approach:
We will use the property: $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$.

Step 3: Detailed Proof:
Let the given integral be:

$$I = \int_{\pi/6}^{\pi/3} \frac{1}{1+\sqrt{\tan x}}\,dx \quad \cdots (1)$$

Here, $a = \frac{\pi}{6}$ and $b = \frac{\pi}{3}$. Their sum is $a+b = \frac{\pi}{6} + \frac{\pi}{3} = \frac{\pi+2\pi}{6} = \frac{3\pi}{6} = \frac{\pi}{2}$.
Applying the property $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$, we get:

$$I = \int_{\pi/6}^{\pi/3} \frac{1}{1+\sqrt{\tan(\frac{\pi}{2}-x)}}\,dx$$

Using the trigonometric identity $\tan(\frac{\pi}{2}-x) = \cot x$:

$$I = \int_{\pi/6}^{\pi/3} \frac{1}{1+\sqrt{\cot x}}\,dx$$

Rewrite $\sqrt{\cot x}$ as $\frac{1}{\sqrt{\tan x}}$:

$$I = \int_{\pi/6}^{\pi/3} \frac{1}{1+\frac{1}{\sqrt{\tan x}}}\,dx = \int_{\pi/6}^{\pi/3} \frac{1}{\frac{\sqrt{\tan x}+1}{\sqrt{\tan x}}}\,dx$$

$$I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\tan x}}{1 + \sqrt{\tan x}}\,dx \quad \cdots (2)$$

Now, we add equation (1) and equation (2):

$$I + I = \int_{\pi/6}^{\pi/3} \frac{1}{1 + \sqrt{\tan x}}\,dx + \int_{\pi/6}^{\pi/3} \frac{\sqrt{\tan x}}{1 + \sqrt{\tan x}}\,dx$$

$$2I = \int_{\pi/6}^{\pi/3} \frac{1 + \sqrt{\tan x}}{1 + \sqrt{\tan x}}\,dx$$

$$2I = \int_{\pi/6}^{\pi/3} 1\,dx$$

Evaluating the simple integral:

$$2I = [x]_{\pi/6}^{\pi/3} = \frac{\pi}{3} - \frac{\pi}{6} = \frac{2\pi - \pi}{6} = \frac{\pi}{6}$$

Solving for I:

$$I = \frac{\pi}{12}$$

Step 4: Final Answer:
We have successfully proven that $I = \frac{\pi}{12}$.

> **Quick Tip**
>
> Whenever you see a definite integral from a to b and the sum $a + b$ is a convenient value like $\pi/2$ or π, you should immediately consider using the property $\int_a^b f(x)\,dx = \int_a^b f(a + b - x)\,dx$. This technique is very common for integrals involving trigonometric functions.

---

6. Do all parts.

a. If function f is defined as

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & \textbf{if } x \neq 0 \\ 0, & \textbf{if } x = 0 \end{cases}$$

then prove that f is continuous.

Correct Answer: Hence Proved.

Solution:

Step 1: Understanding the Concept:
A function $f(x)$ is continuous at a point $x = c$ if three conditions are met: $f(c)$ is defined,

$\lim_{x\to c} f(x)$ exists, and $\lim_{x\to c} f(x) = f(c)$. For the given piecewise function, we need to check continuity for $x \neq 0$ and for $x = 0$.

Step 2: Key Formula or Approach:
For the case $x = 0$, we will use the Squeeze Theorem (or Sandwich Theorem) to evaluate the limit. The theorem states that if $g(x) \leq f(x) \leq h(x)$ for all x in an open interval containing c (except possibly at c itself), and if $\lim_{x\to c} g(x) = L = \lim_{x\to c} h(x)$, then $\lim_{x\to c} f(x) = L$.

Step 3: Detailed Explanation:
Case 1: For $x \neq 0$
The function is defined as $f(x) = x^2 \sin(\frac{1}{x})$. The functions x^2, $\frac{1}{x}$, and $\sin(x)$ are all continuous on their respective domains. Since $x \neq 0$, the composition $\sin(\frac{1}{x})$ is continuous, and the product $x^2 \sin(\frac{1}{x})$ is also continuous. So, $f(x)$ is continuous for all $x \neq 0$.

Case 2: For $x = 0$
We need to check if $\lim_{x\to 0} f(x) = f(0)$. From the function definition, we know that $f(0) = 0$. Now, let's evaluate the limit:

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right)$$

We know that the sine function is bounded between -1 and 1, regardless of its input.

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1 \quad (\text{for } x \neq 0)$$

Multiply the entire inequality by x^2. Since $x^2 \geq 0$, the direction of the inequalities does not change.

$$-x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2$$

Now, we take the limit of the bounding functions as $x \to 0$:

$$\lim_{x\to 0} (-x^2) = 0$$

$$\lim_{x\to 0} (x^2) = 0$$

Since both the lower and upper bounds approach 0, by the Squeeze Theorem, the function in the middle must also approach 0.

$$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$$

We have found that $\lim_{x\to 0} f(x) = 0$ and we know $f(0) = 0$. Since $\lim_{x\to 0} f(x) = f(0)$, the function is continuous at $x = 0$.

Step 4: Final Answer:
Since the function is continuous for all $x \neq 0$ and also continuous at $x = 0$, we can conclude that the function $f(x)$ is continuous for all real numbers. Hence proved.

Quick Tip

The Squeeze Theorem is an essential tool for finding limits of functions that involve products of a term that goes to zero (like x^2) and a bounded function (like $\sin(\frac{1}{x})$ or $\cos(\frac{1}{x})$).

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b. A car is started to move from a point P at time t = 0 and is stopped at the point Q. The distance x metre covered by the car in t second is given by $x = t^2\left(3 - \frac{t}{2}\right)$. Find the time required by the car to reach at the point Q and also find the distance between P and Q.

Correct Answer: Time = 4 seconds, Distance = 16 metres.

Solution:

Step 1: Understanding the Concept:
The problem describes the motion of a car. The key information is that the car "is stopped" at point Q. In physics and calculus, this means that the car's instantaneous velocity is zero at the time it reaches Q. Velocity is the first derivative of the distance function with respect to time.

Step 2: Key Formula or Approach:
1. The distance function is given: $x(t) = t^2(3 - \frac{t}{2})$. 2. The velocity function is the derivative of the distance function: $v(t) = \frac{dx}{dt}$. 3. Set $v(t) = 0$ to find the time 't' when the car stops. 4. Substitute this value of 't' back into the distance function $x(t)$ to find the total distance covered.

Step 3: Detailed Explanation:
First, expand the distance function for easier differentiation:

$$x(t) = 3t^2 - \frac{t^3}{2}$$

Now, find the velocity function $v(t)$ by differentiating $x(t)$ with respect to t:

$$v(t) = \frac{dx}{dt} = \frac{d}{dt}\left(3t^2 - \frac{t^3}{2}\right)$$

$$v(t) = 6t - \frac{3t^2}{2}$$

The car stops at point Q, so its velocity is 0 at that time. We set $v(t) = 0$:

$$6t - \frac{3t^2}{2} = 0$$

Factor out the common term t:

$$t\left(6 - \frac{3t}{2}\right) = 0$$

This gives two possible solutions for t:

$$t = 0 \quad \text{or} \quad 6 - \frac{3t}{2} = 0$$

$t = 0$ corresponds to the starting point P, where the car was at rest. The other solution gives the time to reach point Q.

$$6 = \frac{3t}{2}$$
$$12 = 3t$$
$$t = 4$$

So, the time required for the car to reach point Q is 4 seconds.

Now, to find the distance between P and Q, we substitute $t = 4$ into the distance function $x(t)$:

$$x(4) = 3(4)^2 - \frac{(4)^3}{2}$$
$$x(4) = 3(16) - \frac{64}{2}$$
$$x(4) = 48 - 32 = 16$$

The distance between P and Q is 16 metres.

Step 4: Final Answer:
The time required to reach point Q is 4 seconds, and the distance between P and Q is 16 metres.

Quick Tip

In kinematics problems, "starts from rest" and "stops" both imply that the velocity is zero at that specific time. "Starts at the origin" means the initial position is zero. Carefully translate these phrases into mathematical conditions.

---

c. Find the shortest distance between the lines $\vec{r} = \hat{i} + \hat{j} + \lambda(2\hat{i} - \hat{j} + \hat{k})$ and $\vec{r} = 2\hat{i} + \hat{j} - \hat{k} + \mu(3\hat{i} - 5\hat{j} + 2\hat{k})$.

Correct Answer: $\frac{10}{\sqrt{59}}$ units.

Solution:

Step 1: Understanding the Concept:
The two given lines are in vector form $\vec{r} = \vec{a} + t\vec{b}$. Since their direction vectors are not proportional, the lines are skew (or intersecting). The shortest distance between two skew lines is the length of the perpendicular line segment connecting them.

Step 2: Key Formula or Approach:
The shortest distance d between two skew lines $\vec{r} = \vec{a_1} + \lambda\vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu\vec{b_2}$ is given by the formula:

$$d = \frac{|(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2})|}{|\vec{b_1} \times \vec{b_2}|}$$

Step 3: Detailed Explanation:
From the given equations of the lines, we identify the vectors:

$$\vec{a_1} = \hat{i} + \hat{j} + 0\hat{k} \quad \text{and} \quad \vec{b_1} = 2\hat{i} - \hat{j} + \hat{k}$$

$$\vec{a_2} = 2\hat{i} + \hat{j} - \hat{k} \quad \text{and} \quad \vec{b_2} = 3\hat{i} - 5\hat{j} + 2\hat{k}$$

First, calculate the vector connecting the two points on the lines, $\vec{a_2} - \vec{a_1}$:

$$\vec{a_2} - \vec{a_1} = (2-1)\hat{i} + (1-1)\hat{j} + (-1-0)\hat{k} = \hat{i} + 0\hat{j} - \hat{k}$$

Next, calculate the cross product of the direction vectors, $\vec{b_1} \times \vec{b_2}$:

$$\vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ 3 & -5 & 2 \end{vmatrix} = \hat{i}(-2-(-5)) - \hat{j}(4-3) + \hat{k}(-10-(-3))$$

$$\vec{b_1} \times \vec{b_2} = 3\hat{i} - \hat{j} - 7\hat{k}$$

Now, find the magnitude of this cross product:

$$|\vec{b_1} \times \vec{b_2}| = \sqrt{3^2 + (-1)^2 + (-7)^2} = \sqrt{9+1+49} = \sqrt{59}$$

Next, calculate the scalar triple product for the numerator of the formula:

$$(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2}) = (\hat{i} - \hat{k}) \cdot (3\hat{i} - \hat{j} - 7\hat{k})$$

$$= (1)(3) + (0)(-1) + (-1)(-7) = 3 + 0 + 7 = 10$$

Finally, substitute these values into the shortest distance formula:

$$d = \frac{|10|}{\sqrt{59}} = \frac{10}{\sqrt{59}}$$

Step 4: Final Answer:
The shortest distance between the two lines is $\frac{10}{\sqrt{59}}$ units.

Quick Tip

The numerator of the distance formula, $(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2})$, is a scalar triple product, which can also be computed as the determinant $\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ b_{1x} & b_{1y} & b_{1z} \\ b_{2x} & b_{2y} & b_{2z} \end{vmatrix}$. This can sometimes be a quicker calculation.

---

d. If $y = \cos^{-1} x$, then show that $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 0$.

Solution:

Step 1: Understanding the Concept:
This problem requires finding the second derivative of an inverse trigonometric function and then showing that it satisfies a given second-order differential equation. A useful strategy is to manipulate the first derivative to remove square roots before differentiating a second time.

Step 2: Key Formula or Approach:
1. Find the first derivative, $\frac{dy}{dx}$. 2. Rearrange the equation to eliminate the square root, typically by squaring both sides. 3. Differentiate the rearranged equation implicitly with respect to x. 4. Simplify the resulting equation to match the required form.

Step 3: Detailed Explanation:
Given the function:

$$y = \cos^{-1} x$$

Differentiate with respect to x to find the first derivative:

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}} \quad \text{—(1)}$$

To make the second differentiation easier, let's rearrange this equation to remove the square root.

$$\sqrt{1-x^2}\frac{dy}{dx} = -1$$

Now, square both sides of the equation:

$$(1-x^2)\left(\frac{dy}{dx}\right)^2 = (-1)^2 = 1$$

Differentiate this entire equation with respect to x, using the product rule on the left side:

$$\frac{d}{dx}\left[(1-x^2)\right]\left(\frac{dy}{dx}\right)^2 + (1-x^2)\frac{d}{dx}\left[\left(\frac{dy}{dx}\right)^2\right] = \frac{d}{dx}(1)$$

$$(-2x)\left(\frac{dy}{dx}\right)^2 + (1-x^2)\left[2\left(\frac{dy}{dx}\right)\frac{d^2y}{dx^2}\right] = 0$$

We can see that $2\frac{dy}{dx}$ is a common factor. Since $\frac{dy}{dx}$ is not generally zero (from equation 1), we can divide the entire equation by $2\frac{dy}{dx}$:

$$-x\frac{dy}{dx} + (1-x^2)\frac{d^2y}{dx^2} = 0$$

Rearranging the terms to match the required format:

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 0$$

Step 4: Final Answer:
We have successfully derived the given differential equation. Hence proved.

Quick Tip

When asked to prove differential equations involving inverse trigonometric functions, a very common and effective technique is to differentiate once, rearrange to remove square roots (by squaring), and then differentiate again implicitly. This avoids complex applications of the quotient or chain rule on fractional exponents.

---

e. If x, y, z are three independent events, then prove that $P(X \cap Y \cap Z) = P(X) \cdot P\left(\frac{Y}{X}\right) \cdot P\left(\frac{Z}{X \cap Y}\right)$**.**

Solution:

Step 1: Understanding the Concept:
The equation to be proved is the general multiplication rule (or chain rule) for the probability of the intersection of three events. The problem states that the events are independent, which is a specific condition. We can prove the identity in two ways: by showing it holds true for any set of events, or by using the independence condition to show both sides are equal. We will prove the general rule, which is always true.

Step 2: Key Formula or Approach:
The proof relies on the definition of conditional probability:

$$P(B|A) = \frac{P(A \cap B)}{P(A)} \quad \implies \quad P(A \cap B) = P(A) \cdot P(B|A)$$

Step 3: Detailed Explanation:
We want to prove $P(X \cap Y \cap Z) = P(X) \cdot P(Y|X) \cdot P(Z|X \cap Y)$.
Let's start with the left-hand side (LHS), $P(X \cap Y \cap Z)$. We can group the first two events and write this as:

$$P((X \cap Y) \cap Z)$$

Now, apply the definition of conditional probability, treating $(X \cap Y)$ as a single event 'A' and Z as event 'B'.

$$P((X \cap Y) \cap Z) = P(X \cap Y) \cdot P(Z|(X \cap Y)) \quad \text{—(1)}$$

Next, we can expand the term $P(X \cap Y)$ using the same definition of conditional probability:

$$P(X \cap Y) = P(X) \cdot P(Y|X) \quad \text{—(2)}$$

Now, substitute the expression for $P(X \cap Y)$ from equation (2) into equation (1):

$$P(X \cap Y \cap Z) = [P(X) \cdot P(Y|X)] \cdot P(Z|(X \cap Y))$$

$$P(X \cap Y \cap Z) = P(X) \cdot P(Y|X) \cdot P(Z|X \cap Y)$$

This is the same as the right-hand side (RHS). The identity is true for any three events, whether they are independent or not.

Note on Independence: The problem states that X, Y, and Z are independent. If this condition is applied:

- By definition of independence, $P(X \cap Y \cap Z) = P(X)P(Y)P(Z)$. (LHS)
- Also by independence, $P(Y|X) = P(Y)$ and $P(Z|X \cap Y) = P(Z)$.
- Substituting these into the RHS gives: $P(X) \cdot P(Y) \cdot P(Z)$.

Since both sides simplify to the same expression under the condition of independence, the identity holds true for independent events as well.

Step 4: Final Answer:
By applying the definition of conditional probability twice, we have shown that the identity holds. Hence proved.

> Quick Tip
>
> The multiplication rule for probability is like a chain: the probability of the first event, times the probability of the second given the first has occurred, times the probability of the third given the first two have occurred, and so on. This general rule simplifies to a simple product of probabilities, $P(X)P(Y)P(Z)$, only in the special case where the events are independent.

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7. Do any one part.

a. For matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$ **show that** $A^3 - 6A^2 + 5A + 11I = 0$ **and with the help of this find** A^{-1}**.**

Correct Answer: $A^{-1} = \frac{1}{11}\begin{bmatrix} -3 & 4 & 5 \\ 9 & -1 & -4 \\ 5 & -3 & -1 \end{bmatrix}$

Solution:

Step 1: Understanding the Concept:
This problem has two parts. First, we need to verify that the matrix A satisfies the given polynomial equation, which is its characteristic equation (Cayley-Hamilton Theorem). Second, we use this verified equation to find the inverse of matrix A without using the adjugate method.

Step 2: Verifying the Matrix Equation $A^3 - 6A^2 + 5A + 11I = 0$**:**
First, we calculate A^2 and A^3.

$$A^2 = A \cdot A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix}$$

Now, substitute A^3, A^2, A and I into the equation:

$$A^3 - 6A^2 + 5A + 11I$$

$$= \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix} - 6 \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} + 5 \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} + 11 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix} - \begin{bmatrix} 24 & 12 & 6 \\ -18 & 48 & -84 \\ 42 & -18 & 84 \end{bmatrix} + \begin{bmatrix} 5 & 5 & 5 \\ 5 & 10 & -15 \\ 10 & -5 & 15 \end{bmatrix} + \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}$$

$$= \begin{bmatrix} 8-24+5+11 & 7-12+5+0 & 1-6+5+0 \\ -23+18+5+0 & 27-48+10+11 & -69+84-15+0 \\ 32-42+10+0 & -13+18-5+0 & 58-84+15+11 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

Thus, the equation is verified.

Step 3: Finding A^{-1} using the equation:
We start with the verified equation:

$$A^3 - 6A^2 + 5A + 11I = 0$$

Multiply the entire equation by A^{-1} (post-multiplication or pre-multiplication gives the same result):

$$A^{-1}(A^3 - 6A^2 + 5A + 11I) = A^{-1} \cdot 0$$

$$A^{-1}A^3 - 6A^{-1}A^2 + 5A^{-1}A + 11A^{-1}I = 0$$

Using the properties $A^{-1}A = I$ and $A^{-1}I = A^{-1}$:

$$A^2 - 6A + 5I + 11A^{-1} = 0$$

Now, we isolate the A^{-1} term:

$$11A^{-1} = -A^2 + 6A - 5I$$

$$A^{-1} = \frac{1}{11}(-A^2 + 6A - 5I)$$

Substitute the matrices for A^2, A, and I:

$$A^{-1} = \frac{1}{11} \left(- \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} + 6 \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} - 5 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)$$

$$A^{-1} = \frac{1}{11} \left(\begin{bmatrix} -4 & -2 & -1 \\ 3 & -8 & 14 \\ -7 & 3 & -14 \end{bmatrix} + \begin{bmatrix} 6 & 6 & 6 \\ 6 & 12 & -18 \\ 12 & -6 & 18 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} \right)$$

$$A^{-1} = \frac{1}{11}\begin{bmatrix} -4+6-5 & -2+6-0 & -1+6-0 \\ 3+6-0 & -8+12-5 & 14-18-0 \\ -7+12-0 & 3-6-0 & -14+18-5 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} -3 & 4 & 5 \\ 9 & -1 & -4 \\ 5 & -3 & -1 \end{bmatrix}$$

Step 4: Final Answer:

The matrix equation is verified, and the inverse is $A^{-1} = \frac{1}{11}\begin{bmatrix} -3 & 4 & 5 \\ 9 & -1 & -4 \\ 5 & -3 & -1 \end{bmatrix}$.

> Quick Tip
>
> Using the Cayley-Hamilton theorem to find the inverse is often computationally faster and less prone to sign errors than the traditional method of finding the determinant and the adjugate matrix, especially for 3x3 matrices where you have already calculated A^2.

---

b. Solve the following system of equations by matrix method :
2x + 3y + 3z = 5
x - 2y + z = -4
3x - y - 2z = 3

Correct Answer: x = 1, y = 2, z = -1

Solution:

Step 1: Understanding the Concept:
A system of linear equations can be represented in matrix form as $AX = B$. The solution can be found using the formula $X = A^{-1}B$, provided that the coefficient matrix A is invertible (i.e., its determinant is non-zero).

Step 2: Setting up the Matrix Equation:
The given system of equations can be written as $AX = B$, where:

$$A = \begin{bmatrix} 2 & 3 & 3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}$$

Step 3: Finding the Inverse of the Coefficient Matrix A:
First, we calculate the determinant of A to check for invertibility.

$$\det(A) = |A| = 2((-2)(-2) - (1)(-1)) - 3((1)(-2) - (1)(3)) + 3((1)(-1) - (-2)(3))$$

$$= 2(4+1) - 3(-2-3) + 3(-1+6) = 2(5) - 3(-5) + 3(5) = 10 + 15 + 15 = 40$$

Since $|A| = 40 \neq 0$, the inverse A^{-1} exists.
Next, we find the adjugate of A. The cofactor matrix C is:

$$C_{11} = 5, \quad C_{12} = -(-5) = 5, \quad C_{13} = 5$$

$$C_{21} = -(-6+3) = 3, \quad C_{22} = -4-9 = -13, \quad C_{23} = -(-2-9) = 11$$
$$C_{31} = 3-(-6) = 9, \quad C_{32} = -(2-3) = 1, \quad C_{33} = -4-3 = -7$$

So, the cofactor matrix is $C = \begin{bmatrix} 5 & 5 & 5 \\ 3 & -13 & 11 \\ 9 & 1 & -7 \end{bmatrix}$.

The adjugate is the transpose of the cofactor matrix: $\text{adj}(A) = C^T = \begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}$.

The inverse is $A^{-1} = \frac{1}{|A|}\text{adj}(A)$:

$$A^{-1} = \frac{1}{40}\begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}$$

Step 4: Solving for X:
Now we calculate $X = A^{-1}B$:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{40}\begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}\begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}$$

$$= \frac{1}{40}\begin{bmatrix} (5)(5)+(3)(-4)+(9)(3) \\ (5)(5)+(-13)(-4)+(1)(3) \\ (5)(5)+(11)(-4)+(-7)(3) \end{bmatrix}$$

$$= \frac{1}{40}\begin{bmatrix} 25-12+27 \\ 25+52+3 \\ 25-44-21 \end{bmatrix} = \frac{1}{40}\begin{bmatrix} 40 \\ 80 \\ -40 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Step 5: Final Answer:
By comparing the elements of the resulting matrix, we find the solution: $x = 1, y = 2$, and $z = -1$.

Quick Tip

After finding the solution (x, y, z), it is a very good practice to substitute these values back into one or all of the original equations to quickly verify your answer. For example, in the first equation: $2(1) + 3(2) + 3(-1) = 2 + 6 - 3 = 5$. This confirms the result is correct.

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8. Do any one part.

a. Solve: $\frac{dy}{dx} = \frac{x+y+1}{2x+2y+3}$**.**

Correct Answer: $6y - 3x + \ln|3x + 3y + 4| = C$

Solution:

Step 1: Understanding the Concept:
This is a first-order differential equation that is not directly separable, homogeneous, or linear. However, the terms involving x and y appear as a linear combination $x + y$ in both the numerator and the denominator. This suggests that we can use a substitution to transform the equation into a separable form.

Step 2: Key Formula or Approach:
1. Let $v = x + y$.
2. Differentiate this substitution with respect to x to find a replacement for $\frac{dy}{dx}$.
3. Substitute both v and the expression for $\frac{dy}{dx}$ into the original equation.
4. The resulting equation in terms of v and x will be separable. Solve it by integration.
5. Substitute back $v = x + y$ to get the final solution.

Step 3: Detailed Explanation:
Let the substitution be:

$$v = x + y$$

Differentiating with respect to x:

$$\frac{dv}{dx} = 1 + \frac{dy}{dx} \implies \frac{dy}{dx} = \frac{dv}{dx} - 1$$

Substitute $x + y = v$ and $\frac{dy}{dx}$ into the given equation:

$$\frac{dv}{dx} - 1 = \frac{v+1}{2v+3}$$

Now, solve for $\frac{dv}{dx}$:

$$\frac{dv}{dx} = \frac{v+1}{2v+3} + 1 = \frac{(v+1)+(2v+3)}{2v+3} = \frac{3v+4}{2v+3}$$

This is a separable equation. We can rearrange the terms to integrate:

$$\frac{2v+3}{3v+4}dv = dx$$

Integrate both sides:

$$\int \frac{2v+3}{3v+4}dv = \int dx$$

To solve the integral on the left, we can use algebraic manipulation:

$$\int \frac{2v+3}{3v+4}dv = \int \frac{\frac{2}{3}(3v+4)+3-\frac{8}{3}}{3v+4}dv = \int \left(\frac{2}{3} + \frac{1/3}{3v+4}\right)dv$$

$$= \frac{2}{3}\int dv + \frac{1}{3}\int \frac{1}{3v+4}dv$$

$$= \frac{2}{3}v + \frac{1}{3}\cdot\frac{\ln|3v+4|}{3} = \frac{2}{3}v + \frac{1}{9}\ln|3v+4|$$

So, the equation becomes:

$$\frac{2}{3}v + \frac{1}{9}\ln|3v+4| = x + C_1$$

Substitute back $v = x + y$:

$$\frac{2}{3}(x+y) + \frac{1}{9}\ln|3(x+y)+4| = x + C_1$$

Multiply by 9 to clear the fractions:

$$6(x+y) + \ln|3x+3y+4| = 9x + 9C_1$$
$$6x + 6y + \ln|3x+3y+4| = 9x + C$$
$$6y - 3x + \ln|3x+3y+4| = C$$

Step 4: Final Answer:
The general solution of the differential equation is $6y - 3x + \ln|3x+3y+4| = C$.

> **Quick Tip**
>
> When the numerator and denominator of $\frac{dy}{dx}$ are linear expressions of the form $\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$, check the ratio of coefficients $\frac{a_1}{a_2}$ and $\frac{b_1}{b_2}$. If they are equal, as in this case $(\frac{1}{2})$, a substitution like $v = a_1x + b_1y$ will always transform the equation into a separable one.

---

b. Solve: $(1+y^2)dx = (\tan^{-1} y - x)dy$**.**

Correct Answer: $x = \tan^{-1} y - 1 + Ce^{-\tan^{-1} y}$

Solution:

Step 1: Understanding the Concept:
This differential equation is not easily solvable in the form $\frac{dy}{dx} = f(x,y)$. However, if we rearrange it to express $\frac{dx}{dy}$, it takes the form of a linear differential equation in x.

Step 2: Key Formula or Approach:
The standard form of a first-order linear differential equation in x is:

$$\frac{dx}{dy} + P(y)x = Q(y)$$

The solution is given by:

$$x \cdot (\text{I.F.}) = \int Q(y) \cdot (\text{I.F.}) dy + C$$

where the integrating factor (I.F.) is $e^{\int P(y)dy}$.

Step 3: Detailed Explanation:
The given equation is $(1+y^2)dx = (\tan^{-1} y - x)dy$.
Rearrange it to find $\frac{dx}{dy}$:

$$\frac{dx}{dy} = \frac{\tan^{-1} y - x}{1+y^2}$$

$$\frac{dx}{dy} = \frac{\tan^{-1} y}{1+y^2} - \frac{x}{1+y^2}$$

Bring the term with x to the left side to match the standard form:

$$\frac{dx}{dy} + \frac{1}{1+y^2}x = \frac{\tan^{-1} y}{1+y^2}$$

This is a linear DE with $P(y) = \frac{1}{1+y^2}$ and $Q(y) = \frac{\tan^{-1} y}{1+y^2}$.
First, find the integrating factor (I.F.):

$$\text{I.F.} = e^{\int P(y)dy} = e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1} y}$$

Now, apply the solution formula:

$$x \cdot e^{\tan^{-1} y} = \int \frac{\tan^{-1} y}{1+y^2} \cdot e^{\tan^{-1} y} dy + C$$

To solve the integral on the right, let $t = \tan^{-1} y$, so $dt = \frac{1}{1+y^2}dy$. The integral becomes:

$$\int te^t dt$$

Using integration by parts ($\int udv = uv - \int vdu$) with $u = t$ and $dv = e^t dt$:

$$\int te^t dt = te^t - \int e^t dt = te^t - e^t = e^t(t-1)$$

Substitute back $t = \tan^{-1} y$:

$$\int \frac{\tan^{-1} y}{1+y^2} e^{\tan^{-1} y} dy = e^{\tan^{-1} y}(\tan^{-1} y - 1)$$

Substitute this back into the solution:

$$x \cdot e^{\tan^{-1} y} = e^{\tan^{-1} y}(\tan^{-1} y - 1) + C$$

To get the explicit solution for x, divide the entire equation by $e^{\tan^{-1} y}$:

$$x = \tan^{-1} y - 1 + \frac{C}{e^{\tan^{-1} y}}$$

$$x = \tan^{-1} y - 1 + Ce^{-\tan^{-1} y}$$

Step 4: Final Answer:
The general solution of the differential equation is $x = \tan^{-1} y - 1 + Ce^{-\tan^{-1} y}$.

Quick Tip

If a first-order differential equation looks complicated and is not separable or homogeneous, always try rearranging it into the linear form $\frac{dy}{dx} + P(x)y = Q(x)$. If that doesn't work, try rearranging it into the alternative linear form $\frac{dx}{dy} + P(y)x = Q(y)$. One of these two forms often provides a straightforward path to the solution.

9. Do any one part.

a. Prove that $\int_0^{\pi/2} \sqrt{\frac{1+\cos 4x}{2}}\, dx = 1$.

Solution:

Step 1: Understanding the Concept:
This problem requires evaluating a definite integral. The integrand involves a square root of a trigonometric function, which can be simplified using a half-angle or double-angle identity.

Step 2: Key Formula or Approach:
We will use the trigonometric identity for the cosine of a double angle:

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

By letting $\theta = 2x$, this identity becomes:

$$\cos^2(2x) = \frac{1+\cos 4x}{2}$$

Step 3: Detailed Explanation:
Let the integral be I.

$$I = \int_0^{\pi/2} \sqrt{\frac{1+\cos 4x}{2}}\, dx$$

Using the identity from Step 2, we can simplify the expression under the square root:

$$I = \int_0^{\pi/2} \sqrt{\cos^2(2x)}\, dx$$

$$I = \int_0^{\pi/2} |\cos(2x)|\, dx$$

The presence of the absolute value means we must consider the sign of $\cos(2x)$ over the interval $[0, \pi/2]$.

- For $0 \le x \le \pi/4$, we have $0 \le 2x \le \pi/2$, where $\cos(2x) \ge 0$.
- For $\pi/4 < x \le \pi/2$, we have $\pi/2 < 2x \le \pi$, where $\cos(2x) \le 0$.

Therefore, we must split the integral at $x = \pi/4$:

$$I = \int_0^{\pi/4} \cos(2x)\, dx + \int_{\pi/4}^{\pi/2} (-\cos(2x))\, dx$$

Now, we evaluate each integral:

$$\int \cos(2x)\, dx = \frac{\sin(2x)}{2}$$

For the first part:

$$\left[\frac{\sin(2x)}{2}\right]_0^{\pi/4} = \frac{\sin(2\cdot\pi/4)}{2} - \frac{\sin(0)}{2} = \frac{\sin(\pi/2)}{2} - 0 = \frac{1}{2}$$

For the second part:

$$\left[-\frac{\sin(2x)}{2}\right]_{\pi/4}^{\pi/2} = \left(-\frac{\sin(2\cdot\pi/2)}{2}\right) - \left(-\frac{\sin(2\cdot\pi/4)}{2}\right) = -\frac{\sin(\pi)}{2} + \frac{\sin(\pi/2)}{2} = 0 + \frac{1}{2} = \frac{1}{2}$$

Adding the results from both parts:

$$I = \frac{1}{2} + \frac{1}{2} = 1$$

Step 4: Final Answer:
We have shown that the value of the definite integral is 1.

> Quick Tip
>
> When simplifying $\sqrt{f(x)^2}$, always remember to write it as $|f(x)|$. Forgetting the absolute value is a common error and can lead to incorrect results if $f(x)$ is negative over part of the integration interval.

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b. If $I = \int_0^{\pi} \frac{x\,dx}{a^2\cos^2 x + b^2\sin^2 x}$, find the value of I.

Correct Answer: $I = \frac{\pi^2}{2ab}$

Solution:

Step 1: Understanding the Concept:
This is a standard form of a definite integral that can be solved using a property of definite integrals, often known as the "King's Rule", which helps to eliminate the 'x' in the numerator.

Step 2: Key Formula or Approach:
We use the property: $\int_0^c f(x)\,dx = \int_0^c f(c-x)\,dx$.
Let the given integral be:

$$I = \int_0^{\pi} \frac{x}{a^2\cos^2 x + b^2\sin^2 x}\,dx \quad \cdots (1)$$

Applying the property with $c = \pi$:

$$I = \int_0^{\pi} \frac{\pi - x}{a^2\cos^2(\pi - x) + b^2\sin^2(\pi - x)}\,dx$$

Since $\cos(\pi - x) = -\cos x$ and $\sin(\pi - x) = \sin x$, we have $\cos^2(\pi - x) = \cos^2 x$ and $\sin^2(\pi - x) = \sin^2 x$.

$$I = \int_0^{\pi} \frac{\pi - x}{a^2\cos^2 x + b^2\sin^2 x}\, dx \quad \cdots (2)$$

Step 3: Detailed Explanation:

Add equations (1) and (2):

$$2I = \int_0^{\pi} \frac{x + (\pi - x)}{a^2\cos^2 x + b^2\sin^2 x}\, dx = \int_0^{\pi} \frac{\pi}{a^2\cos^2 x + b^2\sin^2 x}\, dx$$

$$2I = \pi \int_0^{\pi} \frac{1}{a^2\cos^2 x + b^2\sin^2 x}\, dx$$

Let the integrand be $g(x) = \frac{1}{a^2\cos^2 x + b^2\sin^2 x}$. Since $g(\pi - x) = g(x)$, we can use the property $\int_0^{2c} f(x)dx = 2\int_0^c f(x)dx$. Here $2c = \pi \Rightarrow c = \pi/2$.

$$2I = \pi \left(2\int_0^{\pi/2} \frac{1}{a^2\cos^2 x + b^2\sin^2 x}\, dx \right) = 2\pi \int_0^{\pi/2} \frac{1}{a^2\cos^2 x + b^2\sin^2 x}\, dx$$

$$I = \pi \int_0^{\pi/2} \frac{1}{a^2\cos^2 x + b^2\sin^2 x}\, dx$$

Divide the numerator and denominator by $\cos^2 x$:

$$I = \pi \int_0^{\pi/2} \frac{\sec^2 x}{a^2 + b^2\tan^2 x}\, dx$$

Let $t = \tan x$, so $dt = \sec^2 x\, dx$. The limits of integration change from $x = 0 \to t = 0$ and $x = \pi/2 \to t = \infty$.

$$I = \pi \int_0^{\infty} \frac{dt}{a^2 + (bt)^2}$$

This is a standard integral of the form $\int \frac{1}{c^2+u^2} du = \frac{1}{c}\tan^{-1}(\frac{u}{c})$.

$$I = \pi \left[\frac{1}{a}\tan^{-1}\left(\frac{bt}{a}\right) \cdot \frac{1}{b} \right]_0^{\infty} = \frac{\pi}{ab}\left[\tan^{-1}\left(\frac{bt}{a}\right) \right]_0^{\infty}$$

$$I = \frac{\pi}{ab}\left(\lim_{t\to\infty} \tan^{-1}\left(\frac{bt}{a}\right) - \tan^{-1}(0) \right)$$

$$I = \frac{\pi}{ab}\left(\frac{\pi}{2} - 0\right) = \frac{\pi^2}{2ab}$$

Step 4: Final Answer:

The value of the integral is $I = \frac{\pi^2}{2ab}$.

> **Quick Tip**
>
> For definite integrals of the form $\int_0^a x f(x)\, dx$, always try applying the property $\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$ first. It often simplifies the problem by eliminating the x term in the numerator.