

UP Board Class 12 Mathematics - 324(JC) - 2025 Question Paper with Solutions

Time Allowed :3 Hours	Maximum Marks :100	Total Questions :9
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General Instructions

Read the following instructions very carefully and strictly follow them:

1. There are in all nine questions in this question paper.
2. All questions are compulsory.
3. In the beginning of each question, the number of parts to be attempted are clearly mentioned.
4. Marks allotted to the questions are indicated against them.
5. Start solving from the first question and proceed to solve till the last one. Do not waste your time over a question you cannot solve.

1. Do all parts.

Select the correct option of each part and write it on your answer-book.

a. Suppose that $A = \{2, 3, 4, 5\}$ and a relation R on A is defined by $R = \{(a, b) : a, b \in A, a - b = 12\}$. Then the set R is

- (A) $\emptyset$
- (B) Not $\emptyset$
- (C) $\{2, 3\}$
- (D) $\{2, 4, 5\}$

Correct Answer: (A) $\emptyset$

Solution:

The set is $A = \{2, 3, 4, 5\}$. The relation is defined as

$$R = \{(a, b) : a, b \in A, a - b = 12\}.$$

Step 1: Condition Check: For any $a, b \in A$, we require $a - b = 12 \Rightarrow a = b + 12$.

Step 2: Possible Values: Since b must belong to A , the possible values are 2, 3, 4, 5. Then

$$a = b + 12 \in \{14, 15, 16, 17\}.$$

But none of these values belong to A .

Step 3: Conclusion: Therefore, no ordered pair satisfies the condition. Hence

$$R = \emptyset.$$

Final Answer:

$$\boxed{\emptyset}$$

💡 Quick Tip

When dealing with relations on finite sets with a condition involving inequalities or specific values, always check the extreme cases first (maximum and minimum possible values). This can often lead to a quick conclusion without checking every single pair.

b. If the function $f: \mathbb{N} \rightarrow \mathbb{N}$, is defined by $f(x) = x - 1$ for all $x > 2$ and $f(1) = f(2) = 1$, then f is

- (A) one-one and onto
- (B) onto but not one-one
- (C) many one but not onto
- (D) neither one-one nor onto

Correct Answer: (B) onto but not one-one

Solution:

The function $f: \mathbb{N} \rightarrow \mathbb{N}$ is defined by

$$f(x) = \begin{cases} 1, & x = 1 \text{ or } x = 2, \\ x - 1, & x > 2. \end{cases}$$

Step 1: Inspect the mapping on small inputs.

Compute a few values to see the pattern:

$$f(1) = 1, \quad f(2) = 1, \quad f(3) = 2, \quad f(4) = 3, \quad f(5) = 4, \quad \dots$$

Thus the image set is $\{1, 2, 3, 4, \dots\} = \mathbb{N}$.

Step 2: Test injectivity (one-one).

A function is one-one if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ for all $x_1, x_2 \in \mathbb{N}$.

Here,

$$f(1) = 1 = f(2) \quad \text{with} \quad 1 \neq 2.$$

Therefore, two distinct inputs give the same output, so f is *not* injective (it is many-one).

(For completeness: on the subset $\{3, 4, 5, \dots\}$, we have $f(x) = x - 1$, which is strictly increasing and hence injective there; the failure of injectivity arises from the pair 1, 2.)

Step 3: Test surjectivity (onto).

A function $f: \mathbb{N} \rightarrow \mathbb{N}$ is onto if for every $y \in \mathbb{N}$ there exists $x \in \mathbb{N}$ with $f(x) = y$. We show this by cases:

- If $y = 1$, choose $x = 1$ (or $x = 2$); then $f(x) = 1 = y$.
- If $y \geq 2$, choose $x = y + 1$ (note $x > 2$); then $f(x) = x - 1 = (y + 1) - 1 = y$.

Thus every $y \in \mathbb{N}$ has a preimage, so f is surjective (onto).

Step 4: Conclusion.

The function f is *onto but not one-one*. Hence the correct option is (B) onto but not one-one.

💡 Quick Tip

To quickly check if a function is one-one, look for any two distinct inputs that produce the same output. To check if it's onto, determine the set of all possible output values (the range) and see if it covers the entire specified codomain.

c. If $\int x \log x \, dx = \frac{x^2}{2}f(x) - \frac{x^2}{4} + c$, then $f(x)$ is

- (A) $(\log x)^{-1}$
- (B) $2 \log x$
- (C) $\log x$
- (D) $3 \log x$

Correct Answer: (C) $\log x$

Solution:

The given relation is

$$\int x \log x \, dx = \frac{x^2}{2}f(x) - \frac{x^2}{4} + c.$$

Step 1: Differentiate both sides. If two integrals are equal up to a constant, differentiating both sides with respect to x should give the integrands equal.

$$\frac{d}{dx} \left(\int x \log x \, dx \right) = \frac{d}{dx} \left(\frac{x^2}{2}f(x) - \frac{x^2}{4} + c \right).$$

The derivative of the left-hand side gives the integrand:

$$x \log x.$$

Step 2: Differentiate the right-hand side.

$$\frac{d}{dx} \left(\frac{x^2}{2}f(x) - \frac{x^2}{4} + c \right) = xf(x) + \frac{x^2}{2}f'(x) - \frac{x}{2}.$$

Step 3: Equate both sides.

$$x \log x = xf(x) + \frac{x^2}{2}f'(x) - \frac{x}{2}.$$

Step 4: Guess and check with options. Try $f(x) = \log x$. Then

$$f'(x) = \frac{1}{x}.$$

Substitute into the equation:

$$xf(x) + \frac{x^2}{2}f'(x) - \frac{x}{2} = x \log x + \frac{x^2}{2} \cdot \frac{1}{x} - \frac{x}{2} = x \log x + \frac{x}{2} - \frac{x}{2}.$$
$$= x \log x.$$

This matches perfectly.

Final Answer:

$$\boxed{\log x}$$

 Quick Tip

Remember the LIATE rule for choosing 'u' in integration by parts. This priority order (Logarithmic $\wr$ Inverse $\wr$ Algebraic $\wr$ Trigonometric $\wr$ Exponential) simplifies the process and helps avoid complicated integrals.

d. If $y = 5x^2 + 4$, then at the point with x-coordinate 2, the slope is

- (A) $3/2\sqrt{14}$
- (B) $1/2\sqrt{14}$
- (C) 20
- (D) 1

Correct Answer: (C) 20

Solution:

The given curve is

$$y = 5x^2 + 4.$$

To find the slope at $x = 2$, compute the derivative:

$$\frac{dy}{dx} = 10x.$$

Substituting $x = 2$,

$$\left. \frac{dy}{dx} \right|_{x=2} = 10(2) = 20.$$

Hence, the slope of the tangent at the point with $x = 2$ is

$$\boxed{20}.$$

 Quick Tip

In competitive exams, be aware of potential typos in the questions. If a given point (x, y) does not satisfy the equation of the curve, double-check if you only need the x-coordinate for the calculation (like finding a slope). Proceed with the given x-value and see if the result matches an option.

e. The vector function is given by $\vec{f}(t) = t\hat{i} + t^2\hat{j} + 5\hat{k}$, then at point $t = 1$ the slope is

- (A) $\hat{i} + 2\hat{j}$
- (B) $\hat{i} + 3\hat{j}$
- (C) $2\hat{i} + \hat{j} + \hat{k}$
- (D) $\hat{i} + 3\hat{j} + 5\hat{k}$

Correct Answer: (A) $\hat{i} + 2\hat{j}$

Solution:

Step 1: Understanding the Concept:

For a vector function $\vec{f}(t)$ that describes a curve in space, the "slope" at a particular point is represented by the tangent vector to the curve at that point. The tangent vector is found by taking the first derivative of the vector function with respect to the parameter 't'.

Step 2: Key Formula or Approach:

To find the derivative of a vector function $\vec{f}(t) = f_x(t)\hat{i} + f_y(t)\hat{j} + f_z(t)\hat{k}$, we differentiate each component function with respect to 't':

$$\vec{f}'(t) = \frac{d\vec{f}}{dt} = \frac{df_x}{dt}\hat{i} + \frac{df_y}{dt}\hat{j} + \frac{df_z}{dt}\hat{k}$$

Step 3: Detailed Explanation:

The given vector function is:

$$\vec{f}(t) = t\hat{i} + t^2\hat{j} + 5\hat{k}$$

To find the tangent vector (slope), we differentiate $\vec{f}(t)$ with respect to t:

$$\vec{f}'(t) = \frac{d}{dt}(t)\hat{i} + \frac{d}{dt}(t^2)\hat{j} + \frac{d}{dt}(5)\hat{k}$$

Using the power rule for differentiation:

$$\vec{f}'(t) = (1)\hat{i} + (2t)\hat{j} + (0)\hat{k}$$

$$\vec{f}'(t) = \hat{i} + 2t\hat{j}$$

Now, we need to find the tangent vector at the point $t = 1$. We substitute $t = 1$ into the derivative $\vec{f}'(t)$:

$$\vec{f}'(1) = \hat{i} + 2(1)\hat{j}$$

$$\vec{f}'(1) = \hat{i} + 2\hat{j}$$

Step 4: Final Answer:

The slope (tangent vector) at $t = 1$ is $\hat{i} + 2\hat{j}$. So, option (A) is correct.

 Quick Tip

When asked for the "slope" of a vector function, it almost always refers to the tangent vector, which is its first derivative. Remember to differentiate each component of the vector independently with respect to the parameter.

2. Do all parts.

a. Prove that $\sin^{-1} x = \cos^{-1} \sqrt{1 - x^2}$.

Solution:

Step 1: Understanding the Concept:

This question requires proving an identity involving inverse trigonometric functions.

The relationship between $\sin^{-1} x$ and $\cos^{-1} y$ can be established using a right-angled triangle and the fundamental trigonometric identity $\sin^2 \theta + \cos^2 \theta = 1$.

Step 2: Key Formula or Approach:

Let $\theta = \sin^{-1} x$. This implies $\sin \theta = x$.

We will use the identity $\cos \theta = \sqrt{1 - \sin^2 \theta}$ (for the principal value range of $\sin^{-1} x$, which is $[-\pi/2, \pi/2]$, $\cos \theta$ is non-negative).

From $\cos \theta$, we can find θ in terms of $\cos^{-1}$.

Step 3: Detailed Explanation:

Let's start by assuming $\theta = \sin^{-1} x$.

By the definition of the inverse sine function, this means:

$$\sin \theta = x$$

Now, we use the Pythagorean identity $\sin^2 \theta + \cos^2 \theta = 1$.

We can rearrange this to solve for $\cos \theta$:

$$\cos^2 \theta = 1 - \sin^2 \theta$$

Substituting $\sin \theta = x$ into the equation:

$$\cos^2 \theta = 1 - x^2$$

Taking the square root of both sides gives:

$$\cos \theta = \sqrt{1 - x^2}$$

(We take the positive root because the range of $\sin^{-1} x$ is $[-\pi/2, \pi/2]$, and in this interval, $\cos \theta \geq 0$).

Now, we can express θ using the inverse cosine function:

$$\theta = \cos^{-1} \sqrt{1 - x^2}$$

Since we initially let $\theta = \sin^{-1} x$, we can equate the two expressions for θ :

$$\sin^{-1} x = \cos^{-1} \sqrt{1 - x^2}$$

Step 4: Final Answer:

We have successfully shown that both sides of the equation are equal to the same angle θ , thus proving the identity.

💡 Quick Tip

When converting between inverse trigonometric functions, drawing a right-angled triangle can be a very intuitive and fast method. If $\sin \theta = x/1$, the opposite side is x and the hypotenuse is 1. The adjacent side will be $\sqrt{1^2 - x^2}$, from which you can find any other trigonometric ratio.

b. Find the direction cosine of Z-axis.

Correct Answer: $(0, 0, 1)$

Solution:**Step 1: Understanding the Concept:**

Direction cosines of a line (or axis) are the cosines of the angles that the line makes with the positive directions of the coordinate axes (X, Y, and Z axes).

If a line makes angles α, β, γ with the X, Y, and Z axes respectively, then its direction cosines are $l = \cos \alpha$, $m = \cos \beta$, and $n = \cos \gamma$.

Step 2: Key Formula or Approach:

We need to determine the angles that the Z-axis makes with the X, Y, and Z axes.

Angle with X-axis: α

Angle with Y-axis: β

Angle with Z-axis: γ

Then, calculate $\cos \alpha, \cos \beta, \cos \gamma$.

Step 3: Detailed Explanation:

The Z-axis is perpendicular to both the X-axis and the Y-axis.

Therefore, the angle between the Z-axis and the X-axis is $\alpha = 90^\circ$.

The angle between the Z-axis and the Y-axis is $\beta = 90^\circ$.

The angle between the Z-axis and itself is $\gamma = 0^\circ$.

Now, we calculate the cosines of these angles:

$$l = \cos \alpha = \cos(90^\circ) = 0$$

$$m = \cos \beta = \cos(90^\circ) = 0$$

$$n = \cos \gamma = \cos(0^\circ) = 1$$

Step 4: Final Answer:

The direction cosines of the Z-axis are (l, m, n) , which are $(0, 0, 1)$.

💡 Quick Tip

It's helpful to memorize the direction cosines of the coordinate axes: - X-axis: (1, 0, 0)
- Y-axis: (0, 1, 0) - Z-axis: (0, 0, 1) This saves valuable time in exams.

c. Obtain the projection of the vector $\vec{a} = 2\hat{i} + 3\hat{j} + 5\hat{k}$ on the vector $\vec{b} = \hat{i} + 3\hat{j} + \hat{k}$.

Correct Answer: $\frac{16}{\sqrt{11}}$

Solution:

Step 1: Understanding the Concept:

The projection of a vector $\vec{a}$ onto another vector $\vec{b}$ is the scalar length of the component of $\vec{a}$ that lies in the direction of $\vec{b}$.

Step 2: Key Formula or Approach:

The formula for the scalar projection of vector $\vec{a}$ on vector $\vec{b}$ is given by:

$$\text{Projection of } \vec{a} \text{ on } \vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$$

We need to compute the dot product $\vec{a} \cdot \vec{b}$ and the magnitude of $\vec{b}$.

Step 3: Detailed Explanation:

Given vectors are:

$$\vec{a} = 2\hat{i} + 3\hat{j} + 5\hat{k}$$

$$\vec{b} = \hat{i} + 3\hat{j} + \hat{k}$$

First, calculate the dot product $\vec{a} \cdot \vec{b}$:

$$\vec{a} \cdot \vec{b} = (2)(1) + (3)(3) + (5)(1)$$

$$\vec{a} \cdot \vec{b} = 2 + 9 + 5 = 16$$

Next, calculate the magnitude of $\vec{b}$:

$$|\vec{b}| = \sqrt{(1)^2 + (3)^2 + (1)^2}$$

$$|\vec{b}| = \sqrt{1 + 9 + 1} = \sqrt{11}$$

Now, substitute these values into the projection formula:

$$\text{Projection} = \frac{16}{\sqrt{11}}$$

Step 4: Final Answer:

The projection of the vector $\vec{a}$ on the vector $\vec{b}$ is $\frac{16}{\sqrt{11}}$.

 Quick Tip

Remember the distinction: The "scalar projection" is a length (a scalar), calculated as $\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$. The "vector projection" is a vector, calculated as $\left(\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2}\right) \vec{b}$. Pay close attention to what the question asks for.

d. Find the value of $\hat{i} \cdot (\hat{j} \times \hat{k}) + \hat{j} \cdot (\hat{i} \times \hat{k}) + \hat{k} \cdot (\hat{i} \times \hat{j})$.

Correct Answer: 1

Solution:

Step 1: Understanding the Concept:

This problem involves the scalar triple product and the properties of the standard orthogonal unit vectors $\hat{i}, \hat{j}, \hat{k}$.

The expression $\vec{a} \cdot (\vec{b} \times \vec{c})$ is known as the scalar triple product.

Step 2: Key Formula or Approach:

We need to use the cyclic properties of the cross product of unit vectors:

- $\hat{i} \times \hat{j} = \hat{k}$
- $\hat{j} \times \hat{k} = \hat{i}$
- $\hat{k} \times \hat{i} = \hat{j}$

And the anti-cyclic properties:

- $\hat{j} \times \hat{i} = -\hat{k}$
- $\hat{k} \times \hat{j} = -\hat{i}$
- $\hat{i} \times \hat{k} = -\hat{j}$

We also use the dot product properties: $\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$.

Step 3: Detailed Explanation:

Let's evaluate each term of the expression separately.

Term 1: $\hat{i} \cdot (\hat{j} \times \hat{k})$ Using the cyclic property, $\hat{j} \times \hat{k} = \hat{i}$.

$$\hat{i} \cdot (\hat{i}) = 1$$

Term 2: $\hat{j} \cdot (\hat{i} \times \hat{k})$ Using the anti-cyclic property, $\hat{i} \times \hat{k} = -\hat{j}$.

$$\hat{j} \cdot (-\hat{j}) = -(\hat{j} \cdot \hat{j}) = -1$$

Term 3: $\hat{k} \cdot (\hat{i} \times \hat{j})$ Using the cyclic property, $\hat{i} \times \hat{j} = \hat{k}$.

$$\hat{k} \cdot (\hat{k}) = 1$$

Now, add the values of the three terms:

$$\text{Value} = 1 + (-1) + 1 = 1$$

Step 4: Final Answer:

The value of the given expression is 1.

💡 Quick Tip

The scalar triple product $[\hat{i} \ \hat{j} \ \hat{k}] = \hat{i} \cdot (\hat{j} \times \hat{k})$ is equal to the determinant of the matrix formed by these vectors, which is the identity matrix, so its determinant is 1. Swapping any two vectors negates the result. In the second term, $\hat{i}$ and $\hat{j}$ are swapped relative to the cyclic order, hence the -1.

e. Prove that $0 \leq P(E) \leq 1$, where $P(E)$ is the probability of the event E .

Solution:

Step 1: Understanding the Concept:

This is a fundamental axiom of probability theory. The probability of any event must be a value between 0 and 1, inclusive. We need to prove this using the basic definition of probability.

Step 2: Key Formula or Approach:

The classical definition of the probability of an event E is:

$$P(E) = \frac{\text{Number of favorable outcomes for } E}{\text{Total number of outcomes in the sample space}} = \frac{n(E)}{n(S)}$$

We will use the properties of the sets E (event) and S (sample space).

Step 3: Detailed Explanation:

Let S be the sample space, which is the set of all possible outcomes of an experiment.

Let E be an event, which is a subset of the sample space ($E \subseteq S$).

Let $n(S)$ be the total number of outcomes in S and $n(E)$ be the number of outcomes in E .

Part 1: Proving $P(E) \geq 0$

The number of outcomes in any event E , $n(E)$, cannot be negative. The minimum possible number of outcomes is zero (for an impossible event). So, $n(E) \geq 0$.

The total number of outcomes in a sample space, $n(S)$, must be positive for a non-trivial experiment, so $n(S) > 0$.

Therefore, the ratio must be non-negative:

$$P(E) = \frac{n(E)}{n(S)} \geq \frac{0}{n(S)} \implies P(E) \geq 0$$

Part 2: Proving $P(E) \leq 1$

Since the event E is a subset of the sample space S ($E \subseteq S$), the number of elements in E

cannot exceed the number of elements in S . So, $n(E) \leq n(S)$.

Dividing both sides of this inequality by the positive number $n(S)$, we get:

$$\frac{n(E)}{n(S)} \leq \frac{n(S)}{n(S)}$$
$$P(E) \leq 1$$

Part 3: Combining the results

From Part 1 and Part 2, we have $P(E) \geq 0$ and $P(E) \leq 1$.

Combining these gives the final result:

$$0 \leq P(E) \leq 1$$

Step 4: Final Answer:

By using the definitions of an event and sample space, we have proven that the probability of any event E must lie in the range $[0, 1]$.

💡 Quick Tip

This property is one of the core axioms of probability. Remember the two extremes: - $P(E) = 0$ means E is an impossible event. - $P(E) = 1$ means E is a certain event. All other probabilities fall between these two values.

3. Do all parts.

a. If the ordered pairs $(2x - 3, 5)$ and $(x, y - 1)$ are equal, then find the numbers x and y .

Correct Answer: $x = 3, y = 6$

Solution:

Step 1: Understanding the Concept:

Two ordered pairs (a, b) and (c, d) are considered equal if and only if their corresponding components are equal. This means that a must be equal to c , and b must be equal to d .

Step 2: Key Formula or Approach:

Given the equality of ordered pairs $(2x - 3, 5) = (x, y - 1)$, we set up a system of two linear equations by equating the first components and the second components separately.

1. First component equation: $2x - 3 = x$
2. Second component equation: $5 = y - 1$

Step 3: Detailed Explanation:

We solve the two equations to find the values of x and y .

Solving for x :

$$2x - 3 = x$$

Subtract x from both sides:

$$2x - x - 3 = 0$$

$$x - 3 = 0$$

Add 3 to both sides:

$$x = 3$$

Solving for y :

$$5 = y - 1$$

Add 1 to both sides:

$$5 + 1 = y$$

$$y = 6$$

Step 4: Final Answer:

The values are $x = 3$ and $y = 6$.

 Quick Tip

The principle of equality of ordered pairs is fundamental and extends to vectors and matrices. Always equate corresponding elements to form separate, simpler equations to solve.

b. Obtain the differential equation of the family of curves $y = \frac{2ce^{2x}}{1+ce^{2x}}$.

Correct Answer: $\frac{dy}{dx} = y(2 - y)$ or $\frac{dy}{dx} = 2y - y^2$

Solution:

Step 1: Understanding the Concept:

To find the differential equation for a family of curves, we need to eliminate the arbitrary constant (in this case, ' c ') from the given equation. This is typically done by differentiating the equation and then using the original and differentiated equations to eliminate the constant.

Step 2: Key Formula or Approach:

The strategy is to first rearrange the given equation to express the constant ' c ' (or a term containing ' c ') in terms of x and y . Then, differentiate the original equation with respect to x and substitute the expression for the constant to get an equation involving only x , y , and derivatives of y .

Step 3: Detailed Explanation:

The given equation is:

$$y = \frac{2ce^{2x}}{1 + ce^{2x}}$$

First, let's try to isolate the term containing the constant 'c'.

$$y(1 + ce^{2x}) = 2ce^{2x}$$

$$y + yce^{2x} = 2ce^{2x}$$

$$y = 2ce^{2x} - yce^{2x}$$

$$y = ce^{2x}(2 - y)$$

From this, we can express the term ce^{2x} as:

$$ce^{2x} = \frac{y}{2 - y} \quad \text{---(1)}$$

Now, let's differentiate the original equation $y = \frac{2ce^{2x}}{1+ce^{2x}}$ with respect to x using the quotient rule: $\frac{d}{dx} \left(\frac{u}{v} \right) = \frac{vu' - uv'}{v^2}$.

Let $u = 2ce^{2x}$ and $v = 1 + ce^{2x}$.

Then $u' = 4ce^{2x}$ and $v' = 2ce^{2x}$.

$$\frac{dy}{dx} = \frac{(1 + ce^{2x})(4ce^{2x}) - (2ce^{2x})(2ce^{2x})}{(1 + ce^{2x})^2}$$

$$\frac{dy}{dx} = \frac{4ce^{2x} + 4(ce^{2x})^2 - 4(ce^{2x})^2}{(1 + ce^{2x})^2}$$

$$\frac{dy}{dx} = \frac{4ce^{2x}}{(1 + ce^{2x})^2} \quad \text{---(2)}$$

Now, substitute the expression for ce^{2x} from equation (1) into equation (2):

$$\frac{dy}{dx} = \frac{4 \left(\frac{y}{2-y} \right)}{\left(1 + \frac{y}{2-y} \right)^2}$$

Simplify the denominator:

$$1 + \frac{y}{2-y} = \frac{(2-y) + y}{2-y} = \frac{2}{2-y}$$

Substitute this back into the equation for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{\frac{4y}{2-y}}{\left(\frac{2}{2-y} \right)^2} = \frac{\frac{4y}{2-y}}{\frac{4}{(2-y)^2}}$$

$$\frac{dy}{dx} = \frac{4y}{2-y} \times \frac{(2-y)^2}{4}$$

$$\frac{dy}{dx} = y(2-y)$$

Step 4: Final Answer:

The required differential equation is $\frac{dy}{dx} = 2y - y^2$.

 Quick Tip

When forming a differential equation, the primary goal is to eliminate the arbitrary constant. It's often easier to first algebraically manipulate the equation to isolate the constant term before differentiating. This can simplify the substitution step significantly.

c. The modulus of two vectors $\vec{a}$ and $\vec{b}$ are $\sqrt{3}$ and 4 respectively, and $\vec{a} \cdot \vec{b} = 6$. Then find the angle between the vectors $\vec{a}$ and $\vec{b}$.

Correct Answer: $\theta = \frac{\pi}{6}$ radians or 30°

Solution:

Step 1: Understanding the Concept:

The dot product (or scalar product) of two vectors is a scalar quantity that relates the magnitudes of the vectors and the cosine of the angle between them. This relationship can be used to find the angle if the magnitudes and the dot product are known.

Step 2: Key Formula or Approach:

The formula for the dot product of two vectors $\vec{a}$ and $\vec{b}$ is:

$$\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta$$

where $|\vec{a}|$ and $|\vec{b}|$ are the magnitudes (modulus) of the vectors and θ is the angle between them. We can rearrange this formula to solve for $\cos \theta$.

Step 3: Detailed Explanation:

We are given the following information:

- Modulus of $\vec{a}$: $|\vec{a}| = \sqrt{3}$
- Modulus of $\vec{b}$: $|\vec{b}| = 4$
- Dot product: $\vec{a} \cdot \vec{b} = 6$

We substitute these values into the dot product formula:

$$6 = (\sqrt{3})(4) \cos \theta$$

Now, we solve for $\cos \theta$:

$$\cos \theta = \frac{6}{4\sqrt{3}}$$

Simplify the fraction:

$$\cos \theta = \frac{3}{2\sqrt{3}}$$

To rationalize the denominator, multiply the numerator and denominator by $\sqrt{3}$, or recognize that $3 = \sqrt{3} \times \sqrt{3}$:

$$\cos \theta = \frac{\sqrt{3} \times \sqrt{3}}{2\sqrt{3}} = \frac{\sqrt{3}}{2}$$

Now, we find the angle θ whose cosine is $\frac{\sqrt{3}}{2}$. The principal value for the angle is:

$$\theta = \arccos\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6} \text{ radians}$$

This is equivalent to 30° .

Step 4: Final Answer:

The angle between the vectors $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{6}$ or 30° .

💡 Quick Tip

The dot product formula is a crucial tool for problems involving angles between vectors. Remember $\cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$. If the dot product is positive, the angle is acute; if negative, obtuse; if zero, the vectors are orthogonal.

4. If the matrices $A = \begin{bmatrix} 3 & \sqrt{3} & 2 \\ 4 & 2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1/4 \\ 0 & 0 \\ 1/2 & 1/8 \end{bmatrix}$, then prove that $(A')'B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

Solution:

Step 1: Understanding the Concept:

This problem involves matrix operations, specifically the transpose of a matrix and matrix multiplication. The goal is to compute the left-hand side (LHS) of the equation and show that it is equal to the right-hand side (RHS).

Step 2: Key Formula or Approach:

We will use two key properties of matrices: 1. **Double Transpose Property:** The transpose of the transpose of a matrix is the original matrix itself. That is, $(A')' = A$. 2. **Matrix Multiplication:** To find the element in the i-th row and j-th column of the product matrix AB , we take the dot product of the i-th row of A and the j-th column of B .

Step 3: Detailed Explanation:

First, we simplify the left-hand side (LHS) of the equation to be proved.

$$\text{LHS} = (A')'B$$

Using the double transpose property, we know that $(A')' = A$. So, the equation we need to prove becomes:

$$AB = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Now we perform the matrix multiplication $A \times B$.

Matrix A has dimensions 2×3 , and matrix B has dimensions 3×2 . The product AB will be a 2×2 matrix.

$$AB = \begin{bmatrix} 3 & \sqrt{3} & 2 \\ 4 & 2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1/4 \\ 0 & 0 \\ 1/2 & 1/8 \end{bmatrix}$$

Let's calculate each element of the resulting matrix:

- **Element (1,1):** (Row 1 of A) · (Column 1 of B)

$$(3)(0) + (\sqrt{3})(0) + (2)(1/2) = 0 + 0 + 1 = 1$$

- **Element (1,2):** (Row 1 of A) · (Column 2 of B)

$$(3)(1/4) + (\sqrt{3})(0) + (2)(1/8) = \frac{3}{4} + 0 + \frac{2}{8} = \frac{3}{4} + \frac{1}{4} = \frac{4}{4} = 1$$

- **Element (2,1):** (Row 2 of A) · (Column 1 of B)

$$(4)(0) + (2)(0) + (0)(1/2) = 0 + 0 + 0 = 0$$

- **Element (2,2):** (Row 2 of A) · (Column 2 of B)

$$(4)(1/4) + (2)(0) + (0)(1/8) = 1 + 0 + 0 = 1$$

Assembling the resulting matrix:

$$AB = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

This is equal to the right-hand side (RHS) of the equation.

Step 4: Final Answer:

Since LHS = RHS, the statement $(A')'B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is proved.

💡 Quick Tip

Before starting a complex matrix calculation, always check for properties that can simplify the expression, such as $(A')' = A$ or $(AB)' = B'A'$. This can save a significant amount of time and reduce the chances of calculation errors.

4. Do all parts.

a. Find the area of a parallelogram whose adjacent sides are the vectors $\vec{a} = \hat{i} - \hat{j} + 3\hat{k}$ and $\vec{b} = 2\hat{i} - 7\hat{j} + \hat{k}$.

Correct Answer: $15\sqrt{2}$ square units

Solution:

Step 1: Understanding the Concept:

The area of a parallelogram formed by two adjacent vectors $\vec{a}$ and $\vec{b}$ is given by the magnitude of their cross product, i.e., Area = $|\vec{a} \times \vec{b}|$.

Step 2: Key Formula or Approach:

1. Compute the cross product $\vec{a} \times \vec{b}$.
2. Calculate the magnitude of the resulting vector.

The cross product is calculated using the determinant of a matrix:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

Step 3: Detailed Explanation:

Given vectors:

$$\vec{a} = 1\hat{i} - 1\hat{j} + 3\hat{k}$$

$$\vec{b} = 2\hat{i} - 7\hat{j} + 1\hat{k}$$

First, we find the cross product $\vec{a} \times \vec{b}$:

$$\begin{aligned} \vec{a} \times \vec{b} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 3 \\ 2 & -7 & 1 \end{vmatrix} \\ &= \hat{i}((-1)(1) - (3)(-7)) - \hat{j}((1)(1) - (3)(2)) + \hat{k}((1)(-7) - (-1)(2)) \\ &= \hat{i}(-1 + 21) - \hat{j}(1 - 6) + \hat{k}(-7 + 2) \\ &= 20\hat{i} + 5\hat{j} - 5\hat{k} \end{aligned}$$

Next, we find the magnitude of this new vector:

$$\begin{aligned} |\vec{a} \times \vec{b}| &= \sqrt{(20)^2 + (5)^2 + (-5)^2} \\ &= \sqrt{400 + 25 + 25} \\ &= \sqrt{450} = \sqrt{225 \times 2} = 15\sqrt{2} \end{aligned}$$

Step 4: Final Answer:

The area of the parallelogram is $15\sqrt{2}$ square units.

 Quick Tip

When calculating determinants for cross products, be very careful with the signs, especially for the $\hat{j}$ component which is preceded by a minus sign. A small sign error can lead to a completely different result.

b. A box is formed by a 3 m x 8 m rectangular steel-sheet on cutting the squares of length x m from its each corner to form the box without cover. Then find the maximum volume of the box so formed.

Correct Answer: $\frac{200}{27} \text{ m}^3$

Solution:

Step 1: Understanding the Concept:

This is an optimization problem that requires the use of differential calculus. We need to express the volume of the box as a function of the side length ' x ' of the cut-out squares, and then find the value of ' x ' that maximizes this volume.

Step 2: Key Formula or Approach:

1. Determine the dimensions (length, width, height) of the resulting open box in terms of x .
2. Write the volume function $V(x)$.
3. Find the derivative $V'(x)$ and set it to zero to find critical points.
4. Use the second derivative test, $V''(x)$, to confirm that the critical point corresponds to a maximum.

Step 3: Detailed Explanation:

The original sheet has dimensions $L = 8$ m and $W = 3$ m.

When squares of side x are cut from each corner, the sheet is folded up. The dimensions of the resulting box will be:

- Height: $h = x$ - Length: $l = 8 - 2x$ - Width: $w = 3 - 2x$ The volume of the box is $V(x) = l \cdot w \cdot h$.

$$V(x) = (8 - 2x)(3 - 2x)x$$

$$V(x) = (24 - 16x - 6x + 4x^2)x$$

$$V(x) = 4x^3 - 22x^2 + 24x$$

For the dimensions to be positive, we must have $x > 0$, $3 - 2x > 0 \implies x < 1.5$, and $8 - 2x > 0 \implies x < 4$. The valid domain for x is $0 < x < 1.5$.

Now, we find the first derivative to find critical points:

$$V'(x) = \frac{dV}{dx} = 12x^2 - 44x + 24$$

Set $V'(x) = 0$:

$$12x^2 - 44x + 24 = 0$$

Divide by 4:

$$3x^2 - 11x + 6 = 0$$

Factor the quadratic equation:

$$(3x - 2)(x - 3) = 0$$

The possible values for x are $x = \frac{2}{3}$ or $x = 3$.

From our domain $0 < x < 1.5$, the only valid solution is $x = \frac{2}{3}$.

To confirm this is a maximum, we use the second derivative test:

$$V''(x) = 24x - 44$$

$$V''\left(\frac{2}{3}\right) = 24\left(\frac{2}{3}\right) - 44 = 16 - 44 = -28$$

Since $V''\left(\frac{2}{3}\right) < 0$, the volume is maximum at $x = \frac{2}{3}$.

Calculate the maximum volume:

$$\begin{aligned} V\left(\frac{2}{3}\right) &= 4\left(\frac{2}{3}\right)^3 - 22\left(\frac{2}{3}\right)^2 + 24\left(\frac{2}{3}\right) \\ &= 4\left(\frac{8}{27}\right) - 22\left(\frac{4}{9}\right) + 16 \\ &= \frac{32}{27} - \frac{88}{9} + 16 = \frac{32 - 264 + 432}{27} = \frac{200}{27} \end{aligned}$$

Step 4: Final Answer:

The maximum volume of the box is $\frac{200}{27}$ cubic meters.

 Quick Tip

In optimization problems involving geometric shapes, always determine the valid domain for the variable first. This can quickly eliminate extraneous solutions found from the derivative, saving time and preventing errors.

c. A person has a contract of construction. The probability of being a strike is 0.65. The probabilities of completing the construction work on time in both conditions are 0.80 and 0.32 whether the strike is not happened and it is happened respectively. Then find the probability of completing the construction work in due time.

Correct Answer: 0.488

Solution:

Step 1: Understanding the Concept:

This problem can be solved using the Law of Total Probability. This law is used to find the probability of an event when it depends on two or more mutually exclusive and exhaustive

events.

Step 2: Key Formula or Approach:

Let C be the event of completing the work on time.

Let S be the event that a strike happens.

Let S' be the event that a strike does not happen.

The Law of Total Probability states:

$$P(C) = P(C \cap S) + P(C \cap S')$$

Using the conditional probability formula $P(A \cap B) = P(A|B)P(B)$, this becomes:

$$P(C) = P(C|S)P(S) + P(C|S')P(S')$$

Step 3: Detailed Explanation:

From the problem statement, we can extract the following probabilities:

- Probability of a strike: $P(S) = 0.65$ - Probability of no strike: $P(S') = 1 - P(S) = 1 - 0.65 = 0.35$ - Probability of completing on time, given no strike: $P(C|S') = 0.80$ - Probability of completing on time, given a strike happened: $P(C|S) = 0.32$

Now, we apply the Law of Total Probability to find $P(C)$:

$$P(C) = P(C|S) \cdot P(S) + P(C|S') \cdot P(S')$$

$$P(C) = (0.32)(0.65) + (0.80)(0.35)$$

Calculate the individual terms:

$$(0.32)(0.65) = 0.208$$

$$(0.80)(0.35) = 0.280$$

Add them together:

$$P(C) = 0.208 + 0.280 = 0.488$$

Step 4: Final Answer:

The probability of completing the construction work in due time is 0.488.

💡 Quick Tip

Drawing a probability tree can be very helpful for visualizing problems involving conditional probabilities and the law of total probability. The branches would split first into 'Strike' / 'No Strike', and then each of those would split into 'Complete on Time' / 'Not on Time'.

d. Find the area of a triangle $\triangle ABC$ whose vertices are $A(1, 1, 1)$, $B(1, 2, 3)$ and $C(2, 3, 1)$.

Correct Answer: $\frac{\sqrt{21}}{2}$ square units

Solution:

Step 1: Understanding the Concept:

The area of a triangle formed by three points in 3D space can be calculated using the vector cross product. The area is half the magnitude of the cross product of two vectors that form adjacent sides of the triangle.

Step 2: Key Formula or Approach:

The area of $\triangle ABC$ is given by the formula:

$$\text{Area} = \frac{1}{2} |\vec{AB} \times \vec{AC}|$$

We need to: 1. Find the position vectors $\vec{AB}$ and $\vec{AC}$. 2. Calculate their cross product $\vec{AB} \times \vec{AC}$. 3. Find the magnitude of the resulting vector and divide by 2.

Step 3: Detailed Explanation:

The given vertices are A(1, 1, 1), B(1, 2, 3), and C(2, 3, 1).

First, find the vectors representing two sides of the triangle:

$$\vec{AB} = \text{Position Vector of B} - \text{Position Vector of A}$$

$$\vec{AB} = (1 - 1)\hat{i} + (2 - 1)\hat{j} + (3 - 1)\hat{k} = 0\hat{i} + 1\hat{j} + 2\hat{k}$$

$$\vec{AC} = \text{Position Vector of C} - \text{Position Vector of A}$$

$$\vec{AC} = (2 - 1)\hat{i} + (3 - 1)\hat{j} + (1 - 1)\hat{k} = 1\hat{i} + 2\hat{j} + 0\hat{k}$$

Next, compute the cross product $\vec{AB} \times \vec{AC}$:

$$\begin{aligned}\vec{AB} \times \vec{AC} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{vmatrix} \\ &= \hat{i}((1)(0) - (2)(2)) - \hat{j}((0)(0) - (2)(1)) + \hat{k}((0)(2) - (1)(1)) \\ &= \hat{i}(0 - 4) - \hat{j}(0 - 2) + \hat{k}(0 - 1) \\ &= -4\hat{i} + 2\hat{j} - \hat{k}\end{aligned}$$

Now, find the magnitude of this vector:

$$\begin{aligned}|\vec{AB} \times \vec{AC}| &= \sqrt{(-4)^2 + (2)^2 + (-1)^2} \\ &= \sqrt{16 + 4 + 1} = \sqrt{21}\end{aligned}$$

Finally, the area of the triangle is half of this magnitude:

$$\text{Area of } \triangle ABC = \frac{1}{2} |\vec{AB} \times \vec{AC}| = \frac{\sqrt{21}}{2}$$

Step 4: Final Answer:

The area of the triangle is $\frac{\sqrt{21}}{2}$ square units.

💡 Quick Tip

You can use any two adjacent vectors, for example, $\vec{BA}$ and $\vec{BC}$. The magnitude of their cross product will be the same, so the area will also be the same. Choose the vectors that simplify calculations (e.g., those with zero components).

5. Do all parts.

a. If the function $f : [0, \frac{\pi}{2}] \rightarrow \mathbb{R}$ is given by $f(x) = \sin x$ and function $g : [0, \frac{\pi}{2}] \rightarrow \mathbb{R}$ is given by $g(x) = \cos x$, then prove that f and g are one-one but $f + g$ is not one-one.

Solution:

Step 1: Understanding the Concept:

A function is **one-one (injective)** if distinct elements in the domain map to distinct elements in the codomain. That is, if $f(x_1) = f(x_2)$, then $x_1 = x_2$. A common way to prove a differentiable function is one-one on an interval is to show that it is strictly monotonic (either strictly increasing or strictly decreasing) on that interval. This can be checked using its first derivative.

Step 2: Detailed Explanation:

Part 1: Proving $f(x) = \sin(x)$ is one-one on $[0, \frac{\pi}{2}]$.

To check for monotonicity, we find the first derivative of $f(x)$.

$$f'(x) = \frac{d}{dx}(\sin x) = \cos x$$

In the interval $(0, \frac{\pi}{2})$, $f'(x) = \cos x > 0$.

Since the derivative is positive, the function $f(x)$ is strictly increasing on $[0, \frac{\pi}{2}]$.

A strictly monotonic function is always one-one. Therefore, $f(x)$ is one-one.

Part 2: Proving $g(x) = \cos(x)$ is one-one on $[0, \frac{\pi}{2}]$.

We find the first derivative of $g(x)$.

$$g'(x) = \frac{d}{dx}(\cos x) = -\sin x$$

In the interval $(0, \frac{\pi}{2})$, $\sin x > 0$, which means $g'(x) = -\sin x < 0$.

Since the derivative is negative, the function $g(x)$ is strictly decreasing on $[0, \frac{\pi}{2}]$.

A strictly monotonic function is always one-one. Therefore, $g(x)$ is one-one.

Part 3: Proving $(f + g)(x)$ is not one-one on $[0, \frac{\pi}{2}]$.

Let $h(x) = (f + g)(x) = \sin x + \cos x$.

To prove that $h(x)$ is not one-one, we need to find two distinct values x_1 and x_2 in the domain $[0, \frac{\pi}{2}]$ such that $h(x_1) = h(x_2)$.

Let's evaluate $h(x)$ at the endpoints of the interval:

$$h(0) = \sin(0) + \cos(0) = 0 + 1 = 1$$

$$h\left(\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) + \cos\left(\frac{\pi}{2}\right) = 1 + 0 = 1$$

We have $h(0) = h\left(\frac{\pi}{2}\right) = 1$, but $0 \neq \frac{\pi}{2}$.

Since two different inputs (0 and $\frac{\pi}{2}$) produce the same output (1), the function $f + g$ is not one-one.

Step 3: Final Answer:

We have shown that $f(x)$ and $g(x)$ are strictly monotonic and therefore one-one on the given interval, while $f + g$ is not one-one as it has the same value at the endpoints. Hence proved.

💡 Quick Tip

To prove a function is one-one, showing it's strictly monotonic (using its derivative) is a robust method. To prove a function is **not** one-one, you only need to find a single counterexample: two different inputs that give the same output. Testing boundary points is often a good first step.

b. Minimize $Z = 3x + 2y$ by graphical method under the following constraints:
 $x + 2y \leq 10$, $3x + y \leq 15$, $x \geq 0$, $y \geq 0$.

Correct Answer: The minimum value of Z is 0, which occurs at the point (0, 0).

Solution:

Step 1: Understanding the Concept:

This is a Linear Programming Problem (LPP). The graphical method involves plotting the constraints to identify the feasible region, finding the coordinates of the corner points of this region, and then evaluating the objective function Z at these corner points. The minimum (or maximum) value of Z will occur at one of these points.

Step 2: Detailed Explanation:

1. Graph the Constraints:

We treat the inequalities as equations to draw the boundary lines.

- **Line 1:** $x + 2y = 10$. It passes through (10, 0) and (0, 5).
- **Line 2:** $3x + y = 15$. It passes through (5, 0) and (0, 15).
- **Line 3:** $x = 0$ (the y-axis).
- **Line 4:** $y = 0$ (the x-axis).

The inequalities $x \geq 0$ and $y \geq 0$ restrict the feasible region to the first quadrant. For the other two inequalities, shading towards the origin (0,0) satisfies them.

2. Identify the Feasible Region and Corner Points:

The feasible region is the polygon formed by the intersection of these half-planes. The corner points (vertices) are:

- **O (0, 0):** The origin.
- **A (5, 0):** The x-intercept of the line $3x + y = 15$.
- **C (0, 5):** The y-intercept of the line $x + 2y = 10$.
- **B:** The intersection of lines $x + 2y = 10$ and $3x + y = 15$.

To find point B, we solve the system of equations:

$$x + 2y = 10 \quad \text{---(i)}$$

$$3x + y = 15 \quad \text{---(ii)}$$

From (ii), $y = 15 - 3x$. Substitute this into (i):

$$x + 2(15 - 3x) = 10$$

$$x + 30 - 6x = 10$$

$$-5x = -20 \implies x = 4$$

Substitute $x = 4$ back into $y = 15 - 3x$:

$$y = 15 - 3(4) = 15 - 12 = 3$$

So, point B is (4, 3).

3. Evaluate Z at each Corner Point:

The objective function is $Z = 3x + 2y$.

- At O(0, 0): $Z = 3(0) + 2(0) = 0$
- At A(5, 0): $Z = 3(5) + 2(0) = 15$
- At B(4, 3): $Z = 3(4) + 2(3) = 12 + 6 = 18$
- At C(0, 5): $Z = 3(0) + 2(5) = 10$

Step 3: Final Answer:

Comparing the values of Z, the minimum value is 0, which occurs at the corner point O(0, 0).

💡 Quick Tip

For LPP graphical solutions, always find the intersection points of the constraint lines accurately. Create a table to list the corner points and the corresponding values of the objective function Z. This organized approach minimizes calculation errors and clearly shows the optimal solution.

c. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, then show that $A^2 - 5A + 7I = O$. Using this, obtain A^{-1} .

Correct Answer: $A^{-1} = \frac{1}{7} \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$

Solution:

Step 1: Understanding the Concept:

This problem uses the Cayley-Hamilton theorem, which states that a square matrix satisfies its own characteristic equation. First, we must verify the given matrix equation. Then, we can manipulate this equation to find the inverse of the matrix A without using the standard adjoint-determinant method.

Step 2: Detailed Explanation:

Part 1: Show that $A^2 - 5A + 7I = O$

First, calculate A^2 :

$$A^2 = A \cdot A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} (3)(3) + (1)(-1) & (3)(1) + (1)(2) \\ (-1)(3) + (2)(-1) & (-1)(1) + (2)(2) \end{bmatrix}$$
$$A^2 = \begin{bmatrix} 9-1 & 3+2 \\ -3-2 & -1+4 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$$

Next, calculate $5A$:

$$5A = 5 \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix}$$

And $7I$, where I is the 2x2 identity matrix:

$$7I = 7 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

Now, substitute these into the expression:

$$A^2 - 5A + 7I = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$
$$= \begin{bmatrix} 8-15+7 & 5-5+0 \\ -5-(-5)+0 & 3-10+7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

Thus, the equation is verified.

Part 2: Obtain A^{-1} using the equation

Start with the verified equation:

$$A^2 - 5A + 7I = O$$

Multiply the entire equation by A^{-1} from the left (or right):

$$A^{-1}(A^2 - 5A + 7I) = A^{-1}O$$
$$A^{-1}A^2 - 5A^{-1}A + 7A^{-1}I = O$$

Using the properties $A^{-1}A = I$ and $A^{-1}I = A^{-1}$:

$$(A^{-1}A)A - 5I + 7A^{-1} = O$$

$$IA - 5I + 7A^{-1} = O$$

$$A - 5I + 7A^{-1} = O$$

Now, isolate A^{-1} :

$$7A^{-1} = 5I - A$$

$$A^{-1} = \frac{1}{7}(5I - A)$$

Substitute the matrices for I and A:

$$A^{-1} = \frac{1}{7} \left(5 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \right)$$

$$A^{-1} = \frac{1}{7} \left(\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \right)$$

$$A^{-1} = \frac{1}{7} \begin{bmatrix} 5-3 & 0-1 \\ 0-(-1) & 5-2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{7} \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$$

Step 3: Final Answer:

The inverse of matrix A is $A^{-1} = \frac{1}{7} \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$.

 Quick Tip

Using the Cayley-Hamilton theorem to find the inverse of a matrix is a very efficient method. Once you have the characteristic equation in the form of a matrix polynomial that equals zero, you can easily isolate the A^{-1} term by multiplying the entire equation by A^{-1} .

d. Differentiate the function $x^{\cos x}$ with respect to x.

Correct Answer: $x^{\cos x} \left(\frac{\cos x}{x} - \sin x \ln x \right)$

Solution:

Step 1: Understanding the Concept:

The function is of the form $y = [f(x)]^{g(x)}$, where both the base and the exponent are functions of x. This type of function is best differentiated using logarithmic differentiation.

Step 2: Key Formula or Approach:

The process of logarithmic differentiation involves: 1. Let $y = x^{\cos x}$. 2. Take the natural

logarithm ($\ln$) of both sides. 3. Use the logarithm property $\ln(a^b) = b \ln(a)$ to simplify. 4. Differentiate both sides implicitly with respect to x , using the product rule on the right side. 5. Solve for $\frac{dy}{dx}$.

Step 3: Detailed Explanation:

Let the given function be:

$$y = x^{\cos x}$$

Taking the natural logarithm of both sides:

$$\ln y = \ln(x^{\cos x})$$

Using the power rule for logarithms:

$$\ln y = (\cos x)(\ln x)$$

Now, differentiate both sides with respect to x . For the left side, we use the chain rule. For the right side, we use the product rule.

$$\begin{aligned}\frac{d}{dx}(\ln y) &= \frac{d}{dx}((\cos x)(\ln x)) \\ \frac{1}{y} \frac{dy}{dx} &= \left(\frac{d}{dx}(\cos x) \right) (\ln x) + (\cos x) \left(\frac{d}{dx}(\ln x) \right) \\ \frac{1}{y} \frac{dy}{dx} &= (-\sin x)(\ln x) + (\cos x) \left(\frac{1}{x} \right) \\ \frac{1}{y} \frac{dy}{dx} &= \frac{\cos x}{x} - \sin x \ln x\end{aligned}$$

To find $\frac{dy}{dx}$, multiply both sides by y :

$$\frac{dy}{dx} = y \left(\frac{\cos x}{x} - \sin x \ln x \right)$$

Finally, substitute back the original expression for y :

$$\frac{dy}{dx} = x^{\cos x} \left(\frac{\cos x}{x} - \sin x \ln x \right)$$

Step 4: Final Answer:

The derivative of $x^{\cos x}$ is $x^{\cos x} \left(\frac{\cos x}{x} - \sin x \ln x \right)$.

💡 Quick Tip

Logarithmic differentiation is the go-to method for any function of the form $f(x)^{g(x)}$. Don't forget to multiply by 'y' at the end and substitute the original function back into the final expression.

e. Find the differential equation of the family of curves denoted by $y = a \sin(x + b)$, where a and b are arbitrary constants.

Correct Answer: $\frac{d^2y}{dx^2} + y = 0$

Solution:

Step 1: Understanding the Concept:

To form a differential equation from a general solution that contains 'n' arbitrary constants, we must differentiate the equation 'n' times. This creates a system of 'n+1' equations (the original plus the 'n' derivatives). The goal is to algebraically manipulate these equations to eliminate all 'n' arbitrary constants.

Step 2: Key Formula or Approach:

The given equation has two arbitrary constants, 'a' and 'b'. Therefore, we need to differentiate the equation two times to obtain a second-order differential equation free of 'a' and 'b'.

Step 3: Detailed Explanation:

The given equation for the family of curves is:

$$y = a \sin(x + b) \quad \text{---(1)}$$

Differentiate equation (1) with respect to x:

$$\frac{dy}{dx} = \frac{d}{dx}[a \sin(x + b)]$$

Using the chain rule:

$$\frac{dy}{dx} = a \cos(x + b) \cdot \frac{d}{dx}(x + b) = a \cos(x + b) \quad \text{---(2)}$$

Since the constants are not yet eliminated, we differentiate again with respect to x:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}[a \cos(x + b)]$$

Using the chain rule again:

$$\frac{d^2y}{dx^2} = -a \sin(x + b) \cdot \frac{d}{dx}(x + b) = -a \sin(x + b) \quad \text{---(3)}$$

Now we have three equations. We can eliminate the constants 'a' and 'b' by comparing equations (1) and (3). From equation (1), we have $y = a \sin(x + b)$. From equation (3), we have $\frac{d^2y}{dx^2} = -[a \sin(x + b)]$. By substituting the expression for y from (1) into (3), we get:

$$\frac{d^2y}{dx^2} = -y$$

Rearranging the terms to form the standard differential equation format:

$$\frac{d^2y}{dx^2} + y = 0$$

This is the required differential equation, as it is free from the arbitrary constants 'a' and 'b'.

Step 4: Final Answer:

The differential equation of the family of curves is $\frac{d^2y}{dx^2} + y = 0$.

💡 Quick Tip

The number of arbitrary constants in the general solution of a differential equation determines the order of the differential equation. Two constants mean you'll need to differentiate twice to eliminate them, resulting in a second-order equation.

6. Do all parts.

a. Find the shortest distance between the lines $\vec{r} = \hat{i} + 2\hat{j} - 4\hat{k} + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$ and $\vec{r} = 3\hat{i} + 3\hat{j} - 5\hat{k} + \mu(2\hat{i} + 3\hat{j} + 6\hat{k})$.

Correct Answer: $\frac{\sqrt{293}}{7}$ units

Solution:

Step 1: Understanding the Concept:

The given equations are of the form $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$. We first check if the lines are parallel by comparing their direction vectors, $\vec{b}_1$ and $\vec{b}_2$. If they are parallel ($\vec{b}_1 = \vec{b}_2 = \vec{b}$), we use the formula for the shortest distance between parallel lines.

Step 2: Key Formula or Approach:

The direction vectors are $\vec{b}_1 = 2\hat{i} + 3\hat{j} + 6\hat{k}$ and $\vec{b}_2 = 2\hat{i} + 3\hat{j} + 6\hat{k}$. Since $\vec{b}_1 = \vec{b}_2$, the lines are parallel.

The formula for the shortest distance between two parallel lines is:

$$d = \frac{|(\vec{a}_2 - \vec{a}_1) \times \vec{b}|}{|\vec{b}|}$$

where $\vec{a}_1$ and $\vec{a}_2$ are position vectors of points on the lines and $\vec{b}$ is the common direction vector.

Step 3: Detailed Explanation:

From the given equations, we have:

$$\vec{a}_1 = \hat{i} + 2\hat{j} - 4\hat{k}$$

$$\vec{a}_2 = 3\hat{i} + 3\hat{j} - 5\hat{k}$$

$$\vec{b} = 2\hat{i} + 3\hat{j} + 6\hat{k}$$

First, calculate the vector difference $\vec{a}_2 - \vec{a}_1$:

$$\vec{a}_2 - \vec{a}_1 = (3 - 1)\hat{i} + (3 - 2)\hat{j} + (-5 - (-4))\hat{k}$$

$$\vec{a}_2 - \vec{a}_1 = 2\hat{i} + \hat{j} - \hat{k}$$

Next, calculate the cross product $(\vec{a}_2 - \vec{a}_1) \times \vec{b}$:

$$\begin{aligned} (\vec{a}_2 - \vec{a}_1) \times \vec{b} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -1 \\ 2 & 3 & 6 \end{vmatrix} \\ &= \hat{i}((1)(6) - (-1)(3)) - \hat{j}((2)(6) - (-1)(2)) + \hat{k}((2)(3) - (1)(2)) \\ &= \hat{i}(6 + 3) - \hat{j}(12 + 2) + \hat{k}(6 - 2) \\ &= 9\hat{i} - 14\hat{j} + 4\hat{k} \end{aligned}$$

Now, find the magnitude of this cross product:

$$|(\vec{a}_2 - \vec{a}_1) \times \vec{b}| = \sqrt{9^2 + (-14)^2 + 4^2} = \sqrt{81 + 196 + 16} = \sqrt{293}$$

Then, find the magnitude of the direction vector $\vec{b}$:

$$|\vec{b}| = \sqrt{2^2 + 3^2 + 6^2} = \sqrt{4 + 9 + 36} = \sqrt{49} = 7$$

Finally, calculate the shortest distance:

$$d = \frac{\sqrt{293}}{7}$$

Step 4: Final Answer:

The shortest distance between the two parallel lines is $\frac{\sqrt{293}}{7}$ units.

💡 Quick Tip

The first step in any shortest distance problem between two lines is to check if the direction vectors are parallel. If they are, you must use the parallel line formula. If not, you use the formula for skew lines, which is $d = \frac{|(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2)|}{|\vec{b}_1 \times \vec{b}_2|}$. Using the wrong formula is a common mistake.

b. A car is started from a point P at time $t = 0$ and is stopped at the point Q. The distance x metre covered by the car in t second is given by $x = t^2(2 - \frac{t}{3})$. Find the time required by the car to reach at point Q and also find the distance between P and Q.

Correct Answer: Time = 4 seconds, Distance = $\frac{32}{3}$ metres

Solution:

Step 1: Understanding the Concept:

The problem describes the motion of a car with its distance from the start point given as a

function of time. The car stopping at point Q means its velocity becomes zero at that point in time. We need to use calculus to find the velocity from the distance function.

Step 2: Key Formula or Approach:

1. Express the distance x as a polynomial of t .
2. Velocity v is the first derivative of distance with respect to time: $v = \frac{dx}{dt}$.
3. Set the velocity $v = 0$ to find the time t when the car stops.
4. Substitute this value of t back into the distance equation to find the total distance covered.

Step 3: Detailed Explanation:

The distance function is given by:

$$x(t) = t^2 \left(2 - \frac{t}{3} \right) = 2t^2 - \frac{t^3}{3}$$

To find the velocity, we differentiate $x(t)$ with respect to t :

$$v(t) = \frac{dx}{dt} = \frac{d}{dt} \left(2t^2 - \frac{t^3}{3} \right)$$

$$v(t) = 4t - \frac{3t^2}{3} = 4t - t^2$$

The car stops at point Q, so its velocity is zero. We set $v(t) = 0$:

$$4t - t^2 = 0$$

$$t(4 - t) = 0$$

This gives two possible times: $t = 0$ and $t = 4$.

$t = 0$ corresponds to the starting point P. Therefore, the time required for the car to reach point Q is $t = 4$ seconds.

Now, we find the distance between P and Q by substituting $t = 4$ into the distance equation:

$$x(4) = 2(4)^2 - \frac{(4)^3}{3}$$

$$x(4) = 2(16) - \frac{64}{3}$$

$$x(4) = 32 - \frac{64}{3} = \frac{96 - 64}{3} = \frac{32}{3}$$

The distance between P and Q is $\frac{32}{3}$ metres.

Step 4: Final Answer:

The time required to reach point Q is 4 seconds, and the distance between P and Q is $\frac{32}{3}$ metres.

💡 Quick Tip

Remember the relationship between displacement (x), velocity (v), and acceleration (a): $v = \frac{dx}{dt}$ and $a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$. The word "stopped" or "at rest" always implies that the velocity is zero.

c. Find the area enclosed by the curve $y = x^2$ and the line $y = 16$.

Correct Answer: $\frac{256}{3}$ square units

Solution:

Step 1: Understanding the Concept:

The area enclosed between two curves is found by integrating the difference between the upper function and the lower function over the interval defined by their points of intersection.

Step 2: Key Formula or Approach:

1. Find the points of intersection by setting the two equations equal to each other.
2. Set up the definite integral: Area $A = \int_a^b (y_{\text{upper}} - y_{\text{lower}}) dx$, where a and b are the x-coordinates of the intersection points.
3. Evaluate the integral.

Step 3: Detailed Explanation:

The given curves are $y = x^2$ (a parabola) and $y = 16$ (a horizontal line).

To find the points of intersection, we set $x^2 = 16$. This gives $x = \pm 4$. The points of intersection are $(-4, 16)$ and $(4, 16)$.

In the interval $[-4, 4]$, the line $y = 16$ is above the parabola $y = x^2$. So, $y_{\text{upper}} = 16$ and $y_{\text{lower}} = x^2$.

The area is given by the integral:

$$A = \int_{-4}^4 (16 - x^2) dx$$

Since the integrand $(16 - x^2)$ is an even function and the limits of integration are symmetric about 0, we can simplify the calculation:

$$A = 2 \int_0^4 (16 - x^2) dx$$

Now, we evaluate the integral:

$$\begin{aligned} A &= 2 \left[16x - \frac{x^3}{3} \right]_0^4 \\ A &= 2 \left(\left(16(4) - \frac{4^3}{3} \right) - \left(16(0) - \frac{0^3}{3} \right) \right) \\ A &= 2 \left(64 - \frac{64}{3} - 0 \right) \\ A &= 2 \left(\frac{192 - 64}{3} \right) = 2 \left(\frac{128}{3} \right) = \frac{256}{3} \end{aligned}$$

Step 4: Final Answer:

The area enclosed by the curve and the line is $\frac{256}{3}$ square units.

 Quick Tip

Sketching the graphs is always a good first step. It helps you visualize which function is on top and what the limits of integration should be. Also, look for symmetries to simplify the integration process.

d. If $y = \sin^{-1} x$, then prove that $(1 - x^2) \frac{d^2 y}{dx^2} - x \frac{dy}{dx} = 0$.

Solution:

Step 1: Understanding the Concept:

This problem requires finding the first and second derivatives of the given function and then substituting them into the provided differential equation to prove the identity. An alternative, often simpler method, involves implicit differentiation to avoid complex expressions.

Step 2: Key Formula or Approach:

1. Differentiate $y = \sin^{-1} x$ to find $\frac{dy}{dx}$. 2. Rearrange the equation to remove the square root. 3. Differentiate the rearranged equation again with respect to x using the product rule to find a relationship involving $\frac{d^2 y}{dx^2}$. 4. Simplify the result to match the target equation.

Step 3: Detailed Explanation:

Given the function:

$$y = \sin^{-1} x$$

First, we find the first derivative:

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

To avoid dealing with the square root in the next differentiation, we can rearrange this equation:

$$\sqrt{1-x^2} \frac{dy}{dx} = 1$$

Squaring both sides gives:

$$(1-x^2) \left(\frac{dy}{dx} \right)^2 = 1$$

Now, we differentiate this equation with respect to x using the product rule on the left side:

$$\frac{d}{dx} \left[(1-x^2) \left(\frac{dy}{dx} \right)^2 \right] = \frac{d}{dx} (1)$$

$$\frac{d}{dx} (1-x^2) \cdot \left(\frac{dy}{dx} \right)^2 + (1-x^2) \cdot \frac{d}{dx} \left(\frac{dy}{dx} \right)^2 = 0$$

$$(-2x) \left(\frac{dy}{dx} \right)^2 + (1-x^2) \cdot 2 \left(\frac{dy}{dx} \right) \frac{d^2 y}{dx^2} = 0$$

We can divide the entire equation by $2\frac{dy}{dx}$ (since $\frac{dy}{dx} \neq 0$ for $|x| < 1$):

$$-x\frac{dy}{dx} + (1-x^2)\frac{d^2y}{dx^2} = 0$$

Rearranging the terms to match the required form:

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 0$$

Step 4: Final Answer:

We have successfully derived the given differential equation starting from $y = \sin^{-1} x$. Thus, the statement is proven.

💡 Quick Tip

For proofs involving second derivatives of inverse trigonometric or implicit functions, it is often much cleaner to differentiate once, rearrange the equation to eliminate fractions or roots, and then differentiate again implicitly. This avoids complicated applications of the quotient or chain rule.

e. If E_1 and E_2 are mutually exclusive events, then prove that $P(E_1) + P(E_2) = P(E_1 \cup E_2) + P(E_1 \cap E_2)$.

Solution:

Step 1: Understanding the Concept:

We need to prove a relationship between the probabilities of two events, given that they are mutually exclusive. Mutually exclusive events are events that cannot happen at the same time.

Step 2: Key Formula or Approach:

1. Use the definition of mutually exclusive events: If E_1 and E_2 are mutually exclusive, their intersection is the empty set, i.e., $E_1 \cap E_2 = \emptyset$. 2. Use the axiom of probability for the empty set: $P(\emptyset) = 0$. 3. Use the addition axiom for mutually exclusive events: $P(E_1 \cup E_2) = P(E_1) + P(E_2)$. 4. Manipulate these facts to arrive at the desired equation.

Step 3: Detailed Explanation:

The proof is based on the axioms of probability and the definition of mutually exclusive events. By definition, if events E_1 and E_2 are mutually exclusive, then they cannot occur simultaneously. This means their intersection is the null or empty event.

$$E_1 \cap E_2 = \emptyset$$

The probability of the empty event is zero.

$$P(E_1 \cap E_2) = P(\emptyset) = 0$$

One of the fundamental axioms of probability (the addition rule for mutually exclusive events) states that:

$$P(E_1 \cup E_2) = P(E_1) + P(E_2)$$

This equation holds true precisely because the events are mutually exclusive.

Now, let's look at the equation we need to prove:

$$P(E_1) + P(E_2) = P(E_1 \cup E_2) + P(E_1 \cap E_2)$$

We can start from the addition axiom for mutually exclusive events:

$$P(E_1) + P(E_2) = P(E_1 \cup E_2)$$

We can add 0 to the right side of the equation without changing its value:

$$P(E_1) + P(E_2) = P(E_1 \cup E_2) + 0$$

Since we have already established that $P(E_1 \cap E_2) = 0$ for mutually exclusive events, we can substitute this into the equation:

$$P(E_1) + P(E_2) = P(E_1 \cup E_2) + P(E_1 \cap E_2)$$

This completes the proof.

Step 4: Final Answer:

Using the definition and axioms related to mutually exclusive events, we have proven that the given identity holds true.

💡 Quick Tip

The general addition rule for any two events is $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$. Rearranging this gives $P(E_1) + P(E_2) = P(E_1 \cup E_2) + P(E_1 \cap E_2)$. This identity is actually true for *any* pair of events, not just mutually exclusive ones. The condition of being mutually exclusive simplifies $P(E_1 \cap E_2)$ to 0.

7. Do any one part.

a. If $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$, then find A^{-1} .

Correct Answer: $A^{-1} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

Solution:

Step 1: Understanding the Concept:

To find the inverse of a square matrix A , we use the formula $A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$, where $\det(A)$ is the determinant of A and $\text{adj}(A)$ is the adjugate (or adjoint) of A . The inverse exists only if the determinant is non-zero.

Step 2: Key Formula or Approach:

The process involves three main steps: 1. Calculate the determinant of A , $\det(A)$. 2. Find the matrix of cofactors, then transpose it to get the adjugate matrix, $\text{adj}(A)$. 3. Substitute these values into the inverse formula.

Step 3: Detailed Explanation:**1. Calculate the Determinant of A:**

$$\begin{aligned} \det(A) &= 1(4 \cdot 4 - 3 \cdot 3) - 3(1 \cdot 4 - 1 \cdot 3) + 3(1 \cdot 3 - 1 \cdot 4) \\ &= 1(16 - 9) - 3(4 - 3) + 3(3 - 4) \\ &= 1(7) - 3(1) + 3(-1) \\ &= 7 - 3 - 3 = 1 \end{aligned}$$

Since $\det(A) = 1 \neq 0$, the inverse exists.

2. Find the Adjugate of A:

The adjugate is the transpose of the cofactor matrix. Let's find the cofactors C_{ij} :

- $C_{11} = +(16 - 9) = 7$
- $C_{12} = -(4 - 3) = -1$
- $C_{13} = +(3 - 4) = -1$
- $C_{21} = -(3 \cdot 4 - 3 \cdot 3) = -(12 - 9) = -3$
- $C_{22} = +(1 \cdot 4 - 1 \cdot 3) = +(4 - 3) = 1$
- $C_{23} = -(1 \cdot 3 - 1 \cdot 3) = -(3 - 3) = 0$
- $C_{31} = +(3 \cdot 3 - 4 \cdot 3) = +(9 - 12) = -3$
- $C_{32} = -(1 \cdot 3 - 1 \cdot 3) = -(3 - 3) = 0$
- $C_{33} = +(1 \cdot 4 - 1 \cdot 3) = +(4 - 3) = 1$

The cofactor matrix is $C = \begin{bmatrix} 7 & -1 & -1 \\ -3 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$.

The adjugate is the transpose of C :

$$\text{adj}(A) = C^T = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

3. Calculate the Inverse:

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A) = \frac{1}{1} \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Step 4: Final Answer:

The inverse of the matrix A is $A^{-1} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$.

💡 Quick Tip

When calculating cofactors, be meticulous with the sign pattern: $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$. A single sign error will make the entire inverse incorrect. Double-check your determinant calculation first, as if it's zero, you can stop immediately.

b. Solve the following system of equations by matrix method:

$$\begin{aligned} x + y + 2z &= 1 \\ 3x + 2y + z &= 7 \\ 2x + y + 3z &= 2 \end{aligned}$$

Correct Answer: $x = 2, y = 1, z = -1$

Solution:

Step 1: Understanding the Concept:

A system of linear equations can be represented in matrix form as $AX = B$, where A is the matrix of coefficients, X is the column matrix of variables, and B is the column matrix of constants. The solution is found by pre-multiplying both sides by the inverse of A, which gives $X = A^{-1}B$.

Step 2: Key Formula or Approach:

1. Write the system in the form $AX = B$. 2. Calculate the inverse of the coefficient matrix, A^{-1} . 3. Multiply A^{-1} by B to find the solution matrix X.

Step 3: Detailed Explanation:

1. Represent the system in matrix form:

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix}$$

The equation is $AX = B$, and the solution is $X = A^{-1}B$.

2. Find the inverse of A: First, calculate the determinant of A:

$$\begin{aligned}\det(A) &= 1(2 \cdot 3 - 1 \cdot 1) - 1(3 \cdot 3 - 1 \cdot 2) + 2(3 \cdot 1 - 2 \cdot 2) \\ &= 1(6 - 1) - 1(9 - 2) + 2(3 - 4) \\ &= 1(5) - 1(7) + 2(-1) = 5 - 7 - 2 = -4\end{aligned}$$

Since $\det(A) \neq 0$, a unique solution exists.

Next, find the adjugate of A. The cofactors are:

- $C_{11} = +(6 - 1) = 5$
- $C_{12} = -(9 - 2) = -7$
- $C_{13} = +(3 - 4) = -1$
- $C_{21} = -(1 \cdot 3 - 2 \cdot 1) = -(3 - 2) = -1$
- $C_{22} = +(1 \cdot 3 - 2 \cdot 2) = +(3 - 4) = -1$
- $C_{23} = -(1 \cdot 1 - 1 \cdot 2) = -(1 - 2) = 1$
- $C_{31} = +(1 \cdot 1 - 2 \cdot 2) = +(1 - 4) = -3$
- $C_{32} = -(1 \cdot 1 - 2 \cdot 3) = -(1 - 6) = 5$
- $C_{33} = +(1 \cdot 2 - 1 \cdot 3) = +(2 - 3) = -1$

The adjugate matrix is $\text{adj}(A) = C^T = \begin{bmatrix} 5 & -1 & -3 \\ -7 & -1 & 5 \\ -1 & 1 & -1 \end{bmatrix}$.

The inverse is $A^{-1} = \frac{1}{-4} \begin{bmatrix} 5 & -1 & -3 \\ -7 & -1 & 5 \\ -1 & 1 & -1 \end{bmatrix}$.

3. Solve for X:

$$X = A^{-1}B = -\frac{1}{4} \begin{bmatrix} 5 & -1 & -3 \\ -7 & -1 & 5 \\ -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix}$$

$$X = -\frac{1}{4} \begin{bmatrix} (5)(1) + (-1)(7) + (-3)(2) \\ (-7)(1) + (-1)(7) + (5)(2) \\ (-1)(1) + (1)(7) + (-1)(2) \end{bmatrix}$$

$$X = -\frac{1}{4} \begin{bmatrix} 5 - 7 - 6 \\ -7 - 7 + 10 \\ -1 + 7 - 2 \end{bmatrix} = -\frac{1}{4} \begin{bmatrix} -8 \\ -4 \\ 4 \end{bmatrix}$$

$$X = \begin{bmatrix} (-8)/(-4) \\ (-4)/(-4) \\ (4)/(-4) \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

So, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$.

Step 4: Final Answer:

The solution to the system of equations is $x = 2$, $y = 1$, and $z = -1$.

💡 Quick Tip

After finding the solution (x, y, z) , it is a very good practice to substitute these values back into one of the original equations to quickly verify your answer. For example, using the first equation: $2 + 1 + 2(-1) = 3 - 2 = 1$. This confirms the solution is correct.

8. Do any one part.

a. Solve : $\frac{dy}{dx} = \frac{x+2y-3}{2x+y-3}$.

Correct Answer: $x + y - 2 = C(x - y)^3$

Solution:

Step 1: Understanding the Concept:

This is a non-homogeneous first-order differential equation of the form $\frac{dy}{dx} = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$. Since $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$ ($\frac{1}{2} \neq \frac{2}{1}$), the lines intersect. This type of equation can be reduced to a homogeneous differential equation by shifting the origin to the point of intersection of the lines $x + 2y - 3 = 0$ and $2x + y - 3 = 0$.

Step 2: Key Formula or Approach:

1. Find the point of intersection (h, k) of the two lines in the numerator and denominator.
2. Substitute $x = X + h$ and $y = Y + k$. This transforms the equation into a homogeneous form $\frac{dY}{dX} = f\left(\frac{Y}{X}\right)$.
3. Solve the homogeneous equation using the substitution $Y = vX$.
4. Separate variables, integrate, and substitute back to get the solution in terms of x and y .

Step 3: Detailed Explanation:

First, find the intersection of the lines $x + 2y - 3 = 0$ and $2x + y - 3 = 0$.

Solving these simultaneously:

$$x + 2y = 3 \quad (1)$$

$$2x + y = 3 \quad (2)$$

Multiplying (2) by 2 gives $4x + 2y = 6$. Subtracting (1) from this gives $3x = 3 \Rightarrow x = 1$.

Substituting $x = 1$ into (2) gives $2(1) + y = 3 \Rightarrow y = 1$.

The point of intersection is $(h, k) = (1, 1)$.

Now, let $x = X + 1$ and $y = Y + 1$. Then $dx = dX$ and $dy = dY$, so $\frac{dy}{dx} = \frac{dY}{dX}$.

Substituting into the original equation:

$$\frac{dY}{dX} = \frac{(X+1) + 2(Y+1) - 3}{2(X+1) + (Y+1) - 3} = \frac{X+2Y}{2X+Y}$$

This is a homogeneous equation. Let $Y = vX$. Then $\frac{dY}{dX} = v + X \frac{dv}{dX}$.

$$v + X \frac{dv}{dX} = \frac{X + 2vX}{2X + vX} = \frac{1 + 2v}{2 + v}$$

$$X \frac{dv}{dX} = \frac{1 + 2v}{2 + v} - v = \frac{1 + 2v - v(2 + v)}{2 + v} = \frac{1 - v^2}{2 + v}$$

Separate the variables:

$$\frac{2 + v}{1 - v^2} dv = \frac{dX}{X}$$

Using partial fractions for the left side: $\frac{2+v}{(1-v)(1+v)} = \frac{A}{1-v} + \frac{B}{1+v}$.

This gives $2 + v = A(1 + v) + B(1 - v)$. For $v = 1$, $3 = 2A \Rightarrow A = 3/2$. For $v = -1$, $1 = 2B \Rightarrow B = 1/2$.

So the equation becomes:

$$\int \left(\frac{3/2}{1-v} + \frac{1/2}{1+v} \right) dv = \int \frac{dX}{X}$$

Integrating both sides:

$$-\frac{3}{2} \ln |1 - v| + \frac{1}{2} \ln |1 + v| = \ln |X| + C_1$$

$$\ln |1 + v| - 3 \ln |1 - v| = 2 \ln |X| + 2C_1$$

$$\ln \left| \frac{1 + v}{(1 - v)^3} \right| = \ln(X^2) + C_2$$

$$\frac{1 + v}{(1 - v)^3} = e^{C_2} X^2 = C X^2$$

Substitute back $v = Y/X$:

$$\frac{1 + Y/X}{(1 - Y/X)^3} = C X^2 \Rightarrow \frac{X(X + Y)}{X^3} \cdot \frac{X^3}{(X - Y)^3} = C X^2 \Rightarrow \frac{X + Y}{(X - Y)^3} = C$$

Finally, substitute back $X = x - 1$ and $Y = y - 1$:

$$\frac{(x - 1) + (y - 1)}{((x - 1) - (y - 1))^3} = C \Rightarrow \frac{x + y - 2}{(x - y)^3} = C$$

$$x + y - 2 = C(x - y)^3$$

Step 4: Final Answer:

The general solution to the differential equation is $x + y - 2 = C(x - y)^3$.

💡 Quick Tip

To quickly check if a DE of the form $\frac{dy}{dx} = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$ is reducible to homogeneous form, check the ratio of coefficients. If $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$, the method of shifting the origin to the intersection point will work. If the ratios are equal, the equation is solved using the substitution $z = a_1x + b_1y$.

b. Solve : $(1 + x^2)\frac{dy}{dx} + 2xy - 4x^2 = 0$.

Correct Answer: $y(1 + x^2) = \frac{4}{3}x^3 + C$

Solution:

Step 1: Understanding the Concept:

This is a first-order linear differential equation. It can be solved by first arranging it into the standard form $\frac{dy}{dx} + P(x)y = Q(x)$ and then using the method of integrating factors.

Step 2: Key Formula or Approach:

1. Rewrite the equation in the standard form $\frac{dy}{dx} + P(x)y = Q(x)$.
2. Calculate the Integrating Factor (I.F.) using the formula $\text{I.F.} = e^{\int P(x)dx}$.
3. The solution is given by the formula $y \cdot (\text{I.F.}) = \int Q(x) \cdot (\text{I.F.})dx + C$.

Step 3: Detailed Explanation:

The given equation is:

$$(1 + x^2)\frac{dy}{dx} + 2xy - 4x^2 = 0$$

First, rearrange it to isolate the term with $\frac{dy}{dx}$ and y :

$$(1 + x^2)\frac{dy}{dx} + 2xy = 4x^2$$

Now, divide by $(1 + x^2)$ to get the standard form:

$$\frac{dy}{dx} + \frac{2x}{1 + x^2}y = \frac{4x^2}{1 + x^2}$$

Comparing this with $\frac{dy}{dx} + P(x)y = Q(x)$, we have:

$$P(x) = \frac{2x}{1 + x^2} \quad \text{and} \quad Q(x) = \frac{4x^2}{1 + x^2}$$

Next, we calculate the integrating factor:

$$\text{I.F.} = e^{\int P(x)dx} = e^{\int \frac{2x}{1+x^2}dx}$$

Let $u = 1 + x^2$, so $du = 2xdx$. The integral becomes $\int \frac{1}{u}du = \ln|u| = \ln(1 + x^2)$.

$$\text{I.F.} = e^{\ln(1+x^2)} = 1 + x^2$$

Now, we use the solution formula:

$$y \cdot (1 + x^2) = \int \frac{4x^2}{1 + x^2} \cdot (1 + x^2)dx + C$$

$$y \cdot (1 + x^2) = \int 4x^2 dx + C$$

Evaluate the integral:

$$y \cdot (1 + x^2) = 4 \left(\frac{x^3}{3} \right) + C$$
$$y(1 + x^2) = \frac{4}{3}x^3 + C$$

Step 4: Final Answer:

The general solution to the differential equation is $y(1 + x^2) = \frac{4}{3}x^3 + C$.

 Quick Tip

The most crucial step in solving linear differential equations is to correctly identify $P(x)$ and $Q(x)$ by converting the equation to its standard form. A common error is failing to divide the entire equation by the coefficient of $\frac{dy}{dx}$. Always ensure the coefficient of $\frac{dy}{dx}$ is 1 before proceeding.

9. Do any one part.

a. Prove that $\int_0^\pi \sqrt{\left(\frac{1+\cos 2x}{2}\right)} dx = 2$.

Solution:

Step 1: Understanding the Concept:

This problem involves evaluating a definite integral. The key to solving it is to simplify the integrand using a trigonometric identity before performing the integration. We must also be careful with the square root, as $\sqrt{u^2} = |u|$.

Step 2: Key Formula or Approach:

We will use the half-angle trigonometric identity:

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

After substitution, the integral will involve $|\cos x|$, which needs to be handled by splitting the integral over intervals where $\cos x$ is positive and negative.

Step 3: Detailed Explanation:

Let the integral be I .

$$I = \int_0^\pi \sqrt{\frac{1 + \cos 2x}{2}} dx$$

Using the identity $\cos^2 x = \frac{1 + \cos 2x}{2}$, we get:

$$I = \int_0^\pi \sqrt{\cos^2 x} dx$$

$$I = \int_0^{\pi} |\cos x| dx$$

Now, we need to consider the sign of $\cos x$ in the interval $[0, \pi]$.

- For $x \in [0, \frac{\pi}{2}]$, $\cos x \geq 0$, so $|\cos x| = \cos x$.
- For $x \in (\frac{\pi}{2}, \pi]$, $\cos x \leq 0$, so $|\cos x| = -\cos x$.

We split the integral at $x = \frac{\pi}{2}$:

$$I = \int_0^{\pi/2} \cos x dx + \int_{\pi/2}^{\pi} (-\cos x) dx$$

Now, we evaluate each part:

$$\int_0^{\pi/2} \cos x dx = [\sin x]_0^{\pi/2} = \sin(\frac{\pi}{2}) - \sin(0) = 1 - 0 = 1$$

$$\int_{\pi/2}^{\pi} (-\cos x) dx = [-\sin x]_{\pi/2}^{\pi} = (-\sin(\pi)) - (-\sin(\frac{\pi}{2})) = 0 - (-1) = 1$$

Adding the results of the two parts:

$$I = 1 + 1 = 2$$

Step 4: Final Answer:

We have shown that the value of the integral is 2. Hence proved.

💡 Quick Tip

A common mistake is to simplify $\sqrt{\cos^2 x}$ to just $\cos x$. Always remember that the square root of a squared quantity is its absolute value, i.e., $\sqrt{u^2} = |u|$. This often requires splitting the integral into multiple parts based on the sign of the function inside the absolute value.

b. Prove that $\int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}} = \frac{\pi}{12}$.

Solution:

Step 1: Understanding the Concept:

This integral is a classic example that can be solved efficiently using a special property of definite integrals, often called the "King's property". This property is particularly useful when the integrand has a certain symmetry with respect to the sum of the limits.

Step 2: Key Formula or Approach:

We will use the property of definite integrals:

$$\int_a^b f(x) dx = \int_a^b f(a+b-x) dx$$

The strategy is to apply this property to the given integral, add the new form of the integral to the original one, and simplify the result.

Step 3: Detailed Explanation:

Let the given integral be I:

$$I = \int_{\pi/6}^{\pi/3} \frac{1}{1 + \sqrt{\tan x}} dx \quad \text{---(1)}$$

Here, $a = \frac{\pi}{6}$ and $b = \frac{\pi}{3}$. Their sum is $a + b = \frac{\pi}{6} + \frac{\pi}{3} = \frac{\pi + 2\pi}{6} = \frac{\pi}{2}$.

Applying the property $\int_a^b f(x)dx = \int_a^b f(a + b - x)dx$, we get:

$$I = \int_{\pi/6}^{\pi/3} \frac{1}{1 + \sqrt{\tan(\frac{\pi}{2} - x)}} dx$$

Using the trigonometric identity $\tan(\frac{\pi}{2} - x) = \cot x$:

$$I = \int_{\pi/6}^{\pi/3} \frac{1}{1 + \sqrt{\cot x}} dx$$

Rewrite $\sqrt{\cot x}$ as $\frac{1}{\sqrt{\tan x}}$:

$$I = \int_{\pi/6}^{\pi/3} \frac{1}{1 + \frac{1}{\sqrt{\tan x}}} dx = \int_{\pi/6}^{\pi/3} \frac{1}{\frac{\sqrt{\tan x} + 1}{\sqrt{\tan x}}} dx$$

$$I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\tan x}}{1 + \sqrt{\tan x}} dx \quad \text{---(2)}$$

Now, add equation (1) and equation (2):

$$I + I = \int_{\pi/6}^{\pi/3} \frac{1}{1 + \sqrt{\tan x}} dx + \int_{\pi/6}^{\pi/3} \frac{\sqrt{\tan x}}{1 + \sqrt{\tan x}} dx$$

$$2I = \int_{\pi/6}^{\pi/3} \left(\frac{1 + \sqrt{\tan x}}{1 + \sqrt{\tan x}} \right) dx$$

$$2I = \int_{\pi/6}^{\pi/3} 1 dx$$

Now, evaluate the simple integral:

$$2I = [x]_{\pi/6}^{\pi/3} = \frac{\pi}{3} - \frac{\pi}{6} = \frac{2\pi - \pi}{6} = \frac{\pi}{6}$$

Finally, solve for I:

$$I = \frac{1}{2} \cdot \frac{\pi}{6} = \frac{\pi}{12}$$

Step 4: Final Answer:

We have shown that the value of the integral is $\frac{\pi}{12}$. Hence proved.

💡 Quick Tip

Whenever you see an integral of the form $\int_a^b \frac{f(x)}{f(x)+f(a+b-x)} dx$, its value is always $\frac{b-a}{2}$.

The given problem can be converted to this form by writing $\tan x = \frac{\sin x}{\cos x}$, which makes this a very quick check. Here, $a + b = \pi/2$, so $f(a + b - x) = f(\pi/2 - x)$.
