UP Board Class 12 Mathematics - 324(JB) - 2025 Question Paper with Solutions

Time Allowed : 3 Hours	Maximum Marks : 100	Total Questions : 9

General Instructions

Read the following instructions very carefully and strictly follow them:

1. There are in all nine questions in this question paper. All questions are compulsory. In the beginning of each question, the number of parts to be attempted are clearly mentioned. Marks allotted to the questions are indicated against them. Start solving from the first question and proceed to solve till the last one. Do not waste your time over a question you cannot solve.

1. Do all parts.
Select the correct option of each part and write it on your answer-book.

a. The relation R, defined by R = { (T_1, T_2) : T_1 is similar to T_2 }, in the set A of all triangles, is

(A) reflexive and symmetric, but not transitive
(B) reflexive and transitive, but not symmetric
(C) symmetric and transitive, but not reflexive
(D) reflexive, symmetric and also transitive

Correct Answer: (D) reflexive, symmetric and also transitive

Solution:

The relation is defined as

$$R = \{(T_1, T_2) : T_1 \text{ is similar to } T_2\}, \quad T_1, T_2 \in A,$$

where A is the set of all triangles.
Reflexive: Every triangle is similar to itself since all its angles are equal to themselves and the ratio of corresponding sides is 1. Thus $(T_1, T_1) \in R$.
Symmetric: If T_1 is similar to T_2, then by definition T_2 is also similar to T_1. Hence $(T_2, T_1) \in R$.
Transitive: If T_1 is similar to T_2 and T_2 is similar to T_3, then T_1 is similar to T_3. Thus $(T_1, T_3) \in R$.
Therefore, the relation is reflexive, symmetric, and transitive.

$$\boxed{\text{The relation is reflexive, symmetric and transitive (equivalence relation).}}$$

Quick Tip

Relations like "is similar to," "is congruent to," and "is equal to" are classic examples of equivalence relations in mathematics because they satisfy all three conditions: reflexivity, symmetry, and transitivity. Memorizing these examples can save time in exams.

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b. Let [x] represents the greatest integer which is less than or equal to x. Then the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = [x]$ will be

(A) One-one and onto
(B) One-one, but not onto
(C) Onto, but not one-one
(D) Neither one-one nor onto

Correct Answer: (D) Neither one-one nor onto

Solution:

The function is $f(x) = [x]$, where $[x]$ is the greatest integer less than or equal to x.
One-one: Suppose $x_1 = 1.2$ and $x_2 = 1.7$. Then

$$f(1.2) = [1.2] = 1, \quad f(1.7) = [1.7] = 1.$$

Here $x_1 \neq x_2$ but $f(x_1) = f(x_2)$. Hence the function is not injective.
Onto: The codomain is $\mathbb{R}$, but the outputs of $f(x)$ are always integers, i.e., the range is $\mathbb{Z}$. Since $\mathbb{Z} \subsetneq \mathbb{R}$, many real numbers (e.g., 2.5) are not attained as values of f. Thus the function is not surjective.
Conclusion: The function is neither one-one nor onto.

$$\boxed{\text{Neither one-one nor onto}}$$

Step 3: Final Answer:
The function $f(x) = [x]$ is neither one-one nor onto.

Quick Tip

The graph of the greatest integer function is a step function. You can use the Horizontal Line Test to quickly check for injectivity and surjectivity. A horizontal line intersects the graph at multiple points (in fact, infinitely many points for integer y-values), so it's not one-one. The graph only covers integer y-values, not the entire real plane, so it's not onto.

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c. The order of the differential equation $\left(\frac{d^3y}{dx^3}\right)^2 + \log x \left(\frac{d^2y}{dx^2}\right)^3 + 5y = \cos x$ **will be**

(A) 2
(B) 3
(C) 5
(D) 6

Correct Answer: (B) 3

Solution:

The given equation is

$$\left(\frac{d^3y}{dx^3}\right)^2 + \log x \left(\frac{d^2y}{dx^2}\right)^3 + 5y = \cos x.$$

The derivatives present are:

$$\frac{d^3y}{dx^3}, \quad \frac{d^2y}{dx^2}, \quad \text{and } y.$$

The order of a differential equation is determined only by the highest derivative appearing in it, irrespective of the power in which the derivative occurs.
Here the highest derivative is $\frac{d^3y}{dx^3}$, which is of order 3.
Final Answer:

$$\boxed{3}$$

Quick Tip

To find the order of a differential equation, simply look for the derivative with the most "primes" or the highest number in the 'd' term (like d^ny). Don't be distracted by the powers (degree) or other terms like $\log x$ or $\cos x$. The order is determined solely by the highest derivative itself.

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d. If the coordinates of the points P and Q are respectively (2, 3, 0) and (–1, –2, –4), the vector $\vec{PQ}$ will be

(A) $-3\hat{i} - 5\hat{j} + 4\hat{k}$
(B) $3\hat{i} + 5\hat{j} + 4\hat{k}$
(C) $-3\hat{i} - 5\hat{j} - 4\hat{k}$
(D) $3\hat{i} + 5\hat{j} - 4\hat{k}$

Correct Answer: (C) $-3\hat{i} - 5\hat{j} - 4\hat{k}$

Solution:

Coordinates of P are $(2, 3, 0)$ and of Q are $(-1, -2, -4)$. The position vector of P is

$$\vec{OP} = 2\hat{i} + 3\hat{j} + 0\hat{k},$$

and the position vector of Q is

$$\vec{OQ} = -1\hat{i} - 2\hat{j} - 4\hat{k}.$$

The vector $\vec{PQ}$ is given by

$$\vec{PQ} = \vec{OQ} - \vec{OP}.$$

$$\vec{PQ} = \left(-1\hat{i} - 2\hat{j} - 4\hat{k}\right) - \left(2\hat{i} + 3\hat{j} + 0\hat{k}\right).$$

$$\vec{PQ} = -3\hat{i} - 5\hat{j} - 4\hat{k}.$$

Final Answer:

$$\boxed{-3\hat{i} - 5\hat{j} - 4\hat{k}}$$

Quick Tip

A common mistake is to subtract in the wrong order (P - Q instead of Q - P). Always remember the rule: **Terminal minus Initial**. For $\vec{PQ}$, Q is the terminal point and P is the initial point.

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e. If $2X + Y = \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix}$ and $Y = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$, then X will be

(A) $\begin{bmatrix} -1 & -1 \\ -2 & -1 \end{bmatrix}$

(B) $\begin{bmatrix} -1 & -1 \\ -1 & -2 \end{bmatrix}$

(C) $\begin{bmatrix} -2 & -1 \\ -1 & -1 \end{bmatrix}$

(D) $\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}$

Correct Answer: (A) $\begin{bmatrix} -1 & -1 \\ -2 & -1 \end{bmatrix}$

Solution:

Step 1: Understanding the Concept:

This is a matrix algebra problem. We need to solve for the unknown matrix X by treating the equation similarly to a standard algebraic equation, performing matrix subtraction and scalar multiplication.

Step 2: Key Formula or Approach:

The given equation is $2X + Y = A$, where $A = \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix}$.

To find X, we first isolate 2X:

$$2X = A - Y$$

Then, we solve for X by dividing by the scalar 2:

$$X = \frac{1}{2}(A - Y)$$

Step 3: Detailed Explanation:
First, substitute the given matrices into the equation $2X = A - Y$:

$$2X = \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$$

Perform the matrix subtraction by subtracting corresponding elements:

$$2X = \begin{bmatrix} 1-3 & 0-2 \\ -3-1 & 2-4 \end{bmatrix}$$

$$2X = \begin{bmatrix} -2 & -2 \\ -4 & -2 \end{bmatrix}$$

Now, multiply the resulting matrix by the scalar $\frac{1}{2}$ to find X:

$$X = \frac{1}{2}\begin{bmatrix} -2 & -2 \\ -4 & -2 \end{bmatrix}$$

$$X = \begin{bmatrix} \frac{1}{2}(-2) & \frac{1}{2}(-2) \\ \frac{1}{2}(-4) & \frac{1}{2}(-2) \end{bmatrix}$$

$$X = \begin{bmatrix} -1 & -1 \\ -2 & -1 \end{bmatrix}$$

Step 4: Final Answer:
The matrix X is $\begin{bmatrix} -1 & -1 \\ -2 & -1 \end{bmatrix}$. This corresponds to option (A).

> Quick Tip
>
> When solving matrix equations, always perform the addition or subtraction first before dealing with scalar multiplication or division. This helps prevent calculation errors. Double-check your arithmetic, especially with negative numbers.

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2. Do all parts.

a. Find the principal value of cosec$^{-1}(-\sqrt{2})$.

Solution:

Step 1: Understanding the Concept:
The principal value of an inverse trigonometric function is the value that lies within its defined principal value branch.

For $\text{cosec}^{-1}(x)$, the principal value branch is $[-\frac{\pi}{2}, \frac{\pi}{2}] - \{0\}$.

Step 2: Detailed Explanation:
Let $y = \text{cosec}^{-1}(-\sqrt{2})$.
By the definition of an inverse function, this implies:

$$\text{cosec}(y) = -\sqrt{2}$$

We know that $\text{cosec}(y) = \frac{1}{\sin(y)}$.
So, $\sin(y) = -\frac{1}{\sqrt{2}}$.
We know that $\sin(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$.
Since $\sin(-x) = -\sin(x)$, we can write:

$$\sin(-\frac{\pi}{4}) = -\sin(\frac{\pi}{4}) = -\frac{1}{\sqrt{2}}$$

So, $y = -\frac{\pi}{4}$.

Step 3: Final Answer:
We must check if this value lies in the principal value range of $\text{cosec}^{-1}(x)$, which is $[-\frac{\pi}{2}, \frac{\pi}{2}] - \{0\}$.
Since $-\frac{\pi}{2} \le -\frac{\pi}{4} \le \frac{\pi}{2}$ and $-\frac{\pi}{4} \neq 0$, the value is valid.
Therefore, the principal value of $\text{cosec}^{-1}(-\sqrt{2})$ is $-\frac{\pi}{4}$.

Quick Tip

To find the principal value of inverse cosecant, it's often easier to convert the problem into its sine equivalent, i.e., $\text{cosec}^{-1}(x) = \sin^{-1}(\frac{1}{x})$. Then, find the value within the range of $\sin^{-1}(x)$, which is $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

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b. Test whether the function defined by $f(x) = x^2 - \sin(x) + 5$ is continuous at $x = \pi$.

Solution:

Step 1: Understanding the Concept:
A function $f(x)$ is said to be continuous at a point $x = a$ if three conditions are met:
1. $f(a)$ is defined.
2. The limit of the function as x approaches a, $\lim_{x \to a} f(x)$, exists.
3. The limit equals the function's value: $\lim_{x \to a} f(x) = f(a)$.

Step 2: Detailed Explanation:
The given function is $f(x) = x^2 - \sin(x) + 5$. We need to test its continuity at $x = \pi$.

Condition 1: Check if $f(\pi)$ is defined.
Substitute $x = \pi$ into the function:

$$f(\pi) = (\pi)^2 - \sin(\pi) + 5$$

Since $\sin(\pi) = 0$, we get:

$$f(\pi) = \pi^2 - 0 + 5 = \pi^2 + 5$$

The function is defined at $x = \pi$.

Condition 2: Check if the limit exists.
We need to evaluate $\lim_{x\to\pi} f(x)$.

$$\lim_{x\to\pi} (x^2 - \sin(x) + 5)$$

We can apply the limit to each term separately:

$$\lim_{x\to\pi} x^2 - \lim_{x\to\pi} \sin(x) + \lim_{x\to\pi} 5$$
$$= (\pi)^2 - \sin(\pi) + 5 = \pi^2 - 0 + 5 = \pi^2 + 5$$

The limit exists.

Condition 3: Compare the limit and function value.
From our calculations, we have:

$$\lim_{x\to\pi} f(x) = \pi^2 + 5 \quad \text{and} \quad f(\pi) = \pi^2 + 5$$

Since $\lim_{x\to\pi} f(x) = f(\pi)$, all three conditions for continuity are met.

Step 3: Final Answer:
The function $f(x) = x^2 - \sin(x) + 5$ is continuous at $x = \pi$.

Quick Tip

Remember that polynomial functions (like $x^2 + 5$) and basic trigonometric functions (like $\sin x$) are continuous everywhere in their domain. The sum, difference, and product of continuous functions are also continuous. This allows for a quick check without formal limit calculations in many cases.

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c. Evaluate: $\int \mathbf{cosec}\, x(\mathbf{cosec}\, x + \cot x)\, dx$.

Solution:

Step 1: Understanding the Concept:
This problem involves integrating a trigonometric function. The strategy is to simplify the integrand and then apply standard integration formulas.

Step 2: Key Formula or Approach:
The key is to use the distributive property to expand the integrand and then use the following standard integrals:
1. $\int \text{cosec}^2 x \, dx = -\cot x + C$
2. $\int \text{cosec}\, x \cot x \, dx = -\text{cosec}\, x + C$

Step 3: Detailed Explanation:
First, expand the expression inside the integral:

$$\int \text{cosec}\, x(\text{cosec}\, x + \cot x) \, dx = \int (\text{cosec}^2 x + \text{cosec}\, x \cot x) \, dx$$

Now, we can split the integral into two parts using the sum rule for integration:

$$\int \text{cosec}^2 x \, dx + \int \text{cosec}\, x \cot x \, dx$$

Apply the standard integration formulas to each part:
For the first part: $\int \text{cosec}^2 x \, dx = -\cot x$.
For the second part: $\int \text{cosec}\, x \cot x \, dx = -\text{cosec}\, x$.
Combining the results and adding the constant of integration, C:

$$-\cot x - \text{cosec}\, x + C$$

Step 4: Final Answer:
The evaluation of the integral is $-\cot x - \text{cosec}\, x + C$.

Quick Tip
Always simplify the integrand before attempting to integrate. Recognizing standard integral forms like $\int \text{cosec}^2 x \, dx$ and $\int \text{cosec}\, x \cot x \, dx$ is crucial for quickly solving such problems. Memorizing these fundamental formulas saves a lot of time in exams.

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d. If $2P(A) = P(B) = \frac{5}{13}$ and $P(A/B) = \frac{2}{5}$, then find $P(A \cup B)$.

Solution:

Step 1: Understanding the Concept:
This question requires the use of fundamental probability rules, specifically the formula for conditional probability and the addition rule for probabilities.

Step 2: Key Formula or Approach:
1. **Conditional Probability:** $P(A/B) = \frac{P(A \cap B)}{P(B)}$
2. **Addition Rule:** $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Step 3: Detailed Explanation:
Part 1: Find $P(A)$ and $P(B)$.
We are given $2P(A) = P(B) = \frac{5}{13}$.
From this, we can deduce:

$$P(B) = \frac{5}{13}$$
$$2P(A) = \frac{5}{13} \implies P(A) = \frac{1}{2} \times \frac{5}{13} = \frac{5}{26}$$

Part 2: Find $P(A \cap B)$.
Using the conditional probability formula: $P(A/B) = \frac{P(A \cap B)}{P(B)}$.
We are given $P(A/B) = \frac{2}{5}$ and we know $P(B) = \frac{5}{13}$.
Rearranging the formula to solve for $P(A \cap B)$:

$$P(A \cap B) = P(A/B) \times P(B)$$
$$P(A \cap B) = \frac{2}{5} \times \frac{5}{13} = \frac{10}{65} = \frac{2}{13}$$

Part 3: Find $P(A \cup B)$.
Now use the addition rule for probabilities:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Substitute the values we found:

$$P(A \cup B) = \frac{5}{26} + \frac{5}{13} - \frac{2}{13}$$

To add/subtract these fractions, find a common denominator, which is 26.

$$P(A \cup B) = \frac{5}{26} + \frac{5 \times 2}{13 \times 2} - \frac{2 \times 2}{13 \times 2}$$
$$P(A \cup B) = \frac{5}{26} + \frac{10}{26} - \frac{4}{26}$$
$$P(A \cup B) = \frac{5 + 10 - 4}{26} = \frac{11}{26}$$

Step 4: Final Answer:
The value of $P(A \cup B)$ is $\frac{11}{26}$.

> Quick Tip
>
> When solving probability problems, always list the given information and identify the target variable. Then, write down the relevant formulas that connect the knowns to the unknown. This systematic approach prevents confusion and errors.

e. Find the direction-cosines of the y-axis.

Solution:

Step 1: Understanding the Concept:
Direction cosines of a line or a vector are the cosines of the angles that the line makes with the positive directions of the x, y, and z axes. These angles are typically denoted by α, β, and γ, respectively. The direction cosines are $l = \cos\alpha$, $m = \cos\beta$, and $n = \cos\gamma$.

Step 2: Detailed Explanation:
We need to find the direction cosines for the y-axis itself.
Let's determine the angles the y-axis makes with each of the coordinate axes.

1. **Angle with the x-axis (α):** The y-axis is perpendicular to the x-axis. Therefore, the angle between them is 90° or $\frac{\pi}{2}$ radians.

$$\alpha = 90^\circ$$

2. **Angle with the y-axis (β):** The y-axis makes an angle of 0° with itself.

$$\beta = 0^\circ$$

3. **Angle with the z-axis (γ):** The y-axis is perpendicular to the z-axis. Therefore, the angle between them is 90° or $\frac{\pi}{2}$ radians.

$$\gamma = 90^\circ$$

Now, we calculate the cosines of these angles to find the direction cosines (l, m, n).

$$l = \cos\alpha = \cos(90^\circ) = 0$$
$$m = \cos\beta = \cos(0^\circ) = 1$$
$$n = \cos\gamma = \cos(90^\circ) = 0$$

Step 3: Final Answer:
The direction-cosines of the y-axis are (0, 1, 0).

Quick Tip

This is a standard result worth memorizing. The direction cosines for the coordinate axes are:
- **x-axis:** (1, 0, 0) since $\alpha = 0^\circ, \beta = 90^\circ, \gamma = 90^\circ$
- **y-axis:** (0, 1, 0) since $\alpha = 90^\circ, \beta = 0^\circ, \gamma = 90^\circ$
- **z-axis:** (0, 0, 1) since $\alpha = 90^\circ, \beta = 90^\circ, \gamma = 0^\circ$

3. Do all parts.

a. Differentiate $x^{\sin x}$ with respect to x, while $x > 0$.

Correct Answer: $x^{\sin x}\left(\cos x \ln x + \frac{\sin x}{x}\right)$

Solution:

Step 1: Understanding the Concept:
The function is of the form $f(x)^{g(x)}$, where both the base and the exponent are functions of x. This type of function is best differentiated using logarithmic differentiation.

Step 2: Key Formula or Approach:
The process of logarithmic differentiation involves these steps:
1. Let $y = f(x)^{g(x)}$.
2. Take the natural logarithm (ln) of both sides: $\ln y = g(x)\ln(f(x))$.
3. Differentiate both sides with respect to x using the product rule and chain rule.
4. Solve for $\frac{dy}{dx}$ and substitute the original expression for y.
The product rule states that $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$.

Step 3: Detailed Explanation:
Let the given function be y.

$$y = x^{\sin x}$$

Taking the natural logarithm on both sides, we get:

$$\ln y = \ln(x^{\sin x})$$

Using the logarithm property $\ln(a^b) = b\ln a$, we have:

$$\ln y = (\sin x)(\ln x)$$

Now, we differentiate both sides with respect to x. The left side requires the chain rule, and the right side requires the product rule.

$$\begin{aligned}
\frac{d}{dx}(\ln y) &= \frac{d}{dx}(\sin x \cdot \ln x) \\
\frac{1}{y}\frac{dy}{dx} &= (\sin x)\cdot\frac{d}{dx}(\ln x) + (\ln x)\cdot\frac{d}{dx}(\sin x) \\
\frac{1}{y}\frac{dy}{dx} &= (\sin x)\cdot\left(\frac{1}{x}\right) + (\ln x)\cdot(\cos x) \\
\frac{1}{y}\frac{dy}{dx} &= \frac{\sin x}{x} + \cos x \ln x
\end{aligned}$$

To find $\frac{dy}{dx}$, we multiply both sides by y:

$$\frac{dy}{dx} = y\left(\frac{\sin x}{x} + \cos x \ln x\right)$$

Finally, substitute $y = x^{\sin x}$ back into the equation.

$$\frac{dy}{dx} = x^{\sin x}\left(\cos x \ln x + \frac{\sin x}{x}\right)$$

Step 4: Final Answer:
The derivative of $x^{\sin x}$ with respect to x is $x^{\sin x}\left(\cos x \ln x + \frac{\sin x}{x}\right)$.

> **Quick Tip**
>
> Logarithmic differentiation is the standard method for functions of the form $y = [f(x)]^{g(x)}$. The key is to take the natural log of both sides to bring the exponent down, then apply the product rule.

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b. Evaluate : $\int \frac{dx}{\sqrt{x^2-a^2}}$**.**

Correct Answer: $\ln|x + \sqrt{x^2 - a^2}| + C$

Solution:

Step 1: Understanding the Concept:
This question asks for the evaluation of a standard indefinite integral. The integrand is of a specific form whose result is a well-known formula in calculus.

Step 2: Key Formula or Approach:
The integral is a standard form:

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln|x + \sqrt{x^2 - a^2}| + C$$

where C is the constant of integration.
This formula can be derived using trigonometric substitution by setting $x = a\sec\theta$.

Step 3: Detailed Explanation:
This is a direct application of a standard integration formula.
Given the integral:

$$I = \int \frac{dx}{\sqrt{x^2 - a^2}}$$

By comparing this with the standard formula $\int \frac{du}{\sqrt{u^2-a^2}} = \ln|u + \sqrt{u^2 - a^2}| + C$, where $u = x$, we can directly write the result.

$$I = \ln|x + \sqrt{x^2 - a^2}| + C$$

Step 4: Final Answer:
The evaluation of the integral is $\ln|x + \sqrt{x^2 - a^2}| + C$.

> 💡 Quick Tip
>
> Memorizing the three standard integrals involving square roots is crucial for competitive exams: $\int \frac{dx}{\sqrt{a^2-x^2}}$, $\int \frac{dx}{\sqrt{x^2+a^2}}$, and $\int \frac{dx}{\sqrt{x^2-a^2}}$. This saves significant time compared to deriving them from scratch.

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c. Let a relation R be defined in the set $\mathbb{N} \times \mathbb{N}$ as follows: $(a, b)R(c, d)$ if and only if $a + d = b + c$. Prove that R is an equivalence relation.

Correct Answer: The relation R is proven to be reflexive, symmetric, and transitive, and hence it is an equivalence relation.

Solution:

Step 1: Understanding the Concept:
An equivalence relation is a binary relation on a set that satisfies three properties: reflexivity, symmetry, and transitivity. We need to prove that the given relation R on the set $\mathbb{N} \times \mathbb{N}$ satisfies all three properties.
The relation is defined as: $(a, b)R(c, d) \iff a + d = b + c$. This can also be written as $a - b = c - d$.

Step 2: Key Formula or Approach:
To prove that R is an equivalence relation, we must show:
1. **Reflexivity:** $(a, b)R(a, b)$ for all $(a, b) \in \mathbb{N} \times \mathbb{N}$.
2. **Symmetry:** If $(a, b)R(c, d)$, then $(c, d)R(a, b)$ for all $(a, b), (c, d) \in \mathbb{N} \times \mathbb{N}$.
3. **Transitivity:** If $(a, b)R(c, d)$ and $(c, d)R(e, f)$, then $(a, b)R(e, f)$ for all $(a, b), (c, d), (e, f) \in \mathbb{N} \times \mathbb{N}$.

Step 3: Detailed Explanation:
1. Proof of Reflexivity:
We need to check if $(a, b)R(a, b)$.
According to the definition of R, this means we must check if $a + b = b + a$.
Since addition is commutative in the set of natural numbers $\mathbb{N}$, the condition $a + b = b + a$ is always true.
Therefore, R is reflexive.

2. Proof of Symmetry:
Assume that $(a, b)R(c, d)$.

By the definition of R, this means $a + d = b + c$.
We need to prove that $(c, d)R(a, b)$, which means we need to show that $c + b = d + a$.
Starting with our assumption:

$$a + d = b + c$$

By the commutative property of addition, we can rewrite this as:

$$d + a = c + b$$

$$c + b = d + a$$

This is exactly the condition for $(c, d)R(a, b)$.
Therefore, R is symmetric.

3. Proof of Transitivity:
Assume that $(a, b)R(c, d)$ and $(c, d)R(e, f)$.
From $(a, b)R(c, d)$, we have:

$$a + d = b + c \quad \cdots (1)$$

From $(c, d)R(e, f)$, we have:

$$c + f = d + e \quad \cdots (2)$$

We need to prove that $(a, b)R(e, f)$, which means we need to show that $a + f = b + e$.
Adding equation (1) and equation (2):

$$(a + d) + (c + f) = (b + c) + (d + e)$$

$$a + d + c + f = b + c + d + e$$

We can cancel c and d from both sides of the equation.

$$a + f = b + e$$

This is the condition for $(a, b)R(e, f)$.
Therefore, R is transitive.

Step 4: Final Answer:
Since the relation R is reflexive, symmetric, and transitive, it is an equivalence relation on the set $\mathbb{N} \times \mathbb{N}$.

💡 Quick Tip

When proving an equivalence relation, always address the three properties—reflexive, symmetric, and transitive—separately and clearly. For relations involving equations, algebraic manipulation is the key to proving symmetry and transitivity.

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d. Find the projection of the vector $\vec{a} = 2\hat{i} + 3\hat{j} + 2\hat{k}$ on the vector $\vec{b} = \hat{i} + 2\hat{j} + \hat{k}$.

Correct Answer: $\frac{5}{3}(\hat{i} + 2\hat{j} + \hat{k})$ or $\frac{5}{3}\hat{i} + \frac{10}{3}\hat{j} + \frac{5}{3}\hat{k}$

Solution:

Step 1: Understanding the Concept:
The projection of a vector $\vec{a}$ onto a non-zero vector $\vec{b}$ is the orthogonal projection of $\vec{a}$ onto a straight line parallel to $\vec{b}$. This results in a new vector that is parallel to $\vec{b}$.

Step 2: Key Formula or Approach:
The vector projection of $\vec{a}$ onto $\vec{b}$ is given by the formula:

$$\text{proj}_{\vec{b}}\vec{a} = \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2}\right)\vec{b}$$

This formula requires us to compute the dot product $\vec{a} \cdot \vec{b}$ and the squared magnitude of $\vec{b}$.

Step 3: Detailed Explanation:
We are given the vectors:

$$\vec{a} = 2\hat{\imath} + 3\hat{\jmath} + 2\hat{k}$$
$$\vec{b} = \hat{\imath} + 2\hat{\jmath} + \hat{k}$$

First, we calculate the dot product $\vec{a} \cdot \vec{b}$.

$$\vec{a} \cdot \vec{b} = (2)(1) + (3)(2) + (2)(1)$$
$$\vec{a} \cdot \vec{b} = 2 + 6 + 2 = 10$$

Next, we calculate the magnitude of $\vec{b}$, denoted as $|\vec{b}|$.

$$|\vec{b}| = \sqrt{(1)^2 + (2)^2 + (1)^2}$$
$$|\vec{b}| = \sqrt{1 + 4 + 1} = \sqrt{6}$$

Now, we find the square of the magnitude, $|\vec{b}|^2$.

$$|\vec{b}|^2 = (\sqrt{6})^2 = 6$$

Finally, we substitute these values into the projection formula.

$$\text{proj}_{\vec{b}}\vec{a} = \left(\frac{10}{6}\right)(\hat{\imath} + 2\hat{\jmath} + \hat{k})$$

Simplifying the scalar fraction:

$$\text{proj}_{\vec{b}}\vec{a} = \frac{5}{3}(\hat{\imath} + 2\hat{\jmath} + \hat{k})$$

We can also distribute the scalar to get the vector components:

$$\text{proj}_{\vec{b}}\vec{a} = \frac{5}{3}\hat{\imath} + \frac{10}{3}\hat{\jmath} + \frac{5}{3}\hat{k}$$

Step 4: Final Answer:
The projection of vector $\vec{a}$ on vector $\vec{b}$ is $\frac{5}{3}(\hat{\imath} + 2\hat{\jmath} + \hat{k})$.

Quick Tip

Distinguish between scalar projection and vector projection. Scalar projection is just the length, given by $\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$. Vector projection is a vector, found by multiplying the scalar projection by the unit vector of $\vec{b}$, which results in the formula $\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\right)\vec{b}$. Read the question carefully to determine which is required.

---

4. Do all parts.

a. Evaluate : $\int \frac{dx}{(x+1)(x+2)}$.

Solution:

Step 1: Understanding the Concept:
The integral involves a rational function where the denominator is a product of linear factors. This suggests using the method of partial fraction decomposition to break down the integrand into simpler fractions that can be easily integrated.

Step 2: Key Formula or Approach:
We will decompose the fraction $\frac{1}{(x+1)(x+2)}$ into the form:

$$\frac{1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$$

Multiplying both sides by $(x+1)(x+2)$, we get:

$$1 = A(x+2) + B(x+1)$$

To find the constants A and B, we can substitute the roots of the denominator.

Step 3: Detailed Explanation:
First, let's find the values of A and B.
Set $x = -1$:

$$1 = A(-1+2) + B(-1+1) \implies 1 = A(1) \implies A = 1$$

Next, set $x = -2$:

$$1 = A(-2+2) + B(-2+1) \implies 1 = B(-1) \implies B = -1$$

Now, we can rewrite the integral as:

$$\int \frac{dx}{(x+1)(x+2)} = \int \left(\frac{1}{x+1} - \frac{1}{x+2}\right) dx$$

Integrate each term separately:

$$\int \frac{1}{x+1}dx - \int \frac{1}{x+2}dx$$

Using the standard integral $\int \frac{1}{u} du = \ln|u| + C$:

$$= \ln|x+1| - \ln|x+2| + C$$

Using the property of logarithms $\ln a - \ln b = \ln(a/b)$:

$$= \ln\left|\frac{x+1}{x+2}\right| + C$$

Step 4: Final Answer:
The evaluated integral is $\ln\left|\frac{x+1}{x+2}\right| + C$.

> 💡 Quick Tip
>
> When using partial fractions, the "cover-up" method is a fast way to find coefficients for linear factors. To find A (the coefficient for $\frac{1}{x+1}$), cover up the $(x+1)$ term in the original fraction and substitute $x = -1$ into the rest: $A = \frac{1}{-1+2} = 1$. Similarly for B, cover up $(x+2)$ and substitute $x = -2$: $B = \frac{1}{-2+1} = -1$.

---

b. In a leap year, selected at random, find the probability that there are 53 Tuesdays.

Solution:

Step 1: Understanding the Concept:
This question deals with probability. A leap year has 366 days. We need to find the number of weeks and the number of remaining days to determine the probability of having an extra Tuesday.

Step 2: Key Formula or Approach:
Probability is calculated as:

$$P(\text{Event}) = \frac{\text{Number of Favorable Outcomes}}{\text{Total Number of Possible Outcomes}}$$

Step 3: Detailed Explanation:
A leap year has 366 days.
First, we find the number of complete weeks in 366 days by dividing by 7:

$$366 \div 7 = 52 \text{ weeks and } 2 \text{ days}$$

This means a leap year always contains 52 full weeks. Therefore, there are guaranteed to be 52 Tuesdays.
The occurrence of a 53rd Tuesday depends on the two remaining (odd) days.
These two consecutive days can form the following possible pairs:
(Sunday, Monday)
(Monday, Tuesday)
(Tuesday, Wednesday)

(Wednesday, Thursday)
(Thursday, Friday)
(Friday, Saturday)
(Saturday, Sunday)

There are 7 possible outcomes for the pair of odd days.
For there to be 53 Tuesdays, one of these two odd days must be a Tuesday. The favorable outcomes are the pairs that include a Tuesday:

(Monday, Tuesday)
(Tuesday, Wednesday)

There are 2 favorable outcomes.

Step 4: Final Answer:
The total number of possible outcomes is 7, and the number of favorable outcomes is 2.
Therefore, the probability of a leap year having 53 Tuesdays is:

$$P(53 \text{ Tuesdays}) = \frac{2}{7}$$

Quick Tip

Remember the number of odd days for different types of years. A normal year (365 days) has 1 odd day (365 = 52 weeks + 1 day), so the probability of 53 of any specific day is 1/7. A leap year (366 days) has 2 odd days, so the probability is 2/7. This is a common pattern in probability questions.

---

c. If $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, verify that $A'A = I$.

Solution:

Step 1: Understanding the Concept:
This question requires verifying a property of a given matrix. It involves finding the transpose of the matrix A (denoted as A') and then performing matrix multiplication of A' with A. The result should be the identity matrix I. Such a matrix A is called an orthogonal matrix.

Step 2: Key Formula or Approach:
1. Find the transpose of matrix A, A'. The transpose is found by interchanging the rows and columns.
2. Multiply A' by A.
3. Use the trigonometric identity $\sin^2\alpha + \cos^2\alpha = 1$.
4. The identity matrix of order 2 is $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Step 3: Detailed Explanation:
The given matrix is:

$$A = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$$

First, we find the transpose of A, A':

$$A' = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

Next, we compute the product $A'A$:

$$A'A = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$$

Performing the matrix multiplication:

$$A'A = \begin{pmatrix} (\cos\alpha)(\cos\alpha) + (-\sin\alpha)(-\sin\alpha) & (\cos\alpha)(\sin\alpha) + (-\sin\alpha)(\cos\alpha) \\ (\sin\alpha)(\cos\alpha) + (\cos\alpha)(-\sin\alpha) & (\sin\alpha)(\sin\alpha) + (\cos\alpha)(\cos\alpha) \end{pmatrix}$$

$$A'A = \begin{pmatrix} \cos^2\alpha + \sin^2\alpha & \cos\alpha\sin\alpha - \sin\alpha\cos\alpha \\ \sin\alpha\cos\alpha - \cos\alpha\sin\alpha & \sin^2\alpha + \cos^2\alpha \end{pmatrix}$$

Using the identity $\cos^2\alpha + \sin^2\alpha = 1$:

$$A'A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

This is the identity matrix I.

Step 4: Final Answer:
Since $A'A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$, the property is verified.

> 💡 Quick Tip
>
> The matrix given in the question is a rotation matrix. A key property of rotation matrices is that they are orthogonal, which means their transpose is equal to their inverse ($A' = A^{-1}$). Therefore, $A'A = A^{-1}A = I$. Recognizing this can give you confidence in your result before you even start the calculation.

---

d. Show that the given function $f, f(x) = x^3 - 3x^2 + 4x, x \in \mathbb{R}$ is an increasing function in $\mathbb{R}$.

Solution:

Step 1: Understanding the Concept:
To determine if a function is increasing or decreasing over an interval, we use its first derivative. A function $f(x)$ is strictly increasing on an interval if its first derivative, $f'(x)$, is positive

($f'(x) > 0$) for all x in that interval.

Step 2: Key Formula or Approach:
1. Find the first derivative of the function, $f'(x)$.
2. Analyze the sign of $f'(x)$ for all real numbers x. We need to show that $f'(x) > 0$ for all $x \in \mathbb{R}$.
For a quadratic $ax^2 + bx + c$, if $a > 0$ and the discriminant $\Delta = b^2 - 4ac < 0$, the quadratic is always positive.

Step 3: Detailed Explanation:
The given function is:

$$f(x) = x^3 - 3x^2 + 4x$$

First, we find the derivative, $f'(x)$:

$$f'(x) = \frac{d}{dx}(x^3 - 3x^2 + 4x)$$

$$f'(x) = 3x^2 - 6x + 4$$

Now, we need to show that this quadratic expression is always positive for all $x \in \mathbb{R}$. We can do this by checking its discriminant ($\Delta = b^2 - 4ac$).
Here, $a = 3, b = -6, c = 4$.

$$\Delta = (-6)^2 - 4(3)(4)$$

$$\Delta = 36 - 48$$

$$\Delta = -12$$

Since the discriminant $\Delta = -12 < 0$ and the leading coefficient $a = 3 > 0$, the quadratic $3x^2 - 6x + 4$ is always positive for all real values of x.
Alternatively, we can complete the square:

$$f'(x) = 3x^2 - 6x + 4$$

$$f'(x) = 3(x^2 - 2x) + 4$$

$$f'(x) = 3(x^2 - 2x + 1 - 1) + 4$$

$$f'(x) = 3(x - 1)^2 - 3 + 4$$

$$f'(x) = 3(x - 1)^2 + 1$$

Since $(x - 1)^2 \geq 0$ for all $x \in \mathbb{R}$, the minimum value of $3(x - 1)^2$ is 0.
Therefore, the minimum value of $f'(x)$ is $0 + 1 = 1$.
This means $f'(x) \geq 1$ for all $x \in \mathbb{R}$.

Step 4: Final Answer:
Since $f'(x) > 0$ for all $x \in \mathbb{R}$, the function $f(x) = x^3 - 3x^2 + 4x$ is an increasing function in $\mathbb{R}$.

Quick Tip

To quickly check if a quadratic function is always positive, check two conditions: 1. The coefficient of the x^2 term (a) must be positive ($a > 0$), which means the parabola opens upwards. 2. The discriminant ($b^2 - 4ac$) must be negative, which means the parabola has no real roots and never touches or crosses the x-axis. If both are true, the function is always positive.

---

5. Do all parts.

a. Find the value of the determinant $\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix}$**.**

Correct Answer: $1 + a^2 + b^2 + c^2$

Solution:

Step 1: Understanding the Concept:
The problem requires evaluating a 3x3 determinant. Instead of direct expansion, which can be cumbersome, using determinant properties and row/column operations is a more efficient approach.

Step 2: Key Formula or Approach:
The strategy is to simplify the determinant using the following properties: 1. Multiplying a row/column by a constant 'k' multiplies the determinant's value by 'k'.
2. Taking a common factor from any one row or one column.
3. Applying operations like $R_i \to R_i + R_j + R_k$ or $C_i \to C_i - C_j$.

Step 3: Detailed Explanation:
Let the given determinant be Δ.

$$\Delta = \begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix}$$

Multiply Row 1 by a, Row 2 by b, and Row 3 by c. To keep the value of the determinant unchanged, we must also divide by abc.

$$\Delta = \frac{1}{abc} \begin{vmatrix} a(a^2+1) & a^2b & a^2c \\ ab^2 & b(b^2+1) & b^2c \\ c^2a & c^2b & c(c^2+1) \end{vmatrix}$$

Now, take a common from Column 1, b from Column 2, and c from Column 3.

$$\Delta = \frac{abc}{abc} \begin{vmatrix} a^2+1 & a^2 & a^2 \\ b^2 & b^2+1 & b^2 \\ c^2 & c^2 & c^2+1 \end{vmatrix}$$

$$\Delta = \begin{vmatrix} a^2+1 & a^2 & a^2 \\ b^2 & b^2+1 & b^2 \\ c^2 & c^2 & c^2+1 \end{vmatrix}$$

Apply the row operation $R_1 \to R_1 + R_2 + R_3$.

$$\Delta = \begin{vmatrix} 1+a^2+b^2+c^2 & 1+a^2+b^2+c^2 & 1+a^2+b^2+c^2 \\ b^2 & b^2+1 & b^2 \\ c^2 & c^2 & c^2+1 \end{vmatrix}$$

Take the common factor $(1+a^2+b^2+c^2)$ from Row 1.

$$\Delta = (1+a^2+b^2+c^2) \begin{vmatrix} 1 & 1 & 1 \\ b^2 & b^2+1 & b^2 \\ c^2 & c^2 & c^2+1 \end{vmatrix}$$

Now apply the column operations $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$ to simplify.

$$\Delta = (1+a^2+b^2+c^2) \begin{vmatrix} 1 & 1-1 & 1-1 \\ b^2 & (b^2+1)-b^2 & b^2-b^2 \\ c^2 & c^2-c^2 & (c^2+1)-c^2 \end{vmatrix}$$

$$\Delta = (1+a^2+b^2+c^2) \begin{vmatrix} 1 & 0 & 0 \\ b^2 & 1 & 0 \\ c^2 & 0 & 1 \end{vmatrix}$$

The determinant of a lower triangular matrix is the product of its diagonal elements.

$$\Delta = (1+a^2+b^2+c^2) \cdot (1 \cdot 1 \cdot 1)$$

$$\Delta = 1+a^2+b^2+c^2$$

Step 4: Final Answer:
The value of the determinant is $1+a^2+b^2+c^2$.

> **Quick Tip**
>
> For determinants with algebraic entries, always look for simplifications through row/column operations before attempting a direct expansion. The trick of multiplying rows by variables and then taking those variables common from columns is very effective.

---

b. If $y = (\tan^{-1} x)^2$, show that $(x^2+1)^2 \frac{d^2y}{dx^2} + 2x(x^2+1)\frac{dy}{dx} = 2$.

Correct Answer: The given relation is proven by differentiating the function twice and rearranging the terms.

Solution:

Step 1: Understanding the Concept:
This problem involves second-order differentiation. The goal is to find the first and second derivatives of the given function and then show that they satisfy the given differential equation.

Step 2: Key Formula or Approach:
We will use the chain rule for the first derivative and the product rule for the second derivative.
1. Find $\frac{dy}{dx}$.
2. Rearrange the equation to avoid fractions, which simplifies the next differentiation.
3. Differentiate again with respect to x using the product rule.
4. Rearrange the resulting equation to match the required form.

Step 3: Detailed Explanation:
We are given the function:

$$y = (\tan^{-1} x)^2$$

Differentiating with respect to x using the chain rule:

$$\frac{dy}{dx} = 2(\tan^{-1} x)^1 \cdot \frac{d}{dx}(\tan^{-1} x)$$

$$\frac{dy}{dx} = 2(\tan^{-1} x) \cdot \frac{1}{1+x^2}$$

To avoid using the quotient rule for the next step, multiply both sides by $(1 + x^2)$:

$$(1+x^2)\frac{dy}{dx} = 2\tan^{-1} x$$

Now, differentiate both sides again with respect to x. We use the product rule on the left-hand side (LHS).

$$\frac{d}{dx}\left[(1+x^2)\frac{dy}{dx}\right] = \frac{d}{dx}(2\tan^{-1} x)$$

$$(1+x^2)\frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{dy}{dx}\frac{d}{dx}(1+x^2) = 2 \cdot \frac{1}{1+x^2}$$

$$(1+x^2)\frac{d^2y}{dx^2} + \frac{dy}{dx}(2x) = \frac{2}{1+x^2}$$

To eliminate the fraction on the right-hand side, multiply the entire equation by $(1 + x^2)$:

$$(1+x^2)\left[(1+x^2)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx}\right] = (1+x^2)\left(\frac{2}{1+x^2}\right)$$

$$(x^2+1)^2\frac{d^2y}{dx^2} + 2x(x^2+1)\frac{dy}{dx} = 2$$

This is the required expression. Hence, it is shown.

Step 4: Final Answer:
By differentiating the function $y = (\tan^{-1} x)^2$ twice and performing algebraic manipulations, we have shown that $(x^2+1)^2\frac{d^2y}{dx^2} + 2x(x^2+1)\frac{dy}{dx} = 2$.

💡 Quick Tip

When dealing with proofs involving derivatives, it's often easier to rearrange the equation after the first differentiation to remove denominators. This allows you to use the simpler product rule instead of the more complex quotient rule for the second derivative.

---

c. Find the shortest distance between the lines whose vector equations are $\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(\hat{i} - 3\hat{j} + 2\hat{k})$ and $\vec{r} = (4\hat{i} + 5\hat{j} + 6\hat{k}) + \mu(2\hat{i} + 3\hat{j} + \hat{k})$.

Correct Answer: $\frac{3}{\sqrt{19}}$ units

Solution:

Step 1: Understanding the Concept:
The problem asks for the shortest distance between two lines in 3D space given in vector form. The lines are not parallel, so they are skew lines.

Step 2: Key Formula or Approach:
The shortest distance d between two skew lines $\vec{r} = \vec{a_1} + \lambda\vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu\vec{b_2}$ is given by the formula:

$$d = \left| \frac{(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2})}{|\vec{b_1} \times \vec{b_2}|} \right|$$

Step 3: Detailed Explanation:
From the given equations, we identify the vectors:
For the first line:

$$\vec{a_1} = \hat{i} + 2\hat{j} + 3\hat{k}$$
$$\vec{b_1} = \hat{i} - 3\hat{j} + 2\hat{k}$$

For the second line:

$$\vec{a_2} = 4\hat{i} + 5\hat{j} + 6\hat{k}$$
$$\vec{b_2} = 2\hat{i} + 3\hat{j} + \hat{k}$$

First, we find the vector $\vec{a_2} - \vec{a_1}$:

$$\vec{a_2} - \vec{a_1} = (4-1)\hat{i} + (5-2)\hat{j} + (6-3)\hat{k} = 3\hat{i} + 3\hat{j} + 3\hat{k}$$

Next, we calculate the cross product $\vec{b_1} \times \vec{b_2}$:

$$\vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 2 \\ 2 & 3 & 1 \end{vmatrix}$$

$$= \hat{i}((-3)(1) - (2)(3)) - \hat{j}((1)(1) - (2)(2)) + \hat{k}((1)(3) - (-3)(2))$$
$$= \hat{i}(-3-6) - \hat{j}(1-4) + \hat{k}(3+6)$$

$$= -9\hat{i} + 3\hat{j} + 9\hat{k}$$

Now, we find the magnitude of the cross product:

$$|\vec{b_1} \times \vec{b_2}| = \sqrt{(-9)^2 + 3^2 + 9^2} = \sqrt{81 + 9 + 81} = \sqrt{171}$$

$\sqrt{171} = \sqrt{9 \times 19} = 3\sqrt{19}$.

Next, we calculate the dot product $(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2})$:

$$(3\hat{i} + 3\hat{j} + 3\hat{k}) \cdot (-9\hat{i} + 3\hat{j} + 9\hat{k}) = (3)(-9) + (3)(3) + (3)(9)$$

$$= -27 + 9 + 27 = 9$$

Finally, we substitute these values into the shortest distance formula:

$$d = \left| \frac{9}{3\sqrt{19}} \right| = \frac{3}{\sqrt{19}}$$

Step 4: Final Answer:

The shortest distance between the two lines is $\frac{3}{\sqrt{19}}$ units.

> Quick Tip
>
> Before applying the skew lines formula, quickly check if the direction vectors $\vec{b_1}$ and $\vec{b_2}$ are proportional. If they are, the lines are parallel, and a different formula for distance is used. The numerator of the formula is the scalar triple product $[\vec{a_2} - \vec{a_1} \quad \vec{b_1} \quad \vec{b_2}]$.

---

d. Prove that $\tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right) = \frac{\pi}{4} - \frac{1}{2}\cos^{-1} x$**, where** $-\frac{1}{\sqrt{2}} \le x \le 1$**.**

Correct Answer: The identity is proven by substituting $x = \cos(2\theta)$ and simplifying using trigonometric identities.

Solution:

Step 1: Understanding the Concept:

This problem requires proving an identity involving inverse trigonometric functions. The key is to simplify the complex argument of the $\tan^{-1}$ function using a suitable trigonometric substitution.

Step 2: Key Formula or Approach:

The presence of terms $\sqrt{1+x}$ and $\sqrt{1-x}$ suggests the substitution $x = \cos(2\theta)$. This is because:

$$1 + \cos(2\theta) = 2\cos^2\theta$$

$$1 - \cos(2\theta) = 2\sin^2\theta$$

We will also use the identity $\tan(\frac{\pi}{4} - \theta) = \frac{1-\tan\theta}{1+\tan\theta}$.

Step 3: Detailed Explanation:
Let the Left Hand Side (LHS) be y.

$$y = \tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)$$

Let $x = \cos(2\theta)$. This implies $2\theta = \cos^{-1} x$, so $\theta = \frac{1}{2}\cos^{-1} x$.
Now, let's determine the range of θ. Given $-\frac{1}{\sqrt{2}} \le x \le 1$, we have $-\frac{1}{\sqrt{2}} \le \cos(2\theta) \le 1$.
The range of $\cos^{-1}$ is $[0, \pi]$. Therefore, $\cos^{-1}(1) \le 2\theta \le \cos^{-1}(-\frac{1}{\sqrt{2}})$.

$$0 \le 2\theta \le \frac{3\pi}{4} \implies 0 \le \theta \le \frac{3\pi}{8}$$

In this interval, both $\sin\theta$ and $\cos\theta$ are non-negative.
Now substitute $x = \cos(2\theta)$ into the expression:

$$\sqrt{1+x} = \sqrt{1+\cos(2\theta)} = \sqrt{2\cos^2\theta} = \sqrt{2}\cos\theta$$

$$\sqrt{1-x} = \sqrt{1-\cos(2\theta)} = \sqrt{2\sin^2\theta} = \sqrt{2}\sin\theta$$

Substituting these into the LHS:

$$y = \tan^{-1}\left(\frac{\sqrt{2}\cos\theta - \sqrt{2}\sin\theta}{\sqrt{2}\cos\theta + \sqrt{2}\sin\theta}\right) = \tan^{-1}\left(\frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta}\right)$$

Divide the numerator and denominator inside the parenthesis by $\cos\theta$ (which is non-zero in the given range).

$$y = \tan^{-1}\left(\frac{1-\tan\theta}{1+\tan\theta}\right)$$

Using the identity $\tan(\frac{\pi}{4}) = 1$, we can write:

$$y = \tan^{-1}\left(\frac{\tan(\frac{\pi}{4}) - \tan\theta}{1+\tan(\frac{\pi}{4})\tan\theta}\right)$$

This is the formula for $\tan(A-B)$.

$$y = \tan^{-1}\left(\tan\left(\frac{\pi}{4}-\theta\right)\right)$$

Since $0 \le \theta \le \frac{3\pi}{8}$, we have $-\frac{\pi}{8} \le \frac{\pi}{4} - \theta \le \frac{\pi}{4}$. This range is within the principal value branch of $\tan^{-1}$, i.e., $(-\frac{\pi}{2}, \frac{\pi}{2})$.
So, we can simplify $y = \frac{\pi}{4} - \theta$.
Substitute back $\theta = \frac{1}{2}\cos^{-1} x$.

$$y = \frac{\pi}{4} - \frac{1}{2}\cos^{-1} x$$

Thus, LHS = RHS.
Step 4: Final Answer:
The given identity is proved.

Quick Tip

Recognizing the pattern for trigonometric substitution is key. For expressions involving $\sqrt{a^2 - x^2}$, use $x = a\sin\theta$ or $x = a\cos\theta$. For $\sqrt{a^2 + x^2}$, use $x = a\tan\theta$. For expressions like $\sqrt{1 \pm x}$, the substitution $x = \cos(2\theta)$ is extremely useful.

---

e. Find the general solution of the differential equation $x\frac{dy}{dx} + 2y = x^2$ $(x \neq 0)$.

Correct Answer: $y = \frac{x^2}{4} + \frac{C}{x^2}$

Solution:

Step 1: Understanding the Concept:
The given equation is a first-order linear differential equation. We need to find its general solution, which includes an arbitrary constant of integration.

Step 2: Key Formula or Approach:
A first-order linear differential equation is of the form $\frac{dy}{dx} + P(x)y = Q(x)$.
The solution is found using an integrating factor (I.F.), calculated as $I.F. = e^{\int P(x)dx}$.
The general solution is then given by $y \cdot (I.F.) = \int Q(x) \cdot (I.F.)dx + C$.

Step 3: Detailed Explanation:
The given differential equation is:

$$x\frac{dy}{dx} + 2y = x^2$$

First, we rewrite it in the standard form $\frac{dy}{dx} + P(x)y = Q(x)$ by dividing the entire equation by x (since $x \neq 0$).

$$\frac{dy}{dx} + \frac{2}{x}y = x$$

By comparing with the standard form, we have:

$$P(x) = \frac{2}{x}$$

$$Q(x) = x$$

Next, we find the integrating factor (I.F.):

$$I.F. = e^{\int P(x)dx} = e^{\int \frac{2}{x}dx}$$

$$I.F. = e^{2\ln|x|} = e^{\ln(x^2)} = x^2$$

Now, we use the formula for the general solution:

$$y \cdot (I.F.) = \int Q(x) \cdot (I.F.)dx + C$$

$$y \cdot x^2 = \int x \cdot x^2 dx + C$$

$$yx^2 = \int x^3 dx + C$$

Integrating the right-hand side:

$$yx^2 = \frac{x^4}{4} + C$$

Finally, we solve for y by dividing by x^2:

$$y = \frac{x^4}{4x^2} + \frac{C}{x^2}$$

$$y = \frac{x^2}{4} + \frac{C}{x^2}$$

Step 4: Final Answer:
The general solution of the given differential equation is $y = \frac{x^2}{4} + \frac{C}{x^2}$, where C is an arbitrary constant.

> **Quick Tip**
>
> Always ensure a linear differential equation is in its standard form $\frac{dy}{dx} + P(x)y = Q(x)$ before identifying $P(x)$ and $Q(x)$. A common mistake is to forget to divide by the coefficient of the $\frac{dy}{dx}$ term.

---

6. Do all parts.

a. A die was thrown twice and the sum of the numbers which appeared was found to be 6. Find the conditional probability that the number 4 appears at least once.

Solution:

Step 1: Understanding the Concept:
This problem asks for a conditional probability. We are given an event that has already occurred (the sum of the numbers is 6), and we need to find the probability of another event (the number 4 appears at least once) given this condition.

Step 2: Key Formula or Approach:
The formula for conditional probability is $P(A|B) = \frac{P(A \cap B)}{P(B)}$, where:
- A is the event that the number 4 appears at least once.
- B is the event that the sum of the numbers is 6.
In terms of outcomes, this can be calculated as $P(A|B) = \frac{\text{Number of outcomes in } A \cap B}{\text{Number of outcomes in } B}$.

Step 3: Detailed Explanation:
First, let's define the sample space for the given condition (Event B).

Event B: The sum of the numbers is 6.
The possible outcomes for two dice throws that sum to 6 are:

$$B = \{(1,5),(2,4),(3,3),(4,2),(5,1)\}$$

The total number of outcomes for event B is $n(B) = 5$.

Next, let's define Event A.
Event A: The number 4 appears at least once.

Now, we need to find the intersection of A and B ($A \cap B$), which means we look for outcomes in B where the number 4 appears at least once.
From the set of outcomes for B, the ones that contain at least one 4 are:

$$A \cap B = \{(2,4),(4,2)\}$$

The number of outcomes in $A \cap B$ is $n(A \cap B) = 2$.

Step 4: Final Answer:
Using the formula for conditional probability:

$$P(A|B) = \frac{n(A \cap B)}{n(B)} = \frac{2}{5}$$

So, the conditional probability that the number 4 appears at least once, given that the sum is 6, is $\frac{2}{5}$.

> Quick Tip
>
> In conditional probability, the sample space is reduced to the event that is known to have occurred. Instead of considering all 36 possible outcomes of two dice rolls, you only need to consider the 5 outcomes where the sum is 6. This simplifies the problem significantly.

---

b. Maximize $Z = x+2y$ by graphical method under the constraints $x+y \leq 1, -x+y \leq 0, x \geq 0, y \geq 0$.

Solution:

Step 1: Understanding the Concept:
This is a Linear Programming Problem (LPP). The goal is to find the maximum value of a linear objective function Z subject to a set of linear inequalities (constraints). The graphical method involves plotting the constraints to find a feasible region and then testing the corner points of this region in the objective function.

Step 2: Key Formula or Approach:
1. Convert the inequalities into equations to plot the boundary lines.

2. Identify the feasible region that satisfies all constraints.
3. Determine the coordinates of the corner points (vertices) of the feasible region.
4. Evaluate the objective function Z at each corner point.
5. The maximum value of Z will be the largest value among those calculated.

Step 3: Detailed Explanation:
The constraints are:

$x + y \leq 1$
$-x + y \leq 0 \implies y \leq x$
$x \geq 0$
$y \geq 0$

Let's plot the lines:
- $L_1 : x + y = 1$ (passes through (1,0) and (0,1))
- $L_2 : y = x$ (passes through the origin (0,0) at a 45-degree angle)
- $L_3 : x = 0$ (the y-axis)
- $L_4 : y = 0$ (the x-axis)

The feasible region is the area bounded by these lines that satisfies all the inequalities. It is a triangle with vertices at the intersection points of these lines.
Let's find the corner points:
- **Point O:** Intersection of $x = 0$ and $y = 0$. The coordinates are $(0, 0)$.
- **Point A:** Intersection of $x + y = 1$ and $y = x$.
Substituting $y = x$ into the first equation: $x + x = 1 \implies 2x = 1 \implies x = 0.5$. So, $y = 0.5$.
The coordinates are $(0.5, 0.5)$.
- **Point B:** Intersection of $x + y = 1$ and $y = 0$.
Substituting $y = 0$: $x + 0 = 1 \implies x = 1$.
The coordinates are $(1, 0)$.

Now, evaluate the objective function $Z = x + 2y$ at these corner points:
- At O(0, 0): $Z = 0 + 2(0) = 0$
- At A(0.5, 0.5): $Z = 0.5 + 2(0.5) = 0.5 + 1 = 1.5$
- At B(1, 0): $Z = 1 + 2(0) = 1$

Step 4: Final Answer:
Comparing the values of Z, the maximum value is 1.5, which occurs at the point $(0.5, 0.5)$.
Maximum $Z = 1.5$.

Quick Tip

The fundamental theorem of linear programming states that the optimal solution (maximum or minimum) for an LPP, if it exists, will always be at one of the corner points of the feasible region. Therefore, once you have identified the feasible region and its vertices, you only need to check these points.

c. Find the area of the parallelogram whose diagonals are $\vec{a} = 3\hat{i} + \hat{j} - 2\hat{k}$ and $\vec{b} = \hat{i} - 3\hat{j} + 4\hat{k}$.

Solution:

Step 1: Understanding the Concept:
The area of a parallelogram can be determined using the vector cross product. When the diagonals of the parallelogram are given as vectors, there is a specific formula that relates the area to the magnitude of the cross product of these diagonal vectors.

Step 2: Key Formula or Approach:
If $\vec{d_1}$ and $\vec{d_2}$ are the vectors representing the diagonals of a parallelogram, the area of the parallelogram is given by:

$$\text{Area} = \frac{1}{2}|\vec{d_1} \times \vec{d_2}|$$

Step 3: Detailed Explanation:
Here, the diagonal vectors are $\vec{a} = 3\hat{i} + \hat{j} - 2\hat{k}$ and $\vec{b} = \hat{i} - 3\hat{j} + 4\hat{k}$.
First, we need to compute the cross product $\vec{a} \times \vec{b}$.

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & -2 \\ 1 & -3 & 4 \end{vmatrix}$$

$$= \hat{i}((1)(4) - (-2)(-3)) - \hat{j}((3)(4) - (-2)(1)) + \hat{k}((3)(-3) - (1)(1))$$
$$= \hat{i}(4 - 6) - \hat{j}(12 + 2) + \hat{k}(-9 - 1)$$
$$= -2\hat{i} - 14\hat{j} - 10\hat{k}$$

Next, we find the magnitude of this resulting vector:

$$|\vec{a} \times \vec{b}| = \sqrt{(-2)^2 + (-14)^2 + (-10)^2}$$
$$= \sqrt{4 + 196 + 100} = \sqrt{300}$$
$$= \sqrt{100 \times 3} = 10\sqrt{3}$$

Finally, we use the formula for the area:

$$\text{Area} = \frac{1}{2}|\vec{a} \times \vec{b}| = \frac{1}{2}(10\sqrt{3})$$

Step 4: Final Answer:
The area of the parallelogram is $5\sqrt{3}$ square units.

> Quick Tip
>
> It is crucial to distinguish between the formula for the area of a parallelogram given its adjacent sides ($|\vec{a} \times \vec{b}|$) and the one given its diagonals ($\frac{1}{2}|\vec{d_1} \times \vec{d_2}|$). A common mistake is to forget the $\frac{1}{2}$ factor when using diagonals.

d. If θ be the angle between two unit vectors $\hat{a}$ and $\hat{b}$, prove that $\sin\frac{\theta}{2} = \frac{1}{2}|\hat{a} - \hat{b}|$.

Solution:

Step 1: Understanding the Concept:
This problem requires proving a vector identity that relates the angle between two unit vectors to the magnitude of their difference. The proof will utilize the properties of the dot product and trigonometric half-angle identities.

Step 2: Key Formula or Approach:
We will use the following properties: 1. For any vector $\vec{v}$, its squared magnitude is $|\vec{v}|^2 = \vec{v} \cdot \vec{v}$. 2. The dot product of two vectors $\vec{u}$ and $\vec{v}$ is $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}|\cos\theta$. 3. For unit vectors $\hat{a}$ and $\hat{b}$, $|\hat{a}| = 1$ and $|\hat{b}| = 1$. 4. The trigonometric half-angle identity: $1 - \cos\theta = 2\sin^2\left(\frac{\theta}{2}\right)$.

Step 3: Detailed Explanation:
Let's start by considering the expression $|\hat{a} - \hat{b}|^2$.
Using the dot product property, we can write:

$$|\hat{a} - \hat{b}|^2 = (\hat{a} - \hat{b}) \cdot (\hat{a} - \hat{b})$$

Expanding the dot product:

$$= \hat{a} \cdot \hat{a} - \hat{a} \cdot \hat{b} - \hat{b} \cdot \hat{a} + \hat{b} \cdot \hat{b}$$

Since $\hat{a}$ and $\hat{b}$ are unit vectors, $\hat{a} \cdot \hat{a} = |\hat{a}|^2 = 1^2 = 1$ and $\hat{b} \cdot \hat{b} = |\hat{b}|^2 = 1^2 = 1$.
The dot product is commutative, so $\hat{a} \cdot \hat{b} = \hat{b} \cdot \hat{a}$. The expression becomes:

$$|\hat{a} - \hat{b}|^2 = 1 - 2(\hat{a} \cdot \hat{b}) + 1 = 2 - 2(\hat{a} \cdot \hat{b})$$

Now, substitute the definition of the dot product: $\hat{a} \cdot \hat{b} = |\hat{a}||\hat{b}|\cos\theta = (1)(1)\cos\theta = \cos\theta$.

$$|\hat{a} - \hat{b}|^2 = 2 - 2\cos\theta = 2(1 - \cos\theta)$$

Using the half-angle identity $1 - \cos\theta = 2\sin^2(\frac{\theta}{2})$:

$$|\hat{a} - \hat{b}|^2 = 2\left(2\sin^2\left(\frac{\theta}{2}\right)\right) = 4\sin^2\left(\frac{\theta}{2}\right)$$

Taking the square root of both sides:

$$|\hat{a} - \hat{b}| = \sqrt{4\sin^2\left(\frac{\theta}{2}\right)} = 2\left|\sin\left(\frac{\theta}{2}\right)\right|$$

The angle θ between two vectors is usually taken in the range $[0, \pi]$. Therefore, $\frac{\theta}{2}$ is in the range $[0, \frac{\pi}{2}]$, where the sine function is non-negative. So, $\left|\sin\left(\frac{\theta}{2}\right)\right| = \sin\left(\frac{\theta}{2}\right)$.

$$|\hat{a} - \hat{b}| = 2\sin\left(\frac{\theta}{2}\right)$$

Step 4: Final Answer:
Rearranging the equation, we get:

$$\sin\left(\frac{\theta}{2}\right) = \frac{1}{2}|\hat{a} - \hat{b}|$$

This completes the proof.

> Quick Tip
>
> This is a standard and useful vector identity. A similar identity exists for the sum of two unit vectors: $\cos\frac{\theta}{2} = \frac{1}{2}|\hat{a} + \hat{b}|$. Memorizing both can save time in exams. The key to proving them is to start with the squared magnitude and use the properties of the dot product.

---

e. Sand is falling from a pipe at the rate of 12 cm^3/second. The falling sand forms such a cone on the ground that its height is always one-sixth of the radius of its base. At which rate is the height of the cone formed by sand increasing while its height is 4 cm?

Solution:

Step 1: Understanding the Concept:
This is a related rates problem from the application of derivatives. We are given the rate of change of the volume of a cone and a relationship between its height and radius. We need to find the rate of change of its height at a specific instant.

Step 2: Key Formula or Approach:
1. Volume of a cone: $V = \frac{1}{3}\pi r^2 h$.
2. Identify the given quantities: $\frac{dV}{dt} = 12$ cm^3/s and $h = \frac{1}{6}r$.
3. Express the volume V solely in terms of the variable whose rate we want to find (in this case, h).
4. Differentiate the volume equation with respect to time t.
5. Substitute the known values to solve for the unknown rate $\frac{dh}{dt}$.

Step 3: Detailed Explanation:
We are given:
- Rate of change of volume, $\frac{dV}{dt} = 12$.
- Relationship between height h and radius r: $h = \frac{1}{6}r$. This implies $r = 6h$.
- We need to find $\frac{dh}{dt}$ when $h = 4$ cm.

The formula for the volume of a cone is $V = \frac{1}{3}\pi r^2 h$.
To find $\frac{dh}{dt}$, we should express V as a function of h only. Substitute $r = 6h$ into the volume formula:

$$V = \frac{1}{3}\pi(6h)^2 h = \frac{1}{3}\pi(36h^2)h = 12\pi h^3$$

Now, differentiate both sides with respect to time t:

$$\frac{dV}{dt} = \frac{d}{dt}(12\pi h^3)$$

Using the chain rule:

$$\frac{dV}{dt} = 12\pi \cdot (3h^2) \cdot \frac{dh}{dt} = 36\pi h^2 \frac{dh}{dt}$$

Now, substitute the given values: $\frac{dV}{dt} = 12$ and $h = 4$.

$$12 = 36\pi(4)^2 \frac{dh}{dt}$$

$$12 = 36\pi(16) \frac{dh}{dt}$$

$$12 = 576\pi \frac{dh}{dt}$$

Solve for $\frac{dh}{dt}$:

$$\frac{dh}{dt} = \frac{12}{576\pi} = \frac{1}{48\pi}$$

Step 4: Final Answer:
The height of the sand cone is increasing at a rate of $\frac{1}{48\pi}$ cm/second.

> **Quick Tip**
>
> In related rates problems involving geometric shapes, first write down the relevant formula (e.g., volume, area). Then, use the given constraints to express this formula in terms of a single variable. This simplifies the differentiation process and avoids having to deal with multiple rates (like $\frac{dr}{dt}$ and $\frac{dh}{dt}$) in the same equation.

---

7. Do any one part.

a. Solve by matrix method the following system of equations:

$2x + y + z = 1$

$x - 2y - z = \frac{3}{2}$

$3y - 5z = 9$

Solution:

Step 1: Understanding the Concept:
The matrix method for solving a system of linear equations involves representing the system in the form $AX = B$. The solution is then found by calculating $X = A^{-1}B$, where A^{-1} is the inverse of the coefficient matrix A. This method is applicable only if the determinant of A is non-zero.

Step 2: Key Formula or Approach:
1. Write the system of equations in the matrix form $AX = B$.
2. Calculate the determinant of A, $|A|$.
3. Find the adjugate of A, $\text{adj}(A)$.
4. Calculate the inverse of A using the formula $A^{-1} = \frac{1}{|A|}\text{adj}(A)$.

5. Find the solution by computing $X = A^{-1}B$.

Step 3: Detailed Explanation:
The given system of equations can be written as:

$$2x + 1y + 1z = 1$$
$$1x - 2y - 1z = \frac{3}{2}$$
$$0x + 3y - 5z = 9$$

In matrix form $AX = B$, we have:

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -2 & -1 \\ 0 & 3 & -5 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 3/2 \\ 9 \end{bmatrix}$$

First, calculate the determinant of A:

$$|A| = 2((-2)(-5) - (-1)(3)) - 1((1)(-5) - (-1)(0)) + 1((1)(3) - (-2)(0))$$
$$|A| = 2(10 + 3) - 1(-5 - 0) + 1(3 - 0)$$
$$|A| = 2(13) - (-5) + 3 = 26 + 5 + 3 = 34$$

Since $|A| = 34 \neq 0$, a unique solution exists.

Next, find the adjugate of A. The matrix of cofactors is:

$$C_{11} = (-1)^{1+1} \begin{vmatrix} -2 & -1 \\ 3 & -5 \end{vmatrix} = 10 - (-3) = 13$$

$$C_{12} = (-1)^{1+2} \begin{vmatrix} 1 & -1 \\ 0 & -5 \end{vmatrix} = -(-5 - 0) = 5$$

$$C_{13} = (-1)^{1+3} \begin{vmatrix} 1 & -2 \\ 0 & 3 \end{vmatrix} = 3 - 0 = 3$$

$$C_{21} = (-1)^{2+1} \begin{vmatrix} 1 & 1 \\ 3 & -5 \end{vmatrix} = -(-5 - 3) = 8$$

$$C_{22} = (-1)^{2+2} \begin{vmatrix} 2 & 1 \\ 0 & -5 \end{vmatrix} = -10 - 0 = -10$$

$$C_{23} = (-1)^{2+3} \begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix} = -(6 - 0) = -6$$

$$C_{31} = (-1)^{3+1} \begin{vmatrix} 1 & 1 \\ -2 & -1 \end{vmatrix} = -1 - (-2) = 1$$

$$C_{32} = (-1)^{3+2} \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} = -(-2 - 1) = 3$$

$$C_{33} = (-1)^{3+3} \begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix} = -4 - 1 = -5$$

The matrix of cofactors is $\begin{bmatrix} 13 & 5 & 3 \\ 8 & -10 & -6 \\ 1 & 3 & -5 \end{bmatrix}$.
The adjugate of A is the transpose of the cofactor matrix:

$$\text{adj}(A) = \begin{bmatrix} 13 & 8 & 1 \\ 5 & -10 & 3 \\ 3 & -6 & -5 \end{bmatrix}$$

Now, find the inverse of A:

$$A^{-1} = \frac{1}{|A|}\text{adj}(A) = \frac{1}{34}\begin{bmatrix} 13 & 8 & 1 \\ 5 & -10 & 3 \\ 3 & -6 & -5 \end{bmatrix}$$

Finally, calculate the solution $X = A^{-1}B$:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{34}\begin{bmatrix} 13 & 8 & 1 \\ 5 & -10 & 3 \\ 3 & -6 & -5 \end{bmatrix}\begin{bmatrix} 1 \\ 3/2 \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{34}\begin{bmatrix} 13(1) + 8(3/2) + 1(9) \\ 5(1) - 10(3/2) + 3(9) \\ 3(1) - 6(3/2) - 5(9) \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{34}\begin{bmatrix} 13 + 12 + 9 \\ 5 - 15 + 27 \\ 3 - 9 - 45 \end{bmatrix} = \frac{1}{34}\begin{bmatrix} 34 \\ 17 \\ -51 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 34/34 \\ 17/34 \\ -51/34 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \\ -3/2 \end{bmatrix}$$

Step 4: Final Answer:
The solution to the system of equations is $x = 1$, $y = \frac{1}{2}$, and $z = -\frac{3}{2}$.

Quick Tip

The matrix method is systematic but prone to calculation errors, especially with signs in cofactors. To avoid fractions inside the B matrix, you could have multiplied the second equation by 2 initially to get $2x-4y-2z = 3$. The final answer would be the same, but the intermediate calculations might be simpler.

---

b. If $F(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$, prove that $F(x)F(y) = F(x+y)$.

Solution:

Step 1: Understanding the Concept:
This question requires us to prove a property of a function that returns a matrix. We need to perform matrix multiplication on the left-hand side (LHS) and show that the result is identical to the matrix function evaluated at $x + y$, which is the right-hand side (RHS).

Step 2: Key Formula or Approach:
The proof relies on two main components: 1. The rule for matrix multiplication of two 3x3 matrices.
2. The trigonometric sum identities:
- $\cos(x + y) = \cos x \cos y - \sin x \sin y$
- $\sin(x + y) = \sin x \cos y + \cos x \sin y$

Step 3: Detailed Explanation:
We are given the matrix function:

$$F(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The left-hand side (LHS) of the equation to be proved is $F(x)F(y)$.
First, let's write out $F(y)$:

$$F(y) = \begin{bmatrix} \cos y & -\sin y & 0 \\ \sin y & \cos y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now, we perform the matrix multiplication $F(x)F(y)$:

$$F(x)F(y) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos y & -\sin y & 0 \\ \sin y & \cos y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Let's compute each element of the resulting matrix:
- **Row 1, Column 1:** $(\cos x)(\cos y) + (-\sin x)(\sin y) + (0)(0) = \cos x \cos y - \sin x \sin y = \cos(x + y)$
- **Row 1, Column 2:** $(\cos x)(-\sin y) + (-\sin x)(\cos y) + (0)(0) = -(\cos x \sin y + \sin x \cos y) = -\sin(x + y)$
- **Row 1, Column 3:** $(\cos x)(0) + (-\sin x)(0) + (0)(1) = 0$
- **Row 2, Column 1:** $(\sin x)(\cos y) + (\cos x)(\sin y) + (0)(0) = \sin x \cos y + \cos x \sin y = \sin(x+y)$
- **Row 2, Column 2:** $(\sin x)(-\sin y) + (\cos x)(\cos y) + (0)(0) = \cos x \cos y - \sin x \sin y = \cos(x + y)$
- **Row 2, Column 3:** $(\sin x)(0) + (\cos x)(0) + (0)(1) = 0$
- **Row 3, Column 1:** $(0)(\cos y) + (0)(\sin y) + (1)(0) = 0$
- **Row 3, Column 2:** $(0)(-\sin y) + (0)(\cos y) + (1)(0) = 0$
- **Row 3, Column 3:** $(0)(0) + (0)(0) + (1)(1) = 1$

So, the resulting matrix for the LHS is:

$$F(x)F(y) = \begin{bmatrix} \cos(x+y) & -\sin(x+y) & 0 \\ \sin(x+y) & \cos(x+y) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now, let's evaluate the right-hand side (RHS), which is $F(x+y)$. We simply replace x with $(x+y)$ in the definition of $F(x)$:

$$F(x+y) = \begin{bmatrix} \cos(x+y) & -\sin(x+y) & 0 \\ \sin(x+y) & \cos(x+y) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Comparing the results, we see that LHS = RHS.

Step 4: Final Answer:

We have shown that $F(x)F(y) = \begin{bmatrix} \cos(x+y) & -\sin(x+y) & 0 \\ \sin(x+y) & \cos(x+y) & 0 \\ 0 & 0 & 1 \end{bmatrix}$, which is equal to $F(x+y)$.

Therefore, the identity $F(x)F(y) = F(x+y)$ is proved.

Quick Tip

The matrix $F(x)$ is a 2D rotation matrix embedded in 3D space (using homogeneous coordinates). This property, $F(x)F(y) = F(x+y)$, is a fundamental characteristic of rotation matrices. It signifies that performing a rotation by an angle y followed by a rotation by an angle x is equivalent to a single rotation by the angle $(x+y)$.

---

8. Do any one part.

a). Prove that the semi-vertical angle of the cone of given slant height and maximum volume is $\tan^{-1}\sqrt{2}$.

Solution:

Step 1: Understanding the Concept:

This is an optimization problem where we need to find the semi-vertical angle of a cone that maximizes its volume, given that its slant height is constant. We will use calculus, specifically the first and second derivative tests, to find the maximum.

Step 2: Key Formula or Approach:

Let l be the given slant height, h be the height, r be the radius, and θ be the semi-vertical angle of the cone.

The volume of the cone is given by the formula:

$$V = \frac{1}{3}\pi r^2 h$$

The relationships between r, h, l, and θ are:

$$r = l\sin\theta$$
$$h = l\cos\theta$$

We need to express the volume V as a function of a single variable, θ, and then find the value of θ for which V is maximum.

Step 3: Detailed Explanation:
First, substitute the expressions for r and h in terms of l and θ into the volume formula:

$$V(\theta) = \frac{1}{3}\pi(l\sin\theta)^2(l\cos\theta)$$
$$V(\theta) = \frac{1}{3}\pi l^3 \sin^2\theta\cos\theta$$

Since l is a constant, to maximize V, we need to maximize the function $f(\theta) = \sin^2\theta\cos\theta$.
Now, we differentiate V with respect to θ:

$$\frac{dV}{d\theta} = \frac{1}{3}\pi l^3 \frac{d}{d\theta}(\sin^2\theta\cos\theta)$$

Using the product rule $(uv)' = u'v + uv'$, let $u = \sin^2\theta$ and $v = \cos\theta$.

$$u' = 2\sin\theta\cos\theta$$
$$v' = -\sin\theta$$
$$\frac{d}{d\theta}(\sin^2\theta\cos\theta) = (2\sin\theta\cos\theta)(\cos\theta) + (\sin^2\theta)(-\sin\theta)$$
$$= 2\sin\theta\cos^2\theta - \sin^3\theta$$

So,

$$\frac{dV}{d\theta} = \frac{1}{3}\pi l^3(2\sin\theta\cos^2\theta - \sin^3\theta)$$

For maximum or minimum volume, we set $\frac{dV}{d\theta} = 0$:

$$\frac{1}{3}\pi l^3(\sin\theta)(2\cos^2\theta - \sin^2\theta) = 0$$

Since $l \neq 0$ and for a cone, $\theta \in (0, \pi/2)$, $\sin\theta \neq 0$. Thus, we must have:

$$2\cos^2\theta - \sin^2\theta = 0$$
$$2\cos^2\theta = \sin^2\theta$$

Dividing both sides by $\cos^2\theta$ (which is non-zero for $\theta \in (0, \pi/2)$):

$$\tan^2\theta = 2$$
$$\tan\theta = \sqrt{2}$$

This gives the critical point $\theta = \tan^{-1}(\sqrt{2})$.
To confirm this is a maximum, we use the second derivative test. It's often easier to check

the sign of the first derivative around the critical point. Or, we can analyze the derivative of $f(\theta) = \sin^2\theta\cos\theta$. Let's find the second derivative of V.

$$\frac{d^2V}{d\theta^2} = \frac{1}{3}\pi l^3 \frac{d}{d\theta}(2\sin\theta\cos^2\theta - \sin^3\theta)$$
$$\frac{d^2V}{d\theta^2} = \frac{\pi l^3}{3}[2\cos^3\theta - 4\sin^2\theta\cos\theta - 3\sin^2\theta\cos\theta]$$
$$\frac{d^2V}{d\theta^2} = \frac{\pi l^3}{3}[2\cos^3\theta - 7\sin^2\theta\cos\theta]$$
$$\frac{d^2V}{d\theta^2} = \frac{\pi l^3}{3}\cos\theta[2\cos^2\theta - 7\sin^2\theta]$$

At $\tan^2\theta = 2$, we have $\sin^2\theta = 2\cos^2\theta$. Substituting this into the bracket:

$$2\cos^2\theta - 7(2\cos^2\theta) = 2\cos^2\theta - 14\cos^2\theta = -12\cos^2\theta$$

Since $\theta \in (0, \pi/2)$, $\cos\theta > 0$, so $-12\cos^2\theta < 0$.
Therefore, $\frac{d^2V}{d\theta^2} < 0$ at $\tan\theta = \sqrt{2}$, which confirms that the volume is maximum at this angle.

Step 4: Final Answer:
The volume of the cone is maximized when the semi-vertical angle θ satisfies $\tan\theta = \sqrt{2}$. Hence, the semi-vertical angle for the cone of maximum volume is $\theta = \tan^{-1}(\sqrt{2})$.

> Quick Tip
>
> In optimization problems involving geometric shapes, always try to express the quantity to be maximized or minimized (like volume or area) as a function of a single variable (like an angle or a side length). Use the given constraints (like fixed slant height or surface area) to establish relationships between the variables.

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b). Prove that $\int_0^{\pi/4} \log(1 + \tan x)dx = \frac{\pi}{8}\log 2$.

Solution:

Step 1: Understanding the Concept:
This problem involves evaluating a definite integral. The integrand $\log(1+\tan x)$ does not have a simple antiderivative. Therefore, we should look for properties of definite integrals that can simplify the problem. The "King Property" of definite integrals is particularly useful here.

Step 2: Key Formula or Approach:
We will use the following property of definite integrals, often called the King Property:

$$\int_0^a f(x)dx = \int_0^a f(a-x)dx$$

We also need the tangent subtraction formula:

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

And the logarithm properties:

$$\log\left(\frac{m}{n}\right) = \log m - \log n$$

Step 3: Detailed Explanation:
Let the given integral be I.

$$I = \int_0^{\pi/4} \log(1 + \tan x) dx \quad \cdots (1)$$

Using the property $\int_0^a f(x)dx = \int_0^a f(a - x)dx$, with $a = \pi/4$:

$$I = \int_0^{\pi/4} \log\left(1 + \tan\left(\frac{\pi}{4} - x\right)\right) dx$$

Now, we use the tangent subtraction formula for $\tan(\frac{\pi}{4} - x)$, knowing that $\tan(\pi/4) = 1$:

$$\tan\left(\frac{\pi}{4} - x\right) = \frac{\tan(\pi/4) - \tan x}{1 + \tan(\pi/4) \tan x} = \frac{1 - \tan x}{1 + \tan x}$$

Substitute this back into the integral for I:

$$I = \int_0^{\pi/4} \log\left(1 + \frac{1 - \tan x}{1 + \tan x}\right) dx$$

Simplify the expression inside the logarithm:

$$1 + \frac{1 - \tan x}{1 + \tan x} = \frac{(1 + \tan x) + (1 - \tan x)}{1 + \tan x} = \frac{2}{1 + \tan x}$$

So the integral becomes:

$$I = \int_0^{\pi/4} \log\left(\frac{2}{1 + \tan x}\right) dx$$

Using the logarithm property $\log(m/n) = \log m - \log n$:

$$I = \int_0^{\pi/4} (\log 2 - \log(1 + \tan x)) dx$$

Split the integral into two parts:

$$I = \int_0^{\pi/4} \log 2 dx - \int_0^{\pi/4} \log(1 + \tan x) dx \quad \cdots (2)$$

We can see that the second term on the right-hand side is the original integral I from equation (1).

$$I = \int_0^{\pi/4} \log 2 dx - I$$

Now, we solve for I:

$$2I = \int_0^{\pi/4} \log 2 dx$$

Since $\log 2$ is a constant, we can take it out of the integral:

$$2I = \log 2 \int_0^{\pi/4} 1 dx$$

$$2I = \log 2 [x]_0^{\pi/4}$$

$$2I = \log 2 \left(\frac{\pi}{4} - 0\right)$$

$$2I = \frac{\pi}{4} \log 2$$

Finally, divide by 2 to find I:

$$I = \frac{\pi}{8} \log 2$$

Step 4: Final Answer:
We have shown that $\int_0^{\pi/4} \log(1 + \tan x)dx = \frac{\pi}{8} \log 2$.

Quick Tip

For definite integrals of the form $\int_0^a f(x)dx$, always check if applying the property $\int_0^a f(x)dx = \int_0^a f(a - x)dx$ simplifies the integrand or relates the new integral back to the original one. This is a very common and powerful technique, especially with trigonometric and logarithmic functions.

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9. Do any one part.

a). Find the area of the region bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solution:

Step 1: Understanding the Concept:
The area of a region bounded by a curve can be found using a definite integral. The ellipse given by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is symmetric with respect to both the x-axis and the y-axis. We can calculate the area of the region in the first quadrant and then multiply it by 4 to get the total area of the ellipse.

Step 2: Key Formula or Approach:
The area A of a region bounded by a curve $y = f(x)$, the x-axis, and the lines $x = c$ and $x = d$

is given by:

$$A = \int_c^d y\,dx$$

First, we need to express y in terms of x from the equation of the ellipse.
From $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we get:

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} = \frac{a^2 - x^2}{a^2}$$
$$y^2 = \frac{b^2}{a^2}(a^2 - x^2)$$
$$y = \pm\frac{b}{a}\sqrt{a^2 - x^2}$$

For the first quadrant, y is positive, so we take $y = \frac{b}{a}\sqrt{a^2 - x^2}$. The limits of integration for x in the first quadrant are from 0 to a.

Step 3: Detailed Explanation:
The total area A of the ellipse is 4 times the area of the part in the first quadrant.

$$A = 4\int_0^a y\,dx = 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx$$

To evaluate this integral, we use the trigonometric substitution $x = a\sin\theta$.
Then $dx = a\cos\theta\,d\theta$.
We also need to change the limits of integration:
When $x = 0$, $a\sin\theta = 0 \implies \sin\theta = 0 \implies \theta = 0$.
When $x = a$, $a\sin\theta = a \implies \sin\theta = 1 \implies \theta = \frac{\pi}{2}$.
Substituting these into the integral:

$$A = 4\frac{b}{a}\int_0^{\pi/2} \sqrt{a^2 - a^2\sin^2\theta} \cdot (a\cos\theta)\,d\theta$$
$$A = 4\frac{b}{a}\int_0^{\pi/2} \sqrt{a^2(1 - \sin^2\theta)} \cdot (a\cos\theta)\,d\theta$$

Using the identity $1 - \sin^2\theta = \cos^2\theta$:

$$A = 4\frac{b}{a}\int_0^{\pi/2} \sqrt{a^2\cos^2\theta} \cdot (a\cos\theta)\,d\theta$$
$$A = 4\frac{b}{a}\int_0^{\pi/2} (a\cos\theta) \cdot (a\cos\theta)\,d\theta$$
$$A = 4ab\int_0^{\pi/2} \cos^2\theta\,d\theta$$

Now, use the identity $\cos^2\theta = \frac{1+\cos(2\theta)}{2}$:

$$A = 4ab\int_0^{\pi/2} \frac{1 + \cos(2\theta)}{2}\,d\theta$$

$$A = 2ab\left[\theta + \frac{\sin(2\theta)}{2}\right]_0^{\pi/2}$$

$$A = 2ab\left[\left(\frac{\pi}{2} + \frac{\sin(\pi)}{2}\right) - \left(0 + \frac{\sin(0)}{2}\right)\right]$$

$$A = 2ab\left[\left(\frac{\pi}{2} + 0\right) - (0+0)\right]$$

$$A = 2ab\left(\frac{\pi}{2}\right) = \pi ab$$

Step 4: Final Answer:
The area of the region bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is πab.

> Quick Tip
>
> The area of an ellipse with semi-major axis a and semi-minor axis b is given by the formula $A = \pi ab$. This is a standard result worth memorizing. Notice that if $a = b = r$, the ellipse becomes a circle, and the area is πr^2, as expected.

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b). Find the general solution of the differential equation $y - x\frac{dy}{dx} = x + y\frac{dy}{dx}$.

Solution:

Step 1: Understanding the Concept:
The given equation is a first-order ordinary differential equation. We need to identify its type to choose the correct method of solution. By rearranging the equation, we can determine if it is separable, homogeneous, linear, or exact.

Step 2: Key Formula or Approach:
First, we rearrange the equation to isolate the derivative term $\frac{dy}{dx}$.

$$y - x = x\frac{dy}{dx} + y\frac{dy}{dx}$$

$$y - x = (x+y)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{y-x}{y+x}$$

This is a homogeneous differential equation because the numerator and denominator are homogeneous functions of the same degree (degree 1). To solve it, we use the substitution $y = vx$, which implies $\frac{dy}{dx} = v + x\frac{dv}{dx}$.

Step 3: Detailed Explanation:
Substitute $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ into the equation:

$$v + x\frac{dv}{dx} = \frac{vx - x}{vx + x}$$

Factor out x from the numerator and denominator on the right side:

$$v + x\frac{dv}{dx} = \frac{x(v-1)}{x(v+1)} = \frac{v-1}{v+1}$$

Now, we separate the variables v and x.

$$x\frac{dv}{dx} = \frac{v-1}{v+1} - v$$

$$x\frac{dv}{dx} = \frac{v - 1 - v(v+1)}{v+1} = \frac{v - 1 - v^2 - v}{v+1}$$

$$x\frac{dv}{dx} = \frac{-v^2 - 1}{v+1} = -\frac{v^2+1}{v+1}$$

Rearrange to separate the variables:

$$\frac{v+1}{v^2+1}dv = -\frac{1}{x}dx$$

Integrate both sides:

$$\int \frac{v+1}{v^2+1}dv = -\int \frac{1}{x}dx$$

Split the integral on the left side:

$$\int \frac{v}{v^2+1}dv + \int \frac{1}{v^2+1}dv = -\int \frac{1}{x}dx$$

Evaluate each integral:
The first integral is $\frac{1}{2}\ln(v^2+1)$.
The second integral is $\tan^{-1}(v)$.
The integral on the right is $-\ln|x|$.
So, the solution in terms of v and x is:

$$\frac{1}{2}\ln(v^2+1) + \tan^{-1}(v) = -\ln|x| + C$$

where C is the constant of integration.
Now, substitute back $v = \frac{y}{x}$:

$$\frac{1}{2}\ln\left(\left(\frac{y}{x}\right)^2 + 1\right) + \tan^{-1}\left(\frac{y}{x}\right) = -\ln|x| + C$$

Simplify the logarithm term:

$$\frac{1}{2}\ln\left(\frac{y^2+x^2}{x^2}\right) + \tan^{-1}\left(\frac{y}{x}\right) = -\ln|x| + C$$

$$\frac{1}{2}[\ln(y^2+x^2) - \ln(x^2)] + \tan^{-1}\left(\frac{y}{x}\right) = -\ln|x| + C$$

$$\frac{1}{2}\ln(y^2+x^2) - \frac{1}{2}(2\ln|x|) + \tan^{-1}\left(\frac{y}{x}\right) = -\ln|x| + C$$
$$\frac{1}{2}\ln(y^2+x^2) - \ln|x| + \tan^{-1}\left(\frac{y}{x}\right) = -\ln|x| + C$$

The $-\ln|x|$ terms on both sides cancel out.

$$\frac{1}{2}\ln(y^2+x^2) + \tan^{-1}\left(\frac{y}{x}\right) = C$$

Step 4: Final Answer:
The general solution of the differential equation is $\frac{1}{2}\ln(y^2+x^2) + \tan^{-1}\left(\frac{y}{x}\right) = C$.

Quick Tip

A differential equation of the form $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$ is homogeneous. The standard substitution $y = vx$ will always transform it into a separable equation in terms of v and x. Always remember to substitute back to the original variables x and y after integration.
