UP Board Class 12 Mathematics - 324(IZ) - 2025 Question Paper with Solutions

Time Allowed :3 Hours	**Maximum Marks :**100	**Total Questions :**9

General Instructions

Read the following instructions very carefully and strictly follow them:

1. There are in all nine questions in this question paper.
2. All questions are compulsory.
3. In the beginning of each question, the number of parts to be attempted are clearly mentioned.
4. Marks allotted to the questions are indicated against them.
5. Start solving from the first question and proceed to solve till the last one. Do not waste your time over a question you cannot solve.

1. Do all parts.
Select the correct option of each part and write it on your answer-book.

a. At which point is the slope of the curve $y^2 = 4x$ equal to the slope of the line $y = x + 3$?

(A) (1, 2)
(B) (2, 1)
(C) (-1, 2)
(D) (1, -2)

Correct Answer: (A) (1, 2)

Solution:

For the parabola $y^2 = 4x$ (i.e., $y^2 = 4ax$ with $a = 1$), the tangent at (x_1, y_1) is

$$yy_1 = 2(x + x_1).$$

Writing it in slope form,

$$yy_1 = 2x + 2x_1 \Rightarrow y = \frac{2}{y_1}x + \frac{2x_1}{y_1},$$

so the slope at (x_1, y_1) is $m = \dfrac{2}{y_1}$.

The given line $y = x + 3$ has slope 1. Equating slopes,

$$\frac{2}{y_1} = 1 \Rightarrow y_1 = 2.$$

Substitute into the parabola:

$$(2)^2 = 4x_1 \;\Rightarrow\; 4 = 4x_1 \;\Rightarrow\; x_1 = 1.$$

Hence the required point is

$$\boxed{(1,2)}.$$

💡 Quick Tip

The slope of a line in the form $Ax + By + C = 0$ is $-A/B$. The slope of a curve at a point is the value of its derivative at that point. To find a tangent parallel to a line, equate their slopes.

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b. If the vector $2\hat{i}+3\hat{j}+4\hat{k}$ is perpendicular to the vector $5\hat{i}-\lambda\hat{j}+2\hat{k}$, the value of λ is:

(A) 3
(B) 0
(C) 4
(D) 6

Correct Answer: (D) 6

Solution:

Instead of expanding directly, observe the condition for perpendicularity:

$$\vec{a}\cdot\vec{b} = 0.$$

Let

$$\vec{a} = 2\hat{i} + 3\hat{j} + 4\hat{k}, \quad \vec{b} = 5\hat{i} - \lambda\hat{j} + 2\hat{k}.$$

Then,

$$\vec{a}\cdot\vec{b} = (2)(5) + (3)(-\lambda) + (4)(2).$$

Combine terms carefully:

$$= 10 - 3\lambda + 8 = 18 - 3\lambda.$$

For perpendicular vectors:

$$18 - 3\lambda = 0 \;\Rightarrow\; \lambda = 6.$$

Thus,

$$\boxed{\lambda = 6}.$$

💡 Quick Tip

The condition for perpendicular vectors is a dot product of zero. The condition for parallel vectors is that their cross product is zero, or one is a scalar multiple of the other. Remember these fundamental vector properties.

c. If A is a square matrix and $A^2 = A$, then $(A + I)^3 - 7A$ will be:

(A) A
(B) 3A
(C) I - A
(D) I

Correct Answer: (D) I

Solution:

Step 1: Recall the Property of Idempotent Matrices:
If $A^2 = A$, then multiplying both sides by A gives $A^3 = A$. Thus, for all higher powers, $A^n = A$ for $n \geq 1$.
Step 2: Simplify the Expression Directly:
We want to evaluate

$$(A + I)^3 - 7A.$$

Expand the cube using the binomial theorem:

$$(A + I)^3 = A^3 + 3A^2 + 3A + I.$$

Step 3: Apply the Idempotent Property:
Since $A^2 = A$ and $A^3 = A$,

$$(A + I)^3 = A + 3A + 3A + I = 7A + I.$$

Step 4: Subtract $7A$:

$$(A + I)^3 - 7A = (7A + I) - 7A = I.$$

Step 5: Conclusion:
The given expression simplifies to

$$\boxed{I}.$$

> Quick Tip
>
> When simplifying matrix polynomials, always use the given conditions ($A^2 = A$, $A^2 = I$, etc.) to reduce higher powers of the matrix A to simpler forms (like A or I).

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d. The value of $\int \cos^2 x \, dx$ will be:

(A) $-\frac{1}{4} \sin 2x + \frac{x}{4} + C$
(B) $-\frac{1}{2} \sin 2x + \frac{x}{4} + C$
(C) $\cos^2 x - \sin^2 x + C$
(D) $\frac{1}{4} \sin 2x + \frac{x}{2} + C$

Correct Answer: (D) $\frac{1}{4}\sin 2x + \frac{x}{2} + C$

Solution:

Step 1: Use Double Angle Identity:
We know that

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x).$$

Step 2: Integrate Using Substitution:
So,

$$\int \cos^2 x\, dx = \frac{1}{2}\int (1 + \cos 2x)\, dx.$$

$$= \frac{1}{2}\int 1\, dx + \frac{1}{2}\int \cos 2x\, dx.$$

Step 3: Solve Each Part:

$$\frac{1}{2}\int 1\, dx = \frac{x}{2}, \quad \frac{1}{2}\int \cos 2x\, dx = \frac{1}{2}\cdot\frac{\sin 2x}{2}.$$

$$= \frac{x}{2} + \frac{\sin 2x}{4} + C.$$

Step 4: Conclusion:
Thus,

$$\boxed{\int \cos^2 x\, dx = \frac{1}{4}\sin 2x + \frac{x}{2} + C}.$$

Quick Tip

To integrate even powers of sine and cosine like $\sin^2 x$ or $\cos^4 x$, always use the power-reducing formulas derived from the double-angle identities: $\cos^2 x = (1 + \cos 2x)/2$ and $\sin^2 x = (1 - \cos 2x)/2$.

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e. A function $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = 3x$ for all $x \in \mathbb{R}$. Then function f will be:

(A) Not one-one
(B) Not onto
(C) Onto
(D) Many-one

Correct Answer: (C) Onto

Solution:

Step 1: Understanding the Concept:
We need to determine if the function $f(x) = 3x$ is one-to-one (injective) and/or onto (surjective).
- **One-to-one (injective):** A function is one-to-one if every distinct element in the domain maps to a distinct element in the codomain. Formally, $f(x_1) = f(x_2) \implies x_1 = x_2$.
- **Onto (surjective):** A function is onto if every element in the codomain has a corresponding element in the domain. Formally, for every $y \in \mathbb{R}$ (codomain), there exists an $x \in \mathbb{R}$ (domain) such that $f(x) = y$.

Step 2: Detailed Explanation or Calculation:
Checking for One-to-one:
Let $f(x_1) = f(x_2)$ for some $x_1, x_2 \in \mathbb{R}$.
Then, $3x_1 = 3x_2$.
Dividing both sides by 3 gives $x_1 = x_2$.
Since $f(x_1) = f(x_2)$ implies $x_1 = x_2$, the function is **one-to-one**. This eliminates options (A) and (D).

Checking for Onto:
Let y be an arbitrary element in the codomain $\mathbb{R}$. We need to see if we can find an x in the domain $\mathbb{R}$ such that $f(x) = y$.

$$3x = y$$

Solving for x, we get:

$$x = \frac{y}{3}$$

For any real number y, $y/3$ is also a real number. Thus, for any y in the codomain, there exists an $x = y/3$ in the domain that maps to it.
Therefore, the function is **onto**. This eliminates option (B).

Step 3: Final Answer:
The function is both one-to-one and onto. From the given options, (C) "Onto" is a correct description of the function.

> Quick Tip
>
> For a function $f : A \to B$, to check if it's onto, pick an arbitrary element $y \in B$ and try to solve the equation $f(x) = y$ for x. If you can always find a solution for x that is in the domain A, then the function is onto.

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2. Do all parts.

a. If $y = A + Be^x$, then prove that $\frac{d^2y}{dx^2} - \frac{dy}{dx} = 0$, where A and B are constants.

Solution:

Step 1: Understanding the Concept:
To prove that the given function satisfies the differential equation, we need to find the first and second derivatives of the function and substitute them into the equation to show that it holds true.

Step 2: Key Formula or Approach:
1. Find the first derivative, $\frac{dy}{dx}$.
2. Find the second derivative, $\frac{d^2y}{dx^2}$, by differentiating the first derivative.
3. Substitute the expressions for the derivatives into the left-hand side of the given differential equation and show that it simplifies to 0.

Step 3: Detailed Explanation or Calculation:
The given function is $y = A + Be^x$, where A and B are constants.
First Derivative:

$$\frac{dy}{dx} = \frac{d}{dx}(A + Be^x) = \frac{d}{dx}(A) + B\frac{d}{dx}(e^x) = 0 + Be^x = Be^x$$

Second Derivative:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(Be^x) = B\frac{d}{dx}(e^x) = Be^x$$

Now, substitute these derivatives into the expression $\frac{d^2y}{dx^2} - \frac{dy}{dx}$:

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} = (Be^x) - (Be^x) = 0$$

Step 4: Final Answer:
We have shown that $\frac{d^2y}{dx^2} - \frac{dy}{dx} = 0$. Hence proved.

> Quick Tip
>
> An alternative approach is to notice that from the first and second derivatives, we have $\frac{d^2y}{dx^2} = Be^x$ and $\frac{dy}{dx} = Be^x$. This directly implies that $\frac{d^2y}{dx^2} = \frac{dy}{dx}$, which rearranges to the required equation.

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b. Solve the differential equation $\frac{dy}{dx} = \frac{x^2-1}{y^2+1}$.

Solution:

Step 1: Understanding the Concept:
This is a first-order differential equation. We can see that the variables can be separated, with

all y-terms on one side and all x-terms on the other. This method is called separation of variables.

Step 2: Key Formula or Approach:
1. Rearrange the equation so that all terms involving y and dy are on one side, and all terms involving x and dx are on the other.
2. Integrate both sides of the equation with respect to their respective variables.
3. The result is the general solution to the differential equation.

Step 3: Detailed Explanation or Calculation:
The given equation is:

$$\frac{dy}{dx} = \frac{x^2 - 1}{y^2 + 1}$$

Separate the variables by multiplying both sides by $(y^2 + 1)dx$:

$$(y^2 + 1)dy = (x^2 - 1)dx$$

Now, integrate both sides:

$$\int (y^2 + 1)dy = \int (x^2 - 1)dx$$

$$\frac{y^3}{3} + y = \frac{x^3}{3} - x + C$$

where C is the constant of integration.
This is the general solution in implicit form. We can also write it as:

$$y^3 + 3y = x^3 - 3x + 3C$$

Let $K = 3C$, another constant.

$$y^3 + 3y = x^3 - 3x + K$$

Step 4: Final Answer:
The general solution to the differential equation is $\frac{y^3}{3} + y = \frac{x^3}{3} - x + C$.

> **Quick Tip**
>
> The first step in solving a first-order differential equation should always be to check if it's separable. If the equation can be written in the form $f(y)dy = g(x)dx$, this method is usually the simplest.

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c. Show that $f(x) = |x|$ is continuous for all values of x.

Solution:

Step 1: Understanding the Concept:
To show that the absolute value function is continuous everywhere, we can analyze it as a piecewise function and check for continuity at the point where its definition changes, which is at $x = 0$. For all other points, the function is a simple polynomial ($y = x$ or $y = -x$), which is known to be continuous.

Step 2: Key Formula or Approach:
The piecewise definition of $|x|$ is:

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

For continuity at a point c, we must show $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = f(c)$.

Step 3: Detailed Explanation or Calculation:
Case 1: $c > 0$
For any $c > 0$, in a small neighborhood around c, $f(x) = x$.

$$\lim_{x \to c} f(x) = \lim_{x \to c} x = c$$

Also, $f(c) = c$. Since the limit equals the function value, $f(x)$ is continuous for all $c > 0$.
Case 2: $c < 0$
For any $c < 0$, in a small neighborhood around c, $f(x) = -x$.

$$\lim_{x \to c} f(x) = \lim_{x \to c}(-x) = -c$$

Also, $f(c) = -c$. Since the limit equals the function value, $f(x)$ is continuous for all $c < 0$.
Case 3: $c = 0$
We need to check the left-hand limit, right-hand limit, and the function value at $x = 0$.
- **Left-Hand Limit (LHL):** For $x \to 0^-$, we use $f(x) = -x$.

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (-x) = 0$$

- **Right-Hand Limit (RHL):** For $x \to 0^+$, we use $f(x) = x$.

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (x) = 0$$

- **Function Value:** $f(0) = 0$.
Since LHL = RHL = $f(0) = 0$, the function is continuous at $x = 0$.

Step 4: Final Answer:
Since the function is continuous for $x > 0$, $x < 0$, and at $x = 0$, it is continuous for all real values of x. Hence proved.

💡 Quick Tip

For piecewise functions, the only points where continuity might fail are the "break points" where the definition of the function changes. If the function is continuous at these points, and the individual pieces are continuous functions, then the entire function is continuous.

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d. Given any two events A and B are such that $P(A) = \frac{1}{2}, P(B) = \frac{1}{4}$ and $P(A \cap B) = \frac{1}{8}$, then find P(not A and not B).

Solution:

Step 1: Understanding the Concept:
We need to find the probability of the event "not A and not B", which is denoted as $P(A' \cap B')$. We can find this using De Morgan's laws and the rule for the probability of a complement.

Step 2: Key Formula or Approach:
1. De Morgan's Law: $A' \cap B' = (A \cup B)'$.
2. Probability of a complement: $P(E') = 1 - P(E)$.
3. Addition rule for probability: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Step 3: Detailed Explanation or Calculation:
We want to find $P(A' \cap B')$.
Using De Morgan's Law, we can write this as:

$$P(A' \cap B') = P((A \cup B)')$$

Using the rule for complementary events:

$$P((A \cup B)') = 1 - P(A \cup B)$$

Now, we need to find $P(A \cup B)$ using the addition rule:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Substitute the given values:

$$P(A \cup B) = \frac{1}{2} + \frac{1}{4} - \frac{1}{8}$$

To add these fractions, find a common denominator, which is 8:

$$P(A \cup B) = \frac{4}{8} + \frac{2}{8} - \frac{1}{8} = \frac{4 + 2 - 1}{8} = \frac{5}{8}$$

Finally, substitute this value back to find $P(A' \cap B')$:

$$P(A' \cap B') = 1 - P(A \cup B) = 1 - \frac{5}{8} = \frac{3}{8}$$

Step 4: Final Answer:
The probability of P(not A and not B) is $\frac{3}{8}$.

> Quick Tip
>
> De Morgan's laws ($(A \cup B)' = A' \cap B'$ and $(A \cap B)' = A' \cup B'$) are extremely useful for transforming probability problems involving 'and' and 'or' with complements.

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e. If $A = \{1, 2\}$ and $B = \{3, 4, 5\}$, then find all number of relations from A to B.

Solution:

Step 1: Understanding the Concept:
A relation from a set A to a set B is any subset of the Cartesian product $A \times B$. The total number of possible relations is therefore equal to the total number of possible subsets of $A \times B$.

Step 2: Key Formula or Approach:
1. Let $n(A)$ be the number of elements in set A and $n(B)$ be the number of elements in set B.
2. The number of elements in the Cartesian product $A \times B$ is $n(A \times B) = n(A) \times n(B)$.
3. The number of subsets of a set with k elements is 2^k.
4. Therefore, the number of relations from A to B is $2^{n(A) \times n(B)}$.

Step 3: Detailed Explanation or Calculation:
Given the sets:
$A = \{1, 2\}$, so the number of elements in A is $n(A) = 2$.
$B = \{3, 4, 5\}$, so the number of elements in B is $n(B) = 3$.
First, calculate the number of elements in the Cartesian product $A \times B$:

$$n(A \times B) = n(A) \times n(B) = 2 \times 3 = 6$$

A relation is a subset of $A \times B$. The total number of subsets of a set with 6 elements is 2^6.

$$\text{Number of relations} = 2^6 = 64$$

Step 4: Final Answer:
The total number of relations from A to B is 64.

> Quick Tip
>
> Remember the distinction: a relation is a subset of the Cartesian product, while a function is a special type of relation where each element of the first set is paired with exactly one element of the second set. The number of relations is always $2^{n(A)n(B)}$, while the number of functions is $n(B)^{n(A)}$.

3. Do all parts.

a. If $y = A\cos\theta + B\sin\theta$, then prove that $\frac{d^2y}{d\theta^2} = -y$.

Solution:

Step 1: Understanding the Concept:
We are given a function y in terms of θ and asked to prove it satisfies a second-order differential equation. This requires finding the first and second derivatives of y with respect to θ and showing that the second derivative is the negative of the original function.

Step 2: Key Formula or Approach:
1. Find the first derivative, $\frac{dy}{d\theta}$.
2. Find the second derivative, $\frac{d^2y}{d\theta^2}$.
3. Show that the expression for $\frac{d^2y}{d\theta^2}$ is equal to $-y$.

Step 3: Detailed Explanation or Calculation:
The given function is $y = A\cos\theta + B\sin\theta$, where A and B are constants.
First Derivative:
Differentiate y with respect to θ:

$$\frac{dy}{d\theta} = \frac{d}{d\theta}(A\cos\theta + B\sin\theta) = -A\sin\theta + B\cos\theta$$

Second Derivative:
Differentiate $\frac{dy}{d\theta}$ with respect to θ:

$$\frac{d^2y}{d\theta^2} = \frac{d}{d\theta}(-A\sin\theta + B\cos\theta) = -A\cos\theta - B\sin\theta$$

Now, factor out -1 from the expression for the second derivative:

$$\frac{d^2y}{d\theta^2} = -(A\cos\theta + B\sin\theta)$$

We can see that the expression in the parenthesis is the original function y.

$$\frac{d^2y}{d\theta^2} = -y$$

Step 4: Final Answer:
Hence, it is proved that $\frac{d^2y}{d\theta^2} = -y$. This is the differential equation for simple harmonic motion.

> Quick Tip
>
> When asked to prove that a function satisfies a differential equation, always find the required derivatives first. Then, substitute them into the equation. Often, you will be able to substitute the original function back into the expressions for the derivatives to simplify the process.

b. Find the Cartesian equation of a line which passes through point (3, -2, -5) and is parallel to the vector $(3\hat{i} + 2\hat{j} - 2\hat{k})$.

Solution:

Step 1: Understanding the Concept:
The Cartesian equation of a line in 3D space is determined by a point on the line and its direction ratios. The direction ratios of the line will be the same as the components of the vector to which it is parallel.

Step 2: Key Formula or Approach:
The standard Cartesian (or symmetric) form of the equation of a line passing through the point (x_0, y_0, z_0) with direction ratios $\langle a, b, c\rangle$ is:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Step 3: Detailed Explanation or Calculation:
We are given: - The point on the line: $(x_0, y_0, z_0) = (3, -2, -5)$. - The line is parallel to the vector $3\hat{i} + 2\hat{j} - 2\hat{k}$. This means the direction ratios of the line are the components of this vector: $\langle a, b, c\rangle = \langle 3, 2, -2\rangle$.
Now, substitute these values into the standard Cartesian form:

$$\frac{x - 3}{3} = \frac{y - (-2)}{2} = \frac{z - (-5)}{-2}$$

$$\frac{x - 3}{3} = \frac{y + 2}{2} = \frac{z + 5}{-2}$$

Step 4: Final Answer:
The Cartesian equation of the line is $\frac{x-3}{3} = \frac{y+2}{2} = \frac{z+5}{-2}$.

Quick Tip

The vector $a\hat{i} + b\hat{j} + c\hat{k}$ directly gives the direction ratios $\langle a, b, c\rangle$ for a line parallel to it. These values are the denominators in the Cartesian equation. Be careful with the signs of the coordinates in the numerators.

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c. Solve the differential equation $\frac{dy}{dx} = e^x \cos x$.

Solution:

Step 1: Understanding the Concept:
This is a first-order differential equation where the variables are already separated. To solve

for y, we need to integrate the right-hand side with respect to x. The integral of $e^x \cos x$ is a standard type that is solved using integration by parts.

Step 2: Key Formula or Approach:
1. The equation can be written as $dy = e^x \cos x \, dx$.
2. Integrate both sides: $\int dy = \int e^x \cos x \, dx$.
3. Use integration by parts formula: $\int u \, dv = uv - \int v \, du$.
4. A standard integral formula is also available: $\int e^{ax} \cos(bx) dx = \frac{e^{ax}}{a^2+b^2}(a\cos(bx) + b\sin(bx)) + C$.

Step 3: Detailed Explanation or Calculation:
We need to find the integral $I = \int e^x \cos x \, dx$.
Let's use the standard formula with $a = 1, b = 1$:

$$I = \frac{e^x}{1^2 + 1^2}(1\cos(x) + 1\sin(x)) + C$$

$$I = \frac{e^x}{2}(\cos x + \sin x) + C$$

The solution to the differential equation is $y = I$.
Alternatively, using integration by parts:
Let $u = \cos x$ and $dv = e^x dx$. Then $du = -\sin x \, dx$ and $v = e^x$.

$$I = e^x \cos x - \int e^x(-\sin x) \, dx = e^x \cos x + \int e^x \sin x \, dx$$

Apply integration by parts to the new integral: Let $u = \sin x$ and $dv = e^x dx$. Then $du = \cos x \, dx$ and $v = e^x$.

$$I = e^x \cos x + \left(e^x \sin x - \int e^x \cos x \, dx \right)$$

The original integral I reappears on the right side.

$$I = e^x \cos x + e^x \sin x - I$$

$$2I = e^x(\cos x + \sin x)$$

$$I = \frac{e^x}{2}(\cos x + \sin x)$$

So, the solution is $y = \frac{e^x}{2}(\cos x + \sin x) + C$.

Step 4: Final Answer:
The solution of the differential equation is $y = \frac{e^x}{2}(\cos x + \sin x) + C$.

Quick Tip

Integrals of the form $\int e^{ax} \cos(bx) dx$ and $\int e^{ax} \sin(bx) dx$ are cyclic. When using integration by parts, the original integral will reappear after applying the method twice. Memorizing the standard formula for these integrals can save significant time.

d. If $\vec{a}, \vec{b}$ and $\vec{c}$ are vectors and $\vec{a}+\vec{b}+\vec{c}=\vec{0}$, then find the value of $(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})$.

Solution:

Step 1: Understanding the Concept:
We can find the value of the expression involving dot products by using the given vector sum. The key is to take the dot product of the vector sum with itself, which introduces the square of the magnitudes and the dot product terms we are interested in.

Step 2: Key Formula or Approach:
- Given: $\vec{a}+\vec{b}+\vec{c}=\vec{0}$
- Square this equation by taking the dot product with itself: $(\vec{a}+\vec{b}+\vec{c})\cdot(\vec{a}+\vec{b}+\vec{c})=\vec{0}\cdot\vec{0}$
- Use the property $\vec{v}\cdot\vec{v}=|\vec{v}|^2$.

Step 3: Detailed Explanation or Calculation:
Start with the given vector sum:

$$\vec{a}+\vec{b}+\vec{c}=\vec{0}$$

Take the dot product of both sides with $(\vec{a}+\vec{b}+\vec{c})$:

$$(\vec{a}+\vec{b}+\vec{c})\cdot(\vec{a}+\vec{b}+\vec{c})=0$$

Expand the left side:

$$\vec{a}\cdot\vec{a}+\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}+\vec{b}\cdot\vec{a}+\vec{b}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}+\vec{c}\cdot\vec{b}+\vec{c}\cdot\vec{c}=0$$

Using $\vec{v}\cdot\vec{v}=|\vec{v}|^2$ and the commutative property of the dot product ($\vec{u}\cdot\vec{v}=\vec{v}\cdot\vec{u}$):

$$|\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})=0$$

Now, we can solve for the expression we want:

$$2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})=-(|\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2)$$

$$\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}=-\frac{1}{2}(|\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2)$$

Step 4: Final Answer:
The value of $\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}$ is $-\frac{1}{2}(|\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2)$.

Quick Tip

This is a standard and very common vector algebra problem. The technique of "squaring" a vector sum by taking its dot product with itself is extremely useful and should be remembered. It's the vector equivalent of squaring a scalar algebraic expression.

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4. Do all parts.

a. If R_1 and R_2 be two equivalence relations on a set A, then prove that $(R_1 \cap R_2)$ also be an equivalence relation on A.

Solution:

Step 1: Understanding the Concept:
To prove that the intersection of two equivalence relations is also an equivalence relation, we must show that the intersection, $R_1 \cap R_2$, satisfies the three required properties: reflexivity, symmetry, and transitivity. We can use the fact that R_1 and R_2 individually satisfy these properties.

Step 2: Key Definitions:
- **Reflexive:** For all $a \in A$, $(a, a) \in R$.
- **Symmetric:** If $(a, b) \in R$, then $(b, a) \in R$.
- **Transitive:** If $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$.
- **Intersection:** $(a, b) \in R_1 \cap R_2$ means $(a, b) \in R_1$ and $(a, b) \in R_2$.

Step 3: Detailed Explanation:
1. Reflexivity of $R_1 \cap R_2$:
Let $a \in A$. Since R_1 is an equivalence relation, it is reflexive, so $(a, a) \in R_1$. Similarly, since R_2 is reflexive, $(a, a) \in R_2$. Since (a, a) is in both R_1 and R_2, by the definition of intersection, $(a, a) \in R_1 \cap R_2$.
Thus, $R_1 \cap R_2$ is reflexive.

2. Symmetry of $R_1 \cap R_2$:
Let $(a, b) \in R_1 \cap R_2$. This implies $(a, b) \in R_1$ and $(a, b) \in R_2$. Since R_1 is symmetric, $(b, a) \in R_1$. Since R_2 is symmetric, $(b, a) \in R_2$. As (b, a) is in both relations, it must be in their intersection: $(b, a) \in R_1 \cap R_2$.
Thus, $R_1 \cap R_2$ is symmetric.

3. Transitivity of $R_1 \cap R_2$:
Let $(a, b) \in R_1 \cap R_2$ and $(b, c) \in R_1 \cap R_2$. This means $(a, b) \in R_1$, $(b, c) \in R_1$, and also $(a, b) \in R_2$, $(b, c) \in R_2$.

Since R_1 is transitive, $(a, b) \in R_1$ and $(b, c) \in R_1$ implies $(a, c) \in R_1$.

Since R_2 is transitive, $(a, b) \in R_2$ and $(b, c) \in R_2$ implies $(a, c) \in R_2$.

Because (a, c) is in both R_1 and R_2, it must be in their intersection: $(a, c) \in R_1 \cap R_2$.

Thus, $R_1 \cap R_2$ is transitive.

Step 4: Final Answer:
Since $R_1 \cap R_2$ is reflexive, symmetric, and transitive, it is an equivalence relation. Hence proved.

Quick Tip

The structure of proofs involving intersections of sets with certain properties is often the same: assume an element belongs to the intersection, which means it belongs to both sets individually. Apply the known property to each set, and then conclude that the result must also belong to both sets, and therefore to the intersection.

---

b. Find the value of $\int_{-\pi/2}^{\pi/2} \sin^2 x \, dx$.

Solution:

Step 1: Understanding the Concept:
We need to evaluate a definite integral of $\sin^2 x$ over a symmetric interval $[-\pi/2, \pi/2]$. We can use properties of even and odd functions to simplify the integral. Then, we use a trigonometric identity to integrate $\sin^2 x$.

Step 2: Key Formula or Approach:
1. Property of definite integrals over symmetric intervals: If $f(x)$ is an even function ($f(-x) = f(x)$), then $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$.
2. Power-reducing identity: $\sin^2 x = \frac{1-\cos(2x)}{2}$.

Step 3: Detailed Explanation or Calculation:
Let $f(x) = \sin^2 x$. Let's check if it is an even function:

$$f(-x) = \sin^2(-x) = (-\sin x)^2 = \sin^2 x = f(x)$$

Since $f(x)$ is an even function, we can simplify the integral:

$$\int_{-\pi/2}^{\pi/2} \sin^2 x \, dx = 2\int_0^{\pi/2} \sin^2 x \, dx$$

Now, use the power-reducing identity:

$$= 2\int_0^{\pi/2} \frac{1-\cos(2x)}{2} \, dx = \int_0^{\pi/2} (1-\cos(2x)) \, dx$$

Integrate term by term:

$$= \left[x - \frac{\sin(2x)}{2}\right]_0^{\pi/2}$$

Evaluate at the limits:

$$= \left(\frac{\pi}{2} - \frac{\sin(2\cdot\pi/2)}{2}\right) - \left(0 - \frac{\sin(0)}{2}\right)$$

$$= \left(\frac{\pi}{2} - \frac{\sin(\pi)}{2}\right) - (0 - 0)$$

Since $\sin(\pi) = 0$:

$$= \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

Step 4: Final Answer:
The value of the integral is $\frac{\pi}{2}$.

> Quick Tip
>
> Recognizing whether a function is even or odd is a powerful shortcut for definite integrals over symmetric intervals like $[-a, a]$. For an even function, the integral is twice the integral over $[0, a]$. For an odd function, the integral is zero.

---

c. There are two children in a family. It is known that there is at least one child is a boy. Then find the probability that both children are boys.

Solution:

Step 1: Understanding the Concept:
This is a conditional probability problem. We are given some information (the reduced sample space) and asked to find the probability of an event within that new sample space.

Step 2: Key Formula or Approach:
Let A be the event that both children are boys, and B be the event that at least one child is a boy. We want to find $P(A|B)$, the probability of A given B.
The formula for conditional probability is:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Step 3: Detailed Explanation or Calculation:
First, list the entire sample space S for two children (B=Boy, G=Girl):

$$S = \{BB, BG, GB, GG\}$$

Each outcome has a probability of 1/4.
Now, define the events: - Event A: Both children are boys. $A = \{BB\}$. The probability is $P(A) = 1/4$. - Event B: At least one child is a boy. This means we exclude the case where both are girls. $B = \{BB, BG, GB\}$. The probability is $P(B) = 3/4$.
Next, we find the intersection of A and B, $A \cap B$. This is the event where "both children are boys" AND "at least one child is a boy". This is simply the event "both children are boys".

$$A \cap B = \{BB\} \cap \{BB, BG, GB\} = \{BB\}$$

The probability of the intersection is $P(A \cap B) = 1/4$.
Now, apply the conditional probability formula:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1/4}{3/4} = \frac{1}{3}$$

Alternative (Reduced Sample Space) Method:
The given information "at least one child is a boy" reduces our sample space to $S' = \{BB, BG, GB\}$. All three outcomes in this new space are equally likely. We want to find the probability of the event "both children are boys" within this new sample space. The favorable outcome is $\{BB\}$.

$$P(\text{both boys}|\text{at least one boy}) = \frac{\text{Number of favorable outcomes in } S'}{\text{Total number of outcomes in } S'} = \frac{1}{3}$$

Step 4: Final Answer:
The probability that both children are boys is $\frac{1}{3}$.

> Quick Tip
>
> Be careful with conditional probability problems. The phrase "it is known that" or "given that" signals that you are no longer working with the original full sample space. You must restrict your analysis to the outcomes that satisfy the given condition.

---

d. Find the interval in which the given function $f(x) = x^2 - 4x + 6$ is (i) Increasing (ii) Decreasing.

Solution:

Step 1: Understanding the Concept:
A function is increasing where its first derivative is positive ($f'(x) > 0$) and decreasing where its first derivative is negative ($f'(x) < 0$). The points where the derivative is zero or undefined are the critical points, which are the boundaries of these intervals.

Step 2: Key Formula or Approach:
1. Find the first derivative of the function, $f'(x)$.
2. Find the critical points by setting $f'(x) = 0$.
3. Test the sign of $f'(x)$ in the intervals created by the critical points.

Step 3: Detailed Explanation or Calculation:
The given function is $f(x) = x^2 - 4x + 6$.
1. Find the first derivative:

$$f'(x) = \frac{d}{dx}(x^2 - 4x + 6) = 2x - 4$$

2. Find the critical point:
Set the derivative equal to zero:

$$2x - 4 = 0$$
$$2x = 4$$
$$x = 2$$

The critical point is at $x = 2$. This point divides the number line into two intervals: $(-\infty, 2)$ and $(2, \infty)$.
3. Test the intervals:
- Interval 1: $(-\infty, 2)$
Choose a test value in this interval, for example, $x = 0$.

$$f'(0) = 2(0) - 4 = -4$$

Since $f'(x)$ is negative in this interval, the function $f(x)$ is **decreasing** on $(-\infty, 2)$.
- Interval 2: $(2, \infty)$
Choose a test value in this interval, for example, $x = 3$.

$$f'(3) = 2(3) - 4 = 6 - 4 = 2$$

Since $f'(x)$ is positive in this interval, the function $f(x)$ is **increasing** on $(2, \infty)$.

Step 4: Final Answer:
(i) The function is **increasing** on the interval $(2, \infty)$.
(ii) The function is **decreasing** on the interval $(-\infty, 2)$.

Quick Tip

For a quadratic function $ax^2 + bx + c$, the vertex at $x = -b/2a$ is the turning point. This is the critical point where the function changes from decreasing to increasing (if a ¿ 0) or increasing to decreasing (if a ¡ 0). You can use this to quickly find the intervals.

---

5. Do all parts.

a. If the matrix A = $\begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix}$ **satisfies the equation AA' = I, then find the values of x, y and z.**

Correct Answer: $x = \pm\frac{1}{\sqrt{2}}, y = \pm\frac{1}{\sqrt{6}}, z = \pm\frac{1}{\sqrt{3}}$

Solution:

Step 1: Understanding the Concept:
The given equation is $AA' = I$, where A' is the transpose of matrix A and I is the identity matrix.
A matrix A that satisfies this condition is called an orthogonal matrix.
For an orthogonal matrix, the dot product of any row vector with itself is 1, and the dot product of any two distinct row vectors is 0.

Step 2: Key Formula or Approach:

1. Find the transpose of matrix A, denoted as A'.
2. Calculate the product AA'.
3. Equate the resulting matrix with the identity matrix $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.
4. Solve the system of equations obtained by equating the corresponding elements.

Step 3: Detailed Explanation or Calculation:

Given matrix $A = \begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix}$.

The transpose of A is $A' = \begin{bmatrix} 0 & x & x \\ 2y & y & -y \\ z & -z & z \end{bmatrix}$.

Now, we compute the product AA':

$$AA' = \begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix} \begin{bmatrix} 0 & x & x \\ 2y & y & -y \\ z & -z & z \end{bmatrix}$$

$$= \begin{bmatrix} (0)^2 + (2y)^2 + (z)^2 & 0(x) + 2y(y) + z(-z) & 0(x) + 2y(-y) + z(z) \\ x(0) + y(2y) + (-z)(z) & x^2 + y^2 + (-z)^2 & x^2 + y(-y) + (-z)(z) \\ x(0) + (-y)(2y) + z(z) & x(x) + (-y)(y) + z(-z) & x^2 + (-y)^2 + z^2 \end{bmatrix}$$

$$= \begin{bmatrix} 4y^2 + z^2 & 2y^2 - z^2 & -2y^2 + z^2 \\ 2y^2 - z^2 & x^2 + y^2 + z^2 & x^2 - y^2 - z^2 \\ -2y^2 + z^2 & x^2 - y^2 - z^2 & x^2 + y^2 + z^2 \end{bmatrix}$$

Equating AA' with I:

$$\begin{bmatrix} 4y^2 + z^2 & 2y^2 - z^2 & -2y^2 + z^2 \\ 2y^2 - z^2 & x^2 + y^2 + z^2 & x^2 - y^2 - z^2 \\ -2y^2 + z^2 & x^2 - y^2 - z^2 & x^2 + y^2 + z^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

From the non-diagonal elements, we get:
$2y^2 - z^2 = 0 \implies z^2 = 2y^2$ (Equation 1)
From the diagonal elements, we get:
$4y^2 + z^2 = 1$ (Equation 2)
$x^2 + y^2 + z^2 = 1$ (Equation 3)
Substitute $z^2 = 2y^2$ from Eq. 1 into Eq. 2:
$4y^2 + 2y^2 = 1 \implies 6y^2 = 1 \implies y^2 = \frac{1}{6} \implies y = \pm\frac{1}{\sqrt{6}}$.
Now find z^2 using Eq. 1:
$z^2 = 2y^2 = 2(\frac{1}{6}) = \frac{1}{3} \implies z = \pm\frac{1}{\sqrt{3}}$.
Finally, substitute the values of y^2 and z^2 into Eq. 3 to find x:
$x^2 + \frac{1}{6} + \frac{1}{3} = 1 \implies x^2 + \frac{1+2}{6} = 1 \implies x^2 + \frac{3}{6} = 1 \implies x^2 + \frac{1}{2} = 1.$
$x^2 = 1 - \frac{1}{2} = \frac{1}{2} \implies x = \pm\frac{1}{\sqrt{2}}$.

Step 4: Final Answer
The values are $x = \pm\frac{1}{\sqrt{2}}$, $y = \pm\frac{1}{\sqrt{6}}$, and $z = \pm\frac{1}{\sqrt{3}}$.

Quick Tip

The condition $AA' = I$ for a square matrix A signifies that A is an orthogonal matrix. This implies that its row vectors (and column vectors) form an orthonormal set. That is, each row vector has a magnitude of 1, and the dot product of any two different row vectors is 0. This insight can sometimes provide a faster way to set up the equations.

---

b. Find the differential coefficient of $y = x^{\cos x} + (\sin x)^x$ with respect to x.

Correct Answer: $\frac{dy}{dx} = x^{\cos x}\left(\frac{\cos x}{x} - \sin x \ln x\right) + (\sin x)^x(\ln(\sin x) + x \cot x)$

Solution:

Step 1: Understanding the Concept:
The function is a sum of two functions, each of the form $f(x)^{g(x)}$.
This type of function requires logarithmic differentiation. We cannot differentiate it directly using standard power or exponential rules.
We will differentiate each term separately using the sum rule of differentiation.

Step 2: Key Formula or Approach:
1. Let $y = u(x) + v(x)$, where $u(x) = x^{\cos x}$ and $v(x) = (\sin x)^x$.
2. The derivative is $\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$.
3. For functions of the form $f(x)^{g(x)}$, apply logarithms: take the natural log of both sides, use log properties to bring the exponent down, and then differentiate implicitly.
If $u = f^g$, then $\ln u = g \ln f$, and $\frac{1}{u}\frac{du}{dx} = g' \ln f + g\frac{f'}{f}$.

Step 3: Detailed Explanation or Calculation:
Part 1: Differentiating $u = x^{\cos x}$
Take the natural logarithm of both sides:

$$\ln u = \ln(x^{\cos x})$$
$$\ln u = \cos x \ln x$$

Differentiate both sides with respect to x using the product rule:

$$\frac{1}{u}\frac{du}{dx} = (-\sin x)(\ln x) + (\cos x)\left(\frac{1}{x}\right)$$
$$\frac{du}{dx} = u\left(\frac{\cos x}{x} - \sin x \ln x\right)$$

Substitute back $u = x^{\cos x}$:

$$\frac{du}{dx} = x^{\cos x}\left(\frac{\cos x}{x} - \sin x \ln x\right)$$

Part 2: Differentiating $v = (\sin x)^x$
Take the natural logarithm of both sides:

$$\ln v = \ln((\sin x)^x)$$
$$\ln v = x \ln(\sin x)$$

Differentiate both sides with respect to x using the product rule:

$$\frac{1}{v}\frac{dv}{dx} = (1)(\ln(\sin x)) + x\left(\frac{1}{\sin x} \cdot \cos x\right)$$
$$\frac{1}{v}\frac{dv}{dx} = \ln(\sin x) + x \cot x$$
$$\frac{dv}{dx} = v(\ln(\sin x) + x \cot x)$$

Substitute back $v = (\sin x)^x$:

$$\frac{dv}{dx} = (\sin x)^x(\ln(\sin x) + x \cot x)$$

Part 3: Combining the results

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$
$$\frac{dy}{dx} = x^{\cos x}\left(\frac{\cos x}{x} - \sin x \ln x\right) + (\sin x)^x(\ln(\sin x) + x \cot x)$$

Step 4: Final Answer
The differential coefficient is $x^{\cos x}\left(\frac{\cos x}{x} - \sin x \ln x\right) + (\sin x)^x(\ln(\sin x) + x \cot x)$.

Quick Tip

Whenever you encounter a function in the form of $(\text{variable})^{(\text{variable})}$, i.e., $f(x)^{g(x)}$, your first thought should be logarithmic differentiation. Remember that $\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$, but you cannot apply logarithms directly to a sum like $\ln(u+v)$. Always handle each term separately.

---

c. Find the shortest distance between the lines $\vec{r} = (2\hat{i} + \hat{j} – \hat{k}) + \mu(3\hat{i} – 5\hat{j} + 2\hat{k})$ and $\vec{r} = (\hat{i} + \hat{j}) + \lambda(2\hat{i} - \hat{j} + \hat{k})$.

Correct Answer: $\frac{10}{\sqrt{59}}$ units

Solution:

Step 1: Understanding the Concept:
The two given lines are skew lines, meaning they are non-parallel and do not intersect.
The shortest distance between two skew lines is the length of the perpendicular line segment connecting them.

Step 2: Key Formula or Approach:
The shortest distance (d) between two skew lines $\vec{r} = \vec{a_1} + \lambda\vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu\vec{b_2}$ is given by the formula:

$$d = \left| \frac{(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2})}{|\vec{b_1} \times \vec{b_2}|} \right|$$

Step 3: Detailed Explanation or Calculation:
From the given equations of the lines, we identify the vectors:
Line 1: $\vec{a_1} = 2\hat{i} + \hat{j} - \hat{k}$, $\vec{b_1} = 3\hat{i} - 5\hat{j} + 2\hat{k}$
Line 2: $\vec{a_2} = \hat{i} + \hat{j}$, $\vec{b_2} = 2\hat{i} - \hat{j} + \hat{k}$

First, calculate $(\vec{a_2} - \vec{a_1})$:

$$\vec{a_2} - \vec{a_1} = (\hat{i} + \hat{j}) - (2\hat{i} + \hat{j} - \hat{k}) = -\hat{i} + 0\hat{j} + \hat{k}$$

Next, calculate the cross product $(\vec{b_1} \times \vec{b_2})$:

$$\vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -5 & 2 \\ 2 & -1 & 1 \end{vmatrix}$$

$$= \hat{i}((-5)(1) - (2)(-1)) - \hat{j}((3)(1) - (2)(2)) + \hat{k}((3)(-1) - (-5)(2))$$
$$= \hat{i}(-5 + 2) - \hat{j}(3 - 4) + \hat{k}(-3 + 10)$$
$$= -3\hat{i} + \hat{j} + 7\hat{k}$$

Now, calculate the magnitude of the cross product, $|\vec{b_1} \times \vec{b_2}|$:

$$|\vec{b_1} \times \vec{b_2}| = \sqrt{(-3)^2 + 1^2 + 7^2} = \sqrt{9 + 1 + 49} = \sqrt{59}$$

Next, calculate the dot product $(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2})$:

$$(-\hat{i} + \hat{k}) \cdot (-3\hat{i} + \hat{j} + 7\hat{k}) = (-1)(-3) + (0)(1) + (1)(7) = 3 + 7 = 10$$

Finally, substitute these values into the shortest distance formula:

$$d = \left| \frac{10}{\sqrt{59}} \right| = \frac{10}{\sqrt{59}}$$

Step 4: Final Answer
The shortest distance between the two lines is $\frac{10}{\sqrt{59}}$ units.

Quick Tip

Before applying the skew lines formula, quickly check if the lines are parallel by seeing if $\vec{b_1}$ is a scalar multiple of $\vec{b_2}$. If they are, a different formula is needed. If the scalar triple product $(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2})$ equals zero, the lines are coplanar and intersect, so the shortest distance is 0.

---

d. Find the minimum value of Z = 3x + 7y by the graphical method under the following constraints:
$x + y \leq 8, 3x + 5y \geq 0, x \geq 0, y \geq 0$

Correct Answer: The minimum value of Z is 0.

Solution:

Step 1: Understanding the Concept:
This is a Linear Programming Problem (LPP). The goal is to find the minimum value of a linear objective function Z subject to a set of linear inequalities called constraints. The graphical method involves plotting these constraints to identify a feasible region and then testing the corner points of this region in the objective function.

Step 2: Key Formula or Approach:
1. Treat the inequalities as equations to plot the boundary lines.
2. Determine the region that satisfies each inequality.
3. Identify the common region satisfying all constraints, which is the feasible region.
4. Find the coordinates of the vertices (corner points) of the feasible region.
5. Evaluate the objective function $Z = 3x + 7y$ at each vertex.
6. The smallest value of Z among these will be the minimum value.

Step 3: Detailed Explanation or Calculation:
The constraints are:
1. $x + y \leq 8$
2. $3x + 5y \geq 0$
3. $x \geq 0$
4. $y \geq 0$

Graphing the Constraints:
- The constraints $x \geq 0$ and $y \geq 0$ restrict the feasible region to the first quadrant of the Cartesian plane.
- The constraint $3x + 5y \geq 0$ is satisfied by all points in the first quadrant (since x and y are non-negative). Therefore, this constraint is redundant and does not further restrict the feasible region defined by the other constraints.
- To plot $x + y \leq 8$, we first draw the line $x + y = 8$. This line passes through the points (8, 0) and (0, 8). The inequality $\leq$ means the feasible region lies on and below this line.

Identifying the Feasible Region and Corner Points:
The feasible region is the area in the first quadrant bounded by the lines $x = 0$, $y = 0$, and $x + y = 8$. This forms a triangle.
The vertices (corner points) of this triangular feasible region are:
- O(0, 0): The origin.
- A(8, 0): The x-intercept of the line $x + y = 8$.
- B(0, 8): The y-intercept of the line $x + y = 8$.

Evaluating the Objective Function Z at Corner Points:
We now evaluate $Z = 3x + 7y$ at each vertex:
- At O(0, 0): $Z = 3(0) + 7(0) = 0$
- At A(8, 0): $Z = 3(8) + 7(0) = 24$
- At B(0, 8): $Z = 3(0) + 7(8) = 56$

Step 4: Final Answer
Comparing the values of Z (0, 24, 56), the minimum value is 0, which occurs at the point (0, 0).

Quick Tip

In a graphical LPP, the optimal solution (maximum or minimum) for a bounded feasible region is always found at one of its corner points. Always identify all constraints; sometimes a constraint might be redundant (as $3x + 5y \geq 0$ was here, given $x \geq 0, y \geq 0$), which can simplify the problem.

---

e. If a function f: R → {x ∈ R : x ∈ (-1, 1)} is defined as $f(x) = \frac{x}{1+|x|}$, x ∈ R, then prove that f is one-one and onto.

Correct Answer: The function f is proven to be both one-one (injective) and onto (surjective).

Solution:

Step 1: Understanding the Concept:
To prove that a function is one-one and onto (bijective), we need to satisfy two conditions:
1. **One-one (Injective):** For any two elements x_1, x_2 in the domain, if $f(x_1) = f(x_2)$, then it must be that $x_1 = x_2$. Alternatively, a strictly monotonic function is always one-one.
2. **Onto (Surjective):** For every element y in the codomain, there must exist at least one element x in the domain such that $f(x) = y$. This means the range of the function must be equal to its codomain.
The given function is $f(x) = \frac{x}{1+|x|}$, with domain $\mathbb{R}$ and codomain $(-1, 1)$.
First, we can write f(x) as a piecewise function:

$$f(x) = \begin{cases} \frac{x}{1+x} & \text{if } x \geq 0 \\ \frac{x}{1-x} & \text{if } x < 0 \end{cases}$$

Step 2: Proving f is One-One (Injective):
Let $x_1, x_2 \in \mathbb{R}$ such that $f(x_1) = f(x_2)$.
Case 1: $x_1 \geq 0$ **and** $x_2 \geq 0$**.**

$$\frac{x_1}{1+x_1} = \frac{x_2}{1+x_2} \implies x_1(1+x_2) = x_2(1+x_1) \implies x_1 + x_1x_2 = x_2 + x_2x_1 \implies x_1 = x_2$$

Case 2: $x_1 < 0$ **and** $x_2 < 0$**.**

$$\frac{x_1}{1-x_1} = \frac{x_2}{1-x_2} \implies x_1(1-x_2) = x_2(1-x_1) \implies x_1 - x_1x_2 = x_2 - x_2x_1 \implies x_1 = x_2$$

Case 3: $x_1 \geq 0$ **and** $x_2 < 0$**.**
If $x_1 \geq 0$, then $f(x_1) = \frac{x_1}{1+x_1} \geq 0$.
If $x_2 < 0$, then $f(x_2) = \frac{x_2}{1-x_2} < 0$.
Therefore, $f(x_1)$ can never be equal to $f(x_2)$ in this case (unless both are 0, which means $x_1 = x_2 = 0$, violating $x_2 < 0$).
Since in all possible cases, $f(x_1) = f(x_2)$ implies $x_1 = x_2$, the function f is one-one.

Step 3: Proving f is Onto (Surjective):
Let y be an arbitrary element from the codomain $(-1, 1)$. We need to find an $x \in \mathbb{R}$ such that $f(x) = y$.
Case 1: $y \in [0, 1)$**.**
We expect x to be non-negative, so we use $f(x) = \frac{x}{1+x}$.

$$y = \frac{x}{1+x} \implies y(1+x) = x \implies y + yx = x \implies y = x(1-y) \implies x = \frac{y}{1-y}$$

Since $0 \leq y < 1$, the numerator $y \geq 0$ and the denominator $1 - y > 0$, which means $x \geq 0$. So, a valid x exists in the domain.
Case 2: $y \in (-1, 0)$**.**
We expect x to be negative, so we use $f(x) = \frac{x}{1-x}$.

$$y = \frac{x}{1-x} \implies y(1-x) = x \implies y - yx = x \implies y = x(1+y) \implies x = \frac{y}{1+y}$$

Since $-1 < y < 0$, the numerator $y < 0$ and the denominator $1 + y > 0$, which means $x < 0$. So, a valid x exists in the domain.
Since for any y in the codomain $(-1, 1)$, we can find a corresponding x in the domain $\mathbb{R}$, the function f is onto.

Step 4: Final Answer
The function f is both one-one and onto, hence it is a bijection from $\mathbb{R}$ to $(-1, 1)$.

> 💡 Quick Tip
>
> An alternative way to prove a differentiable function is one-one is to show its derivative is always positive or always negative. For $f(x) = \frac{x}{1+|x|}$, you can find that $f'(x) = \frac{1}{(1+|x|)^2}$ for $x \neq 0$, which is always positive. A function that is always increasing (or decreasing) must be one-one. This calculus-based approach is often faster than the algebraic case-by-case proof.

---

6. Do all parts.

a. Show that ($3\hat{i}$ - $4\hat{j}$ - $4\hat{k}$), ($2\hat{i}$ - $\hat{j}$ + $\hat{k}$) and ($\hat{i}$ - $3\hat{j}$ - $5\hat{k}$) are the position vectors of vertices of a right angle triangle.

Solution:

Step 1: Understanding the Concept:
To show that three given position vectors form the vertices of a right-angled triangle, we can find the vectors representing the sides of the triangle.
If any two of these side vectors are perpendicular to each other, the triangle is a right-angled triangle.
Two vectors are perpendicular if their dot product (scalar product) is zero.

Step 2: Key Formula or Approach:
Let the position vectors of the vertices A, B, and C be $\vec{a}$, $\vec{b}$, and $\vec{c}$ respectively.
The vectors representing the sides are $\vec{AB} = \vec{b} - \vec{a}$, $\vec{BC} = \vec{c} - \vec{b}$, and $\vec{AC} = \vec{c} - \vec{a}$.
The condition for a right-angled triangle is that the dot product of any two side vectors is zero, for example, $\vec{AB} \cdot \vec{AC} = 0$.

Step 3: Detailed Explanation or Calculation:
Let the given position vectors be:
$\vec{a} = 3\hat{i} - 4\hat{j} - 4\hat{k}$
$\vec{b} = 2\hat{i} - \hat{j} + \hat{k}$
$\vec{c} = \hat{i} - 3\hat{j} - 5\hat{k}$

Now, let's find the vectors for the sides of the triangle:
$\vec{AB} = \vec{b} - \vec{a} = (2\hat{i} - \hat{j} + \hat{k}) - (3\hat{i} - 4\hat{j} - 4\hat{k})$
$\vec{AB} = (2-3)\hat{i} + (-1-(-4))\hat{j} + (1-(-4))\hat{k} = -\hat{i} + 3\hat{j} + 5\hat{k}$

$\vec{BC} = \vec{c} - \vec{b} = (\hat{i} - 3\hat{j} - 5\hat{k}) - (2\hat{i} - \hat{j} + \hat{k})$
$\vec{BC} = (1-2)\hat{i} + (-3-(-1))\hat{j} + (-5-1)\hat{k} = -\hat{i} - 2\hat{j} - 6\hat{k}$

$\vec{AC} = \vec{c} - \vec{a} = (\hat{i} - 3\hat{j} - 5\hat{k}) - (3\hat{i} - 4\hat{j} - 4\hat{k})$
$\vec{AC} = (1-3)\hat{i} + (-3-(-4))\hat{j} + (-5-(-4))\hat{k} = -2\hat{i} + \hat{j} - \hat{k}$

Next, we calculate the dot products of these side vectors:
$\vec{AB} \cdot \vec{BC} = (-\hat{i} + 3\hat{j} + 5\hat{k}) \cdot (-\hat{i} - 2\hat{j} - 6\hat{k}) = (-1)(-1) + (3)(-2) + (5)(-6) = 1 - 6 - 30 = -35$
$\vec{BC} \cdot \vec{AC} = (-\hat{i} - 2\hat{j} - 6\hat{k}) \cdot (-2\hat{i} + \hat{j} - \hat{k}) = (-1)(-2) + (-2)(1) + (-6)(-1) = 2 - 2 + 6 = 6$
$\vec{AB} \cdot \vec{AC} = (-\hat{i} + 3\hat{j} + 5\hat{k}) \cdot (-2\hat{i} + \hat{j} - \hat{k}) = (-1)(-2) + (3)(1) + (5)(-1) = 2 + 3 - 5 = 0$

Step 4: Final Answer:
Since the dot product $\vec{AB} \cdot \vec{AC} = 0$, the sides represented by vectors $\vec{AB}$ and $\vec{AC}$ are perpendicular to each other.
Therefore, the triangle formed by the given position vectors is a right-angled triangle, with the right angle at vertex A.

> Quick Tip
>
> To verify if vectors form a right-angled triangle, calculating dot products of the side vectors is often faster than calculating magnitudes and checking the Pythagorean theorem. If any dot product is zero, the sides are perpendicular.

---

b. If $x\sqrt{1+y} + y\sqrt{1+x} = 0$, $-1 < x < 1$, then prove that $\frac{dy}{dx} = -\frac{1}{(1+x)^2}$.

Solution:

Step 1: Understanding the Concept:
The problem asks to find the derivative of y with respect to x from an implicit equation. The strategy is to first simplify the equation algebraically to express y explicitly as a function of x, and then differentiate.

Step 2: Key Formula or Approach:
The main approach involves algebraic manipulation to isolate y, followed by differentiation using the quotient rule.
Quotient Rule: If $y = \frac{u(x)}{v(x)}$, then $\frac{dy}{dx} = \frac{u'v - uv'}{v^2}$.

Step 3: Detailed Explanation or Calculation:
Given the equation:

$$x\sqrt{1+y} + y\sqrt{1+x} = 0$$

Rearrange the terms:

$$x\sqrt{1+y} = -y\sqrt{1+x}$$

Square both sides to eliminate the square roots:

$$(x\sqrt{1+y})^2 = (-y\sqrt{1+x})^2$$

$$x^2(1+y) = y^2(1+x)$$

Expand the terms:

$$x^2 + x^2y = y^2 + y^2x$$

Rearrange the equation to group terms:

$$x^2 - y^2 + x^2y - y^2x = 0$$

Factorize the terms. The first two terms form a difference of squares, and we can factor xy from the last two terms:

$$(x-y)(x+y) + xy(x-y) = 0$$

Factor out the common term $(x-y)$:

$$(x-y)(x+y+xy) = 0$$

This implies either $x - y = 0$ or $x + y + xy = 0$.
If $x-y=0$, then $y = x$. Substituting this into the original equation gives $x\sqrt{1+x}+x\sqrt{1+x} = 2x\sqrt{1+x} = 0$, which is only true for $x = 0$ or $x = -1$. Since we are looking for a general relation for $-1 < x < 1$, we consider the other case.
So, we use the relation:

$$x + y + xy = 0$$

Now, we express y as a function of x:

$$y + xy = -x$$

$$y(1+x) = -x$$

$$y = -\frac{x}{1+x}$$

Now, we differentiate y with respect to x using the quotient rule:

$$\frac{dy}{dx} = -\left[\frac{\frac{d}{dx}(x)\cdot(1+x) - x\cdot\frac{d}{dx}(1+x)}{(1+x)^2}\right]$$

$$\frac{dy}{dx} = -\left[\frac{1\cdot(1+x) - x\cdot 1}{(1+x)^2}\right]$$

$$\frac{dy}{dx} = -\left[\frac{1+x-x}{(1+x)^2}\right]$$

$$\frac{dy}{dx} = -\frac{1}{(1+x)^2}$$

Step 4: Final Answer:
We have successfully shown that if $x\sqrt{1+y} + y\sqrt{1+x} = 0$, then $\frac{dy}{dx} = -\frac{1}{(1+x)^2}$.

💡 Quick Tip

For implicit functions involving square roots, squaring both sides can simplify the expression significantly. However, be mindful that squaring can introduce extraneous solutions, so it's good practice to check the context of the problem, as we did by discarding the $y = x$ case.

c. There are 10 white and 5 black balls in a bag. Two balls are drawn one by one. First ball is not placed back before the second is taken out. Assume that the taking out of each ball from the bag is equally likely. What is the probability that both balls taken out are white?

Solution:

Step 1: Understanding the Concept:
This is a problem of conditional probability, specifically drawing without replacement. The outcome of the second draw depends on the outcome of the first draw because the first ball is not returned to the bag.

Step 2: Key Formula or Approach:
The probability of two dependent events A and B occurring in sequence is given by the multiplication rule of probability:
$P(\text{A and B}) = P(A) \times P(B|A)$
Where $P(A)$ is the probability of the first event, and $P(B|A)$ is the probability of the second event given that the first event has occurred.
Let A be the event that the first ball is white, and B be the event that the second ball is white.

Step 3: Detailed Explanation or Calculation:
First, identify the initial state of the bag:
Number of white balls = 10
Number of black balls = 5
Total number of balls = 10 + 5 = 15

Event A: The first ball drawn is white.
The probability of drawing a white ball first is:

$$P(A) = \frac{\text{Number of white balls}}{\text{Total number of balls}} = \frac{10}{15}$$

Event B—A: The second ball drawn is white, given the first was white.
After drawing one white ball without replacement, the state of the bag changes:
Number of white balls remaining = 10 - 1 = 9
Total number of balls remaining = 15 - 1 = 14
The probability of drawing a second white ball is:

$$P(B|A) = \frac{\text{Number of remaining white balls}}{\text{Total number of remaining balls}} = \frac{9}{14}$$

Now, we find the probability that both balls are white:

$$P(\text{Both white}) = P(A) \times P(B|A)$$

$$P(\text{Both white}) = \frac{10}{15} \times \frac{9}{14}$$

Simplify the fractions:

$$P(\text{Both white}) = \frac{2}{3} \times \frac{9}{14} = \frac{18}{42}$$

Reduce the fraction to its simplest form by dividing the numerator and denominator by their greatest common divisor, which is 6:

$$P(\text{Both white}) = \frac{18 \div 6}{42 \div 6} = \frac{3}{7}$$

Step 4: Final Answer:
The probability that both balls taken out are white is $\frac{3}{7}$.

> **Quick Tip**
>
> For "without replacement" problems, remember to adjust both the number of favorable outcomes and the total number of outcomes for each subsequent draw. An alternative method is using combinations: $P = \frac{\binom{10}{2}}{\binom{15}{2}}$, which gives the same result.

---

d. Prove that $|\vec{a} \cdot \vec{b}| \leq |\vec{a}||\vec{b}|$ is always true for any two vectors $\vec{a}$ and $\vec{b}$.

Solution:

Step 1: Understanding the Concept:
The inequality $|\vec{a} \cdot \vec{b}| \leq |\vec{a}||\vec{b}|$ is known as the Cauchy-Schwarz inequality for vectors in Euclidean space. It relates the dot product of two vectors to the product of their magnitudes.

Step 2: Key Formula or Approach:
The proof relies on the geometric definition of the dot product (scalar product) of two vectors $\vec{a}$ and $\vec{b}$:

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$$

where $|\vec{a}|$ and $|\vec{b}|$ are the magnitudes of the vectors, and θ is the angle between them.

Step 3: Detailed Explanation or Calculation:
We start with the definition of the dot product:

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$$

Take the absolute value (or modulus) of both sides of the equation:

$$|\vec{a} \cdot \vec{b}| = \Big| |\vec{a}||\vec{b}| \cos\theta \Big|$$

Since the magnitudes $|\vec{a}|$ and $|\vec{b}|$ are non-negative scalars, we can take them out of the absolute value sign:

$$|\vec{a} \cdot \vec{b}| = |\vec{a}||\vec{b}||\cos\theta|$$

Now, we consider the property of the cosine function. For any angle θ, the value of $\cos\theta$ lies in the range $[-1, 1]$.
This means that the absolute value of $\cos\theta$ must be in the range $[0, 1]$.

$$0 \leq |\cos\theta| \leq 1$$

Since $|\cos\theta| \leq 1$, we can multiply both sides of this inequality by the non-negative quantity $|\vec{a}||\vec{b}|$ without changing the inequality direction:

$$|\vec{a}||\vec{b}||\cos\theta| \leq |\vec{a}||\vec{b}| \times 1$$

Substituting $|\vec{a} \cdot \vec{b}|$ back in place of $|\vec{a}||\vec{b}||\cos\theta|$, we get:

$$|\vec{a} \cdot \vec{b}| \leq |\vec{a}||\vec{b}|$$

Step 4: Final Answer:
The inequality is proven based on the fundamental properties of the dot product and the cosine function. The equality holds when $|\cos\theta| = 1$, which means $\theta = 0$ or $\theta = \pi$ (i.e., when the vectors are collinear).

> Quick Tip
>
> This is a fundamental vector inequality. Remembering the proof, which simply relies on the definition of the dot product and the fact that $|\cos\theta| \leq 1$, is very useful for vector-based problems in physics and mathematics.

---

e. Solve: $\int \frac{3x-5}{x^3-x^2-x+1} dx$

Solution:

Step 1: Understanding the Concept:
This is an indefinite integral of a rational function. The standard method for solving such integrals is to use partial fraction decomposition, which involves breaking down the complex fraction into a sum of simpler fractions that are easier to integrate.

Step 2: Key Formula or Approach:
1. Factor the denominator completely.
2. Express the rational function as a sum of partial fractions. For a denominator of the form $(x-a)^2(x-b)$, the decomposition is $\frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{x-b}$.
3. Solve for the unknown coefficients (A, B, C).

4. Integrate the resulting simpler fractions using standard integration formulas: $\int \frac{1}{u}du = \ln|u|$ and $\int u^n du = \frac{u^{n+1}}{n+1}$.

Step 3: Detailed Explanation or Calculation:
1. Factor the denominator:

$$x^3 - x^2 - x + 1$$

Factor by grouping:

$$x^2(x-1) - 1(x-1) = (x^2-1)(x-1) = (x-1)(x+1)(x-1) = (x-1)^2(x+1)$$

So the integral is:

$$\int \frac{3x-5}{(x-1)^2(x+1)}dx$$

2. Decompose into partial fractions:

$$\frac{3x-5}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$$

Multiply both sides by the denominator $(x-1)^2(x+1)$:

$$3x - 5 = A(x-1)(x+1) + B(x+1) + C(x-1)^2$$

3. Solve for A, B, and C:
We can substitute convenient values of x to find the coefficients.
Let $x = 1$:

$$3(1) - 5 = A(0) + B(1+1) + C(0) \implies -2 = 2B \implies B = -1$$

Let $x = -1$:

$$3(-1) - 5 = A(0) + B(0) + C(-1-1)^2 \implies -8 = C(-2)^2 \implies -8 = 4C \implies C = -2$$

To find A, let's use another value, for example $x = 0$:

$$3(0) - 5 = A(0-1)(0+1) + B(0+1) + C(0-1)^2$$

$$-5 = A(-1)(1) + B(1) + C(1) \implies -5 = -A + B + C$$

Substitute the values of B and C we found:

$$-5 = -A + (-1) + (-2) \implies -5 = -A - 3 \implies -2 = -A \implies A = 2$$

4. Integrate:
Now, substitute the coefficients back into the integral:

$$\int \left(\frac{2}{x-1} - \frac{1}{(x-1)^2} - \frac{2}{x+1} \right) dx$$

Integrate term by term:

$$= 2\int \frac{1}{x-1}dx - \int \frac{1}{(x-1)^2}dx - 2\int \frac{1}{x+1}dx$$

$$= 2\ln|x-1| - \int (x-1)^{-2}dx - 2\ln|x+1|$$

$$= 2\ln|x-1| - \frac{(x-1)^{-1}}{-1} - 2\ln|x+1| + C$$
$$= 2\ln|x-1| + \frac{1}{x-1} - 2\ln|x+1| + C$$

Step 4: Final Answer:
The solution to the integral is:

$$2\ln|x-1| - 2\ln|x+1| + \frac{1}{x-1} + C$$

This can also be written using logarithm properties as:

$$2\ln\left|\frac{x-1}{x+1}\right| + \frac{1}{x-1} + C$$

Quick Tip

When setting up partial fractions, always remember the rule for repeated factors: for a factor $(x-a)^k$, you need k terms: $\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_k}{(x-a)^k}$. This is a common point of error.

---

7. Do any one part.

a. If A = $\begin{bmatrix} 3 & 3 & 1 \\ 3 & 4 & 1 \\ 4 & 3 & 1 \end{bmatrix}$**, then verify that A(adj A) = |A| I and find** $\mathbf{A^{-1}}$**.**

Solution:

Step 1: Understanding the Concept:
This question involves two key properties of matrices. First, the relationship between a square matrix, its adjugate (adj A), its determinant (|A|), and the identity matrix (I), which is given by A(adj A) = (adj A)A = |A|I. Second, using this property to find the inverse of the matrix A, given by the formula $A^{-1} = \frac{1}{|A|}$ (adj A).

Step 2: Key Formula or Approach:
1. Calculate the determinant of A, |A|.
2. Find the matrix of cofactors of A.
3. Find the adjugate of A, adj(A), which is the transpose of the cofactor matrix.
4. Calculate the product A(adj A).
5. Calculate the product |A|I and verify it is equal to A(adj A).
6. Calculate the inverse A^{-1} using the formula $A^{-1} = \frac{1}{|A|}\text{adj}(A)$.

Step 3: Detailed Explanation or Calculation:
1. Calculate the Determinant $|\mathbf{A}|$:

$$|\mathrm{A}| = 3(4\cdot 1 - 1\cdot 3) - 3(3\cdot 1 - 1\cdot 4) + 1(3\cdot 3 - 4\cdot 4)$$

$$|\mathrm{A}| = 3(4-3) - 3(3-4) + 1(9-16)$$

$$|\mathrm{A}| = 3(1) - 3(-1) + 1(-7) = 3 + 3 - 7 = -1$$

Since $|\mathrm{A}| \neq 0$, the inverse of A exists.

2. Find the Adjugate, adj(A):
First, we find the cofactor matrix C.

$$C_{11} = +(4-3) = 1, \quad C_{12} = -(3-4) = 1, \quad C_{13} = +(9-16) = -7$$

$$C_{21} = -(3-3) = 0, \quad C_{22} = +(3-4) = -1, \quad C_{23} = -(9-12) = 3$$

$$C_{31} = +(3-4) = -1, \quad C_{32} = -(3-3) = 0, \quad C_{33} = +(12-9) = 3$$

The cofactor matrix is $\mathrm{C} = \begin{bmatrix} 1 & 1 & -7 \\ 0 & -1 & 3 \\ -1 & 0 & 3 \end{bmatrix}$.
The adjugate of A is the transpose of C:

$$\mathrm{adj(A)} = C^T = \begin{bmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \\ -7 & 3 & 3 \end{bmatrix}$$

3. Verify $\mathbf{A(adj\ A)} = |\mathbf{A}|\mathbf{I}$:
Calculate A(adj A):

$$A(\mathrm{adj\ A}) = \begin{bmatrix} 3 & 3 & 1 \\ 3 & 4 & 1 \\ 4 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \\ -7 & 3 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3(1)+3(1)+1(-7) & 3(0)+3(-1)+1(3) & 3(-1)+3(0)+1(3) \\ 3(1)+4(1)+1(-7) & 3(0)+4(-1)+1(3) & 3(-1)+4(0)+1(3) \\ 4(1)+3(1)+1(-7) & 4(0)+3(-1)+1(3) & 4(-1)+3(0)+1(3) \end{bmatrix}$$

$$= \begin{bmatrix} 3+3-7 & 0-3+3 & -3+0+3 \\ 3+4-7 & 0-4+3 & -3+0+3 \\ 4+3-7 & 0-3+3 & -4+0+3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Calculate $|\mathrm{A}|$I:

$$|\mathrm{A}|I = (-1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Since A(adj A) = $|\mathrm{A}|$I, the property is verified.

4. Find $\mathbf{A^{-1}}$:

$$A^{-1} = \frac{1}{|\mathrm{A}|}\mathrm{adj(A)} = \frac{1}{-1}\begin{bmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \\ -7 & 3 & 3 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -1 & 0 & 1 \\ -1 & 1 & 0 \\ 7 & -3 & -3 \end{bmatrix}$$

Step 4: Final Answer:

The property A(adj A) = |A|I is verified, as both sides equal $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

The inverse of the matrix A is $A^{-1} = \begin{bmatrix} -1 & 0 & 1 \\ -1 & 1 & 0 \\ 7 & -3 & -3 \end{bmatrix}$.

Quick Tip

The relationship A(adj A) = |A|I is a fundamental theorem in matrix algebra. It provides a direct method to calculate the inverse of a matrix. Always start by calculating the determinant; if it's zero, the matrix is singular, and the inverse does not exist.

---

b. Solve the system of equations
x + y + z = 2,
2x + y - 3z = 0,
x - y + z - 4 = 0 by matrix method.

Solution:

Step 1: Understanding the Concept:
A system of linear equations can be represented in matrix form as AX = B, where A is the coefficient matrix, X is the column matrix of variables, and B is the column matrix of constants. The solution can be found using the matrix inverse method, where $X = A^{-1}B$, provided that A is non-singular ($|A| \neq 0$).

Step 2: Key Formula or Approach:
1. Write the system of equations in the matrix form AX = B.
2. Calculate the determinant of the coefficient matrix A, |A|.
3. If $|A| \neq 0$, find the inverse of A, A^{-1}, using $A^{-1} = \frac{1}{|A|}\text{adj}(A)$.
4. Solve for the variables by calculating $X = A^{-1}B$.

Step 3: Detailed Explanation or Calculation:
1. Express in Matrix Form AX = B:
First, rewrite the third equation in standard form: $x - y + z = 4$.
The system is:

$$x + y + z = 2$$

$$2x + y - 3z = 0$$
$$x - y + z = 4$$

In matrix form:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -3 \\ 1 & -1 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}$$

2. Calculate the Determinant |A|:

$$|\mathrm{A}| = 1(1 \cdot 1 - (-3) \cdot (-1)) - 1(2 \cdot 1 - (-3) \cdot 1) + 1(2 \cdot (-1) - 1 \cdot 1)$$
$$|\mathrm{A}| = 1(1 - 3) - 1(2 + 3) + 1(-2 - 1)$$
$$|\mathrm{A}| = 1(-2) - 1(5) + 1(-3) = -2 - 5 - 3 = -10$$

Since |A| = -10 $\neq$ 0, a unique solution exists.

3. Find the Inverse $\mathbf{A^{-1}}$:

First, find the adjugate of A. The cofactor matrix C is:

$$C_{11} = +(1 - 3) = -2, \quad C_{12} = -(2 - (-3)) = -5, \quad C_{13} = +(-2 - 1) = -3$$
$$C_{21} = -(1 - (-1)) = -2, \quad C_{22} = +(1 - 1) = 0, \quad C_{23} = -(-1 - 1) = 2$$
$$C_{31} = +(-3 - 1) = -4, \quad C_{32} = -(-3 - 2) = 5, \quad C_{33} = +(1 - 2) = -1$$

So, C = $\begin{bmatrix} -2 & -5 & -3 \\ -2 & 0 & 2 \\ -4 & 5 & -1 \end{bmatrix}$.

The adjugate is adj(A) = C^T:

$$\mathrm{adj(A)} = \begin{bmatrix} -2 & -2 & -4 \\ -5 & 0 & 5 \\ -3 & 2 & -1 \end{bmatrix}$$

The inverse is $\mathrm{A}^{-1} = \frac{1}{|\mathrm{A}|}\mathrm{adj(A)}$:

$$A^{-1} = -\frac{1}{10} \begin{bmatrix} -2 & -2 & -4 \\ -5 & 0 & 5 \\ -3 & 2 & -1 \end{bmatrix}$$

4. Solve for X:

$$X = A^{-1}B = -\frac{1}{10} \begin{bmatrix} -2 & -2 & -4 \\ -5 & 0 & 5 \\ -3 & 2 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}$$
$$X = -\frac{1}{10} \begin{bmatrix} (-2)(2) + (-2)(0) + (-4)(4) \\ (-5)(2) + (0)(0) + (5)(4) \\ (-3)(2) + (2)(0) + (-1)(4) \end{bmatrix}$$
$$X = -\frac{1}{10} \begin{bmatrix} -4 + 0 - 16 \\ -10 + 0 + 20 \\ -6 + 0 - 4 \end{bmatrix} = -\frac{1}{10} \begin{bmatrix} -20 \\ 10 \\ -10 \end{bmatrix}$$

$$X = \begin{bmatrix} (-20)/(-10) \\ 10/(-10) \\ (-10)/(-10) \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Step 4: Final Answer:
The solution to the system of equations is x = 2, y = -1, and z = 1.

> Quick Tip
>
> After finding the solution using the matrix method, it's always a good practice to substitute the values of x, y, and z back into the original equations to verify that they are correct. This quick check can help you catch any calculation errors.

---

8. Do any one part.

a. Prove that: $\int_0^{\pi/4} \log_e(1 + \tan\theta)d\theta = \frac{\pi}{8}\log_e 2$

Solution:

Step 1: Understanding the Concept:
This is a problem of evaluating a definite integral. A common and powerful technique for integrals of the form $\int_0^a f(x)dx$ is to use the property $\int_0^a f(x)dx = \int_0^a f(a-x)dx$. This property is sometimes called the "King Property".

Step 2: Key Formula or Approach:
1. Let the given integral be I.
2. Apply the property $\int_0^a f(\theta)d\theta = \int_0^a f(a-\theta)d\theta$, where $a = \pi/4$.
3. Use the trigonometric identity $\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$.
4. Simplify the resulting integral and add it to the original integral I to solve for I.

Step 3: Detailed Explanation or Calculation:
Let the integral be I:

$$I = \int_0^{\pi/4} \log_e(1 + \tan\theta)d\theta \quad \cdots (1)$$

Using the property $\int_0^a f(\theta)d\theta = \int_0^a f(a-\theta)d\theta$, with $a = \pi/4$:

$$I = \int_0^{\pi/4} \log_e(1 + \tan(\pi/4 - \theta))d\theta$$

We know the identity $\tan(\pi/4-\theta) = \frac{\tan(\pi/4)-\tan\theta}{1+\tan(\pi/4)\tan\theta} = \frac{1-\tan\theta}{1+\tan\theta}$.
Substituting this into the integral:

$$I = \int_0^{\pi/4} \log_e \left(1 + \frac{1-\tan\theta}{1+\tan\theta}\right) d\theta$$

Simplify the term inside the logarithm:

$$I = \int_0^{\pi/4} \log_e \left(\frac{(1+\tan\theta)+(1-\tan\theta)}{1+\tan\theta}\right) d\theta$$

$$I = \int_0^{\pi/4} \log_e \left(\frac{2}{1+\tan\theta}\right) d\theta$$

Using the logarithm property $\log(a/b) = \log a - \log b$:

$$I = \int_0^{\pi/4} (\log_e 2 - \log_e(1+\tan\theta)) d\theta$$

Split the integral into two parts:

$$I = \int_0^{\pi/4} \log_e 2 \, d\theta - \int_0^{\pi/4} \log_e(1+\tan\theta) d\theta \quad \cdots (2)$$

From equation (1), we know that the second term is I. So, we have:

$$I = \int_0^{\pi/4} \log_e 2 \, d\theta - I$$

$$2I = \int_0^{\pi/4} \log_e 2 \, d\theta$$

Since $\log_e 2$ is a constant:

$$2I = \log_e 2 \int_0^{\pi/4} 1 \, d\theta = \log_e 2 [\theta]_0^{\pi/4}$$

$$2I = \log_e 2(\pi/4 - 0) = \frac{\pi}{4} \log_e 2$$

Solving for I:

$$I = \frac{\pi}{8} \log_e 2$$

Step 4: Final Answer:
We have successfully proven that $\int_0^{\pi/4} \log_e(1+\tan\theta) d\theta = \frac{\pi}{8} \log_e 2$.

Quick Tip

The "King Property" ($\int_0^a f(x)dx = \int_0^a f(a-x)dx$) is extremely useful for definite integrals, especially when the integrand involves trigonometric functions with limits like 0 to $\pi/2$ or 0 to $\pi/4$. Adding the original integral I to the transformed integral is a common trick that often leads to simplification.

b. Prove that the radius of the right circular cylinder of maximum curved surface inscribed in a cone is half of the radius of the cone.

Solution:

Step 1: Understanding the Concept:
This is an optimization problem that falls under the application of derivatives. We need to find the dimensions of an inscribed cylinder that maximize its curved surface area. The key is to express the quantity to be maximized (curved surface area) as a function of a single variable and then use calculus to find the maximum.

Step 2: Key Formula or Approach:
1. Define the dimensions of the cone (radius R, height H) and the inscribed cylinder (radius r, height h).
2. Use similar triangles from a cross-sectional view to establish a relationship between r, h, R, and H.
3. Write the formula for the curved surface area (CSA) of the cylinder, $S = 2\pi rh$.
4. Use the relationship from step 2 to express S as a function of a single variable, say r.
5. Find the first derivative of S with respect to r ($\frac{dS}{dr}$) and set it to zero to find critical points.
6. Use the second derivative test ($\frac{d^2S}{dr^2}$) to confirm that the critical point corresponds to a maximum.

Step 3: Detailed Explanation or Calculation:
Let the cone have a fixed radius R and height H. Let the cylinder inscribed within it have a variable radius r and height h.
Consider a vertical cross-section of the cone and cylinder. This forms two similar right-angled triangles. The larger triangle has height H and base R. The smaller triangle, which is above the cylinder, has height $H - h$ and base r.
By the property of similar triangles:

$$\frac{H-h}{H} = \frac{r}{R}$$

We can express h in terms of r (and the constants R, H):

$$1 - \frac{h}{H} = \frac{r}{R} \implies \frac{h}{H} = 1 - \frac{r}{R} \implies h = H\left(1 - \frac{r}{R}\right)$$

The curved surface area (CSA) of the cylinder is given by $S = 2\pi rh$.
Substitute the expression for h into the formula for S to get S as a function of r:

$$S(r) = 2\pi r\left[H\left(1 - \frac{r}{R}\right)\right]$$

$$S(r) = 2\pi H\left(r - \frac{r^2}{R}\right)$$

To find the maximum area, we differentiate $S(r)$ with respect to r and set the derivative to zero:

$$\frac{dS}{dr} = \frac{d}{dr}\left[2\pi H\left(r - \frac{r^2}{R}\right)\right]$$

$$\frac{dS}{dr} = 2\pi H\left(1 - \frac{2r}{R}\right)$$

Set $\frac{dS}{dr} = 0$ for critical points:

$$2\pi H\left(1 - \frac{2r}{R}\right) = 0$$

Since $2\pi H \neq 0$, we must have:

$$1 - \frac{2r}{R} = 0 \implies 1 = \frac{2r}{R} \implies 2r = R \implies r = \frac{R}{2}$$

To confirm this is a maximum, we use the second derivative test:

$$\frac{d^2S}{dr^2} = \frac{d}{dr}\left[2\pi H\left(1 - \frac{2r}{R}\right)\right]$$

$$\frac{d^2S}{dr^2} = 2\pi H\left(0 - \frac{2}{R}\right) = -\frac{4\pi H}{R}$$

Since R and H are positive dimensions, $\frac{d^2S}{dr^2}$ is negative. A negative second derivative indicates that the function has a local maximum at the critical point.

Step 4: Final Answer:
The curved surface area of the inscribed cylinder is maximum when its radius r is equal to $\frac{R}{2}$, which is half the radius of the cone. This completes the proof.

> Quick Tip
>
> In optimization problems involving inscribed shapes, the key is almost always to find a relationship between the dimensions of the inner and outer shapes using similar triangles (for cones/pyramids) or trigonometry (for spheres). This allows you to reduce the function you're optimizing to a single variable.

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9. Do any one part.

a. (i) If $y = 600e^{-7x} + 500e^{7x}$, then show that $\frac{d^2y}{dx^2} = 49y$.

Solution:

Step 1: Understanding the Concept:
The problem requires finding the second derivative of the given function y with respect to x and then showing that the resulting expression is equal to 49 times the original function y.

This involves applying the rules of differentiation for exponential functions.

Step 2: Key Formula or Approach:
The key formula for differentiation is the chain rule applied to exponential functions:

$$\frac{d}{dx}(e^{kx}) = k \cdot e^{kx}$$

We will apply this rule twice to find the second derivative.

Step 3: Detailed Explanation or Calculation:
Given the function:

$$y = 600e^{-7x} + 500e^{7x}$$

First, find the first derivative, $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{d}{dx}(600e^{-7x} + 500e^{7x})$$

$$\frac{dy}{dx} = 600 \cdot (-7)e^{-7x} + 500 \cdot (7)e^{7x}$$

$$\frac{dy}{dx} = -4200e^{-7x} + 3500e^{7x}$$

Next, find the second derivative, $\frac{d^2y}{dx^2}$, by differentiating $\frac{dy}{dx}$:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(-4200e^{-7x} + 3500e^{7x})$$

$$\frac{d^2y}{dx^2} = -4200 \cdot (-7)e^{-7x} + 3500 \cdot (7)e^{7x}$$

$$\frac{d^2y}{dx^2} = 29400e^{-7x} + 24500e^{7x}$$

Now, let's factor out 49 from this expression:

$$\frac{d^2y}{dx^2} = 49(600e^{-7x} + 500e^{7x})$$

We can see that the expression in the parenthesis is the original function y.
Therefore,

$$\frac{d^2y}{dx^2} = 49y$$

Step 4: Final Answer:
By differentiating the function y twice, we have shown that $\frac{d^2y}{dx^2}$ is equal to $49y$, thus verifying the given relation.

Quick Tip

For functions of the form $y = Ae^{kx} + Be^{-kx}$, the second derivative will always be $\frac{d^2y}{dx^2} = k^2y$. Recognizing this pattern can save time in multiple-choice questions or during verification steps.

a. (ii) Find the area bounded by the ellipse $\frac{x^2}{16} + \frac{y^2}{25} = 1$.

Solution:

Step 1: Understanding the Concept:
The problem is to find the total area enclosed by an ellipse. The equation of an ellipse is given in standard form, from which we can identify the lengths of the semi-major and semi-minor axes.

Step 2: Key Formula or Approach:
The standard equation of an ellipse centered at the origin is:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where a is the length of the semi-horizontal axis and b is the length of the semi-vertical axis. The area of such an ellipse is given by the formula:

$$\text{Area} = \pi ab$$

Step 3: Detailed Explanation or Calculation:
The given equation of the ellipse is:

$$\frac{x^2}{16} + \frac{y^2}{25} = 1$$

Comparing this to the standard form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we can identify a^2 and b^2:

$$a^2 = 16 \implies a = \sqrt{16} = 4$$

$$b^2 = 25 \implies b = \sqrt{25} = 5$$

Now, we use the formula for the area of an ellipse:

$$\text{Area} = \pi ab$$

Substitute the values of a and b:

$$\text{Area} = \pi(4)(5)$$

$$\text{Area} = 20\pi$$

Step 4: Final Answer:
The area bounded by the given ellipse is 20π square units.

Quick Tip

Memorize the formula for the area of an ellipse, Area $= \pi ab$. Be careful to use the values of a and b (the semi-axes lengths), not a^2 and b^2. For a circle, $a = b = r$, and the formula correctly simplifies to Area $= \pi r^2$.

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b. Solve the integral $\int \frac{\sec^2 2\theta}{(\cot\theta - \tan\theta)^2} d\theta$

Solution:

Step 1: Understanding the Concept:
This problem involves solving an indefinite integral containing trigonometric functions. The key to solving it is to simplify the integrand using trigonometric identities before performing the integration, which will likely involve a substitution.

Step 2: Key Formula or Approach:
The solution uses the following key trigonometric identities: 1. $\cot\theta = \frac{\cos\theta}{\sin\theta}$ and $\tan\theta = \frac{\sin\theta}{\cos\theta}$ 2. $\cos^2\theta - \sin^2\theta = \cos 2\theta$ 3. $2\sin\theta\cos\theta = \sin 2\theta$ 4. $\frac{\cos 2\theta}{\sin 2\theta} = \cot 2\theta$ 5. $\frac{d}{d\theta}(\tan 2\theta) = 2\sec^2 2\theta$
The integration will be performed using the method of substitution.

Step 3: Detailed Explanation or Calculation:
Let the integral be I. First, simplify the denominator:

$$\cot\theta - \tan\theta = \frac{\cos\theta}{\sin\theta} - \frac{\sin\theta}{\cos\theta} = \frac{\cos^2\theta - \sin^2\theta}{\sin\theta\cos\theta}$$

Using the double angle identities, $\cos^2\theta - \sin^2\theta = \cos 2\theta$ and $\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$:

$$\cot\theta - \tan\theta = \frac{\cos 2\theta}{\frac{1}{2}\sin 2\theta} = 2\cot 2\theta$$

Now substitute this back into the integral:

$$I = \int \frac{\sec^2 2\theta}{(2\cot 2\theta)^2} d\theta = \int \frac{\sec^2 2\theta}{4\cot^2 2\theta} d\theta$$

Rewrite the integrand in terms of tan and sec. Since $\cot 2\theta = \frac{1}{\tan 2\theta}$:

$$I = \frac{1}{4}\int \frac{\sec^2 2\theta}{1/\tan^2 2\theta} d\theta = \frac{1}{4}\int \tan^2 2\theta \sec^2 2\theta d\theta$$

Now, we use the substitution method. Let $u = \tan 2\theta$. Then, the derivative is $du = \frac{d}{d\theta}(\tan 2\theta) d\theta = 2\sec^2 2\theta d\theta$.
This gives us $\sec^2 2\theta d\theta = \frac{1}{2} du$.
Substitute u and du into the integral:

$$I = \frac{1}{4}\int u^2 \left(\frac{1}{2} du\right) = \frac{1}{8}\int u^2 du$$

Now, integrate with respect to u:

$$I = \frac{1}{8}\left(\frac{u^3}{3}\right) + C = \frac{u^3}{24} + C$$

Finally, substitute back $u = \tan 2\theta$:

$$I = \frac{\tan^3 2\theta}{24} + C$$

Step 4: Final Answer:
The solution to the integral is $\frac{\tan^3 2\theta}{24} + C$.

Quick Tip

When faced with a complex trigonometric integral, the first step should always be to simplify the integrand as much as possible using identities. Look for expressions that can be simplified into double angles (like $\cot\theta - \tan\theta$) or that are derivatives of other parts of the expression, suggesting a substitution.
