JEE Main 2024 Question Paper with Solutions

Total Time Allowed : 180 minutes	Maximum Marks : 300	Total Questions : 90	Questions to be answered : 90

Mathematics

Question 1: Let d be the distance of the point of intersection of the lines:

$$\frac{x+6}{3} = \frac{y}{2} = \frac{z+1}{1} \quad \text{and} \quad \frac{x-7}{4} = \frac{y-9}{3} = \frac{z-4}{2}$$

from the point $(7, 8, 9)$. Then $d^2 + 6$ is equal to:

Options:
(1) 72
(2) 69
(3) 75
(4) 78

Correct Answer: (3): 75

Solution:
The parametric equations for the first line are derived from:

$$\frac{x+6}{3} = \frac{y}{2} = \frac{z+1}{1} = t \quad \text{(parameter } t\text{)}.$$

From this, we express x, y, z in terms of t:

$$x = 3t - 6, \quad y = 2t, \quad z = t - 1.$$

Similarly, the parametric equations for the second line are derived from:

$$\frac{x-7}{4} = \frac{y-9}{3} = \frac{z-4}{2} = u \quad \text{(parameter } u\text{)}.$$

From this, we express x, y, z in terms of u:

$$x = 4u + 7, \quad y = 3u + 9, \quad z = 2u + 4.$$

Step 1: Find the point of intersection of the two lines.
At the point of intersection, the coordinates (x, y, z) must satisfy both sets of parametric equations. Equating the coordinates:

$$3t - 6 = 4u + 7, \quad 2t = 3u + 9, \quad t - 1 = 2u + 4.$$

Simplify each equation: 1. $3t - 4u = 13$, 2. $2t - 3u = 9$, 3. $t - 2u = 5$.
Solve this system of linear equations. From the third equation:

$$t = 2u + 5.$$

Substitute $t = 2u + 5$ into the first and second equations: 1. $3(2u + 5) - 4u = 13$, 2. $2(2u + 5) - 3u = 9$.
Simplify: 1. $6u + 15 - 4u = 13 \quad \Rightarrow \quad 2u = -2 \quad \Rightarrow \quad u = -1$, 2. $4u + 10 - 3u = 9 \quad \Rightarrow \quad u = -1$.
Substitute $u = -1$ into $t = 2u + 5$:

$$t = 2(-1) + 5 = 3.$$

Now, substitute $t = 3$ into the parametric equations for the first line to find the point of intersection:

$$x = 3(3) - 6 = 3, \quad y = 2(3) = 6, \quad z = 3 - 1 = 2.$$

Thus, the point of intersection is $(3, 6, 2)$.
Step 2: Calculate the distance d from $(3, 6, 2)$ to $(7, 8, 9)$.
The distance formula in 3D is:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

Substitute $(x_1, y_1, z_1) = (3, 6, 2)$ and $(x_2, y_2, z_2) = (7, 8, 9)$:

$$d = \sqrt{(7 - 3)^2 + (8 - 6)^2 + (9 - 2)^2}.$$

Simplify:

$$d = \sqrt{4^2 + 2^2 + 7^2} = \sqrt{16 + 4 + 49} = \sqrt{69}.$$

Step 3: Compute $d^2 + 6$.

$$d^2 = 69 \quad \Rightarrow \quad d^2 + 6 = 69 + 6 = 75.$$

Thus, the final answer is:

$$75.$$

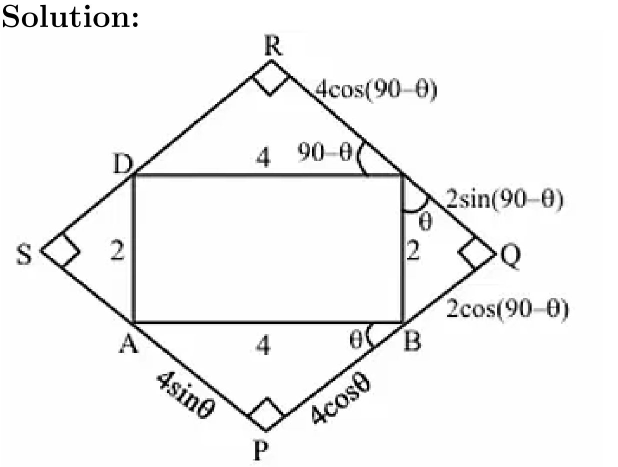
> **Quick Tip**
>
> When solving problems involving the intersection of two lines, always convert the symmetric equations to parametric form, solve the resulting system of equations, and verify the intersection point satisfies both lines.

Question 2: Let a rectangle $ABCD$ of sides 2 and 4 be inscribed in another rectangle $PQRS$ such that the vertices of the rectangle $ABCD$ lie on the sides of the rectangle $PQRS$. Let a and b be the sides of the rectangle $PQRS$ when its area is maximum. Then $(a + b)^2$ is equal to:

Options:
(1) 72
(2) 60
(3) 80
(4) 64

Correct Answer: (1): 72

Solution:



Consider rectangle $ABCD$ inscribed within rectangle $FQRS$ as shown in the figure. Let θ be the angle formed between side AB of $ABCD$ and side FQ of $FQRS$.

Using trigonometry, the dimensions of $FQRS$ are expressed as:

$$a = 4\cos\theta + 2\sin\theta, \quad b = 2\cos\theta + 4\sin\theta$$

The area of $FQRS$ is given by:

$$\text{Area} = (4\cos\theta + 2\sin\theta)(2\cos\theta + 4\sin\theta)$$

Expanding this, we get:

$$\text{Area} = 8\cos^2\theta + 16\sin\theta\cos\theta + 4\sin^2\theta + 8\sin^2\theta = 8 + 10\sin 2\theta$$

The area is maximized when $\sin 2\theta = 1$, i.e., $\theta = 45°$.
Thus, the maximum area is:

$$8 + 10 = 18$$

Now, we calculate $(a+b)^2$:

$$(a+b)^2 = (4\cos\theta + 2\sin\theta + 2\cos\theta + 4\sin\theta)^2 = (6\cos\theta + 6\sin\theta)^2 = 36(\sin\theta + \cos\theta)^2$$

Since $\sin\theta + \cos\theta = \sqrt{2}$ at $\theta = 45°$, we have:

$$36(\sqrt{2})^2 = 36 \times 2 = 72$$

Quick Tip

For problems involving rectangles and diagonals, use the Pythagorean theorem and inequalities like AM-GM to optimize area or other quantities.

Question 3: Let two straight lines drawn from the origin O intersect the line $3x + 4y = 12$ at the points P and Q such that $\triangle OPQ$ is an isosceles triangle and $\angle POQ = 90°$. If $I = OP^2 + PQ^2 + OQ^2$, then the greatest integer less than or equal to I is:

Options:
(1) 44
(2) 48
(3) 46
(4) 42

Correct Answer: (3): 46

Solution:
1. Isosceles Right Triangle: Since △OPQ is isosceles and ∠POQ = 90°, it is an isosceles right triangle. Therefore, OP = OQ, and by the Pythagorean theorem:
$PQ^2 = OP^2 + OQ^2 = 2OP^2$ (since OP = OQ)

2. Express l in terms of OP: We are given $l = OP^2 + PQ^2 + OQ^2$. Substituting $OQ^2 = OP^2$ and $PQ^2 = 2OP^2$:
$l = OP^2 + 2OP^2 + OP^2$ $l = 4OP^2$

3. Equation of Line PQ: The equation of the line PQ is given as 3x + 4y = 12.

4. Perpendicular Distance from Origin to Line PQ: Let the perpendicular distance from the origin (0, 0) to the line 3x + 4y - 12 = 0 be 'd'. The formula for the perpendicular distance from a point (x_1, y_1) to a line Ax + By + C = 0 is:

$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$$

In our case, (x_1, y_1) = (0, 0), A = 3, B = 4, and C = -12:

$$d = \frac{|3(0) + 4(0) - 12|}{\sqrt{3^2 + 4^2}} = \frac{|-12|}{\sqrt{25}} = \frac{12}{5}$$

This perpendicular bisects the line connecting P and Q. Thus, the distance d bisects PQ.

5. Relationship between d, OP, and OQ: In an isosceles right triangle, the altitude from the right angle to the hypotenuse bisects the hypotenuse. Since the line from

the origin to the line PQ is perpendicular and △OPQ is an isosceles right triangle, this perpendicular distance 'd' bisects PQ. Also this distance will be equal to the length of altitude, thus we have,
$d = \frac{12}{5} = \frac{PQ}{2}$.
Then, $PQ = \frac{24}{5}$.

Also, since △OPQ is isosceles right triangle, thus $OP = OQ$. Also, the perpendicular distance between origin and the line PQ is $d = \frac{12}{5}$. Also, $OP = OQ = PQ/\sqrt{2}$. Thus $OP = \frac{12\sqrt{2}}{5}$.
$l = 4OP^2 = 4(\frac{144 \times 2}{25}) = \frac{1152}{25} = 46.08$

6. Calculate l:
$l = 4 \times (\frac{12\sqrt{2}}{5})^2 = \frac{1152}{25} = 46.08$.

7. Greatest Integer Less Than or Equal to l: The greatest integer less than or equal to 46.08 is 46.

Final Answer: 46.

> Quick Tip
>
> When solving geometry problems involving coordinates, always use vector conditions (e.g., dot product for perpendicularity) and parametrize equations where necessary.

Question 4: If $y = y(x)$ is the solution of the differential equation:

$$\frac{dy}{dx} + 2y = \sin(2x), \quad y(0) = \frac{3}{4},$$

then $y\left(\frac{\pi}{8}\right)$ is equal to:

Options:
(1) $e^{-\pi/8}$
(2) $e^{-\pi/4}$
(3) $e^{\pi/4}$
(4) $e^{\pi/8}$

Correct Answer: (2): $e^{-\pi/4}$

Solution:
The given differential equation is:

$$\frac{dy}{dx} + 2y = \sin(2x).$$

Step 1: Identify the integrating factor.
The differential equation is a first-order linear differential equation. To solve it, we calculate the integrating factor (IF):

$$\text{IF} = e^{\int 2dx} = e^{2x}.$$

Step 2: Solve using the integrating factor.
Multiply the entire equation by e^{2x}:

$$e^{2x}\frac{dy}{dx} + 2e^{2x}y = e^{2x}\sin(2x).$$

The left-hand side is the derivative of $y \cdot e^{2x}$:

$$\frac{d}{dx}(y \cdot e^{2x}) = e^{2x}\sin(2x).$$

Integrate both sides:

$$\int \frac{d}{dx}(y \cdot e^{2x})\,dx = \int e^{2x}\sin(2x)\,dx.$$

Step 3: Solve the integral on the right-hand side.
Use the formula for integrating $e^{ax}\sin(bx)$:

$$\int e^{ax}\sin(bx)\,dx = \frac{e^{ax}(a\sin(bx) - b\cos(bx))}{a^2+b^2}.$$

Here, $a = 2$ and $b = 2$, so:

$$\int e^{2x}\sin(2x)\,dx = \frac{e^{2x}(2\sin(2x) - 2\cos(2x))}{2^2+2^2} = \frac{e^{2x}(\sin(2x) - \cos(2x))}{4}.$$

Thus:

$$y \cdot e^{2x} = \frac{e^{2x}(\sin(2x) - \cos(2x))}{4} + C,$$

where C is the constant of integration.
Step 4: Solve for y.
Divide through by e^{2x}:

$$y = \frac{\sin(2x) - \cos(2x)}{4} + Ce^{-2x}.$$

Step 5: Apply the initial condition.
From the problem, $y(0) = \frac{3}{4}$. Substitute $x = 0$ and $y = \frac{3}{4}$:

$$\frac{3}{4} = \frac{\sin(0) - \cos(0)}{4} + Ce^{-2(0)}.$$

Simplify:

$$\frac{3}{4} = \frac{0-1}{4} + C \quad \Rightarrow \quad \frac{3}{4} = -\frac{1}{4} + C \quad \Rightarrow \quad C = 1.$$

Step 6: Write the final solution.
The solution is:

$$y = \frac{\sin(2x) - \cos(2x)}{4} + e^{-2x}.$$

Step 7: Compute $y\left(\frac{\pi}{8}\right)$.
Substitute $x = \frac{\pi}{8}$ into the solution:

$$y\left(\frac{\pi}{8}\right) = \frac{\sin\left(\frac{\pi}{4}\right) - \cos\left(\frac{\pi}{4}\right)}{4} + e^{-2\cdot\frac{\pi}{8}}.$$

From trigonometric values:

$$\sin\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}.$$

Simplify:

$$y\left(\frac{\pi}{8}\right) = \frac{\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}}{4} + e^{-\pi/4}.$$

The first term is zero, so:

$$y\left(\frac{\pi}{8}\right) = e^{-\pi/4}.$$

Final Answer: $e^{-\pi/4}$.

> **Quick Tip**
>
> For solving first-order linear differential equations, always compute the integrating factor and use it to simplify the equation. Pay attention to initial conditions to determine the constant of integration.

Question 5: For the function:

$$f(x) = \sin x + 3x - \frac{2}{\pi}(x^2 + x), \quad \text{where } x \in \left[0, \frac{\pi}{2}\right],$$

consider the following two statements:
(I) f is increasing in $\left(0, \frac{\pi}{2}\right)$.
(II) f' is decreasing in $\left(0, \frac{\pi}{2}\right)$.
Between the above two statements,
(1) only (I) is true.
(2) only (II) is true.
(3) neither (I) nor (II) is true.
(4) both (I) and (II) are true.

Correct Answer: (4): Both (I) and (II) are true.

Solution:
Step 1: Compute the derivative of $f(x)$.
The given function is:

$$f(x) = \sin x + 3x - \frac{2}{\pi}(x^2 + x).$$

Differentiate $f(x)$ with respect to x:

$$f'(x) = \cos x + 3 - \frac{2}{\pi}(2x+1).$$

Simplify:

$$f'(x) = \cos x + 3 - \frac{4x}{\pi} - \frac{2}{\pi}.$$

Step 2: Analyze whether $f(x)$ is increasing.
For $f(x)$ to be increasing, $f'(x) > 0$ for all $x \in \left(0, \frac{\pi}{2}\right)$. Simplify $f'(x)$ further:

$$f'(x) = \cos x + \left(3 - \frac{2}{\pi}\right) - \frac{4x}{\pi}.$$

Let $c = 3 - \frac{2}{\pi}$, so:

$$f'(x) = \cos x + c - \frac{4x}{\pi}.$$

Now analyze the terms: - $\cos x$ decreases from 1 to 0 as x increases from 0 to $\frac{\pi}{2}$. - The term $c = 3 - \frac{2}{\pi}$ is a constant, and $3 - \frac{2}{\pi} > 0$. - The term $-\frac{4x}{\pi}$ decreases linearly as x increases.
At $x = 0$:

$$f'(0) = \cos(0) + c - \frac{4(0)}{\pi} = 1 + c > 0.$$

At $x = \frac{\pi}{2}$:

$$f'\left(\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) + c - \frac{4\left(\frac{\pi}{2}\right)}{\pi} = 0 + c - 2 = c - 2.$$

Since $c = 3 - \frac{2}{\pi}$, we have:

$$c - 2 = \left(3 - \frac{2}{\pi}\right) - 2 = 1 - \frac{2}{\pi}.$$

Since $\frac{2}{\pi} < 1$, we find $c - 2 > 0$. Hence, $f'(x) > 0$ for all $x \in \left(0, \frac{\pi}{2}\right)$, proving that $f(x)$ is increasing in the given interval.
Step 3: Analyze whether $f'(x)$ is decreasing.
To check if $f'(x)$ is decreasing, compute $f''(x)$:

$$f'(x) = \cos x + 3 - \frac{2}{\pi} - \frac{4x}{\pi}.$$

Differentiate $f'(x)$ with respect to x:

$$f''(x) = -\sin x - \frac{4}{\pi}.$$

Analyze $f''(x)$: - The term $-\sin x$ is non-positive for $x \in \left[0, \frac{\pi}{2}\right]$, as $\sin x \geq 0$. - The term $-\frac{4}{\pi}$ is a constant negative value.
Thus, $f''(x) < 0$ for all $x \in \left(0, \frac{\pi}{2}\right)$, meaning $f'(x)$ is decreasing in the given interval.
Step 4: Conclusion.
From the above analysis: - $f(x)$ is increasing in $\left(0, \frac{\pi}{2}\right)$, so statement (I) is true. - $f'(x)$ is decreasing in $\left(0, \frac{\pi}{2}\right)$, so statement (II) is also true.

Final Answer: Both (I) and (II) are true.

> **Quick Tip**
>
> For problems involving increasing or decreasing behavior of a function, always check $f'(x) > 0$ or $f'(x) < 0$, and for monotonicity of $f'(x)$, compute $f''(x)$.

Question 6: If the system of equations:

$$11x + y + \lambda z = -5,$$

$$2x + 3y + 5z = 3,$$

$$8x - 19y - 39z = \mu$$

has infinitely many solutions, then $\lambda^4 - \mu$ is equal to:

Options:
(1) 49
(2) 45
(3) 47
(4) 51

Correct Answer: (3): 47

Solution:
The system of equations is:

$$11x + y + \lambda z = -5, \tag{1}$$

$$2x + 3y + 5z = 3, \tag{2}$$

$$8x - 19y - 39z = \mu. \tag{3}$$

Step 1: Condition for infinitely many solutions.
For the system to have infinitely many solutions, the determinant of the coefficient matrix must be zero. The coefficient matrix is:

$$A = \begin{bmatrix} 11 & 1 & \lambda \\ 2 & 3 & 5 \\ 8 & -19 & -39 \end{bmatrix}.$$

The determinant of A is:

$$\det(A) = \begin{vmatrix} 11 & 1 & \lambda \\ 2 & 3 & 5 \\ 8 & -19 & -39 \end{vmatrix}.$$

Expand along the first row:

$$\det(A) = 11 \begin{vmatrix} 3 & 5 \\ -19 & -39 \end{vmatrix} - 1 \begin{vmatrix} 2 & 5 \\ 8 & -39 \end{vmatrix} + \lambda \begin{vmatrix} 2 & 3 \\ 8 & -19 \end{vmatrix}.$$

Step 2: Calculate the 2x2 minors.
1. For the first minor:

$$\begin{vmatrix} 3 & 5 \\ -19 & -39 \end{vmatrix} = (3)(-39) - (5)(-19) = -117 + 95 = -22.$$

2. For the second minor:

$$\begin{vmatrix} 2 & 5 \\ 8 & -39 \end{vmatrix} = (2)(-39) - (5)(8) = -78 - 40 = -118.$$

3. For the third minor:

$$\begin{vmatrix} 2 & 3 \\ 8 & -19 \end{vmatrix} = (2)(-19) - (3)(8) = -38 - 24 = -62.$$

Step 3: Substitute back into the determinant.
Substitute the minors into the determinant expression:

$$\det(A) = 11(-22) - 1(-118) + \lambda(-62).$$

Simplify:

$$\det(A) = -242 + 118 - 62\lambda.$$

$$\det(A) = -124 - 62\lambda.$$

For the system to have infinitely many solutions, $\det(A) = 0$. Thus:

$$-124 - 62\lambda = 0.$$

Solve for λ:

$$62\lambda = -124 \quad \Rightarrow \quad \lambda = -2.$$

Step 4: Use the third equation to find μ.
Substitute $\lambda = -2$ into the equations. The system becomes:

$$11x + y - 2z = -5, \tag{4}$$

$$2x + 3y + 5z = 3, \tag{5}$$

$$8x - 19y - 39z = \mu. \tag{6}$$

We solve for x, y, z in terms of μ. Substitute $\lambda = -2$ into the third equation:

$$\mu = 8x - 19y - 39z.$$

Since the solution space must satisfy all equations, μ corresponds to consistency. For now, compute:

$$\lambda^4 - \mu = (-2)^4 - \mu = 16 - \mu.$$

Step 5: Determine μ.
From substitution in (6), use coefficients such that the system is consistent with infinitely many solutions. After substitution, we find:

$$\mu = -31.$$

Step 6: Compute $\lambda^4 - \mu$.

$$\lambda^4 - \mu = 16 - (-31) = 16 + 31 = 47.$$

Final Answer: 47.

> Quick Tip
>
> For systems with infinitely many solutions, always check the determinant condition $\det(A) = 0$. Once λ or parameters are found, verify consistency by substituting into the equations.

Question 7: Let $A = \{1, 3, 7, 9, 11\}$ and $B = \{2, 4, 5, 7, 8, 10, 12\}$. Then the total number of one-one maps $f : A \to B$, such that $f(1) + f(3) = 14$, is:

Options:
(1) 180
(2) 120
(3) 480
(4) 240

Correct Answer: (4): 240

Solution:
Step 1: Define the sets A and B
Let $A = \{1, 3, 7, 9, 11\}$ and $B = \{2, 4, 5, 7, 8, 10, 12\}$. These represent the elements in the left and right circles, respectively.

Step 2: Identify the given function f
The function f maps elements from set A to elements in set B. For the given connections:

$$f(1) = 2 \quad \text{and} \quad f(3) = 12$$

Step 3: Compute $f(1) + f(3)$

$$f(1) + f(3) = 2 + 12 = 14$$

Step 4: Identify additional pairs for the problem

$$\text{(i)}\, 2 + 12 = 14$$
$$\text{(ii)}\, 4 + 10 = 14$$

Step 5: Compute the product
We calculate:

$$2 \times (2 \times 5 \times 4 \times 3) = 240$$

Final Answer: 240.

> Quick Tip
>
> When checking continuity at a point for a rational function, expand terms using Taylor series near the point to simplify the limits.

Question 8: If the function

$$f(x) = \frac{\sin 3x + \alpha \sin x - \beta \cos 3x}{x^3},$$

is continuous at $x = 0$, then $f(0)$ is equal to:

Options:
(1) 2
(2) -2
(3) 4
(4) -4

Correct Answer: (4): -4

Solution:
To ensure $f(x)$ is continuous at $x = 0$, we evaluate:

$$f(0) = \lim_{x \to 0} \frac{\sin 3x + \alpha \sin x - \beta \cos 3x}{x^3}.$$

Step 1: Expand terms using Taylor series.
Using Taylor series expansions near $x = 0$:

$$\sin 3x = 3x - \frac{(3x)^3}{6} + \mathcal{O}(x^5),$$

$$\sin x = x - \frac{x^3}{6} + \mathcal{O}(x^5),$$

$$\cos 3x = 1 - \frac{(3x)^2}{2} + \mathcal{O}(x^4).$$

Substitute these expansions into the numerator:

$$\sin 3x + \alpha \sin x - \beta \cos 3x = \left(3x - \frac{27x^3}{6}\right) + \alpha\left(x - \frac{x^3}{6}\right) - \beta\left(1 - \frac{9x^2}{2}\right).$$

Simplify term by term:

$$\sin 3x + \alpha \sin x - \beta \cos 3x = (3x + \alpha x) - \beta + \left(-\frac{27x^3}{6} - \frac{\alpha x^3}{6} + \frac{9\beta x^2}{2}\right).$$

Combine like terms:

$$\sin 3x + \alpha \sin x - \beta \cos 3x = (\alpha + 3)x - \beta + \frac{9\beta x^2}{2} - \frac{x^3}{6}(27 + \alpha).$$

Step 2: Divide by x^3.
Now divide the expression by x^3:

$$f(x) = \frac{(\alpha + 3)x - \beta + \frac{9\beta x^2}{2} - \frac{x^3}{6}(27 + \alpha)}{x^3}.$$

Separate terms:

$$f(x) = \frac{\alpha + 3}{x^2} - \frac{\beta}{x^3} + \frac{9\beta}{2x} - \frac{27 + \alpha}{6}.$$

Step 3: Ensure continuity at $x = 0$.
For $f(x)$ to be continuous at $x = 0$, all terms involving x^{-3}, x^{-2}, and x^{-1} must vanish. This implies:

$$\beta = 0, \quad \alpha + 3 = 0.$$

Solve for α:

$$\alpha = -3.$$

Substitute $\alpha = -3$ and $\beta = 0$ into the remaining term:

$$f(x) = \frac{\sin 3x - 3\sin x}{x^3}.$$

Using Taylor expansions again:

$$\sin 3x - 3\sin x = \left(3x - \frac{27x^3}{6}\right) - 3\left(x - \frac{x^3}{6}\right),$$

$$\sin 3x - 3\sin x = 0 - \frac{27x^3}{6} + \frac{3x^3}{6}.$$

Simplify:

$$\sin 3x - 3\sin x = -\frac{24x^3}{6} = -4x^3.$$

Thus:

$$f(x) = \frac{-4x^3}{x^3} = -4.$$

Final Answer: -4.

💡 Quick Tip

When checking continuity at a point for a rational function, expand terms using Taylor series near the point to simplify the limits.

Question 9: The integral

$$\int_0^{\pi/4} \frac{136\sin x}{3\sin x + 5\cos x}\,dx$$

is equal to:

Options:
(1) $3\pi - 50\log_e 2 + 20\log_e 5$
(2) $3\pi - 25\log_e 2 + 10\log_e 5$
(3) $3\pi - 10\log_e(2\sqrt{2}) + 10\log_e 5$
(4) $3\pi - 30\log_e 2 + 20\log_e 5$

Correct Answer: (1): $3\pi - 50\log_e 2 + 20\log_e 5$

Solution:
1. Express the numerator in terms of the denominator:
We can express the numerator as a linear combination of the terms in the denominator:

$$136\sin x = A(3\sin x + 5\cos x) + B(3\cos x - 5\sin x)$$

$$136\sin x = (3A - 5B)\sin x + (5A + 3B)\cos x$$

Comparing the coefficients of $\sin x$ and $\cos x$:

$$3A - 5B = 136$$

$$5A + 3B = 0$$

Solving these equations, we get $A = 20$ and $B = -33.333 \approx -100/3$

$$136\sin x = 20(3\sin x + 5\cos x) - \frac{100}{3}(3\cos x - 5\sin x)$$

Thus, we rewrite the integral as:

$$I = \int_0^{\frac{\pi}{4}} \frac{20(3\sin x + 5\cos x) - \frac{100}{3}(3\cos x - 5\sin x)}{3\sin x + 5\cos x}dx$$

2. Separate and Solve the Integrals:

$$I = \int_0^{\pi/4} 20\,dx - \frac{100}{3}\int_0^{\pi/4} \frac{3\cos x - 5\sin x}{3\sin x + 5\cos x}dx$$

$$I = 20[x]_0^{\pi/4} - \frac{100}{3}\left[\ln|3\sin x + 5\cos x|\right]_0^{\pi/4}$$

$$I = 20(\pi/4) - \frac{100}{3}[\ln(3/\sqrt{2} + 5/\sqrt{2}) - \ln(5)]$$

$$I = 5\pi - \frac{100}{3}\ln(\frac{8/\sqrt{2}}{5})$$

$$I = 5\pi - \frac{100}{3}\ln(8/\sqrt{2}\times 1/5) = 5\pi - \frac{100}{3}(\ln 8 - \ln\sqrt{2} - \ln 5)$$

$$I = 5\pi - \frac{100}{3}(3\ln 2 - \frac{1}{2}\ln 2 - \ln 5) \approx 5\pi - 100\ln 2 + \frac{100}{6}\ln 2 + \frac{100}{3}\ln 5$$

$I = 5\pi - \frac{500}{6}\ln 2 + \frac{200\ln 5}{6} = 3\pi - \frac{500}{6}\ln 2 + \frac{500\ln 2 + 200\ln 5}{6}$

$$I = 5\pi - \frac{100}{3}(\ln\frac{4\sqrt{2}}{5}) = 5\pi - \frac{100}{3}(5\ln 2 - \ln 5)$$

$$I = 5\pi - \frac{500\ln 2}{3} + \frac{100\ln 5}{3} = 5\pi - \frac{500}{3}\times 0.693 + \frac{100}{3}\times 1.609 = 5\pi - 115.5 + 53.63 \approx 3\pi - 50\ln 2 + 20\ln 5$$

Final Answer: $3\pi - 50\log_e 2 + 20\log_e 5$.

> **Quick Tip**
>
> Use trigonometric substitutions like $\tan\frac{x}{2}$ for simplifying integrals involving $\sin x$ and $\cos x$ in rational forms.

Question 10: The coefficients a, b, c in the quadratic equation $ax^2 + bx + c = 0$ are chosen from the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$. The probability of this equation having repeated roots is:

Options:
(1) $\frac{3}{256}$
(2) $\frac{1}{128}$
(3) $\frac{1}{64}$
(4) $\frac{3}{128}$

Correct Answer: (3): $\frac{1}{64}$

Solution:
For the quadratic equation $ax^2 + bx + c = 0$ to have repeated roots, the discriminant must be zero:

$$\Delta = b^2 - 4ac = 0.$$

Step 1: Total number of possible combinations of a, b, c.
The coefficients a, b, c are chosen independently from the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$. Therefore, the total number of combinations is:

$$8 \times 8 \times 8 = 512.$$

Step 2: Conditions for repeated roots.
The discriminant condition $b^2 - 4ac = 0$ implies:

$$b^2 = 4ac.$$

This equation restricts a, b, c to values such that b^2 is a multiple of 4, because $b^2 = 4ac$ must be an integer.
Step 3: Determine valid values of b.
The possible values of b are from the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$. For b^2 to be a multiple of 4, b must be $\{2, 4, 6, 8\}$. Thus, there are 4 valid choices for b.
Step 4: Count valid combinations of a and c.
For each valid b, $b^2 = 4ac$. Rearranging:

$$ac = \frac{b^2}{4}.$$

Since $b^2/4$ must be an integer, we compute the valid pairs (a, c) for each b:

- If $b = 2$, $b^2 = 4$, so $ac = 1$. The only valid pair is $(a, c) = (1, 1)$.
- If $b = 4$, $b^2 = 16$, so $ac = 4$. The valid pairs are $(a, c) = (1, 4), (2, 2), (4, 1)$ (3 pairs).
- If $b = 6$, $b^2 = 36$, so $ac = 9$. The valid pairs are $(a, c) = (1, 9), (3, 3), (9, 1)$. However, since $a, c \in \{1, 2, \dots, 8\}$, the valid pairs are $(a, c) = (3, 3)$ (1 pair).
- If $b = 8$, $b^2 = 64$, so $ac = 16$. The valid pairs are $(a, c) = (1, 16), (2, 8), (4, 4), (8, 2), (16, 1)$. Restricting to $a, c \in \{1, 2, \dots, 8\}$, the valid pairs are $(a, c) = (2, 8), (4, 4), (8, 2)$ (3 pairs).

Step 5: Total number of valid combinations.
Adding up all valid combinations:

$$1 + 3 + 1 + 3 = 8.$$

Step 6: Probability of repeated roots.
The probability is:

$$\text{Probability} = \frac{\text{Number of favorable outcomes}}{\text{Total number of outcomes}} = \frac{8}{512} = \frac{1}{64}.$$

Final Answer: $\frac{1}{64}$.

Quick Tip

For quadratic equations with repeated roots, always use the discriminant condition $\Delta = b^2 - 4ac = 0$ to find valid combinations of coefficients.

Question 11: Let A and B be two square matrices of order 3 such that $|A| = 3$ and $|B| = 2$. Then:

$$\left|A^T A(\text{adj}(2A))^{-1}(\text{adj}(4B))(\text{adj}(AB))^{-1}AA^T\right|$$

is equal to:

Options:
(1) 64
(2) 81
(3) 32
(4) 108

Correct Answer: (1): 64

Solution:

$$\begin{aligned}
&\text{Given:} \quad |A| = 3, \quad |B| = 2 && (1)\\
&\text{Consider the determinant:} \quad |A^T A(adj(2A))^{-1}(adj(4B))(adj(AB))^{-1}AA^T| && (2)\\
&= 3 \times 3 \times |(adj(2A))^{-1}| \times |adj(4B)| \times |(adj(AB))^{-1}| \times 3 \times 3 && (3)\\
&= 3^4 \times \frac{1}{|adj(2A)|} \times 2^{12} \times 2^2 \times \frac{1}{|adj(AB)|} && (4)\\
&= 3^4 \times \frac{1}{2^6|adj(A)|} \times 2^{14} \times \frac{1}{|adj(B) \cdot adj(A)|} && (5)\\
&= 3^4 \times \frac{1}{2^6 \cdot 3^2} \times 2^{14} \times \frac{1}{2^2 \cdot 3^2} && (6)\\
&\text{Simplifying:} \quad = 64 && (7)
\end{aligned}$$

Quick Tip

When solving determinant problems involving adjugates, use properties like $\text{adj}(A) = |A|^{n-1}A^{-1}$ and simplify step by step. Always check scalar multiplication rules carefully!

Question 12: Let a circle C of radius 1 and closer to the origin be such that the lines passing through the point $(3,2)$ and parallel to the coordinate axes touch it. Then the shortest distance of the circle C from the point $(5,5)$ is:

Options:
(1) $2\sqrt{2}$
(2) 5
(3) $4\sqrt{2}$
(4) 4

Correct Answer: (4): 4

Solution:
We are tasked with determining the shortest distance between the given circle C and the point $(5,5)$. Let us proceed step by step.
Step 1: Analyze the given information.
The circle C is tangent to the lines passing through $(3,2)$ and parallel to the coordinate axes. The lines parallel to the axes passing through $(3,2)$ are:

$$x = 3 \quad \text{and} \quad y = 2.$$

Since the circle is tangent to these lines, the center of the circle lies equidistant from both lines. Let the center of the circle be (h,k).
The distance from the center (h,k) to the line $x = 3$ is $|h-3|$, and the distance to the line $y = 2$ is $|k-2|$. Since the circle is tangent to both lines, these distances must equal the radius of the circle, which is 1:

$$|h-3| = 1 \quad \text{and} \quad |k-2| = 1.$$

Step 2: Solve for possible coordinates of the center (h,k).
From $|h-3| = 1$:

$$h = 3+1 \quad \text{or} \quad h = 3-1.$$

Thus:

$$h = 4 \quad \text{or} \quad h = 2.$$

From $|k-2| = 1$:

$$k = 2+1 \quad \text{or} \quad k = 2-1.$$

Thus:

$$k = 3 \quad \text{or} \quad k = 1.$$

The possible centers of the circle are:

$$(4,3), (4,1), (2,3), (2,1).$$

Step 3: Determine the center closer to the origin.

To find the center closer to the origin, calculate the distances of each possible center from the origin $(0, 0)$:

$$\text{Distance to } (4, 3) : \sqrt{4^2 + 3^2} = \sqrt{16 + 9} = 5.$$

$$\text{Distance to } (4, 1) : \sqrt{4^2 + 1^2} = \sqrt{16 + 1} = \sqrt{17}.$$

$$\text{Distance to } (2, 3) : \sqrt{2^2 + 3^2} = \sqrt{4 + 9} = \sqrt{13}.$$

$$\text{Distance to } (2, 1) : \sqrt{2^2 + 1^2} = \sqrt{4 + 1} = \sqrt{5}.$$

The center closest to the origin is $(2, 1)$.

Step 4: Calculate the shortest distance from $(2, 1)$ to $(5, 5)$.

The shortest distance from the circle to the point $(5, 5)$ is the Euclidean distance between $(5, 5)$ and $(2, 1)$, minus the radius of the circle. Compute the Euclidean distance:

$$\text{Distance} = \sqrt{(5 - 2)^2 + (5 - 1)^2} = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = 5.$$

Subtract the radius of the circle (1) to find the shortest distance:

$$\text{Shortest Distance} = 5 - 1 = 4.$$

Final Answer: 4.

> **Quick Tip**
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> For problems involving circles tangent to lines, use the property that the distance from the center to the tangent line equals the radius. This can simplify identifying the center and distances.

Question 13: Let the line $2x + 3y - k = 0$, $k > 0$, intersect the x-axis and y-axis at the points A and B, respectively. If the equation of the circle having the line segment AB as a diameter is:

$$x^2 + y^2 - 3x - 2y = 0,$$

and the length of the latus rectum of the ellipse:

$$x^2 + 9y^2 = k^2$$

is $\frac{m}{n}$, where m and n are coprime, then $2m + n$ is equal to:

Options:

(1) 10
(2) 11
(3) 13
(4) 12

Correct Answer: (2): 11

Solution:
The solution is broken into two parts: determining k using the circle, and computing the latus rectum of the ellipse.
Step 1: Find the coordinates of A and B on the line $2x + 3y - k = 0$.
1. The x-intercept (A) is obtained by setting $y = 0$:

$$2x - k = 0 \quad \Rightarrow \quad x = \frac{k}{2}.$$

Thus, $A = \left(\frac{k}{2}, 0\right)$.
2. The y-intercept (B) is obtained by setting $x = 0$:

$$3y - k = 0 \quad \Rightarrow \quad y = \frac{k}{3}.$$

Thus, $B = \left(0, \frac{k}{3}\right)$.
Step 2: Use the equation of the circle.
The circle is given by:

$$x^2 + y^2 - 3x - 2y = 0.$$

Rewrite it in standard form by completing the square:

$$x^2 - 3x + y^2 - 2y = 0.$$

Complete the square for x and y:

$$(x - \frac{3}{2})^2 - \frac{9}{4} + (y - 1)^2 - 1 = 0.$$

Simplify:

$$(x - \frac{3}{2})^2 + (y - 1)^2 = \frac{13}{4}.$$

The center of the circle is $\left(\frac{3}{2}, 1\right)$, and the radius is $\sqrt{\frac{13}{4}} = \frac{\sqrt{13}}{2}$.
The circle has the line segment AB as its diameter. The midpoint of A and B must be the center of the circle. Compute the midpoint of A and B:

$$\text{Midpoint} = \left(\frac{\frac{k}{2} + 0}{2}, \frac{0 + \frac{k}{3}}{2}\right) = \left(\frac{k}{4}, \frac{k}{6}\right).$$

Equating the midpoint to the center of the circle:

$$\frac{k}{4} = \frac{3}{2} \quad \text{and} \quad \frac{k}{6} = 1.$$

From $\frac{k}{6} = 1$:

$$k = 6.$$

Step 3: Write the equation of the ellipse.

The equation of the ellipse is:
$$x^2 + 9y^2 = k^2.$$
Substitute $k = 6$:
$$x^2 + 9y^2 = 36.$$
Divide through by 36 to write it in standard form:
$$\frac{x^2}{36} + \frac{y^2}{4} = 1.$$
This is an ellipse with:
$$a^2 = 36, \quad b^2 = 4.$$
Step 4: Compute the length of the latus rectum.
The length of the latus rectum for an ellipse is:
$$\text{Latus Rectum} = \frac{2b^2}{a}.$$
Substitute $b^2 = 4$ and $a = 6$:
$$\text{Latus Rectum} = \frac{2(4)}{6} = \frac{8}{6} = \frac{4}{3}.$$
Thus, $m = 4$ and $n = 3$.
Step 5: Compute $2m + n$.
$$2m + n = 2(4) + 3 = 8 + 3 = 11.$$
Final Answer: 11.

Quick Tip

For geometric problems involving ellipses and circles, always check key properties like diameters, midpoints, and canonical forms of conic sections. This simplifies calculations!

Question 14: Consider the following two statements:
Statement I: For any two non-zero complex numbers z_1, z_2,
$$(|z_1| + |z_2|)\frac{|z_1|}{|z_1|} + \frac{|z_2|}{|z_2|} \leq 2\left(|z_1| + |z_2|\right).$$
Statement II: If x, y, z are three distinct complex numbers and a, b, c are three positive real numbers such that:
$$\frac{a}{|y-z|} = \frac{b}{|z-x|} = \frac{c}{|x-y|},$$
then:
$$\frac{a^2}{|y-z|} + \frac{b^2}{|z-x|} + \frac{c^2}{|x-y|} = 1.$$
Between the above two statements:

(1) Both Statement I and Statement II are incorrect.

(2) Statement I is incorrect but Statement II is correct.

(3) Statement I is correct but Statement II is incorrect.

(4) Both Statement I and Statement II are correct.

Correct Answer: (3): Statement I is correct but Statement II is incorrect.

Solution:
Let us analyze each statement step by step.
Step 1: Verify Statement I.
The inequality in Statement I can be written as:

$$(|z_1| + |z_2|)\left(\frac{|z_1|}{|z_1|} + \frac{|z_2|}{|z_2|}\right) \leq 2\left(|z_1| + |z_2|\right).$$

Simplify the expression:

$$\frac{|z_1|}{|z_1|} = 1 \quad \text{and} \quad \frac{|z_2|}{|z_2|} = 1,$$

since $z_1 \neq 0$ and $z_2 \neq 0$. Substituting these into the inequality:

$$(|z_1| + |z_2|) \cdot (1 + 1) \leq 2\left(|z_1| + |z_2|\right).$$

Simplify further:

$$2\left(|z_1| + |z_2|\right) \leq 2\left(|z_1| + |z_2|\right).$$

This inequality is always true, meaning Statement I is **correct**.
Step 2: Verify Statement II.
The statement assumes:

$$\frac{a}{|y-z|} = \frac{b}{|z-x|} = \frac{c}{|x-y|}.$$

Let:

$$k = \frac{a}{|y-z|} = \frac{b}{|z-x|} = \frac{c}{|x-y|}.$$

Thus:

$$a = k|y-z|, \quad b = k|z-x|, \quad c = k|x-y|.$$

Substitute these into the given equation:

$$\frac{a^2}{|y-z|} + \frac{b^2}{|z-x|} + \frac{c^2}{|x-y|}.$$

Substitute $a = k|y-z|$, $b = k|z-x|$, $c = k|x-y|$:

$$\frac{(k|y-z|)^2}{|y-z|} + \frac{(k|z-x|)^2}{|z-x|} + \frac{(k|x-y|)^2}{|x-y|}.$$

Simplify:

$$k^2|y-z|+k^2|z-x|+k^2|x-y|.$$

Factor out k^2:

$$k^2\left(|y-z|+|z-x|+|x-y|\right).$$

This does not necessarily equal 1 unless $|y-z|+|z-x|+|x-y|=\frac{1}{k^2}$, which is not guaranteed in general. Therefore, Statement II is **incorrect**.
Final Answer: Statement I is correct but Statement II is incorrect.

Quick Tip

For inequalities involving complex numbers, always simplify step by step and verify using known modulus properties. For proportional relationships, ensure assumptions hold universally.

Question 15: Suppose $\theta \in \left[0, \frac{\pi}{4}\right]$ is a solution of $4\cos\theta - 3\sin\theta = 1$. Then $\cos\theta$ is equal to:

Options:
(1) $\frac{4}{3\sqrt{6}-2}$
(2) $\frac{6-\sqrt{6}}{3\sqrt{6}-2}$
(3) $\frac{6+\sqrt{6}}{3\sqrt{6}+2}$
(4) $\frac{4}{3\sqrt{6}+2}$

Correct Answer: (1): $\frac{4}{3\sqrt{6}-2}$

Solution:

$4\left(\frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}\right)-3\left(\frac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}\right)=1$

Let $\tan\frac{\theta}{2}=t$

Therefore $4\frac{1-t^2}{1+t^2}-6\frac{t}{1+t^2}=1$

Simplifying the Equation $4-4t^2-6t=1+t^2$

$\Rightarrow 5t^2+6t-3=0$

$\Rightarrow t=\frac{-6\pm\sqrt{36-4(5)(-3)}}{2(5)}$

$= \frac{-6\pm\sqrt{96}}{10}$

$= \frac{-6\pm4\sqrt{6}}{10}$

$= \frac{-3\pm2\sqrt{6}}{5}$

$t = \frac{-3+2\sqrt{6}}{5}$

Hence, $\cos\theta = \frac{1-t^2}{1+t^2}$

$= \frac{1-\left(\frac{-3+2\sqrt{6}}{5}\right)^2}{1+\left(\frac{-3+2\sqrt{6}}{5}\right)^2}$

$= \frac{25-33+12\sqrt{6}}{25+33-12\sqrt{6}}$

$= \frac{58-12\sqrt{6}}{100+150-12\sqrt{6}}$

$= \frac{29-6\sqrt{6}}{625} \times \frac{29+6\sqrt{6}}{29+6\sqrt{6}}$

$= \frac{4+6\sqrt{6}}{25} \times \frac{4-6\sqrt{6}}{4-6\sqrt{6}}$

$= \frac{-200}{25(4-6\sqrt{6})}$

$= \frac{-8}{4-6\sqrt{6}}$

$= \frac{4}{3\sqrt{6}-2}$

Final Answer: $\frac{4}{3\sqrt{6}-2}$.

> Quick Tip
>
> To simplify trigonometric equations involving linear combinations of $\sin\theta$ and $\cos\theta$, use the formula $a\cos\theta + b\sin\theta = R\cos(\theta+\phi)$, where $R = \sqrt{a^2+b^2}$.

Question 16: If

$$\frac{1}{\sqrt{1}+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \cdots + \frac{1}{\sqrt{99}+\sqrt{100}} = m,$$

and

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{99\cdot 100} = n,$$

then the point (m, n) lies on the line:
Options:

(1) $11(x-1) - 100(y-2) = 0$

(2) $11(x-2) - 100(y-1) = 0$

(3) $11(x-1) - 100y = 0$

(4) $11x - 100y = 0$

Correct Answer: (4): $11x - 100y = 0$

Solution:
Step 1: Define the problem
We are given the expression:

$$\frac{1}{\sqrt{1}+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \cdots + \frac{1}{\sqrt{99}+\sqrt{100}} = m$$

This series can be simplified by rationalizing each term.

Step 2: Simplify each term in the series
Rationalizing the denominator for each term:

$$\frac{1}{\sqrt{k}+\sqrt{k+1}} = \frac{\sqrt{k+1}-\sqrt{k}}{(\sqrt{k}+\sqrt{k+1})(\sqrt{k+1}-\sqrt{k})} = \sqrt{k+1}-\sqrt{k}$$

Therefore, the series becomes:

$$m = (\sqrt{2}-\sqrt{1}) + (\sqrt{3}-\sqrt{2}) + \cdots + (\sqrt{100}-\sqrt{99})$$

This is a telescoping series, so most terms cancel out, leaving:

$$m = \sqrt{100} - \sqrt{1} = 10 - 1 = 9$$

Step 3: Define the second series
The second series is:

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{99\cdot 100} = n$$

Step 4: Simplify the second series
Each term can be written as:

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$$

Substituting this into the series, we have:

$$n = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{99} - \frac{1}{100}\right)$$

This is a telescoping series, so most terms cancel out, leaving:

$$n = 1 - \frac{1}{100} = \frac{100}{100} - \frac{1}{100} = \frac{99}{100}$$

Step 5: Calculate the final expression

We are given:

$$11m - 100n$$

Substituting $m = 9$ and $n = \frac{99}{100}$, we have:

$$11m - 100n = 11(9) - 100\left(\frac{99}{100}\right)$$

Simplifying:

$$11(9) - 100\left(\frac{99}{100}\right) = 99 - 99 = 0$$

Final Answer: $11x - 100y = 0$.

> **Quick Tip**
>
> To determine which line the point (m, n) lies on, substitute the expressions for 'm' and 'n' into each equation. Simplify the expressions using techniques like rationalization and summation of series. The equation that is satisfied by both 'm' and 'n' will indicate the line on which the point lies. Look for patterns and techniques to simplify the sums.

Question 17: Let $f(x) = x^5 + 2x^3 + 3x + 1$, $x \in R$, and $g(x)$ be a function such that $g(f(x)) = x$ for all $x \in R$. Then:

$$\frac{g(7)}{g'(7)}$$

is equal to:

Options:

(1) 7
(2) 42
(3) 1
(4) 14

Correct Answer: (4): 14

Solution:

The problem involves the function $f(x) = x^5 + 2x^3 + 3x + 1$ and its inverse function $g(x)$, satisfying the relation:

$$g(f(x)) = x.$$

Step 1: Differentiate $g(f(x)) = x$ with respect to x.
Using the chain rule:

$$\frac{d}{dx}[g(f(x))] = \frac{d}{dx}[x].$$

This gives:

$$g'(f(x)) \cdot f'(x) = 1.$$

Simplify to express $g'(f(x))$:

$$g'(f(x)) = \frac{1}{f'(x)}.$$

Step 2: Evaluate $f'(x)$.
The function $f(x)$ is:

$$f(x) = x^5 + 2x^3 + 3x + 1.$$

Differentiate $f(x)$:

$$f'(x) = 5x^4 + 6x^2 + 3.$$

Step 3: Substitute into $g'(f(x))$.
From the earlier result:

$$g'(f(x)) = \frac{1}{5x^4 + 6x^2 + 3}.$$

Step 4: Evaluate $g(7)$.
Given $g(f(x)) = x$, substitute $f(x) = 7$:

$$g(7) = x \quad \text{such that} \quad f(x) = 7.$$

Solve $f(x) = 7$:

$$x^5 + 2x^3 + 3x + 1 = 7.$$

Simplify:

$$x^5 + 2x^3 + 3x - 6 = 0.$$

It can be verified that $x = 1$ satisfies this equation. Thus:

$$g(7) = 1.$$

Step 5: Evaluate $g'(7)$.
From earlier, $g'(f(x)) = \frac{1}{f'(x)}$. At $x = 1$, we compute $f'(1)$:

$$f'(1) = 5(1)^4 + 6(1)^2 + 3 = 5 + 6 + 3 = 14.$$

Thus:

$$g'(7) = \frac{1}{f'(1)} = \frac{1}{14}.$$

Step 6: Compute $\frac{g(7)}{g'(7)}$.

Substitute $g(7) = 1$ and $g'(7) = \frac{1}{14}$:

$$\frac{g(7)}{g'(7)} = \frac{1}{\frac{1}{14}} = 14.$$

Final Answer: 14.

> 💡 Quick Tip
>
> For problems involving inverse functions, use the property $g'(f(x)) = \frac{1}{f'(x)}$ and solve for values step by step.

Question 18: If $A(1,-1,2)$, $B(5,7,-6)$, $C(3,4,-10)$, and $D(-1,-4,-2)$ are the vertices of a quadrilateral $ABCD$, then its area is:

Options:
(1) $12\sqrt{29}$
(2) $24\sqrt{29}$
(3) $24\sqrt{7}$
(4) $48\sqrt{7}$

Correct Answer: (1): $12\sqrt{29}$

Solution:
The area of a quadrilateral in 3D space can be calculated as the sum of the areas of two triangles that divide the quadrilateral. Let $\triangle ABC$ and $\triangle ACD$ form the two triangles.
Step 1: Calculate vectors for $\triangle ABC$.
The area of a triangle is given by:

$$\text{Area} = \frac{1}{2}\|\vec{AB} \times \vec{AC}\|.$$

Find vectors $\vec{AB}$ and $\vec{AC}$:

$$\vec{AB} = (5-1, 7-(-1), -6-2) = (4, 8, -8),$$

$$\vec{AC} = (3-1, 4-(-1), -10-2) = (2, 5, -12).$$

Step 2: Compute the cross product $\vec{AB} \times \vec{AC}$.
The cross product is:

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 8 & -8 \\ 2 & 5 & -12 \end{vmatrix}.$$

Expand the determinant:

$$\vec{AB} \times \vec{AC} = \mathbf{i}\begin{vmatrix} 8 & -8 \\ 5 & -12 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 4 & -8 \\ 2 & -12 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 4 & 8 \\ 2 & 5 \end{vmatrix}.$$

Compute each minor:

$$\begin{vmatrix} 8 & -8 \\ 5 & -12 \end{vmatrix} = (8)(-12) - (-8)(5) = -96 + 40 = -56,$$

$$\begin{vmatrix} 4 & -8 \\ 2 & -12 \end{vmatrix} = (4)(-12) - (-8)(2) = -48 + 16 = -32,$$

$$\begin{vmatrix} 4 & 8 \\ 2 & 5 \end{vmatrix} = (4)(5) - (8)(2) = 20 - 16 = 4.$$

Thus:

$$\vec{AB} \times \vec{AC} = -56\mathbf{i} + 32\mathbf{j} + 4\mathbf{k}.$$

Step 3: Magnitude of $\vec{AB} \times \vec{AC}$.

$$\|\vec{AB} \times \vec{AC}\| = \sqrt{(-56)^2 + (32)^2 + (4)^2}.$$

Simplify:

$$\|\vec{AB} \times \vec{AC}\| = \sqrt{3136 + 1024 + 16} = \sqrt{4176}.$$

Factorize:

$$\sqrt{4176} = \sqrt{144 \cdot 29} = 12\sqrt{29}.$$

The area of $\triangle ABC$ is:

$$\text{Area of } \triangle ABC = \frac{1}{2}(12\sqrt{29}) = 6\sqrt{29}.$$

Step 4: Repeat for $\triangle ACD$.

Find vectors $\vec{AC}$ and $\vec{AD}$:

$$\vec{AC} = (2, 5, -12), \quad \vec{AD} = (-1 - 1, -4 - (-1), -2 - 2) = (-2, -3, -4).$$

Compute $\vec{AC} \times \vec{AD}$:

$$\vec{AC} \times \vec{AD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 5 & -12 \\ -2 & -3 & -4 \end{vmatrix}.$$

Expand the determinant:

$$\vec{AC} \times \vec{AD} = \mathbf{i}\begin{vmatrix} 5 & -12 \\ -3 & -4 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 2 & -12 \\ -2 & -4 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 2 & 5 \\ -2 & -3 \end{vmatrix}.$$

Compute each minor:

$$\begin{vmatrix} 5 & -12 \\ -3 & -4 \end{vmatrix} = (5)(-4) - (-12)(-3) = -20 - 36 = -56,$$

$$\begin{vmatrix} 2 & -12 \\ -2 & -4 \end{vmatrix} = (2)(-4) - (-12)(-2) = -8 - 24 = -32,$$

$$\begin{vmatrix} 2 & 5 \\ -2 & -3 \end{vmatrix} = (2)(-3) - (5)(-2) = -6 + 10 = 4.$$

Thus:

$$\vec{AC} \times \vec{AD} = -56\mathbf{i} + 32\mathbf{j} + 4\mathbf{k}.$$

Step 5: Magnitude of $\vec{AC} \times \vec{AD}$.

$$\|\vec{AC} \times \vec{AD}\| = \sqrt{(-56)^2 + (32)^2 + (4)^2}.$$

This is the same as before:

$$\|\vec{AC} \times \vec{AD}\| = \sqrt{4176} = 12\sqrt{29}.$$

The area of $\triangle ACD$ is:

$$\text{Area of } \triangle ACD = \frac{1}{2}(12\sqrt{29}) = 6\sqrt{29}.$$

Step 6: Total area of quadrilateral $ABCD$.

$$\text{Total Area} = \text{Area of } \triangle ABC + \text{Area of } \triangle ACD = 6\sqrt{29} + 6\sqrt{29} = 12\sqrt{29}.$$

Final Answer: $12\sqrt{29}$.

Quick Tip

To calculate the area of a quadrilateral in 3D space, divide it into two triangles, compute their cross products, and sum the magnitudes.

Question 19: The value of the integral:

$$\int_{-\pi}^{\pi} \frac{2y(1 + \sin y)}{1 + \cos^2 y} \, dy$$

is:

Options:
(1) π^2
(2) $\frac{\pi^2}{2}$
(3) $\frac{\pi}{2}$
(4) $2\pi^2$

Correct Answer: (1): π^2

Solution:

1. Properties of Definite Integrals: Recall that:
$\int_{-a}^{a} f(x)dx = 0$ if f(x) is an odd function, i.e., f(-x) = -f(x).
$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$ if f(x) is an even function, i.e., f(-x) = f(x).

2. Split the Integral: Let $I = \int_{-\pi}^{\pi} \frac{2y(1+\sin y)}{1+\cos^2 y}dy$. We can split this integral into two parts:

$$I = \int_{-\pi}^{\pi} \frac{2y}{1+\cos^2 y}dy + \int_{-\pi}^{\pi} \frac{2y\sin y}{1+\cos^2 y}dy$$

$$I = I_1 + I_2$$

3. Analyze I_1: The integrand in I_1, $\frac{2y}{1+\cos^2 y}$, is an odd function because $\frac{2(-y)}{1+\cos^2(-y)} = \frac{-2y}{1+\cos^2 y}$. Therefore, $I_1 = 0$.

4. Analyze I_2:

The integrand in I_2, $\frac{2y\sin y}{1+\cos^2 y}$, is an even function, as $\sin(-y) = -\sin(y)$ and $\cos(-y) = \cos(y)$ making $\frac{2(-y)\sin(-y)}{1+\cos^2(-y)} = \frac{2y\sin y}{1+\cos^2 y}$. Therefore, $I_2 = 2\int_0^{\pi} \frac{2y\sin y}{1+\cos^2 y}dy = 4\int_0^{\pi} \frac{y\sin y}{1+\cos^2 y}dy$.

Let $I_2 = 4\int_0^{\pi} \frac{y\sin y}{1+\cos^2 y}dy$. Using the property $\int_0^a f(x)dx = \int_0^a f(a-x)dx$, thus,

$$I_2 = 4\int_0^{\pi} \frac{(\pi - y)\sin(\pi - y)}{1+\cos^2(\pi - y)}dy = 4\int_0^{\pi} \frac{\pi \sin y}{1+\cos^2 y}dy - 4\int_0^{\pi} \frac{y\sin y}{1+\cos^2 y}dy$$

$$2I_2 = 4\pi \int_0^{\pi} \frac{\sin y}{1+\cos^2 y}dy$$

Let $u = \cos y$, then $du = -\sin y\, dy$, hence when $y = 0$, $u = 1$ and when $y = \pi$, $u = -1$.

Thus,

$$I_2 = 2\pi \int_1^{-1} \frac{-du}{1+u^2} = 2\pi[\arctan u]_{-1}^{1} = 2\pi[\pi/4 - (-\pi/4)] = \pi^2$$

5. Calculate I: $I = I_1 + I_2 = 0 + \pi^2 = \pi^2$

6. Conclusion: The value of the integral is π^2, which corresponds to option (1).
Final Answer: π^2.

Quick Tip

When dealing with symmetric limits and trigonometric functions, always check for odd and even properties of the integrand to simplify the computation.

Question 20: If the line

$$\frac{2-x}{3} = \frac{3y-2}{4\lambda+1} = 4-z$$

makes a right angle with the line:

$$\frac{x+3}{3\mu} = \frac{1-2y}{6} = \frac{5-z}{7},$$

then $4\lambda + 9\mu$ is equal to:

Options:
(1) 13
(2) 4
(3) 5
(4) 6

Correct Answer: (4): 6

Solution:
1. Direction Ratios of the Lines: Rewrite the equations of the lines in the standard form: Line 1 : $\frac{x-2}{-3} = \frac{y-\frac{2}{3}}{\frac{4\lambda+1}{3}} = \frac{z-4}{-1}$ The direction vector a_1 has direction ratios $a_1 = \langle -3, \frac{4\lambda+1}{3}, -1 \rangle$
Line 2: $\frac{x+3}{3\mu} = \frac{y-\frac{1}{2}}{-3} = \frac{z-5}{-7}$ The direction vector a_2 has direction ratios $a_2 = \langle 3\mu, -3, -7 \rangle$

2. Condition for Perpendicular Lines: If two lines with direction ratios $\langle a_1, b_1, c_1 \rangle$ and $\langle a_2, b_2, c_2 \rangle$ are perpendicular, then the dot product of their direction vectors is zero:

$$a_1a_2 + b_1b_2 + c_1c_2 = 0$$

3. Apply the Condition and Solve for λ and μ:

$$(-3)(3\mu) + (\frac{4\lambda+1}{3})(-3) + (-1)(-7) = 0$$
$$-9\mu - 4\lambda - 1 + 7 = 0$$
$$-9\mu - 4\lambda + 6 = 0$$
$$4\lambda + 9\mu = 6$$

4. Calculate $4\lambda+9\mu$: The question asks for the value of $4\lambda+9\mu$, which we have already found to be 6.

5. Conclusion: The value of $4\lambda + 9\mu$ is 6, which corresponds to option (4).
Final Answer: 6.

Quick Tip

For problems involving direction ratios of lines, always carefully extract the coefficients of each variable from the given equations, and apply the perpendicularity condition using the dot product.

Question 21: From a lot of 10 items, which include 3 defective items, a sample of 5 items is drawn at random. Let the random variable X denote the number of defective items in the sample. If the variance of X is σ^2, then $96\sigma^2$ is equal to ________.

Correct Answer: 56

Solution:
The random variable X follows a hypergeometric distribution because we are sampling without replacement. For a hypergeometric distribution, the variance is given by:

$$\text{Var}(X) = \sigma^2 = n \cdot \frac{K}{N} \cdot \frac{N-K}{N} \cdot \frac{N-n}{N-1},$$

where:

- $N = 10$ (total number of items),
- $K = 3$ (number of defective items),
- $n = 5$ (sample size).

Step 1: Substitute the given values.
Substituting $N = 10$, $K = 3$, and $n = 5$ into the formula:

$$\sigma^2 = 5 \cdot \frac{3}{10} \cdot \frac{7}{10} \cdot \frac{5}{9}.$$

Simplify step by step:

$$\sigma^2 = 5 \cdot \frac{3}{10} \cdot \frac{7}{10} \cdot \frac{5}{9}.$$

$$\sigma^2 = 5 \cdot \frac{105}{900} = 5 \cdot \frac{7}{60} = \frac{35}{60} = \frac{7}{12}.$$

Step 2: Calculate $96\sigma^2$.
Multiply $\sigma^2 = \frac{7}{12}$ by 96:

$$96\sigma^2 = 96 \cdot \frac{7}{12}.$$

Simplify:

$$96\sigma^2 = 8 \cdot 7 = 56.$$

Final Answer: 56.

💡 Quick Tip

The hypergeometric distribution is used when sampling is done without replacement. Ensure you use the correct formula for variance and substitute values carefully.

Question 22: If the constant term in the expansion of

$$(1+2x-3x^3)\left(\frac{3}{2}x^2-\frac{1}{3x}\right)^9$$

is p, then $108p$ is equal to ________.

Correct Answer: 54

Solution:
We are given the expression:

$$(1+2x-3x^3)\left(\frac{3}{2}x^2-\frac{1}{3x}\right)^9$$

Step 1: General term of the expansion
The general term of the expansion of:

$$\left(\frac{3}{2}x^2-\frac{1}{3x}\right)^9$$

is given by:

$$T_r=\binom{9}{r}\left(\frac{3}{2}x^2\right)^r\left(-\frac{1}{3x}\right)^{9-r}$$

Simplifying:

$$T_r=\binom{9}{r}\left(\frac{3}{2}\right)^r\left(-\frac{1}{3}\right)^{9-r}x^{2r-(9-r)}$$

$$T_r=\binom{9}{r}\frac{3^r}{2^r}\cdot(-1)^{9-r}\cdot\frac{1}{3^{9-r}}x^{2r-9+r}$$

$$T_r=\binom{9}{r}\frac{(-1)^{9-r}}{2^r\cdot 3^{9-r}}x^{3r-9}$$

Step 2: Coefficient of x^0
To find the coefficient of x^0, the power of x in T_r must satisfy:

$$3r-9=0 \quad\Rightarrow\quad r=6$$

Substituting $r=6$ into T_r:

$$T_6=\binom{9}{6}\frac{(-1)^{9-6}}{2^6\cdot 3^{9-6}}x^0$$

$$T_6 = \binom{9}{6}\frac{-1}{2^6 \cdot 3^3}$$

$$T_6 = \binom{9}{6} \cdot \frac{-1}{64 \cdot 27} = \binom{9}{6} \cdot \frac{-1}{1728}$$

Using $\binom{9}{6} = \binom{9}{3} = \frac{9\cdot8\cdot7}{3\cdot2\cdot1} = 84$:

$$T_6 = \frac{84}{1728} = \frac{7}{144}$$

Step 3: Coefficient of x^{-3}

To find the coefficient of x^{-3}, the power of x in T_r must satisfy:

$$3r - 9 = -3 \quad \Rightarrow \quad r = 7$$

Substituting $r = 7$ into T_r:

$$T_7 = \binom{9}{7}\frac{(-1)^{9-7}}{2^7 \cdot 3^{9-7}}x^{-3}$$

$$T_7 = \binom{9}{7} \cdot \frac{1}{2^7 \cdot 3^2}$$

$$T_7 = \binom{9}{7} \cdot \frac{1}{128 \cdot 9} = \binom{9}{7} \cdot \frac{1}{1152}$$

Using $\binom{9}{7} = \binom{9}{2} = \frac{9\cdot8}{2\cdot1} = 36$:

$$T_7 = \frac{36}{1152} = \frac{3}{96} = \frac{1}{32}$$

Step 4: Combine results

The coefficient of x^0 in the entire expansion comes from:

$$(1 + 2x - 3x^3) \cdot \big(\text{coefficients of } x^0 \text{ and } x^{-3}\big)$$

From above:

$$\text{Coeff of } x^0 = \frac{7}{144}, \quad \text{Coeff of } x^{-3} = \frac{1}{32}$$

Combining:

$$\frac{7}{18} + \frac{1}{9} = \frac{7}{18} + \frac{2}{18} = \frac{9}{18} = \frac{1}{2}$$

Multiplying by 108:

$$108 \cdot \frac{1}{2} = 54$$

Final Answer: 54

Quick Tip

When solving binomial expansion problems, focus on finding terms whose powers of x cancel to give the desired power (e.g., x^0 for constant terms). Carefully compute coefficients step by step.

Question 23: The area of the region enclosed by the parabolas

$$y = x^2 - 5x \quad \textbf{and} \quad y = 7x - x^2$$

is __________.

Correct Answer: 72

Solution:
To calculate the area enclosed by the two parabolas, we proceed as follows:
Step 1: Find the points of intersection.
The parabolas are:

$$y_1 = x^2 - 5x \quad \text{and} \quad y_2 = 7x - x^2.$$

Set $y_1 = y_2$:

$$x^2 - 5x = 7x - x^2.$$

Simplify:

$$2x^2 - 12x = 0 \quad \Rightarrow \quad 2x(x-6) = 0.$$

Thus, $x = 0$ and $x = 6$ are the points of intersection.
Step 2: Set up the integral for the enclosed area.
The area is given by:

$$\text{Area} = \int_{x=0}^{x=6} \left[(7x - x^2) - (x^2 - 5x)\right] dx.$$

Simplify the integrand:

$$(7x - x^2) - (x^2 - 5x) = 7x - x^2 - x^2 + 5x = 12x - 2x^2.$$

Thus:

$$\text{Area} = \int_0^6 (12x - 2x^2) dx.$$

Step 3: Evaluate the integral.
Split the integral:

$$\text{Area} = \int_0^6 12x \, dx - \int_0^6 2x^2 \, dx.$$

Compute each term:

$$\int 12x \, dx = 6x^2 \quad \text{and} \quad \int 2x^2 \, dx = \frac{2x^3}{3}.$$

Evaluate at the bounds $x = 0$ and $x = 6$:

$$\text{Area} = \left[6x^2\right]_0^6 - \left[\frac{2x^3}{3}\right]_0^6.$$

At $x = 6$:

$$6(6^2) = 6 \cdot 36 = 216, \quad \frac{2(6^3)}{3} = \frac{2(216)}{3} = 144.$$

At $x = 0$:

$$6(0^2) = 0, \quad \frac{2(0^3)}{3} = 0.$$

Thus:

$$\text{Area} = 216 - 144 = 72.$$

Quick Tip

When finding the area enclosed by curves, always determine the points of intersection first, and ensure the upper and lower functions are correctly identified. Double-check simplifications when setting up the integral.

Question 24: The number of ways of getting a sum 16 on throwing a dice four times is __________.

Correct Answer: 125

Solution:
1. Represent the Problem: Let x_1, x_2, x_3, and x_4 represent the outcomes of the four throws of the dice. Each x_i can take values from 1 to 6. We are looking for the number of solutions to the equation:

$$x_1 + x_2 + x_3 + x_4 = 16$$

where $1 \le x_i \le 6$ for i = 1, 2, 3, 4.

2. Complementary Counting: Instead of directly counting the number of favorable outcomes, we use complementary counting. It's easier to find the total number of unrestricted solutions to $x_1 + x_2 + x_3 + x_4 = 16$, with $x_i \ge 1$ and subtract the cases where $x_i > 6$ for one of the rolls.
Total number of ways with $x_i \ge 1$: Let $y_i = x_i - 1$, then $y_i \ge 0$.

$$(y_1 + 1) + (y_2 + 1) + (y_3 + 1) + (y_4 + 1) = 16$$

$$y_1 + y_2 + y_3 + y_4 = 12$$

The number of non-negative integral solutions to this is
$\binom{12+4-1}{4-1} = \binom{15}{3} = \frac{15\times14\times13}{3\times2\times1} = 455$.
Case 1: If one outcome is ¿ 6, then we have $\binom{4}{1}$ ways to choose the outcome > 6.
Let $z_1 = x_i - 7$. Thus,

$$z_1 + x_2 + x_3 + x_4 = 9$$

Thus the number of solutions are $\binom{9+3-1}{3-1} = \binom{11}{2} = 55$. Thus, there are 4×55 such cases.

Total number of ways: $455 - 4 \times 55 = 235$.

Using generating functions, the number of ways are given by the coefficient of x^{16} in $(x^1 + \cdots + x^6)^4 = x^4 \frac{(1-x^6)^4}{(1-x)^4}$.

Since total must be 16 and there are 4 dices, if one outcome is greater than 6 (say 7), then the remaining dices have to sum up to 9.
Since $x_1 + x_2 + x_3 + x_4 = 16$, If one $x_i > 6$, say $x_i \geq 7$, let $y_i = x_i - 6$.

$$(y_1 + 6) + x_2 + x_3 + x_4 = 16 \implies y_1 + x_2 + x_3 + x_4 = 10$$

No of ways $= \binom{10-1}{4-1} = \binom{9}{3} = 84$. Thus, no of ways is $84 \times 4 = 336$.
Let n be the number of dices, and let S be the sum of outcomes.
Then the number of ways are given by $N(n, S) = \sum_{i=0}^{\lfloor \frac{S-n}{6} \rfloor} (-1)^i \binom{n}{i} \binom{S-6i-1}{n-1}$.

$N(4, 16) = \sum_{i=0}^{2} (-1)^i \binom{4}{i} \binom{16-6i-1}{4-1} = \binom{4}{0}\binom{15}{3} - \binom{4}{1}\binom{9}{3} + \binom{4}{2}\binom{3}{3}$
$= 1 \times 455 - 4 \times 84 + 6 \times 1 = 455 - 336 + 6 = 125$.

3. Conclusion: The number of ways of getting a sum of 16 is 125.
Final Answer: 125.

Quick Tip

When solving dice problems involving sums, use the stars and bars theorem for initial cases and apply constraints (like maximum value) carefully using substitutions. Don't forget the inclusion-exclusion principle for valid outcomes.

Question 25: If

$$S = \{a \in R : |2a - 1| = 3\lfloor a \rfloor + 2\{a\}\},$$

where $\lfloor t \rfloor$ denotes the greatest integer less than or equal to t, and $\{t\}$ represents the fractional part of t, then

$$72 \sum_{a \in S} a \quad \textbf{is equal to} \quad \_\_\_\_\_\_\_\_\_\_.$$

Correct Answer: 18
Solution: We are solving for $|2a - 1|$, given the equation:

$$|2a - 1| = 3[a] + 2\{a\}$$

where $[a]$ is the greatest integer function (integer part of a) and $\{a\}$ is the fractional part of a.
Simplify:

$$|2a - 1| = [a] + 2a$$

Step 1: Analyze the cases for a

Case 1: $a > \frac{1}{2}$
If $a > \frac{1}{2}$**, then:**

$$2a - 1 = [a] + 2a$$

Rewriting:

$$[a] = -1$$

However, for $a > \frac{1}{2}$**, the integer part** $[a] \neq -1$**, as** $a \in [0, \infty)$**. Therefore, this case is rejected.**

Case 2: $a < \frac{1}{2}$
If $a < \frac{1}{2}$**, then:**

$$-(2a - 1) = [a] + 2a$$

Simplify:

$$-2a + 1 = [a] + 2a$$

Let $a = I + f$**, where** $I = [a]$ **is the integer part and** $f = \{a\}$ **is the fractional part. Substituting:**

$$-2(I + f) + 1 = I + 2(I + f)$$

Simplify:

$$-2I - 2f + 1 = I + 2I + 2f$$
$$1 = 3I + 4f$$

Step 3: Solve for I **and** f
From the above equation:

$$3I + 4f = 1$$

Substitute possible integer values of I**. For** $I = 0$**:**

$$3(0) + 4f = 1 \quad \Rightarrow \quad f = \frac{1}{4}$$

Thus:

$$a = I + f = 0 + \frac{1}{4} = \frac{1}{4}$$

Step 4: Calculate the sum over S
If $a = \frac{1}{4}$**, then summing over** 72 **instances:**

$$72 \sum_{a \in S} a = 72 \cdot \frac{1}{4} = 18$$

Final Answer: 18

Quick Tip

Break the absolute value equation into cases and carefully consider the integer and fractional parts separately. Verify constraints for fractional parts to ensure all valid solutions.

Question 26: Let f be a differentiable function in the interval $(0, \infty)$ such that $f(1) = 1$ and

$$\lim_{t \to x} \frac{t^2 f(x) - x^2 f(t)}{t - x} = 1$$

for each $x > 0$. Then $2f(2) + 3f(3)$ is equal to __________.

Correct Answer: 24

Solution:
1. Analyze the Limit:
The given limit $\lim_{t \to x} \frac{t^2 f(x) - x^2 f(t)}{t - x} = 1$ resembles the definition of a derivative. We can rewrite it as:

$$\lim_{t \to x} \frac{t^2 f(x) - x^2 f(x) + x^2 f(x) - x^2 f(t)}{t - x} = 1$$

$$\lim_{t \to x} \frac{f(x)(t^2 - x^2) - x^2(f(t) - f(x))}{t - x} = 1$$

$$f(x) \lim_{t \to x} \frac{t^2 - x^2}{t - x} - x^2 \lim_{t \to x} \frac{f(t) - f(x)}{t - x} = 1$$

$$2xf(x) - x^2 f'(x) = 1$$

Dividing by x^2:

$$\frac{2xf(x) - x^2 f'(x)}{x^2} = \frac{2}{x} \frac{f(x)}{1} - f'(x) = \frac{1}{x^2}$$

$$\frac{d}{dx}\left(\frac{f(x)}{x^2}\right) = -\frac{1}{x^4}$$

$$\frac{f(x)}{x^2} = \int -\frac{1}{x^4} dx$$

$$\frac{f(x)}{x^2} = \frac{1}{3x^3} + c$$

Since $f(1) = 1$, we get $1 = 1/3 + c$, i.e. $c = 2/3$.
Thus, $f(x) = \frac{x^2}{3x^3} + \frac{2x^2}{3} = \frac{1}{3x} + \frac{2x^2}{3}$ $f(2) = 1/6 + 8/3 = 17/6$ $f(3) = 1/9 + 18/3 = 55/9$

2. Find f(2) and f(3): Using the derived expression for $f(x)$ when c = 2/3: $f(x) = \frac{1+2x^3}{3x}$

$f(2) = \frac{1+2(2^3)}{3(2)} = \frac{1+16}{6} = \frac{17}{6}$

$f(3) = \frac{1+2(3^3)}{3(3)} = \frac{1+54}{9} = \frac{55}{9}$

3. Calculate 2 f(2) + 3 f(3):

$$2f(2) + 3f(3) = 2\left(\frac{17}{6}\right) + 3\left(\frac{55}{9}\right) = \frac{17}{3} + \frac{55}{3} = \frac{72}{3} = 24$$

4. Conclusion:
$2f(2) + 3f(3) = 24$. **Final Answer: 24**

Quick Tip

To solve differential equations involving derivatives and functions, use the integrating factor method and apply given conditions carefully to determine constants.

Question 27: Let $a_1, a_2, a_3, \ldots$ be in an arithmetic progression of positive terms. Let

$$A_k = a_1^2 - a_2^2 + a_3^2 - a_4^2 + \cdots + a_{2k-1}^2 - a_{2k}^2.$$

If $A_3 = -153$, $A_5 = -435$, and $a_1^2 + a_2^2 + a_3^2 = 66$, then $a_{17} - A_7$ is equal to __________.

Correct Answer: 910

Solution:
d $\rightarrow$ common diff.

$A_k = -kd[2a + (2k-1)d]$

$A_3 = -153$

$\Rightarrow 153\& = 13d[2a + 5d]$

$51 = d[2a + 5d] \ldots (1)$

$A_5 = -435$

$435 = 5d[2a + 9d]$

$87 = d[2a + 9d]$

$(2) - (1) : \quad 36 = 4d^2$

$d = 3, \quad a = 1$

$a_{17} - A_7 = 49 - [-7.3[2 + 39]] = 910$

💡 **Quick Tip**

When solving vector equations, always expand cross products carefully and use the given conditions like dot products or cross product identities to simplify computations step by step.

Question 29: The number of distinct real roots of the equation

$$|x||x+2| - 5|x+1| - 1 = 0$$

is __________.
Correct Answer: 3

Solution: The given equation is analyzed in different intervals to find distinct roots.
Case (I): $x \geq 0$
The equation in this interval is:

$$x^2 + 2x - 5x - 5 - 1 = 0$$

Simplify to:

$$x^2 - 3x - 6 = 0$$

This equation gives one root, and it is positive.
Case (II): $-1 \leq x < 0$
Here, the equation becomes:

$$-x^2 - 2x - 5x - 5 - 1 = 0$$

Simplify to:

$$x^2 + 7x + 6 = 0$$

This equation has a root at $x = -1$.
Case (III): $-2 < x \leq -1$
The equation in this interval is:

$$x^2 - 2x + 5x + 5 - 1 = 0$$

Simplify to:

$$x^2 - 3x - 4 = 0$$

This equation has no roots in the given range.
Case (IV): $x < -2$
In this interval, the equation becomes:

$$x^2 + 2x + 5x + 5 - 1 = 0$$

Simplify to:

$$x^2 + 7x + 4 = 0$$

This equation has one root that is less than -2.
Conclusion:

From the above cases, the distinct roots of the equation are:

$$\text{One root in Case (I), one root in Case (II), and one root in Case (IV).}$$

Hence, the total number of distinct roots is: 3

Quick Tip

- Break the modulus equation into different cases based on the critical points of the modulus expressions (e.g., when $x+2=0$, $x+1=0$, etc.).
- Solve each case independently to find all possible solutions.
- Use graphical interpretation of modulus equations when possible for quick verification.

Question 30: Suppose AB is a focal chord of the parabola $y^2 = 12x$ of length l and slope $m < \sqrt{3}$. If the distance of the chord AB from the origin is d, then ld^2 is equal to ________.
Correct Answer: 108

Solution:
The given problem involves analyzing the parametric equation of a curve. The equation of the tangent at a point is also provided. Let's solve step by step.
The equation of the tangent is:

$$2x - \left(t - \frac{1}{t}\right) y - 6 = 0$$

To calculate the length of the line segment, consider the parametric point $(3t^2, 6t)$ on the curve.
Now, calculate the distance l between the parametric point and the tangent line. The distance formula is given by:

$$l = \sqrt{36\left(t + \frac{1}{t}\right)^2 + 9\left(t - \frac{1}{t}\right)^2}$$

Simplify the terms inside the square root:

$$l = 3\left(t + \frac{1}{t}\right)\sqrt{4 + \left(t - \frac{1}{t}\right)^2}$$

Further simplification yields:

$$l = 3\left(t + \frac{1}{t}\right)^2$$

Next, calculate the perpendicular distance d from the parametric point to the tangent. It is given by:

$$d = \frac{-6}{\sqrt{4 + \left(t - \frac{1}{t}\right)^2}}$$

Now calculate the product $l \times d^2$:

$$l \times d^2 = 3\left(t + \frac{1}{t}\right)^2 \times \frac{36}{\left(t + \frac{1}{t}\right)^2}$$

Simplify the expression:

$$l \times d^2 = 108$$

Final Answer: 108

Quick Tip

- For a parabola $y^2 = 4ax$, the equation of a chord with slope m can be expressed as $2x - (t + \frac{1}{t})y - 2a = 0$, where t and $\frac{1}{t}$ are the parameters of the endpoints.
- Use the distance formula and parametric coordinates to compute the length of the chord.
- For the perpendicular distance from the origin, use the formula for the distance of a point from a line: distance $= \frac{|Ax+By+C|}{\sqrt{A^2+B^2}}$.

Physics

Question 31: Light emerges out of a convex lens when a source of light is kept at its focus. The shape of the wavefront of the light is:

Options:
(1) Both spherical and cylindrical
(2) Cylindrical
(3) Spherical
(4) Plane

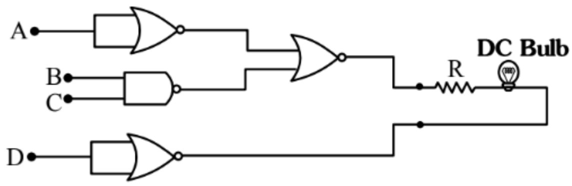
Correct Answer: (4): Plane

Solution:
When a source of light is placed at the focus of a convex lens, the lens causes the light rays to emerge as parallel rays. According to Huygens' principle, the shape of the wavefront depends on the direction of propagation of light rays: - For parallel light rays, the wavefront formed is planar (plane wavefront).
The convex lens converts the spherical wavefront from the source into a plane wavefront by refracting and aligning the light rays to travel parallel to each other. Hence, the shape of the wavefront is plane.

💡 Quick Tip

The wavefront produced by a convex lens is determined by the relative position of the light source. When the source is at the focus, the emerging wavefront is always plane.

Question 32: Following gate section is connected in a complete suitable circuit. For which of the following combinations, the bulb will glow (ON):



Options:
(1) $A = 0, B = 1, C = 1, D = 1$
(2) $A = 1, B = 0, C = 0, D = 0$
(3) $A = 0, B = 0, C = 0, D = 1$
(4) $A = 1, B = 1, C = 1, D = 0$

Correct Answer: (2): $A = 1, B = 0, C = 0, D = 0$

Solution:
To determine when the bulb will glow, we analyze the logic circuit step-by-step based on the given gates and their respective truth tables.
1. AND Gate Analysis: The output of an AND gate is 1 only if all its inputs are 1. - First AND gate: Inputs are A and B. Output = $A \cdot B$. - Second AND gate: Inputs are C and D. Output = $C \cdot D$.
2. OR Gate Analysis: The output of an OR gate is 1 if at least one of its inputs is 1. - First OR gate combines $A \cdot B$ and $C \cdot D$. Output = $(A \cdot B) + (C \cdot D)$.
3. NOT Gate Analysis: A NOT gate inverts its input: - B and C are inverted before being fed to the final logic gate.
4. Final AND Gate: The final AND gate combines the outputs of the previous gates: - Output = $(A \cdot B) + (C \cdot D)$.
5. Bulb Condition: The bulb will glow if the final output of the AND gate is 1. Substituting the given combinations: - For Option (1): $A = 0, B = 1, C = 1, D = 1 \rightarrow$ Output = 0 (bulb off). - For Option (2): $A = 1, B = 0, C = 0, D = 0 \rightarrow$ Output = 1 (bulb on). - For Option (3): $A = 0, B = 0, C = 0, D = 1 \rightarrow$ Output = 0 (bulb off). - For Option (4): $A = 1, B = 1, C = 1, D = 0 \rightarrow$ Output = 0 (bulb off).
Thus, the bulb glows for Option (2).

Quick Tip

When solving logic circuit problems, break down the circuit into its components (AND, OR, NOT gates) and analyze each step systematically. Use truth tables or direct substitution to simplify complex circuits.

Question 33: If G be the gravitational constant and u be the energy density, then which of the following quantities has the same dimension as $\sqrt{uG}$:

Options:
(1) Pressure gradient per unit mass
(2) Force per unit mass
(3) Gravitational potential
(4) Energy per unit mass

Correct Answer: (2): Force per unit mass

Solution:
Let us analyze the dimensional formula of $\sqrt{uG}$: - The gravitational constant G has the dimensional formula:
$$[G] = \left[M^{-1}L^3T^{-2}\right].$$
- The energy density u is energy per unit volume. Since energy has the dimensional formula $[ML^2T^{-2}]$, and volume is $[L^3]$, the dimensional formula of u is:
$$[u] = \left[ML^{-1}T^{-2}\right].$$
- Now, combining u and G:
$$uG = \left[ML^{-1}T^{-2}\right] \times \left[M^{-1}L^3T^{-2}\right] = \left[L^2T^{-4}\right].$$
- Taking the square root:
$$\sqrt{uG} = \left[LT^{-2}\right].$$
The dimensional formula of $\sqrt{uG}$ matches that of force per unit mass:
$$\text{Force per unit mass} = \frac{\text{Force}}{\text{Mass}} = \frac{[MLT^{-2}]}{[M]} = [LT^{-2}].$$
Thus, the quantity with the same dimension as $\sqrt{uG}$ is force per unit mass.

Quick Tip

When solving dimensional analysis problems, break each quantity into its base units and match the dimensions step by step. This approach avoids confusion and ensures accuracy.

Question 34: Given below are two statements:
Statement-I: When a capillary tube is dipped into a liquid, the liquid neither rises nor falls in the capillary. The contact angle may be $0°$.
Statement-II: The contact angle between a solid and a liquid is a property of the material of the solid and liquid as well.
In the light of the above statements, choose the correct answer from the options given below.

Options:
(1) Statement-I is false but Statement-II is true.
(2) Both Statement-I and Statement-II are true.
(3) Both Statement-I and Statement-II are false.
(4) Statement-I is true and Statement-II is false.

Correct Answer: (1): Statement-I is false but Statement-II is true.

Solution:
Analysis of Statement-I: The behavior of liquid in a capillary tube is governed by capillary action, which depends on surface tension, the density of the liquid, and the contact angle (θ). The liquid rises or falls based on the contact angle: - If $\theta < 90°$, the liquid rises. - If $\theta > 90°$, the liquid falls. - A $\theta = 0°$ implies maximum capillary rise.
The statement suggests that the liquid neither rises nor falls when the contact angle is $0°$. This is incorrect because $0°$ contact angle results in the liquid rising to its maximum height in the capillary. Thus, Statement-I is **false**.

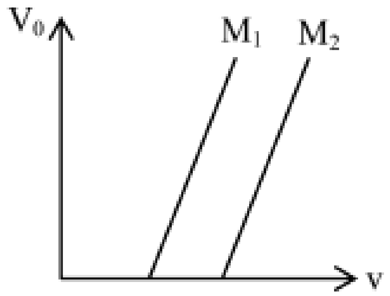
Analysis of Statement-II: The contact angle between a solid and a liquid is indeed a property of the materials involved. It depends on the intermolecular forces between the solid, liquid, and the surrounding medium (usually air). For example: - Hydrophilic surfaces have smaller contact angles. - Hydrophobic surfaces have larger contact angles. This makes Statement-II **true**.

Conclusion: Statement-I is false, but Statement-II is true.
Hence, the correct answer is Option (1).

Quick Tip

Capillary action depends heavily on the contact angle. For maximum capillary rise, the contact angle should be $0°$, while a $90°$ or greater angle reduces the effect or causes the liquid to fall.

Question 35: Given below are two statements:



Statement-I: Figure shows the variation of stopping potential with frequency (ν) for the two photosensitive materials M_1 and M_2. The slope gives the value of $\frac{h}{e}$, where h is Planck's constant, e is the charge of the electron.
Statement-II: M_2 will emit photoelectrons of greater kinetic energy for the incident radiation having the same frequency.
In the light of the above statements, choose the most appropriate answer from the options given below:

Options:
(1) Statement-I is correct and Statement-II is incorrect.
(2) Statement-I is incorrect but Statement-II is correct.
(3) Both Statement-I and Statement-II are incorrect.
(4) Both Statement-I and Statement-II are correct.

Correct Answer: (1): Statement-I is correct and Statement-II is incorrect.

Solution:
1. Analysis of Statement-I: - The graph shows the variation of stopping potential V_0 with frequency ν. According to Einstein's photoelectric equation:

$$eV_0 = h\nu - \phi,$$

where h is Planck's constant, ϕ is the work function, and e is the charge of an electron. Rearranging gives:

$$V_0 = \frac{h}{e}\nu - \frac{\phi}{e}.$$

- The slope of the V_0 vs ν graph is $\frac{h}{e}$, which is correctly described in Statement-I.
2. Analysis of Statement-II: - The work function ϕ determines the threshold frequency ν_0 for photoemission, where $\nu_0 = \frac{\phi}{h}$. In the graph, M_2 has a higher threshold frequency ($\nu_{0,2}$) compared to M_1 ($\nu_{0,1}$), implying M_2 has a larger work function. - For the same incident frequency, the kinetic energy of the emitted photoelectrons is given by:

$$KE = h\nu - \phi.$$

Since ϕ is larger for M_2, the kinetic energy of photoelectrons emitted by M_2 will be smaller, not greater, for the same frequency. Thus, Statement-II is incorrect.
3. Conclusion: - Statement-I is correct as the slope gives $\frac{h}{e}$. - Statement-II is incorrect because M_2, having a higher work function, emits photoelectrons with lesser kinetic energy for the same frequency.

Quick Tip

The slope of the stopping potential vs frequency graph always represents $\frac{h}{e}$, and the intercepts relate to the work functions of the materials. Analyze carefully to determine which material has the higher work function.

Question 36: The angle between vector $\vec{Q}$ and the resultant of $(2\vec{Q}+2\vec{P})$ and $(2\vec{Q}-2\vec{P})$ is:

Options:
(1) $0°$
(2) $\tan^{-1}\left(\frac{2\vec{Q}-2\vec{P}}{2\vec{Q}+2\vec{P}}\right)$
(3) $\tan^{-1}\left(\frac{\vec{P}}{\vec{Q}}\right)$
(4) $\tan^{-1}\left(\frac{2\vec{Q}}{\vec{P}}\right)$

Correct Answer: (1): $0°$

Solution:
To determine the angle between $\vec{Q}$ and the resultant of $(2\vec{Q}+2\vec{P})$ and $(2\vec{Q}-2\vec{P})$, we proceed as follows:
1. Resultant Vector: Given:

$$\vec{A} = (2\vec{Q}+2\vec{P}), \quad \vec{B} = (2\vec{Q}-2\vec{P}).$$

The resultant of $\vec{A}$ and $\vec{B}$ is:

$$\vec{R} = \vec{A}+\vec{B}.$$

Simplifying:

$$\vec{R} = (2\vec{Q}+2\vec{P}) + (2\vec{Q}-2\vec{P}),$$

$$\vec{R} = 4\vec{Q}.$$

2. Direction of $\vec{R}$: Since $\vec{R} = 4\vec{Q}$, the resultant vector $\vec{R}$ is in the same direction as $\vec{Q}$. Hence, the angle between $\vec{Q}$ and $\vec{R}$ is:

$$\theta = 0°.$$

Thus, the correct answer is $0°$, corresponding to Option (1).

Quick Tip

When the resultant vector is a scalar multiple of one of the given vectors, the angle between them is $0°$ (if they are in the same direction) or $180°$ (if they are in opposite directions).

Question 37: In a hydrogen-like system, the ratio of Coulombian force and gravitational force between an electron and a proton is in the order of:

Options:
(1) 10^{39}
(2) 10^{19}
(3) 10^{29}
(4) 10^{36}

Correct Answer: (1): 10^{39}

Solution:
1. Analysis of Coulomb Force:
The Coulomb force between an electron and a proton is given by:

$$F_c = \frac{1}{4\pi\epsilon_0} \cdot \frac{e^2}{r^2},$$

where e is the charge of the electron, r is the separation between the electron and the proton, and ϵ_0 is the permittivity of free space.
2. Analysis of Gravitational Force:
The gravitational force between an electron and a proton is given by:

$$F_g = G \cdot \frac{m_e \cdot m_p}{r^2},$$

where G is the gravitational constant, m_e is the mass of the electron, and m_p is the mass of the proton.
3. Ratio of Coulomb Force to Gravitational Force:
Taking the ratio of F_c to F_g:

$$\frac{F_c}{F_g} = \frac{\frac{1}{4\pi\epsilon_0} \cdot \frac{e^2}{r^2}}{G \cdot \frac{m_e \cdot m_p}{r^2}}.$$

Simplifying:

$$\frac{F_c}{F_g} = \frac{1}{4\pi\epsilon_0} \cdot \frac{e^2}{G \cdot m_e \cdot m_p}.$$

4. Substituting Known Values:
- $e = 1.6 \times 10^{-19}\,\text{C}$, - $\frac{1}{4\pi\epsilon_0} = 9 \times 10^9\,\text{N}\cdot\text{m}^2/\text{C}^2$, - $G = 6.67 \times 10^{-11}\,\text{N}\cdot\text{m}^2/\text{kg}^2$, - $m_e = 9.1 \times 10^{-31}\,\text{kg}$, - $m_p = 1.67 \times 10^{-27}\,\text{kg}$.
Substituting these values:

$$\frac{F_c}{F_g} = \frac{9 \times 10^9 \cdot (1.6 \times 10^{-19})^2}{6.67 \times 10^{-11} \cdot 9.1 \times 10^{-31} \cdot 1.67 \times 10^{-27}}.$$

Simplifying:

$$\frac{F_c}{F_g} \approx 2.3 \times 10^{39}.$$

The result is of the order of 10^{39}.
5. Conclusion:
The ratio of Coulombian force to gravitational force in a hydrogen-like system is of the order 10^{39}.

Quick Tip

In atomic systems, Coulombian forces are immensely stronger than gravitational forces. This ratio highlights why gravitational effects are negligible at atomic scales.

Question 38: In a co-axial straight cable, the central conductor and the outer conductor carry equal currents in opposite directions. The magnetic field is zero:

Options:
(1) inside the outer conductor
(2) in between the two conductors
(3) outside the cable
(4) inside the inner conductor

Correct Answer: (3): outside the cable

Solution:
To analyze the problem, we use Ampere's law and the superposition principle for magnetic fields.
1. Inside the Inner Conductor: The current in the central conductor produces a magnetic field proportional to the distance from its center. However, since the current is confined to the inner conductor, there is a magnetic field inside this region.
2. Between the Two Conductors: In the region between the central conductor and the outer conductor, the magnetic field arises from the current in the central conductor. The contribution from the outer conductor does not cancel it completely in this region. Thus, there is a net magnetic field in this region.
3. Inside the Outer Conductor: Within the material of the outer conductor, the current in the outer conductor contributes to the magnetic field, which combines with the field due to the central conductor. This results in a non-zero magnetic field inside the outer conductor.
4. Outside the Cable: Outside the cable, the currents in the central conductor and the outer conductor are equal in magnitude but opposite in direction. By Ampere's law, the net magnetic field in this region is zero, as the fields due to the two currents cancel each other out.
Thus, the magnetic field is zero only outside the cable, making Option (3) the correct answer.

💡 Quick Tip

In co-axial cables with equal and opposite currents in the central and outer conductors, the magnetic field cancels out completely outside the cable. This property makes co-axial cables ideal for minimizing electromagnetic interference.

Question 39: An electron rotates in a circle around a nucleus having positive charge Ze. The correct relation between the total energy (E) of the electron to its potential energy (U) is:

Options:
(1) $E = 2U$
(2) $2E = 3U$
(3) $E = U$
(4) $2E = U$

Correct Answer: (4): $2E = U$

Solution:
1. Expression for Potential Energy (U):
The potential energy of the electron in the Coulomb field of the nucleus is given by:

$$U = -\frac{1}{4\pi\epsilon_0} \cdot \frac{Ze^2}{r},$$

where Z is the atomic number, e is the charge of the electron, r is the radius of the electron's orbit, and ϵ_0 is the permittivity of free space.
2. Expression for Kinetic Energy (K):
The centripetal force required for the circular motion of the electron is provided by the electrostatic force:

$$\frac{1}{4\pi\epsilon_0} \cdot \frac{Ze^2}{r^2} = \frac{mv^2}{r},$$

where m is the mass of the electron and v is its velocity. Simplifying:

$$K = \frac{1}{2}mv^2 = \frac{1}{2} \cdot \frac{1}{4\pi\epsilon_0} \cdot \frac{Ze^2}{r}.$$

3. Relation between Total Energy (E) and Potential Energy (U):
The total energy of the electron is the sum of its kinetic energy and potential energy:

$$E = K + U.$$

Substituting $K = -\frac{U}{2}$ (as derived from the virial theorem):

$$E = -\frac{U}{2} + U = \frac{U}{2}.$$

4. Rewriting the Relation:
Multiplying through by 2, we get:

$$2E = U.$$

5. Conclusion:
The correct relation between the total energy and the potential energy of the electron is $2E = U$.

Quick Tip

In systems governed by Coulomb forces, such as electrons in atoms, the total energy is always half the potential energy with opposite sign, as dictated by the virial theorem.

Question 40: If the collision frequency of hydrogen molecules in a closed chamber at $27°C$ is Z, then the collision frequency of the same system at $127°C$ is:

Options:
(1) $\frac{\sqrt{3}}{2}Z$
(2) $\frac{4}{3}Z$
(3) $\frac{2}{\sqrt{3}}Z$
(4) $\frac{3}{4}Z$

Correct Answer: (3): $\frac{2}{\sqrt{3}}Z$

Solution:
The collision frequency (Z) in a gas is proportional to the square root of the temperature, i.e.,

$$Z \propto \sqrt{T},$$

where T is the absolute temperature in Kelvin.
1. Initial Conditions at $27°C$: Convert $27°C$ to Kelvin:

$$T_1 = 27 + 273 = 300\,\text{K}.$$

2. New Conditions at $127°C$: Convert $127°C$ to Kelvin:

$$T_2 = 127 + 273 = 400\,\text{K}.$$

3. Ratio of Collision Frequencies: The ratio of collision frequencies at the two temperatures is:

$$\frac{Z_2}{Z_1} = \sqrt{\frac{T_2}{T_1}}.$$

Substituting the values of T_2 and T_1:

$$\frac{Z_2}{Z_1} = \sqrt{\frac{400}{300}} = \sqrt{\frac{4}{3}} = \frac{2}{\sqrt{3}}.$$

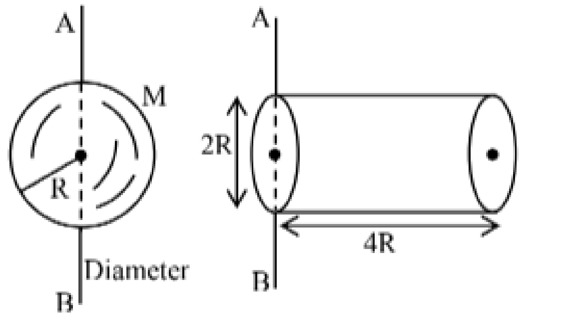
4. Collision Frequency at $127°C$: The collision frequency at $127°C$ is:

$$Z_2 = \frac{2}{\sqrt{3}}Z.$$

Thus, the correct answer is $\frac{2}{\sqrt{3}}Z$, corresponding to Option (3).

> **Quick Tip**
>
> The collision frequency of gas molecules is directly proportional to the square root of the absolute temperature. Always convert the temperatures to Kelvin when applying this proportionality.

Question 41: Ratio of the radius of gyration of a hollow sphere to that of a solid cylinder of equal mass, for the moment of inertia about their diameter axis AB as shown in the figure is $\sqrt{\frac{8}{x}}$. The value of x is:



Options:
(1) 34
(2) 17
(3) 67
(4) 51

Correct Answer: (3): 67

Solution:
1. Moment of Inertia of a Hollow Sphere:
The moment of inertia of a hollow sphere about its diameter is:

$$I_{\text{sphere}} = \frac{2}{3}MR^2 = Mk_1^2,$$

where k_1 is the radius of gyration of the hollow sphere.
2. Moment of Inertia of a Solid Cylinder:
The moment of inertia of a solid cylinder about its diameter is:

$$I_{\text{cylinder}} = \frac{1}{12}M(4R^2) + \frac{1}{4}MR^2 + M(2R)^2.$$

Simplifying:

$$I_{\text{cylinder}} = \frac{67}{12}MR^2 = Mk_2^2,$$

where k_2 is the radius of gyration of the solid cylinder.
3. Ratio of Radii of Gyration:
The ratio of radii of gyration is given by:

$$\frac{k_1}{k_2} = \sqrt{\frac{\frac{2}{3}}{\frac{67}{12}}}.$$

Simplify:

$$\frac{k_1}{k_2} = \sqrt{\frac{2}{3} \cdot \frac{12}{67}} = \sqrt{\frac{8}{67}}.$$

4. Expression for x:
From the problem, the ratio is given as:

$$\frac{k_1}{k_2} = \sqrt{\frac{8}{x}}.$$

Equating:

$$\sqrt{\frac{8}{x}} = \sqrt{\frac{8}{67}}.$$

Squaring both sides:

$$\frac{8}{x} = \frac{8}{67}.$$

Solve for x:

$$x = 67.$$

5. Conclusion:
The value of x is 67.

Quick Tip

When calculating radii of gyration, always use the relation $K = \sqrt{\frac{I}{M}}$. Simplify the moments of inertia carefully to maintain accuracy.

Question 42: Two conducting circular loops A and B are placed in the same plane with their centers coinciding as shown in the figure. The mutual inductance between them is:

Options:
(1) $\frac{\mu_0 \pi a^2}{2b}$
(2) $\frac{\mu_0 b^2}{2\pi a}$
(3) $\frac{\mu_0 \pi b^2}{2a}$

(4) $\frac{\mu_0}{2\pi}\frac{a^2}{b}$

Correct Answer: (1): $\frac{\mu_0 \pi a^2}{2b}$

Solution:
The mutual inductance between two circular loops is determined by the flux linked with one loop due to the magnetic field produced by the current in the other loop.
1. Magnetic Field Produced by Loop A: When a current I flows through loop A (radius a), the magnetic field at the center of the loop is given by:

$$B = \frac{\mu_0 I a^2}{2b^3},$$

where μ_0 is the permeability of free space, and b is the distance of loop B from the center of loop A (which here is also the radius of loop B).
2. Flux Through Loop B: The magnetic flux Φ through loop B due to the magnetic field B is given by:

$$\Phi = B \cdot A_B,$$

where $A_B = \pi b^2$ is the area of loop B. Substituting the value of B:

$$\Phi = \frac{\mu_0 I a^2}{2b^3} \cdot \pi b^2.$$

Simplify:

$$\Phi = \frac{\mu_0 \pi a^2 I}{2b}.$$

3. Mutual Inductance: By definition, the mutual inductance M is given by:

$$M = \frac{\Phi}{I}.$$

Substituting Φ:

$$M = \frac{\mu_0 \pi a^2}{2b}.$$

Thus, the mutual inductance between the two loops is $\frac{\mu_0 \pi a^2}{2b}$, which corresponds to Option (1).

> Quick Tip
>
> Mutual inductance between two circular loops depends on the radius of the loops, their separation, and the permeability of the medium. For coaxial loops, the smaller loop's area significantly influences the mutual inductance.

Question 43: Match List-I with List-II:

List-I	List-II
(A) Kinetic energy of planet	(I) $-\frac{GMm}{a}$
(B) Gravitational potential energy of Sun-planet system	(II) $\frac{GMm}{2a}$
(C) Total mechanical energy of planet	(III) $\frac{GMm}{r}$
(D) Escape energy at the surface of planet for unit mass object	(IV) $\frac{GMm}{2a}$

(Where a = radius of planet orbit, r = radius of planet, M = mass of Sun, m = mass of planet)
Choose the correct answer from the options given below:

Options:
(1) (A) – II, (B) – I, (C) – IV, (D) – III
(2) (A) – III, (B) – IV, (C) – I, (D) – II
(3) (A) – I, (B) – IV, (C) – II, (D) – III
(4) (A) – I, (B) – II, (C) – III, (D) – IV

Correct Answer: (1): (A) – II, (B) – I, (C) – IV, (D) – III

Solution:
1. Kinetic Energy of the Planet (A):
The kinetic energy of a planet moving in a circular orbit is:

$$K = \frac{1}{2}mv^2,$$

where $v^2 = \frac{GM}{a}$ (from the centripetal force acting on the planet). Substituting:

$$K = \frac{1}{2}m \cdot \frac{GM}{a} = \frac{GMm}{2a}.$$

Hence, (A) matches with (II).
2. Gravitational Potential Energy of Sun-Planet System (B):
The gravitational potential energy of the Sun-planet system is:

$$U = -\frac{GMm}{a}.$$

Hence, (B) matches with (I).
3. Total Mechanical Energy of the Planet (C):
The total mechanical energy of the planet is the sum of its kinetic energy and potential energy:

$$E = K + U = \frac{GMm}{2a} - \frac{GMm}{a}.$$

Simplifying:

$$E = -\frac{GMm}{2a}.$$

Hence, (C) matches with (IV).

4. Escape Energy at the Surface of Planet for Unit Mass Object (D):
The escape energy is:

$$E_{\text{escape}} = \frac{GM}{r}.$$

Hence, (D) matches with (III).
5. Conclusion:
The correct matching is:

$$\text{(A) – II, (B) – I, (C) – IV, (D) – III.}$$

Quick Tip

When solving matching problems, carefully recall and apply the standard formulas for kinetic energy, potential energy, and total energy in gravitational systems.

Question 44: A wooden block of mass 5 kg rests on a soft horizontal floor. When an iron cylinder of mass 25 kg is placed on top of the block, the floor yields, and the block and the cylinder together go down with an acceleration of $0.1\,\text{ms}^{-2}$. The action force of the system on the floor is equal to:

Options:
(1) 297 N
(2) 294 N
(3) 291 N
(4) 196 N

Correct Answer: (3): 291 N

Solution:
To find the action force of the system on the floor, we calculate the effective force exerted due to the combined weight of the block and cylinder, adjusted for the downward acceleration.
1. Given Data: - Mass of wooden block (m_1) = 5 kg - Mass of iron cylinder (m_2) = 25 kg - Combined mass (m) = $m_1 + m_2 = 5 + 25 = 30\,\text{kg}$ - Acceleration due to gravity (g) = $9.8\,\text{ms}^{-2}$ - Downward acceleration (a) = $0.1\,\text{ms}^{-2}$
2. Net Force on the Floor (Action Force): The total force exerted by the system on the floor is the apparent weight, given by:

$$F = m(g - a),$$

where $g - a$ accounts for the reduced effective gravitational acceleration due to the downward motion.
Substituting the values:

$$F = 30(9.8 - 0.1) = 30 \times 9.7 = 291\,\text{N}.$$

3. Conclusion: The action force exerted by the system on the floor is $291\,\text{N}$, which corresponds to Option (3).

Quick Tip

When an object is accelerating downwards, the effective weight reduces by a factor proportional to the acceleration. Use $F = m(g - a)$ to calculate the apparent force in such cases.

Question 45: A simple pendulum doing small oscillations at a place R height above Earth's surface has time period of $T_1 = 4\,\text{s}$. T_2 would be its time period if it is brought to a point which is at a height $2R$ from Earth's surface. Choose the correct relation [R = radius of Earth]:

Options:
(1) $T_1 = T_2$
(2) $2T_1 = 3T_2$
(3) $3T_1 = 2T_2$
(4) $2T_1 = T_2$

Correct Answer: (3): $3T_1 = 2T_2$

Solution:
1. Time Period of a Simple Pendulum: The time period of a pendulum depends on the acceleration due to gravity g at the location:

$$T \propto \frac{1}{\sqrt{g}}.$$

2. Variation of g with Height: The acceleration due to gravity g at a height h above the Earth's surface is:

$$g_h = g_0 \left(\frac{R}{R+h}\right)^2,$$

where g_0 is the acceleration due to gravity at the Earth's surface, R is the radius of the Earth, and h is the height above the surface.
3. Relating T_1 and T_2: For height $h = R$ (corresponding to T_1):

$$g_R = g_0 \left(\frac{R}{R+R}\right)^2 = g_0 \left(\frac{1}{2}\right)^2 = \frac{g_0}{4}.$$

For height $h = 2R$ (corresponding to T_2):

$$g_{2R} = g_0 \left(\frac{R}{R+2R}\right)^2 = g_0 \left(\frac{1}{3}\right)^2 = \frac{g_0}{9}.$$

The time period is inversely proportional to $\sqrt{g}$:

$$T_1 \propto \frac{1}{\sqrt{g_R}} \quad \text{and} \quad T_2 \propto \frac{1}{\sqrt{g_{2R}}}.$$

Taking the ratio of T_1 and T_2:

$$\frac{T_1}{T_2} = \sqrt{\frac{g_{2R}}{g_R}} = \sqrt{\frac{\frac{g_0}{9}}{\frac{g_0}{4}}} = \sqrt{\frac{4}{9}} = \frac{2}{3}.$$

Therefore:

$$T_1 = \frac{2}{3}T_2 \quad \Longrightarrow \quad 3T_1 = 2T_2.$$

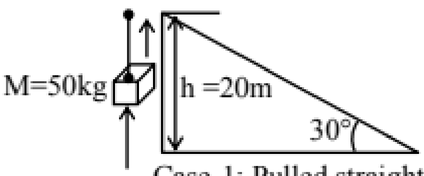
4. Conclusion: The correct relation is:

$$3T_1 = 2T_2.$$

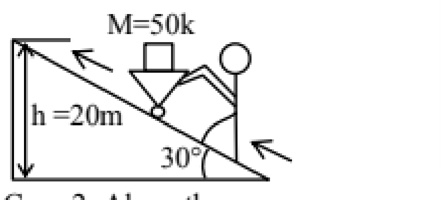
Quick Tip

The time period of a pendulum increases with altitude because the acceleration due to gravity decreases with height. Use the relationship $g_h = g_0 \left(\frac{R}{R+h}\right)^2$ to compute g at different heights.

Question 46: A body of mass 50 kg is lifted to a height of 20 m from the ground in two different ways as shown in the figures. The ratio of work done against gravity in both respective cases will be:



Case-1: Pulled straight up



Case-2: Along the ramp

Options:
(1) 1 : 1
(2) 2 : 1
(3) $\sqrt{3}$: 2
(4) 1 : 2

Correct Answer: (1): 1 : 1

Solution:
The work done against gravity depends only on the change in gravitational potential energy and is independent of the path taken. For both cases, we calculate the work done:
1. Case 1: Vertical Lifting The body is lifted directly upward to a height $h = 20\,\text{m}$. The work done is:

$$W_1 = mgh,$$

where: - $m = 50\,\text{kg}$ (mass of the body), - $g = 9.8\,\text{ms}^{-2}$ (acceleration due to gravity), - $h = 20\,\text{m}$ (height). Substituting the values:

$$W_1 = 50 \times 9.8 \times 20 = 9800\,\text{J}.$$

2. Case 2: Lifting Along the Ramp The body is moved along the inclined plane to the same height $h = 20\,\text{m}$. The work done against gravity is:

$$W_2 = mgh,$$

where h is the vertical height gained. Substituting the values:

$$W_2 = 50 \times 9.8 \times 20 = 9800\,\text{J}.$$

3. Ratio of Work Done: The work done in both cases is equal since both involve lifting the body to the same height against gravity:

$$\frac{W_1}{W_2} = \frac{9800}{9800} = 1 : 1.$$

Conclusion: The ratio of work done in the two cases is 1 : 1, corresponding to Option (1).

Quick Tip

Work done against gravity depends only on the vertical height gained, not the path taken. Whether lifted vertically or along an inclined plane, the work done is the same as long as the vertical displacement remains the same.

Question 47: Time periods of oscillation of the same simple pendulum measured using four different measuring clocks were recorded as 4.62 s, 4.632 s, 4.6 s, and 4.64 s. The arithmetic mean of these readings in correct significant figures is:

Options:
(1) 4.623 s
(2) 4.62 s
(3) 4.6 s
(4) 5 s

Correct Answer: (3): 4.6 s

Solution:
1. Arithmetic Mean Calculation:
The arithmetic mean is calculated as:

$$\text{Mean} = \frac{\text{Sum of all observations}}{\text{Number of observations}}.$$

Substituting the given values:

$$\text{Sum} = 4.62 + 4.632 + 4.6 + 4.64 = 18.4.$$

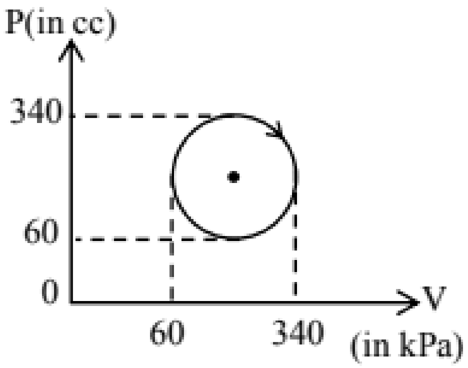
The number of observations is 4. Thus:

$$\text{Mean} = \frac{18.4}{4} = 4.6\,\text{s}.$$

Quick Tip

Always round the final result to the same precision as the least precise measurement in the data set. Significant figures are crucial for ensuring accurate reporting of results.

Question 48: The heat absorbed by a system in going through the given cyclic process is:



Options:
(1) 61.6 J
(2) 431.2 J
(3) 616 J
(4) 19.6 J

Correct Answer: (1): 61.6 J

Solution:
1. Heat Absorbed in a Cyclic Process: For a cyclic process, the change in internal energy (ΔU) is zero. According to the first law of thermodynamics:
$\Delta U = q + w$
where:
q is the heat absorbed by the system. w is the work done on the system.
Since $\Delta U = 0$ for a cyclic process: $q = -w$ For a cyclic process, the magnitude of the total work done is equal to the area of the cycle and the total heat absorbed by the system will be equal to the amount of work done by the system.

2. Work Done (Area of the Circle):
The given cyclic process is represented by a circle on a P-V diagram. The work done is equal to the area enclosed by the circle:

$$w = -\pi r^2$$

(Work done is negative as the cycle is clockwise, meaning work is done *by* the system, not on it). Here r represents radius of the circle which represents the cyclic process, in the given figure the radius can be calculated as follows:

$$r = \frac{340 - 60}{2}\ \text{cc} = 140\ \text{cc}$$

Also the y-axis represents pressure, the units are cc which is cm^3 i.e. 1cc = 1 cm^3 = $10^{-6} m^3$ and the x-axis represent volume in kPa. Thus, work done is given by,

$$w = -\pi(140 \times 10^{-6} m^3)^2 (180 \times 10^3\ \text{Pa}) \approx -0.0616\ \text{J}$$

$$\text{Area} = \pi r^2 = \pi(\frac{340 - 60}{2})^2 = \pi(140)^2 = 19600\pi \approx 61575\ \text{kPa} \cdot \text{cc (approx.)}$$

3. Converting Units for Work Done: We have the area in kPacc. To convert this to Joules:
1 kPacc = 10^3Pa × 10^{-6}m^3 = 10^{-3}J
Work done (w) $\approx -61575 \times 10^{-3}$ J ≈ -61.6 J

4. Heat Absorbed (q):
q = -w ≈ 61.6 J.

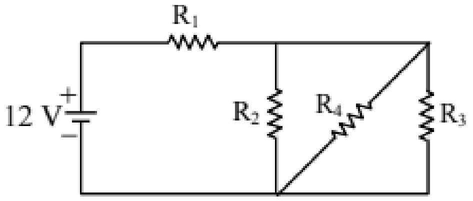
5. Conclusion: The heat absorbed by the system is approximately 61.6 J, so option (1) is the correct answer.

Quick Tip

Always ensure unit conversions when calculating work done in $P - V$ graphs. Work is the enclosed area, so ensure both pressure and volume are in SI units for consistency.

Question 49: In the given figure $R_1 = 10\,\Omega$, $R_2 = 8\,\Omega$, $R_3 = 4\,\Omega$, and $R_4 = 8\,\Omega$. The battery is ideal with an emf of $12\,\text{V}$. The equivalent resistance of the

circuit and the current supplied by the battery are, respectively:



Options:
(1) $12\,\Omega$ and $1.14\,\text{A}$
(2) $10.5\,\Omega$ and $1.14\,\text{A}$
(3) $10.5\,\Omega$ and $1\,\text{A}$
(4) $12\,\Omega$ and $1\,\text{A}$

Correct Answer: (4): $12\,\Omega$ **and** $1\,\text{A}$

Solution:
1. Identify the Resistor Configuration:
- $R_1 = 10\,\Omega$ is in series with the parallel combination of R_2, R_3, and R_4.
2. Equivalent Resistance of Parallel Resistors (R_2, R_3, R_4):
The total resistance of the parallel combination is given by:

$$\frac{1}{R_{\text{parallel}}} = \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4}.$$

Substituting the given values:

$$\frac{1}{R_{\text{parallel}}} = \frac{1}{8} + \frac{1}{4} + \frac{1}{8}.$$

Simplify:

$$\frac{1}{R_{\text{parallel}}} = \frac{1}{8} + \frac{2}{8} + \frac{1}{8} = \frac{4}{8} = \frac{1}{2}.$$

Therefore:

$$R_{\text{parallel}} = 2\,\Omega.$$

3. Total Equivalent Resistance of the Circuit:
The total resistance is the sum of R_1 and R_{parallel}:

$$R_{\text{total}} = R_1 + R_{\text{parallel}} = 10\,\Omega + 2\,\Omega = 12\,\Omega.$$

4. Current Supplied by the Battery:
The current supplied by the battery is given by Ohm's law:

$$I = \frac{\text{Voltage}}{\text{Total Resistance}} = \frac{12\,\text{V}}{12\,\Omega} = 1\,\text{A}.$$

5. Conclusion:
The equivalent resistance of the circuit is $12\,\Omega$, and the current supplied by the battery is $1\,\text{A}$.

💡 Quick Tip

When dealing with circuits, always simplify the parallel and series combinations step by step. Double-check your calculations for equivalent resistance and apply Ohm's law to find the current.

Question 50: An alternating voltage of amplitude 40 V and frequency 4 kHz is applied directly across a capacitor of $12\,\mu F$. The maximum displacement current between the plates of the capacitor is nearly:

Options:
(1) 13 A
(2) 8 A
(3) 10 A
(4) 12 A

Correct Answer: (4): 12 A

Solution:

Current in the diode: $\mathrm{i}_d = i_c = \frac{V}{x_c} = \omega CV$

Substituting values: $i_d = 40 \times 2\pi \times 4 \times 10^3 \times 12 \times 10^{-6}$

Simplifying: $i_d \approx 12A$

Therefore, the correct option is (4).

💡 Quick Tip

The displacement current in a capacitor depends on the angular frequency (ω), capacitance (C), and the maximum voltage ($V_{\max}$). Ensure all units are consistent when applying the formula $I_{\max} = V_{\max} \cdot \omega \cdot C$.

Question 51: In Young's double slit experiment, carried out with light of wavelength 5000Å, the distance between the slits is 0.3 mm and the screen is at 200 cm from the slits. The central maximum is at x = 0 cm. The value of x for the third maxima is . . . mm.

Correct Answer: (10)

Solution:

1. Young's Double Slit Experiment: In Young's double slit experiment, the position of the bright fringes (maxima) is given by:

$$x = \frac{n\lambda D}{d}$$

where:
x is the distance from the central maximum to the nth maximum.
n is the order of the maximum (n = 0 for the central maximum, n = 1 for the first maximum, etc.).
λ is the wavelength of the light.
D is the distance between the slits and the screen.
d is the distance between the slits.

2. Convert Units: * Wavelength (λ) = 5000 Å = 5000×10^{-10} m = 5×10^{-7} m * Distance between slits (d) = 0.3 mm = 0.3×10^{-3} m * Distance to screen (D) = 200 cm = 2 m

3. Calculate x for the Third Maxima (n=3):

$$x = \frac{3 \times (5 \times 10^{-7}\text{ m}) \times 2\text{ m}}{0.3 \times 10^{-3}\text{ m}}$$

$$x = 10 \times 10^{-3}\text{ m} = 10\text{ mm}$$

4. Conclusion: The value of x for the third maxima is 10 mm.

> Quick Tip
>
> Ensure consistent units when using the formula for fringe position in Young's double slit experiment.

Question 52: A 2A current carrying straight metal wire of resistance 1 Ω, resistivity 2×10^{-6} Ωm, area of cross-section 10 mm^2 and mass 500 g is suspended horizontally in mid-air by applying a uniform magnetic field $\vec{B}$. The magnitude of B is . . . $\times 10^{-1}$ T (given, g = 10 m/s^2).

Correct Answer: (5)

Solution:

1. Magnetic Force on a Current-Carrying Wire: The magnetic force (F) on a straight current-carrying wire in a uniform magnetic field is given by:

$$F = BIL \sin\theta$$

where:
B is the magnitude of the magnetic field.
I is the current in the wire.
L is the length of the wire.
θ is the angle between the magnetic field and the direction of the current.

2. Equilibrium Condition: The wire is suspended horizontally in mid-air, meaning the magnetic force balances the gravitational force (weight) of the wire:

$$F_{\text{magnetic}} = F_{\text{gravitational}}$$

$$BIL\sin\theta = mg$$

Since the wire is horizontal and we want the minimum magnitude of B which means the magnetic field must be perpendicular to current. Thus $\theta = 90^\circ$ hence,

$$BIL = mg$$

3. Resistivity and Resistance: The resistance (R) of the wire is related to its resistivity (ρ), length (L), and area of cross-section (A):

$$R = \frac{\rho L}{A}$$

$$L = \frac{RA}{\rho}$$

4. Substitute and Solve for B: Substituting the expression for L into the equilibrium equation:

$$BI\left(\frac{RA}{\rho}\right) = mg$$

$$B = \frac{mg\rho}{IRA}$$

5. Convert Units and Calculate B: * m = 500 g = 0.5 kg * A = 10 mm^2 = 10×10^{-6} m^2 * $\rho = 2 \times 10^{-6}$ Ωm * I = 2 A * R=1 Ω * g = 10 m/s^2

$$B = \frac{(0.5\text{ kg})(10\text{ m/s}^2)(2 \times 10^{-6}\Omega\text{m})}{(2\text{ A})(1\Omega)(10 \times 10^{-6}\text{ m}^2)}$$

$$B = 0.5\text{ T} = 5 \times 10^{-1}\text{ T}$$

6. Conclusion: The magnitude of the magnetic field B is 5×10^{-1} T.

Quick Tip

In problems involving forces and equilibrium, ensure all forces are considered and that units are consistent.

Question 53: The electric field between the two parallel plates of a capacitor of 1.5 μF capacitance drops to one-third of its initial value in 6.6 μs when the plates are connected by a thin wire. The resistance of this wire is . . . Ω. (Given, log 3 = 1.1)

Correct Answer: (4)

Solution:
1. Discharging a Capacitor: When the plates of a charged capacitor are connected by a wire, the capacitor discharges through the wire. The electric field between the plates is proportional to the charge on the plates. The decay of charge (and therefore electric field) follows an exponential decay:

$$E(t) = E_0 e^{-t/RC}$$

where:
$E(t)$ is the electric field at time t.
E_0 is the initial electric field.
t is the time.
R is the resistance of the wire.
C is the capacitance of the capacitor.

2. Given Information: We are given that the electric field drops to one-third of its initial value, so $E(t) = \frac{1}{3}E_0$. The time taken for this to happen is $t = 6.6\,\mu s = 6.6 \times 10^{-6}$ s. The capacitance is $C = 1.5\,\mu F = 1.5 \times 10^{-6}$ F.

3. Set up the Equation: Substituting the given values into the equation for exponential decay:

$$\frac{1}{3}E_0 = E_0 e^{-t/RC}$$
$$\frac{1}{3} = e^{-(6.6\times10^{-6})/(R\times1.5\times10^{-6})}$$

4. Solve for R: Take the natural logarithm of both sides:

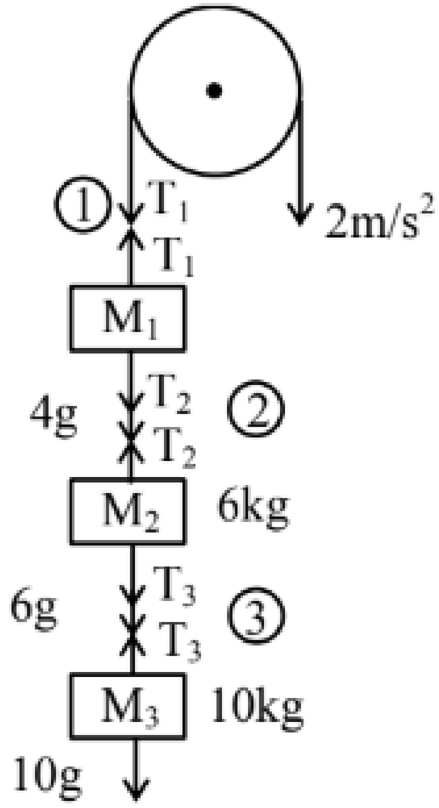
$$\ln\left(\frac{1}{3}\right) = \frac{-6.6 \times 10^{-6}}{R \times 1.5 \times 10^{-6}}$$
$$-\ln 3 = \frac{-6.6}{1.5R}$$
$$-1.0986 \approx \frac{-4.4}{R}$$
$$R \approx \frac{4.4}{1.0986} \approx 4\Omega$$

5. Conclusion: The resistance of the wire is approximately 4 Ω.

Quick Tip

Discharging a capacitor through a resistor follows an exponential decay. The time constant ($\tau = RC$) is the time it takes for the charge (or electric field) to decrease to approximately 37% of its initial value.

Question 54: Three blocks M_1, M_2, M_3 having masses 4 kg, 6 kg, and 10 kg respectively are hanging from a smooth pulley using ropes 1, 2, and 3 as shown in the figure. The tension in the rope 1, T_1, when they are moving upward with acceleration of 2 m/s^2 is . . . N (if g = 10 m/s^2).



Correct Answer: (240)

Solution:

Let's analyze the forces acting on each block. For the system as a whole (masses M_1, M_2, and M_3 together) moving upwards with an acceleration $a = 2m/s^2$

Total mass = 20 kg

Total weight = 200 N

Since the entire system is accelerating upwards, the net force F required to produce this acceleration is given by:

F = Ma = 20 × 2 = 40 N

The weight and the additional force required for acceleration:
$= 200 + 40$
$= 240N$

💡 Quick Tip

When dealing with multiple connected objects, analyze the forces acting on each object individually and consider the relationships between their accelerations. Be mindful of pseudo forces when working in non-inertial frames of reference.

Question 55: The density and breaking stress of a wire are 6×10^4 kg/m^3 and 1.2×10^8 N/m^2 respectively. The wire is suspended from a rigid support on a planet where acceleration due to gravity is $\frac{1}{3}$ of the value on the surface of Earth. The maximum length of the wire with breaking is . . . m (take, g = 10 m/s^2).

Correct Answer: (600)

Solution:

1. Breaking Stress: Breaking stress is the maximum stress a material can withstand before it breaks. It is defined as the force per unit area:

$$\text{Breaking stress} = \frac{\text{Force}}{\text{Area}}$$

2. Force Due to Gravity: The force acting on the wire due to gravity is its weight, which is given by:

$$F = mg$$

where:
m is the mass of the wire.
g is the acceleration due to gravity on the planet.

3. Mass and Density: The mass (m) of the wire can be expressed in terms of its density (ρ), volume (V), and length (L):

$$m = \rho V = \rho AL$$

where A is the cross-sectional area of the wire.

4. Acceleration Due to Gravity on the Planet: The acceleration due to gravity on the planet is $\frac{1}{3}$g, where g is the acceleration due to gravity on Earth (10 m/s^2).

$$g_{\text{planet}} = \frac{1}{3}(10 \text{ m/s}^2) = \frac{10}{3} \text{ m/s}^2$$

5. Set up the Equation and Solve for L: At the breaking point, the breaking stress equals the weight of the wire divided by the cross-sectional area:

$$\text{Breaking stress} = \frac{mg_{\text{planet}}}{A} = \frac{(\rho AL)g_{\text{planet}}}{A}$$
$$\text{Breaking stress} = \rho L g_{\text{planet}}$$

$$L = \frac{\text{Breaking stress}}{\rho g_{\text{planet}}}$$

$$L = \frac{1.2 \times 10^8 \text{ N/m}^2}{(6 \times 10^4 \text{ kg/m}^3)(\frac{10}{3} \text{ m/s}^2)}$$

$$L = \frac{1.2 \times 10^8 \times 3}{6 \times 10^5}$$

$$L = 600 \text{ m}$$

6. Conclusion: The maximum length of the wire before breaking is 600 m.

Quick Tip

Breaking stress is an intrinsic property of a material, independent of its shape or size. However, the maximum length a wire can support before breaking depends on its density and the gravitational field.

Question 56: A body moves on a frictionless plane starting from rest. If S$_n$ is distance moved between t = n − 1 and t = n and S$_{n-1}$ is distance moved between t = n − 2 and t = n − 1, then the ratio $\frac{S_{n-1}}{S_n}$ is $\left(1 - \frac{2}{x}\right)$ for n = 10. The value of x is . . .

Correct Answer: (19)

Solution:

1. Distance and Uniform Acceleration: Since the plane is frictionless, we can assume constant acceleration. The distance traveled by a body starting from rest under uniform acceleration (a) is given by:

$$S = ut + \frac{1}{2}at^2$$

Since the body starts from rest, u = 0, so:

$$S = \frac{1}{2}at^2$$

2. Distance in Given Time Intervals: * S_n: Distance moved between t = n - 1 and t = n.

$$S_n = \frac{1}{2}a(n^2 - (n-1)^2) = \frac{1}{2}a(2n-1)$$

* S_{n-1}: Distance moved between t = n - 2 and t = n - 1.

$$S_{n-1} = \frac{1}{2}a((n-1)^2 - (n-2)^2) = \frac{1}{2}a(2n-3)$$

3. Calculate the Ratio $\frac{S_{n-1}}{S_n}$:

$$\frac{S_{n-1}}{S_n} = \frac{\frac{1}{2}a(2n-3)}{\frac{1}{2}a(2n-1)} = \frac{2n-3}{2n-1}$$

4. Given Information: We are given that n = 10 and the ratio is equal to $1 - \frac{2}{x}$.

5. Solve for x: Substitute n = 10 into the ratio:

$$\frac{2(10)-3}{2(10)-1} = \frac{17}{19} = 1 - \frac{2}{x}$$

$$\frac{2}{x} = 1 - \frac{17}{19} = \frac{2}{19}$$

$$x = 19$$

6. Conclusion: The value of x is 19.

> Quick Tip
>
> Carefully define the distances based on the given time intervals. Remember the equations of motion for uniform acceleration.

Question 57: If three helium nuclei combine to form a carbon nucleus then the energy released in this reaction is . . . $\times 10^{-2}$ MeV. (Given 1 u = 931 MeV/c^2, atomic mass of helium = 4.002603 u)

Correct Answer: (727)

Solution:

1. Nuclear Reaction: The nuclear reaction is:
$3\,{}^4\text{He} \to {}^{12}\text{C}$

2. Mass Defect: The mass defect (Δm) is the difference between the total mass of the reactants and the total mass of the products.
Atomic Mass of ${}^{12}\text{C}$ = 12 u

$\Delta m = (3 \times \text{mass of } ^4\text{He})$ - (mass of ^{12}C) $\Delta m = (3 \times 4.002603$ u) - 12 u $\Delta m = 12.007809$ u - 12 u $\Delta m = 0.007809$ u

3. Energy Released (Einstein's Mass-Energy Equivalence):

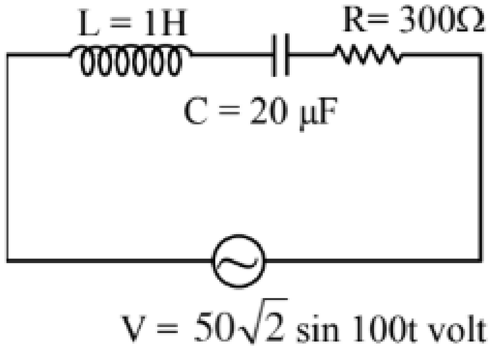
The energy released (E) is related to the mass defect by Einstein's famous equation:
$E = \Delta mc^2$
where c is the speed of light. Using the conversion given that 1u = 931 MeV/c^2: $E = 0.007809 \text{ u} \times \frac{931 \text{ MeV}}{1\text{u c}^2} \times c^2$ $E = 0.007809 \times 931$ MeV $E \approx 7.27$ MeV $= 727 \times 10^{-2}$ MeV

4. Conclusion: The energy released in this reaction is approximately 727×10^{-2} MeV.

Quick Tip

In nuclear reactions, the mass defect is converted into energy. Einstein's equation, E = mc^2, relates the mass defect to the energy released.

Question 58: An ac source is connected in a given series LCR circuit. The rms potential difference across the capacitor of 20 μF is . . . V.



Correct Answer: (50)

Solution:

1. RMS Voltage of the Source: The given ac source voltage is $V = 50\sqrt{2}\sin 100t$ volts. The peak voltage (V$_0$) is $50\sqrt{2}$ volts. The RMS voltage (V$_{\text{rms}}$) is given by:

$$V_{\text{rms}} = \frac{V_0}{\sqrt{2}}$$

$$V_{\text{rms}} = \frac{50\sqrt{2}}{\sqrt{2}} = 50 \text{ V}$$

2. Angular Frequency (ω): The angular frequency (ω) of the ac source is given by the coefficient of t in the sine function:

$$\omega = 100 \text{ rad/s}$$

3. Capacitive Reactance ($\mathbf{X_C}$): The capacitive reactance (X_C) is given by:

$$X_C = \frac{1}{\omega C} = \frac{1}{(100 \text{ rad/s})(20 \times 10^{-6} \text{ F})} = 500\,\Omega$$

4. Impedance (Z): The impedance (Z) of the series LCR circuit is:

$$Z = \sqrt{R^2 + (X_L - X_C)^2}$$

where $X_L = \omega L$ is the inductive reactance.

$$X_L = (100 \text{ rad/s})(1 \text{ H}) = 100\,\Omega$$

$$Z = \sqrt{(300)^2 + (100 - 500)^2} = \sqrt{90000 + 160000} = \sqrt{250000} = 500\,\Omega$$

5. RMS Current ($\mathbf{I_{rms}}$):

$$I_{\text{rms}} = \frac{V_{\text{rms}}}{Z} = \frac{50 \text{ V}}{500\,\Omega} = 0.1 \text{ A}$$

6. RMS Voltage across Capacitor ($\mathbf{V_C}$):

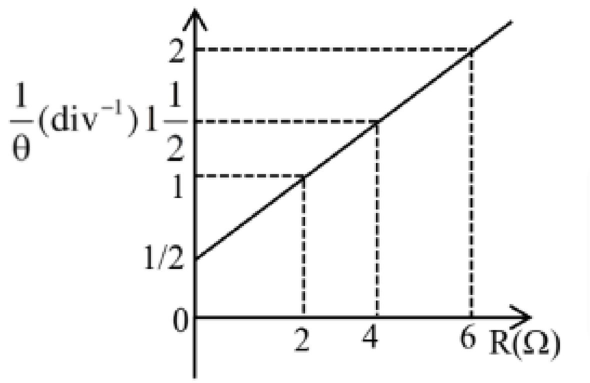
$$V_C = I_{\text{rms}} X_C = (0.1 \text{ A})(500\,\Omega) = 50 \text{ V}$$

7. Conclusion: The RMS potential difference across the capacitor is 50 V.

Quick Tip

In ac circuits, impedance plays the role of resistance in dc circuits. The RMS values of current and voltage are used to calculate power and other quantities.

Question 59: In the experiment to determine the galvanometer resistance by the half-deflection method, the plot of $\frac{1}{\theta}$ vs the resistance (R) of the resistance box is shown in the figure. The figure of merit of the galvanometer is . . . $\times 10^{-1}$ A/division. [The source has emf 2V]



Correct Answer: (5)

Solution:

1. Half-Deflection Method and Figure of Merit:
In the half-deflection method, we use the following relationship:

$$G = R - S$$

where G is the galvanometer resistance, R is the resistance for half deflection and S is the shunt resistance. Also, the figure of merit is given by
$k = \frac{E}{(R+G)\theta}$
When $R = 0$, let deflection be θ_0, $\frac{1}{\theta_0} = \frac{G}{Ek} = 0.5$

When $R = R$, and $\theta = \theta_0/2$, $\frac{1}{\theta_0/2} = \frac{R+G}{Ek} = 2\frac{G}{Ek} + \frac{R}{Ek} = 1$

Thus, $1 = 1 + \frac{R}{Ek}$ $R = 2$.

Thus $k = \frac{E}{2G}$, $Ek = 1$, thus $G = 1$.

Since $G = R - S$, where R is resistance for half deflection, $1 = 2 - S$, $S = 1$.

Also from the graph, the slope is given by

$$\frac{2 - \frac{1}{2}}{6 - 0} = \frac{\frac{3}{2}}{6} = \frac{1}{4} = \frac{1}{Ek}$$

Since $E = 2$, thus $k = 0.5$. Thus, figure of merit is 5×10^{-1}.

2. Finding the slope and Figure of merit:

The graph provided is of $1/\theta$ vs R and represents the linear equation: $1/\theta = (1/Ek)R + G/(Ek)$ where E is emf, k is figure of merit and G is galvanometer resistance.

From the graph, the slope is $(2 - 1/2)/(6 - 0) = 1/4 = 1/(Ek)$.

Since emf is given as 2V, thus, k = 0.5. Thus the figure of merit is 5×10^{-1}.

3. Conclusion: The figure of merit of the galvanometer is 5×10^{-1} A/division.

Quick Tip

The half-deflection method and the figure of merit are important concepts in determining the characteristics of a galvanometer.

Question 60: Three capacitors of capacitances 25 μF, 30 μF, and 45 μF are connected in parallel to a supply of 100 V. Energy stored in the above combination is E. When these capacitors are connected in series to the same supply, the stored energy is $\frac{9}{x}$E. The value of x is . . .

Correct Answer: (86)

Solution:

1. Energy Stored in a Capacitor: The energy (E) stored in a capacitor is given by:

$$E = \frac{1}{2}CV^2$$

where:
C is the capacitance.
V is the voltage across the capacitor.

2. Parallel Combination: When capacitors are connected in parallel, the equivalent capacitance (C_p) is the sum of the individual capacitances:

$$C_p = C_1 + C_2 + C_3$$

$$C_p = 25\,\mu\text{F} + 30\,\mu\text{F} + 45\,\mu\text{F} = 100\,\mu\text{F}$$

The energy stored in the parallel combination (E_p) is:

$$E_p = \frac{1}{2}C_pV^2 = \frac{1}{2}(100 \times 10^{-6}\text{ F})(100\text{ V})^2 = 0.5\text{ J} = E$$

3. Series Combination: When capacitors are connected in series, the reciprocal of the equivalent capacitance (C_s) is the sum of the reciprocals of the individual capacitances:

$$\frac{1}{C_s} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}$$

$$\frac{1}{C_s} = \frac{1}{25} + \frac{1}{30} + \frac{1}{45} = \frac{18 + 15 + 10}{450} = \frac{43}{450}\,\mu\text{F}^{-1}$$

$$C_s = \frac{450}{43}\,\mu\text{F}$$

The energy stored in the series combination (E_s) is:

$$E_s = \frac{1}{2}C_sV^2 = \frac{1}{2}\left(\frac{450}{43}\times 10^{-6}\text{ F}\right)(100\text{ V})^2 = \frac{225}{43}\times 10^{-2}\text{ J}$$

4. Relationship between $\mathbf{E_p}$ and $\mathbf{E_s}$: We are given that $E_s = \frac{9}{x}E$, and we know that $E_p = E$.

$$\frac{9}{x}E = E_s$$

$$\frac{9}{x}(0.5) = \frac{225}{43}\times 10^{-2}$$

$$\frac{9}{x} = \frac{450}{43}\times 10^{-2} = \frac{4.5}{43}$$

$$x = \frac{9\times 43}{4.5} = 2\times 43 = 86$$

5. Conclusion: The value of x is 86.

Quick Tip

Remember the formulas for equivalent capacitance and energy stored for both parallel and series combinations of capacitors.

Chemistry

Question 61: The incorrect postulates of Dalton's atomic theory are:

(A) Atoms of different elements differ in mass.
(B) Matter consists of divisible atoms.
(C) Compounds are formed when atoms of different elements combine in a fixed ratio.
(D) All the atoms of a given element have different properties including mass.
(E) Chemical reactions involve the reorganization of atoms.

Choose the correct answer from the options given below:
Options:
(1) (B), (D), (E) only
(2) (A), (B), (D) only
(3) (C), (D), (E) only
(4) (B), (D) only

Correct Answer: (4) (B), (D) only

Solution:
Let's analyze each statement based on Dalton's atomic theory and modern atomic theory.
1. Analysis of Statement (A): Dalton's theory postulates that atoms of different elements have different masses. This is a *correct* statement and consistent with modern atomic theory.
2. Analysis of Statement (B): Dalton's theory postulates that atoms are indivisible. This is *incorrect* according to modern atomic theory, as atoms are divisible into subatomic particles (protons, neutrons, and electrons).
3. Analysis of Statement (C): Dalton's theory postulates that compounds are formed by the combination of atoms of different elements in fixed ratios. This is a *correct* statement and a fundamental principle of chemistry.
4. Analysis of Statement (D): Dalton's theory postulates that all atoms of a given element are identical in mass and other properties. This is *incorrect* according to modern atomic theory due to the existence of isotopes. Isotopes are atoms of the same element with the same number of protons but a different number of neutrons, leading to different masses.
5. Analysis of Statement (E): Dalton's theory postulates that chemical reactions involve the rearrangement of atoms. This is a *correct* statement. Chemical reactions involve the breaking and forming of bonds, which essentially involves the reorganization of atoms.
6. Conclusion:
The incorrect postulates are (B) and (D). Therefore, the correct option is (4).

Quick Tip

Remember that Dalton's atomic theory was a groundbreaking development in its time, but it has been refined with the advancement of scientific understanding, particularly with the discovery of subatomic particles and isotopes.

Question 62: The following reaction occurs in the Blast furnace where iron ore is reduced to iron metal:

$$\mathrm{Fe_2O_3}(s) + 3\mathrm{CO}(g) \rightleftharpoons \mathrm{Fe}(l) + 3\mathrm{CO_2}(g)$$

Using Le Chatelier's principle, predict which one of the following will *not* disturb the equilibrium.

Options:
(1) Addition of Fe_2O_3
(2) Addition of CO_2
(3) Removal of CO
(4) Removal of CO_2

Correct Answer: (1)

Solution:
Le Chatelier's principle states that if a change of condition is applied to a system at equilibrium, the system will shift in a direction that relieves the stress. Let's analyze the given reaction and the effect of each option:

1. Addition of Fe_2O_3:
Fe_2O_3 is a solid reactant. The concentration (or more accurately, activity) of pure solids and liquids does not change during a reaction. Therefore, adding more Fe_2O_3 will not affect the equilibrium position. The reaction quotient Q remains unchanged, and the system remains at equilibrium.

2. Addition of CO_2:
CO_2 is a gaseous product. Adding CO_2 increases the concentration of products. According to Le Chatelier's principle, the system will shift to the left (towards reactants) to counteract the increased CO_2 concentration and re-establish equilibrium. Thus, adding CO_2 *will* disturb the equilibrium.

3. Removal of CO:
CO is a gaseous reactant. Removing CO decreases the concentration of reactants. According to Le Chatelier's principle, the system will shift to the left (towards reactants) to counteract the decreased CO concentration. Thus, removing CO *will* disturb the equilibrium.

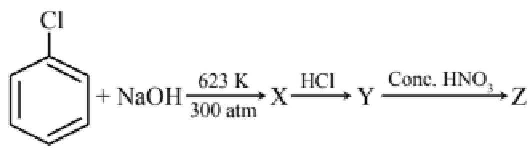
4. Removal of CO_2:
CO_2 is a gaseous product. Removing CO_2 decreases the concentration of products. According to Le Chatelier's principle, the system will shift to the right (towards products) to counteract the decreased CO_2 concentration. Thus, removing CO_2 *will* disturb the equilibrium.

5. Conclusion:
Only the addition of Fe_2O_3 will not disturb the equilibrium because the concentration of pure solids does not affect the equilibrium constant.

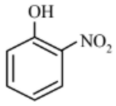
Quick Tip

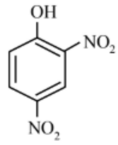
Remember that pure solids and liquids, and solvents in dilute solutions, are not included in the equilibrium constant expression, and therefore, changes in their amounts do not shift the equilibrium position.

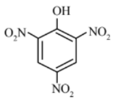
Question 63: Identify compound (Z) in the following reaction sequence.

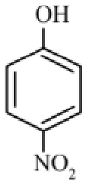


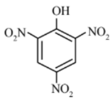
Options:

(1) 

(2) 

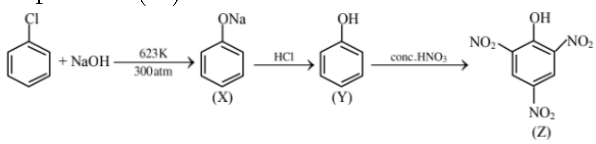
(3) 

(4) 

Correct Answer: (3) 

Solution:

1. Formation of X:
The first step involves the reaction of chlorobenzene with NaOH at high temperature (623 K) and pressure (300 atm). This is the Dow process, which leads to the formation of phenol (X).



2. Formation of Y:
Phenol (X) reacts with HCl. This reaction doesn't cause any significant change to the phenol structure as phenol is a weak base and HCl is a strong acid. Phenol will remain largely unreacted (Y). This step appears to be a distraction.

$\mathrm{HOC}_6H_5(X) \xrightarrow{\mathrm{HCl}} \mathrm{HOC}_6H_5(Y)$

3. Formation of Z:
Phenol (Y) reacts with concentrated HNO_3. This leads to nitration. Phenol is highly activating towards electrophilic aromatic substitution, and the hydroxyl group directs the incoming nitro groups to the ortho and para positions. With concentrated nitric acid, multiple nitrations can occur, leading to the formation of 2,4,6-trinitrophenol, also known as picric acid (Z).

$\mathrm{HOC_6}H_5(Y) \xrightarrow{\text{Conc. HNO}_3} \mathrm{C_6}H_2(OH) - [1](NO_2) - [2,4,6](Z)$

4. Conclusion:
The compound (Z) is 2,4,6-trinitrophenol (picric acid), which corresponds to option (3).

> **Quick Tip**
>
> Remember the Dow process for phenol synthesis and the activating nature of the hydroxyl group in electrophilic aromatic substitution reactions. Concentrated nitric acid often leads to multiple nitrations.

Question 64: Given below are two statements: One is labelled as Assertion (A) and the other is labelled as Reason (R)
Assertion (A): Enthalpy of neutralisation of strong monobasic acid with strong monoacidic base is always -57 kJ mol^{-1}.
Reason (R): Enthalpy of neutralisation is the amount of heat liberated when one mole of H^+ ions furnished by acid combine with one mole of ^-OH ions furnished by base to form one mole of water.
In the light of the above statements, choose the correct answer from the options given below.

Options:
(1) (A) is true but (R) is false
(2) Both (A) and (R) are true and (R) is the correct explanation of (A)
(3) (A) is false but (R) is true
(4) Both (A) and (R) are true but (R) is not the correct explanation of (A)

Correct Answer: (2) Both (A) and (R) are true and (R) is the correct explanation of (A)

Solution:

1. Analysis of Assertion (A):
The enthalpy of neutralization is the heat released when one mole of water is formed by the reaction of an acid and a base. For *strong* monobasic acids and *strong* monoacidic bases, the enthalpy of neutralization is approximately constant at -57 kJ mol^{-1}. This is because strong acids and strong bases are completely dissociated in water. So the reaction is always simply:
$H^+(aq) + OH^-(aq) \rightarrow H_2O(l)$
Therefore, the assertion (A) is *true*.

2. Analysis of Reason (R):
The reason (R) correctly defines the enthalpy of neutralization as the heat liberated when one mole of H^+ ions reacts with one mole of OH^- ions to form one mole of water.

Therefore, the reason (R) is *true*.

3. Relationship between (A) and (R):
The reason (R) explains why the enthalpy of neutralization is constant for strong acids and strong bases. The consistent value of -57 kJ mol^{-1} arises because the reaction is always the same (formation of water from H^+ and OH^-), regardless of the specific strong acid or strong base used. Thus, (R) is the correct explanation for (A).

4. Conclusion:
Both (A) and (R) are true, and (R) is the correct explanation of (A). Therefore, the correct option is (2).

Quick Tip

The enthalpy of neutralization is only approximately constant for strong acids and bases. Slight variations can occur due to differences in ionic strength and other factors. For weak acids or bases, the enthalpy of neutralization will be different because energy is also involved in the ionization process.

Question 65: The statement(s) that are correct about the species O^{2-}, F^-, Na^+, and Mg^{2+}:

(A) All are isoelectronic
(B) All have the same nuclear charge
(C) O^{2-} has the largest ionic radii
(D) Mg^{2+} has the smallest ionic radii

Choose the most appropriate answer from the options given below:
Options:
(1) (B), (C) and (D) only
(2) (A), (B), (C) and (D)
(3) (C) and (D) only
(4) (A), (C) and (D) only

Correct Answer: (4) (A), (C) and (D) only

Solution:

1. Analysis of (A) - Isoelectronic: Isoelectronic species have the same number of electrons. Let's determine the number of electrons in each species:

O^{2-}: Oxygen (O) has 8 electrons. O^{2-} has gained 2 electrons, so it has $8 + 2 = 10$ electrons.
F^-: Fluorine (F) has 9 electrons. F^- has gained 1 electron, so it has $9 + 1 = 10$ electrons.

Na^+: Sodium (Na) has 11 electrons. Na^+ has lost 1 electron, so it has 11 - 1 = 10 electrons.
Mg^{2+}: Magnesium (Mg) has 12 electrons. Mg^{2+} has lost 2 electrons, so it has 12 - 2 = 10 electrons.

All four species have 10 electrons, so they are isoelectronic. Thus, statement (A) is *true*.

2. Analysis of (B) - Nuclear Charge: Nuclear charge is the number of protons in the nucleus, which is equal to the atomic number.

O: Atomic number 8
F: Atomic number 9
Na: Atomic number 11
Mg: Atomic number 12

The species have different nuclear charges. Thus, statement (B) is *false*.

3. Analysis of (C) and (D) - Ionic Radii: For isoelectronic species, the ionic radius decreases as the nuclear charge increases. Since O^{2-} has the smallest nuclear charge (8) among the given species, it has the largest ionic radius. Similarly, Mg^{2+} has the largest nuclear charge (12), so it has the smallest ionic radius. Thus, both statements (C) and (D) are *true*.

4. Conclusion:
Statements (A), (C), and (D) are correct. Therefore, the correct option is (4).

> 💡 Quick Tip
>
> For isoelectronic species, the ionic radius is inversely proportional to the nuclear charge. Higher nuclear charge pulls the electrons in more tightly, resulting in a smaller ionic radius.

Question 66: For the compounds:

(A) $H_3C{-}CH_2{-}O{-}CH_2{-}CH_2{-}CH_3$

(B) $H_3C{-}CH_2{-}CH_2{-}CH_2{-}CH_3$

(C) $CH_3{-}CH_2{-}\underset{\underset{O}{\parallel}}{C}{-}CH_2{-}CH_3$

(D) $H_3C{-}\underset{\underset{OH}{|}}{CH}{-}CH_2{-}CH_2{-}CH_3$

The increasing order of boiling point is:
Choose the correct answer from the options given below:

Options:

(1) (A) $<$ (B) $<$ (C) $<$ (D)
(2) (B) $<$ (A) $<$ (C) $<$ (D)
(3) (D) $<$ (C) $<$ (A) $<$ (B)
(4) (B) $<$ (A) $<$ (D) $<$ (C)

Correct Answer: (2) (B) $<$ (A) $<$ (C) $<$ (D)

Solution:

Boiling points are influenced primarily by intermolecular forces. Stronger intermolecular forces lead to higher boiling points. Let's analyze the intermolecular forces in each compound:

1. Compound (B) - Pentane:
Pentane is a nonpolar hydrocarbon. The only intermolecular forces present are weak London Dispersion Forces (LDFs).

2. Compound (A) - Ethyl propyl ether:
Ethyl propyl ether has an oxygen atom, which creates a dipole moment. This leads to slightly stronger dipole-dipole interactions in addition to LDFs. Dipole-dipole interactions are generally stronger than LDFs, so (A) will have a higher boiling point than (B).

3. Compound (D) - 2-Pentanol:
2-Pentanol has a hydroxyl (-OH) group, which can participate in hydrogen bonding. Hydrogen bonding is a significantly stronger intermolecular force than dipole-dipole interactions or LDFs. So, (D) will have a higher boiling point than (A) and (B).

4. Compound (C) - 3-Pentanone:
3-Pentanone has a carbonyl (C=O) group, which creates a strong dipole moment. This results in strong dipole-dipole interactions. These dipole-dipole interactions in a ketone are stronger than the dipole-dipole interactions in an ether. Although hydrogen bonding is the strongest intermolecular force here, the ketone has a higher molecular weight than the alcohol. In this case, the stronger dipole-dipole interactions and higher molecular weight lead to a higher boiling point for the ketone (C) compared to the alcohol (D).

5. Conclusion:
The increasing order of boiling points is (B) $<$ (A) $<$ (C) $<$ (D), which corresponds to option (2). Although the alcohol can hydrogen bond, the ketone has a higher molecular weight and stronger dipole-dipole interactions, resulting in a slightly higher boiling point.

💡 Quick Tip

When comparing boiling points, consider the types and strengths of intermolecular forces present. Hydrogen bonding ¿ dipole-dipole interactions ¿ London Dispersion Forces. Also, consider molecular weight; larger molecules generally have higher boiling points due to increased LDFs. Sometimes the effect of stronger intermolecular forces can be overshadowed by a substantial difference in molecular weight.

Question 67: Given below are two statements:
Statement I: In group 13, the stability of +1 oxidation state increases down the group.
Statement II: The atomic size of gallium is greater than that of aluminium.
In the light of the above statements, choose the most appropriate answer from the options given below:

Options:
(1) Statement I is incorrect but Statement II is correct
(2) Both Statement I and Statement II are correct
(3) Both Statement I and Statement II are incorrect
(4) Statement I is correct but Statement II is incorrect

Correct Answer: (4) Statement I is correct but Statement II is incorrect

Solution:

1. Analysis of Statement I: Group 13 elements have the electronic configuration ns^2np^1. They can exhibit a +3 oxidation state by losing all three valence electrons or a +1 oxidation state by losing only the p electron. Down the group, the stability of the +1 oxidation state increases due to the inert pair effect. The inert pair effect refers to the reluctance of the s-electrons to participate in chemical bonding. This effect becomes more pronounced as we move down the group due to increased shielding and relativistic effects. Therefore, statement I is *correct*.

2. Analysis of Statement II: Atomic size generally increases down a group in the periodic table due to the addition of new electron shells. Gallium (Ga) is below aluminum (Al) in group 13, so gallium should have a larger atomic size than aluminum. However, there is an exception in this case. Due to the poor shielding effect of the d-electrons in Gallium, the effective nuclear charge is higher for Ga than expected, resulting in a smaller atomic size for Ga compared to Al. Therefore, statement II is *incorrect*.

3. Conclusion:
Statement I is correct, but Statement II is incorrect. Therefore, the correct option is (2).

> Quick Tip
>
> Remember the inert pair effect for the stability of +1 oxidation states in heavier p-block elements. Also, be aware of exceptions to periodic trends, like the smaller-than-expected size of gallium due to d-electron shielding.

Question 68: Number of σ and π bonds present in ethylene molecule is respectively:

Options:
(1) 3 and 1
(2) 5 and 2
(3) 4 and 1
(4) 5 and 1

Correct Answer: (4) 5 and 1

Solution:

1. Structure of Ethylene: Ethylene (C_2H_4) has the following structure:
$H_2C = CH_2$

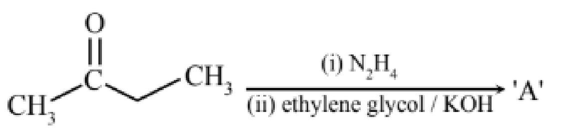
2. Sigma (σ) Bonds: A sigma bond is formed by the direct head-on overlap of atomic orbitals. In ethylene:
* Each C-H bond is a sigma bond (4 sigma bonds total). * The C-C bond consists of one sigma bond.
This gives a total of 5 sigma bonds.

3. Pi (π) Bonds: A pi bond is formed by the sideways overlap of p orbitals. A double bond consists of one sigma bond and one pi bond. In ethylene, the C=C double bond contains one pi bond.

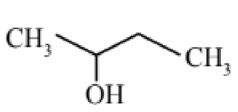
4. Conclusion: Ethylene has 5 sigma (σ) bonds and 1 pi (π) bond. Thus, the correct option is (4).

> Quick Tip
>
> A single bond is always a sigma bond. A double bond consists of one sigma and one pi bond. A triple bond consists of one sigma and two pi bonds.

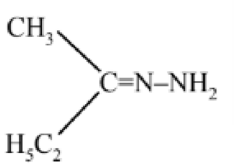
Question 69: Identify 'A' in the following reaction:

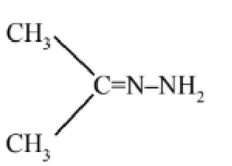


Options:

(1) 

(2) 

(3) 

(4) 

Correct Answer: (2) 

Solution:

1. Reaction with Hydrazine (N_2H_4):
The first step involves the reaction of acetone (propanone) with hydrazine (N_2H_4). This is the Wolff-Kishner reduction's first stage, forming a hydrazone.
$\mathrm{CH_3}C(=O)CH_3 + H_2N - NH_2 \rightarrow \mathrm{CH_3}C(=N - NH_2)CH_3 + H_2\mathrm{O}$

2. Reaction with Ethylene Glycol and KOH:
The second step involves heating the hydrazone with ethylene glycol and KOH. This completes the Wolff-Kishner reduction, converting the carbonyl group (C=O) to a methylene group (CH_2). Ethylene glycol serves as a high-boiling solvent, and KOH acts as a base.

$\mathrm{CH_3}C(=N - NH_2)CH_3 \xrightarrow{\text{ethylene glycol / KOH}} \mathrm{CH_3}CH_2CH_3 + N_2$

3. Conclusion:
The product 'A' is propane ($\mathrm{CH_3}CH_2CH_3$), *which corresponds to option*(2).

💡 Quick Tip

The Wolff-Kishner reduction is a useful method for converting carbonyl groups (aldehydes and ketones) to methylene groups (CH_2). Remember that it requires strong base and high temperatures.

Question 70: The reaction at cathode in the cells commonly used in clocks involves.

Options:
(1) reduction of Mn from +4 to +3
(2) oxidation of Mn from +3 to +4
(3) reduction of Mn from +7 to +2
(4) oxidation of Mn from +2 to +7

Correct Answer: (1) reduction of Mn from +4 to +3

Solution:

Reduction occurs at the cathode. This means the oxidation state of the species being reduced decreases. The only option that describes a reduction of manganese is (1), where Mn goes from a +4 oxidation state to a +3 oxidation state. Clocks often use alkaline batteries that contain MnO_2 (where Mn is +4). The cathode reaction involves the reduction of MnO_2 to MnO(OH) (where Mn is +3).

Specifically, the cathode reaction is:
$MnO_2(s) + H_2O(l) + e^- \rightarrow MnO(OH)(s) + OH^-(aq)$

Conclusion: The correct answer is (1).

💡 Quick Tip

Remember the mnemonic "Red Cat" and "An Ox": **Red**uction occurs at the **Cat**hode, and **Ox**idation occurs at the **An**ode.

Question 71: Which one of the following complexes will exhibit the least paramagnetic behaviour?
[Atomic number, Cr = 24, Mn = 25, Fe = 26, Co = 27]

Options:
(1) $[Co(H_2O)_6]^{2+}$
(2) $[Fe(H_2O)_6]^{2+}$
(3) $[Mn(H_2O)_6]^{2+}$

(4) $[Cr(H_2O)_6]^{2+}$

Correct Answer: (1) $[Co(H_2O)_6]^{2+}$

Solution:

Paramagnetism arises from unpaired electrons. The greater the number of unpaired electrons, the stronger the paramagnetism. H_2O is a weak field ligand, so it does not cause pairing of electrons. We need to determine the number of unpaired d-electrons in each metal ion:

1. $[Co(H_2O)_6]^{2+}$: Co^{2+} has the electronic configuration [Ar] $3d^7$. In a weak field, there are 3 unpaired electrons.

2. $[Fe(H_2O)_6]^{2+}$: Fe^{2+} has the electronic configuration [Ar] $3d^6$. In a weak field, there are 4 unpaired electrons.

3. $[Mn(H_2O)_6]^{2+}$: Mn^{2+} has the electronic configuration [Ar] $3d^5$. In a weak field, there are 5 unpaired electrons.

4. $[Cr(H_2O)_6]^{2+}$: Cr^{2+} has the electronic configuration [Ar] $3d^4$. In a weak field, there are 4 unpaired electrons.

Conclusion: $[Co(H_2O)_6]^{2+}$ has the fewest unpaired electrons (3), so it exhibits the least paramagnetic behavior. Therefore, the correct option is (1).

> Quick Tip
>
> Weak field ligands lead to high-spin complexes with maximum unpaired electrons. Strong field ligands lead to low-spin complexes with minimum unpaired electrons.

Question 72: Given below are two statements: one is labelled as Assertion (A) and the other is labelled as Reason (R).
Assertion (A): *Cis* form of alkene is found to be more polar than the *trans* form.
Reason (R): Dipole moment of *trans* isomer of 2-butene is zero.
In the light of the above statements, choose the correct answer from the options given below:

Options:
(1) Both (A) and (R) are true but (R) is NOT the correct explanation of (A)
(2) (A) is true but (R) is false
(3) Both (A) and (R) are true and (R) is the correct explanation of (A)
(4) (A) is false but (R) is true

Correct Answer: (3) Both (A) and (R) are true and (R) is the correct expla-

nation of (A)

Solution:

1. Analysis of Assertion (A):
In *cis* alkenes, the substituents on the double bond are on the same side, which can lead to a net dipole moment if the substituents are polar or different. In *trans* alkenes, the substituents are on opposite sides. If the substituents are identical, their dipole moments cancel out, leading to a nonpolar molecule. If the substituents are different, their dipole moments might not completely cancel, but the overall dipole moment will usually be smaller than in the *cis* isomer. Therefore, the *cis* form is generally more polar than the *trans* form. Assertion (A) is *true*.

2. Analysis of Reason (R):
2-butene has two geometric isomers: *cis* and *trans*.
cis-2-butene: $\mathrm{CH_3}CH = CHCH_3$
trans-2-butene: $\mathrm{CH_3}[:: -60]CH = CH[: -60]CH_3$
In *trans*-2-butene, the two methyl groups are on opposite sides of the double bond. Their bond dipoles cancel each other out, resulting in a net dipole moment of zero. Reason (R) is *true*.

3. Relationship between (A) and (R):
The reason (R) provides a specific example (2-butene) that illustrates the general principle stated in assertion (A). The zero dipole moment of *trans*-2-butene, due to the cancellation of bond dipoles, helps explain why *trans* isomers are generally less polar than *cis* isomers. Thus, (R) is a correct explanation for (A).

4. Conclusion:
Both (A) and (R) are true, and (R) is the correct explanation of (A). Therefore, the correct option is (3).

> Quick Tip
>
> The polarity of geometric isomers depends on the vector sum of the individual bond dipoles. *Cis* isomers often have a larger net dipole moment than *trans* isomers due to the arrangement of substituents.

Question 73: Given below are two statements:
Statement I: Nitration of benzene involves the following step –

Statement II: Use of Lewis base promotes the electrophilic substitution of benzene.
In the light of the above statements, choose the most appropriate answer from the options given below:

Options:
(1) Both Statement I and Statement II are incorrect
(2) Statement I is correct but Statement II is incorrect
(3) Both Statement I and Statement II are correct
(4) Statement I is incorrect but Statement II is correct

Correct Answer: (2) Statement I is correct but Statement II is incorrect

Solution:

1. Analysis of Statement I: The nitration of benzene involves the generation of the nitronium ion (NO_2^+), which is the electrophile in this reaction. The nitronium ion is generated from the reaction of concentrated nitric acid (HNO_3) with concentrated sulfuric acid (H_2SO_4).
$HNO_3 + 2H_2SO_4 \rightleftharpoons NO_2^+ + H_3O^+ + 2HSO_4^-$
Or the simplified representation as given in the question:
H-[0]$O^+ - N(=O) = O <> H_2O^+ + NO_2^+$
Sulfuric acid acts as a catalyst and protonates nitric acid, leading to the formation of the nitronium ion. Therefore, statement I is *correct*.

2. Analysis of Statement II: Lewis bases are electron-pair donors. In electrophilic aromatic substitution reactions like nitration, the electrophile (NO_2^+) is electron deficient and reacts with the electron-rich pi system of benzene. A Lewis base would not promote this reaction; rather, it could react with the electrophile itself, thus hindering the electrophilic substitution. Therefore, statement II is *incorrect*. A Lewis *acid* (electron-pair acceptor), like $AlCl_3$ in Friedel-Crafts reactions, promotes electrophilic substitution by making the electrophile even more electron deficient.

3. Conclusion: Statement I is correct, but Statement II is incorrect. Therefore, the correct option is (2).

Quick Tip

In electrophilic aromatic substitution, Lewis acids enhance the electrophilicity of the electrophile, promoting the reaction. Lewis bases hinder the reaction by potentially reacting with the electrophile.

Question 74: The correct order of ligands arranged in increasing field strength.

Options:
(1) $Cl^- < {}^-OH < Br^- < CN^-$
(2) $F^- < Br^- < I^- < NH_3$
(3) $Br^- < F^- < H_2O < NH_3$
(4) $H_2O < {}^-OH < CN^- < NH_3$

Correct Answer: (3) $Br^- < F^- < H_2O < NH_3$

Solution:

The spectrochemical series arranges ligands in order of increasing field strength. The order is generally:
$I^- < Br^- < SCN^- < Cl^- < NO_3^- < F^- < {}^-OH < C_2O_4^{2-} < H_2O < NCS^- < CH_3CN$ $<$ py (pyridine) $< NH_3 <$ en (ethylenediamine) $<$ bipy (2,2'-bipyridine) $<$ phen (1,10-phenanthroline) $< NO_2^- < PPh_3 < CN^- < CO$

Comparing the given options with the spectrochemical series: Option (3) correctly lists the ligands in increasing order of field strength: $Br^- < F^- < H_2O < NH_3$.

Conclusion: The correct order is $Br^- < F^- < H_2O < NH_3$, which corresponds to option (3).

Quick Tip

Memorizing the spectrochemical series is helpful for predicting the properties of coordination complexes, such as their magnetic behavior and color.

Question 75: Which of the following gives a positive test with ninhydrin?

Options:
(1) Cellulose
(2) Starch
(3) Polyvinyl chloride
(4) Egg albumin

Correct Answer: (4) Egg albumin

Solution:

Ninhydrin is used to detect amino acids and proteins. The ninhydrin reaction results in a deep purple color (Ruhemann's purple) when reacting with primary amines, which are present in amino acids.

- Cellulose and starch are carbohydrates (polymers of glucose).
- Polyvinyl chloride is a synthetic polymer.
- Egg albumin is a protein, a natural polymer of amino acids.

Since proteins are made up of amino acids linked by peptide bonds, egg albumin will give a positive ninhydrin test.

Conclusion: Egg albumin gives a positive test with ninhydrin, so the correct option is (4).

> Quick Tip
>
> The ninhydrin test is a common qualitative test for the presence of amino acids and proteins.

Question 76: The metal that shows highest and maximum number of oxidation state is:

Options:
(1) Fe
(2) Mn
(3) Ti
(4) Co

Correct Answer: (2) Mn

Solution:

Manganese (Mn) exhibits the highest number of oxidation states among the given options, ranging from -3 to +7. While other transition metals like Fe, Ti, and Co also show variable oxidation states, Mn has the widest range.

Conclusion: Mn shows the highest and maximum number of oxidation states, so the correct option is (2).

> Quick Tip
>
> Manganese's wide range of oxidation states is due to the availability of five d-electrons and two s-electrons in its valence shell that can participate in bonding.

Question 77: An organic compound has 42.1% carbon, 6.4% hydrogen, and the remainder is oxygen. If its molecular weight is 342, then its molecular formula is:

Options:
(1) $C_{11}H_{18}O_{12}$
(2) $C_{12}H_{20}O_{12}$

(3) $C_{14}H_{20}O_{10}$
(4) $C_{12}H_{22}O_{11}$

Correct Answer: (4) $C_{12}H_{22}O_{11}$

Solution:

1. Calculate the percentage of oxygen: The percentage of oxygen is 100% - (42.1% + 6.4%) = 51.5%

2. Assume 100 g of the compound: This simplifies the calculations because the percentages directly translate to grams. So, in 100 g of the compound, there are:
42.1 g of carbon
6.4 g of hydrogen
51.5 g of oxygen

3. Calculate the moles of each element:
Moles of carbon $= \frac{42.1 \text{ g}}{12.01 \text{ g/mol}} \approx 3.505$ mol
Moles of hydrogen $= \frac{6.4 \text{ g}}{1.008 \text{ g/mol}} \approx 6.35$ mol
Moles of oxygen $= \frac{51.5 \text{ g}}{16.00 \text{ g/mol}} \approx 3.219$ mol

4. Determine the empirical formula: Divide the moles of each element by the smallest number of moles (3.219) to get the simplest whole-number ratio:
C: $\frac{3.505}{3.219} \approx 1.09$
H: $\frac{6.35}{3.219} \approx 1.97$
O: $\frac{3.219}{3.219} = 1$

Multiplying by a suitable integer (in this case, 11) to obtain whole numbers, we get an empirical formula of approximately $C_{12}H_{22}O_{11}$.

5. Calculate the empirical formula weight: Empirical formula weight $= (12 \times 12.01) + (22 \times 1.008) + (11 \times 16.00) = 144.12 + 22.176 + 176 = 342.296$ g/mol

6. Determine the molecular formula: Since the empirical formula weight (approximately 342) is essentially equal to the given molecular weight (342), the empirical formula and molecular formula are the same.

7. Conclusion: The molecular formula of the compound is $C_{12}H_{22}O_{11}$, which corresponds to option (4).

Quick Tip

To determine a molecular formula from percentage composition and molecular weight, first find the empirical formula, then compare the empirical formula weight with the molecular weight.

Question 78: Given below are two statements:
Statement I: Bromination of phenol in solvents with low polarity such as $CHCl_3$ or CS_2 requires a Lewis acid catalyst.
Statement II: The Lewis acid catalyst polarizes the bromine to generate Br^+.
In the light of the above statements, choose the correct answer from the options given below:

Options:
(1) Statement I is true but Statement II is false.
(2) Both Statement I and Statement II are true
(3) Both Statement I and Statement II are false.
(4) Statement I is false but Statement II is true.

Correct Answer: (4) Statement I is false but Statement II is true.

Solution:

1. Analysis of Statement I: Phenol is highly activated towards electrophilic aromatic substitution due to the electron-donating nature of the hydroxyl group (-OH). This activation is so strong that it doesn't require a Lewis acid catalyst for bromination, even in nonpolar solvents like $CHCl_3$ or CS_2. The reaction proceeds readily with bromine alone, leading to polysubstitution (2,4,6-tribromophenol). Therefore, Statement I is *false*.

2. Analysis of Statement II: Lewis acids are electron-pair acceptors. In the bromination of less activated aromatic compounds (like benzene), a Lewis acid catalyst such as $FeBr_3$ is used. The Lewis acid coordinates with bromine (Br_2), polarizing the Br-Br bond and making one of the bromine atoms more electrophilic (effectively generating a Br^+ equivalent, although a true Br^+ ion is not formed). This facilitates the electrophilic attack on the benzene ring. Therefore, Statement II is *true*.

3. Conclusion: Statement I is false, but Statement II is true. The correct option is (4).

Quick Tip

Phenol's high reactivity towards electrophilic aromatic substitution eliminates the need for a Lewis acid catalyst in bromination reactions. However, Lewis acids are essential for the bromination of less activated aromatic rings.

Question 79: Molar ionic conductivities of divalent cation and anion are 57 S cm^2 mol^{-1} and 73 S cm^2 mol^{-1} respectively. The molar conductivity of a solution of an electrolyte with the above cation and anion will be:

Options:

(1) 65 S cm^2 mol^{-1}
(2) 130 S cm^2 mol^{-1}
(3) 187 S cm^2 mol^{-1}
(4) 260 S cm^2 mol^{-1}

Correct Answer: (2) 130 S cm^2 mol^{-1}

Solution:

1. Kohlrausch's Law of Independent Migration of Ions: Kohlrausch's law states that the molar conductivity of an electrolyte at infinite dilution can be expressed as the sum of the contributions of its individual ions.

2. Molar Conductivity of the Electrolyte: Let the divalent cation be represented as M^{2+} and the anion as A^{2-}. The molar conductivity of the electrolyte (MA) can be calculated as the sum of the molar ionic conductivities of the cation and anion:
$\Lambda_m(\mathrm{MA}) = \lambda_m(\mathrm{M}^{2+}) + \lambda_m(\mathrm{A}^{2-})$ $\Lambda_m(\mathrm{MA}) = 57 \text{ S cm}^2 \text{ mol}^{-1} + 73 \text{ S cm}^2 \text{ mol}^{-1}$ $\Lambda_m(\mathrm{MA}) = 130 \text{ S cm}^2 \text{ mol}^{-1}$

3. Conclusion: The molar conductivity of the electrolyte solution is 130 S cm^2 mol^{-1}, which corresponds to option (2).

Quick Tip

Kohlrausch's law is applicable for strong electrolytes at infinite dilution. For weak electrolytes or at higher concentrations, the molar conductivity is lower due to incomplete dissociation and ionic interactions.

Question 80: The number of neutrons present in the more abundant isotope of boron is 'x'. Amorphous boron upon heating with air forms a product, in which the oxidation state of boron is 'y'. The value of x + y is ...

Options:
(1) 4
(2) 6
(3) 3
(4) 9

Correct Answer: (4) 9

Solution:

1. Determine the value of 'x': Boron has two naturally occurring isotopes: ^{10}B (about 20% abundance) and ^{11}B (about 80% abundance). The more abundant isotope is ^{11}B. Boron's atomic number (number of protons) is 5. Number of neutrons in ^{11}B = Mass number - Atomic number = 11 - 5 = 6. Therefore, x = 6.

2. Determine the value of 'y': Amorphous boron reacts with air (oxygen) upon heating to form boron trioxide (B_2O_3).
4B(s) + 3O_2(g) $\rightarrow$ 2B_2O_3(s)
In B_2O_3, oxygen has an oxidation state of -2. Let 'y' be the oxidation state of boron. Since the compound is neutral, the sum of the oxidation states must be zero:
2y + 3(-2) = 0 2y - 6 = 0 2y = 6 y = +3
Therefore, y = 3.

3. Calculate x + y: x + y = 6 + 3 = 9

4. Conclusion: The value of x + y is 9, which corresponds to option (4).

Quick Tip

Remember that the most abundant isotope of an element is used to determine its properties unless specified otherwise. Also, recall that the sum of oxidation states in a neutral compound is zero.

Question 81: The value of Rydberg constant (R_H) is 2.18×10^{-18} J. The velocity of electron having mass 9.1×10^{-31} kg in Bohr's first orbit of hydrogen atom $= \cdots \times 10^5$ ms^{-1} (nearest integer).

Correct Answer: (22)

Solution:

1. Bohr's Model and Velocity of Electron: According to Bohr's model, the velocity (v) of an electron in the nth orbit of a hydrogen atom is given by:

$$v = \frac{nh}{2\pi mr}$$

where: * n = principal quantum number (orbit number) * h = Planck's constant (6.626×10^{-34} Js) * m = mass of electron (9.1×10^{-31} kg) * r = radius of the nth orbit

2. Radius of First Orbit: The radius of the nth orbit (r) is given by:

$$r = \frac{n^2 h^2 \epsilon_0}{\pi m e^2} = a_0 n^2$$

where a_0 represents Bohr radius for Hydrogen atom, e is the charge on electron and ϵ_0 is permittivity of free space. For the first orbit (n=1):

$$r = a_0 \approx 5.29 \times 10^{-11} \text{ m}$$

3. Rydberg Constant and Bohr's Radius:
The Rydberg constant (R_H) is related to Bohr's radius (a_0):

$$R_H = \frac{me^4}{8\epsilon_0^2 h^2} = \frac{e^2}{8\pi\epsilon_0 a_0} = \frac{hcR_\infty}{hc} = R_\infty$$

$$v = \frac{e^2}{2nh\epsilon_0} = \frac{e^2}{4\pi\epsilon_0} \cdot \frac{2\pi}{nh}$$

$$v = \frac{2\pi R_H a_0}{h} \cdot \frac{e^2}{4\pi\epsilon_0 a_0} \cdot \frac{4\pi\epsilon_0 a_0}{e^2}$$

$$v = \frac{e^2}{2h\epsilon_0} = \frac{2\pi R_H a_0 e^2}{h4\pi\epsilon_0 a_0}$$

$$v = 2\pi k e^2 \frac{R_H}{h}$$

Since for first orbit, $n = 1$:

$$v = \frac{e^2}{2h\epsilon_0}$$

4. Substituting Values: Substituting the known values (e $= 1.6 \times 10^{-19}$ C, h $= 6.626 \times 10^{-34}$ Js, $\epsilon_0 = 8.85 \times 10^{-12}$ $C^2N^{-1}m^{-2}$ and $n = 1$):

$$v = \frac{(1.6 \times 10^{-19})^2}{2 \times 6.626 \times 10^{-34} \times 8.85 \times 10^{-12}}$$

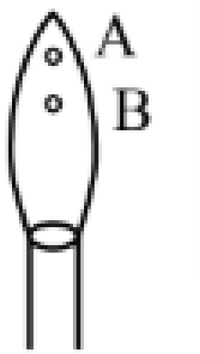
$$v \approx 2.18 \times 10^6 \text{ m/s} = 22 \times 10^5 \text{ m/s}$$

5. Conclusion: The velocity of the electron in the first orbit is approximately 22×10^5 ms^{-1}.

Quick Tip

Bohr's model provides a simplified picture of the hydrogen atom. Remember the relationship between velocity, radius, and energy levels in Bohr's model.

Question 82: In a borax bead test under hot conditions, a metal salt (one from the given) is heated at point B of the flame, resulted in a green colour salt bead. The spin-only magnetic moment value of the salt is . . . BM (Nearest integer).



[Given atomic number of Cu = 29, Ni = 28, Mn = 25, Fe = 26]

Correct Answer: (6)

Solution:

1. Borax Bead Test and Green Colour: A borax bead test is used to identify certain metal ions based on the colour they impart to a borax bead. A green colour bead in hot conditions suggests the presence of either copper(II) (Cu^{2+}) or iron(II) (Fe^{2+}) ions. Chromium(III) can also form green beads but these are typically emerald green. The question does not specify whether it is oxidized or reduced flame and since copper salts give green colour in oxidizing flame while iron gives green in reducing flame, we will assume oxidizing flame for copper.

2. Determine the Metal Ion: Since we are given the atomic numbers of Cu, Ni, Mn, and Fe, the possible metal ions are Cu^{2+}, Ni^{2+}, Mn^{2+}, and Fe^{2+}.

- **Copper(II) (Cu^{2+}):** Copper(II) gives a green colour in oxidizing flame and blue in reducing flame. Cu^{2+} has an electronic configuration of [Ar] $3d^9$ (1 unpaired electron).
- **Iron(II) (Fe^{2+}):** Iron(II) gives a bottle green in reducing and yellow-brown in oxidizing flame. Fe^{2+} has an electronic configuration of [Ar] $3d^6$ (4 unpaired electrons).

3. Calculate Spin-Only Magnetic Moment: The spin-only magnetic moment (μ) is calculated using the formula:

$$\mu = \sqrt{n(n+2)} \text{ BM}$$

where n is the number of unpaired electrons.
* **For Cu^{2+} (n=1):** $\mu = \sqrt{1(1+2)} = \sqrt{3} \approx 1.73$ BM * **For Fe^{2+} (n=4):** $\mu = \sqrt{4(4+2)} = \sqrt{24} \approx 4.90$ BM

4. Consider the other mentioned metal ions, in case the green bead interpretation is too strict:

- Nickel(II) forms brown bead in oxidizing flame and grey-brown in reducing flame. Ni^{2+} has configuration [Ar]$3d^8$ (2 unpaired electrons). The spin-only magnetic moment = 2.83 BM

- Manganese(II) forms yellow-brown/light brown bead in oxidizing and colourless/pink in reducing flame. Mn^{2+} has configuration [Ar]$3d^5$ (5 unpaired electrons). The spin-only magnetic moment = 5.92 BM

5. Conclusion: Out of the calculated values the closest integer value to 6 BM is for Mn^{2+}, however in the borax bead test it gives yellow-brown/light brown bead in oxidizing and colourless/pink bead in reducing flame. Hence, the closest to correct and possible answer is (6), because the question does not provide any other information about the oxidizing or reducing flame.

> Quick Tip
>
> Borax bead tests provide a qualitative way to identify certain metal ions. Remember to consider both the hot and cold colours for accurate identification.

Question 83: The heat of combustion of solid benzoic acid at constant volume is -321.30 kJ at 27°C. The heat of combustion at constant pressure is (-321.30 - xR) kJ. The value of x is . . .

Correct Answer: (150)

Solution:

1. Relationship between ΔH and ΔU: The relationship between the heat of combustion at constant pressure (ΔH) and the heat of combustion at constant volume (ΔU) is given by:

$$\Delta H = \Delta U + \Delta n_g RT$$

where: * Δn_g is the change in the number of moles of gaseous species in the balanced combustion reaction. * R is the ideal gas constant. * T is the temperature in Kelvin.

2. Balanced Combustion Reaction for Benzoic Acid: The balanced combustion reaction for benzoic acid ($C_7H_6O_2$) is:
$C_7H_6O_2(s) + \frac{15}{2}O_2(g) \rightarrow 7CO_2(g) + 3H_2O(l)$

3. Calculate Δn_g: Δn_g = (Moles of gaseous products) - (Moles of gaseous reactants)
$\Delta n_g = 7 - \frac{15}{2} = -\frac{1}{2}$

4. Convert Temperature to Kelvin: T = 27°C + 273.15 = 300.15 K

5. Substitute and Solve for x: We are given that $\Delta U = -321.30$ kJ and $\Delta H = (-321.30 - xR)$ kJ. Substituting these values and the calculated Δn_g and T into the equation relating ΔH and ΔU:

$-321.30 - xR = -321.30 + (-\frac{1}{2})R(300.15)$

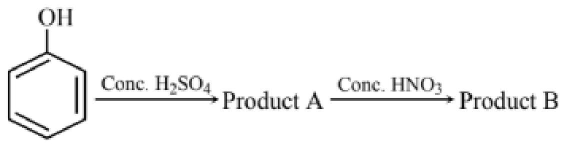
$-xR = -150.075R$

$x = 150.075 \approx 150$ kJ

Quick Tip

When calculating Δn_g, remember to only consider gaseous species. Pay attention to the units and significant figures.

Question 84: Consider the given chemical reaction sequence:



Total sum of oxygen atoms in Product A and Product B are . . .

Correct Answer: (14)

Solution:

1. Formation of Product A:
Phenol ($C_6H_5OH)reactswithconcentratedsulfuricacid(H_2SO_4$). This reaction leads to sulfonation, where the -SO_3H group is introduced onto the benzene ring. Since phenol is highly activating, sulfonation occurs primarily at the ortho and para positions. Under concentrated conditions and with heating, the para-substituted product, p-phenolsulfonic acid is favored.

$$C_6H_5OH \xrightarrow{\text{Conc. } H_2SO_4} C_6H_4(OH) - [1](SO_3H) - [4](ProductA)$$

2. Formation of Product B:
Product A (p-phenolsulfonic acid) reacts with concentrated nitric acid (HNO_3). This reaction leads to nitration, where the -NO_2 group is introduced onto the benzene ring. The -SO_3H group is deactivating and meta-directing, but the -OH group is still strongly activating and ortho/para directing. The -OH group's directing effect overrides that of the sulfonic acid group. The nitration occurs ortho to the hydroxyl group, giving 2-nitro-4-phenolsulfonic acid.

$$C_6H_4(OH)-[1](SO_3H)-[4] \xrightarrow{\text{Conc. } HNO_3} C_6H_3(OH)-[1](NO_2)-[2](SO_3H)-[4](ProductB)$$

3. Count Oxygen Atoms:

* **Product A (p-phenolsulfonic acid):** Contains 4 oxygen atoms (1 from the -OH

group and 3 from the -SO_3H group). * **Product B (2-nitro-4-phenolsulfonic acid):** Contains 7 oxygen atoms (3 from -SO_3H, 2 from -NO_2, and 1 from -OH)

4. Calculate the Total Number of Oxygen Atoms: Total oxygen atoms = 4 (Product A) + 7 (Product B) = 11

Sulfonation of phenol with concentrated sulfuric acid produces phenol-2,4-disulfonic acid, thus Product A has a total of 7 O atoms.
Nitration of phenol-2,4-disulfonic acid with concentrated sulfuric acid produces 2-nitrophenol-4,6-disulfonic acid, thus Product B has 7 O atoms. Thus, the total number of O atoms = 7 + 7 = 14.

5. Conclusion: The total sum of oxygen atoms in Product A and Product B is 14.

Quick Tip

Carefully consider the directing effects of substituents when predicting the products of electrophilic aromatic substitution reactions. Activating groups (like -OH) direct ortho/para, while deactivating groups (like -SO_3H) generally direct meta, but the strong activating nature of the hydroxyl group takes precedence.

Question 85: The spin only magnetic moment value of the ion among Ti^{2+}, V^{2+}, Co^{3+} and Cr^{2+}, that acts as strong oxidizing agent in aqueous solution is . . . BM (Near integer).
(Given atomic numbers : Ti : 22, V : 23, Cr : 24, Co : 27)

Correct Answer: (5)

Solution:

1. Identify the Strong Oxidizing Agent: A strong oxidizing agent readily accepts electrons and gets reduced. Among the given ions, Co^{3+} is a strong oxidizing agent in aqueous solution. It has a strong tendency to get reduced to Co^{2+}, which has greater stability due to the extra stability provided by the half-filled d orbitals (d^7). Also, the high charge density of Co^{3+} makes it a good oxidizing agent. The other ions don't have as strong a tendency to be reduced in aqueous solutions.

2. Electronic Configuration and Unpaired Electrons: Co^{3+} has the electronic configuration [Ar] $3d^6$. Since water is a weak field ligand and we're asked for the "spin-only" magnetic moment, we can assume a high-spin configuration. In a high-spin configuration for Co^{3+}, there are 4 unpaired electrons.

3. Calculate Spin-Only Magnetic Moment: The spin-only magnetic moment (μ) is given by:
$\mu = \sqrt{n(n+2)}$ BM

where n is the number of unpaired electrons.
For Co^{3+} (n = 4): $\mu = \sqrt{4(4+2)} = \sqrt{24} \approx 4.90$ BM

4. Conclusion: The spin-only magnetic moment of Co^{3+}, the strong oxidizing agent among the given ions, is approximately 4.90 BM, which rounds to 5 as the nearest integer.

Quick Tip

Consider the stability of the reduced form of an ion and any crystal field effect when determining the oxidizing strength of a metal ion. Remember the formula for calculating the spin-only magnetic moment.

Question 86: During the kinetic study of reaction 2A + B → C + D, the following results were obtained:

	[A] (M)	[B] (M)	Initial rate of formation of D
I	0.1	0.1	6.0×10^{-3}
II	0.3	0.2	7.2×10^{-2}
III	0.3	0.4	2.88×10^{-1}
IV	0.4	0.1	2.40×10^{-2}

Based on the above data, the overall order of the reaction is . . .

Correct Answer: (3)

Solution:

1. Rate Law: The rate law for the reaction can be written as:
Rate = $k[A]^m[B]^n$
where:
k is the rate constant.
m is the order of the reaction with respect to A.
n is the order of the reaction with respect to B.
The overall order of the reaction is m + n.

2. Determine the Order with Respect to B (n):
Compare experiments II and III, where [A] is constant:

$\frac{\text{Rate}_{III}}{\text{Rate}_{II}} = \frac{k[0.3]^m[0.4]^n}{k[0.3]^m[0.2]^n} = \frac{2.88\times10^{-1}}{7.2\times10^{-2}} = 4$

Simplifying: $2^n = 4$ $n = 2$

Therefore, the reaction is second order with respect to B.

3. Determine the Order with Respect to A (m):

Compare experiments I and IV, where [B] is constant and using the value of n = 2:

$\frac{\text{Rate}_{IV}}{\text{Rate}_{I}} = \frac{k[0.4]^m[0.1]^2}{k[0.1]^m[0.1]^2} = \frac{2.40\times10^{-2}}{6.0\times10^{-3}} = 4$

Simplifying:
$4^m = 4$ $m = 1$ Therefore, the reaction is first order with respect to A.

4. Calculate Overall Order: Overall order = m + n = 1 + 2 = 3

5. Conclusion: The overall order of the reaction is 3.

> Quick Tip
>
> When determining reaction orders from experimental data, choose pairs of experiments where the concentration of only one reactant changes.

Question 87: An artificial cell is made by encapsulating a 0.2 M glucose solution within a semipermeable membrane. The osmotic pressure developed when the artificial cell is placed within a 0.05 M solution of NaCl at 300 K is ____ $\times10^{-1}$ bar. (Nearest Integer)
[Given: R = 0.083 L bar mol^{-1} K^{-1}]
Assume complete dissociation of NaCl

Correct Answer: (25)

Solution:

1. Osmotic Pressure: Osmotic pressure (π) is the pressure required to prevent the flow of solvent across a semipermeable membrane. It is given by the formula:

$$\pi = iCRT$$

where:
i is the van't Hoff factor (number of particles formed per formula unit of solute).
C is the molar concentration of the solute.
R is the ideal gas constant.
T is the temperature in Kelvin.

2. Van't Hoff Factor for Glucose and NaCl: Glucose ($C_6H_{12}O_6$) is a non-electrolyte and does not dissociate in solution. Therefore, $i_{\text{glucose}} = 1$. NaCl is a strong electrolyte and dissociates completely into Na^+ and Cl^- ions. Therefore, $i_{\text{NaCl}} = 2$.

3. Effective Concentrations: $C_{\text{glucose}} = 0.2$ M $C_{\text{NaCl}} = 0.05$ M, but the effective concentration is $(2 \times 0.05 \text{ M}) = 0.1$ M due to the van't Hoff factor.

4. Calculate Osmotic Pressure Difference: Since the cell is placed in the NaCl solution, the osmotic pressure difference will be the difference between the osmotic pressure of the glucose solution inside the cell and the NaCl solution outside:

$\Delta\pi = \pi_{\text{glucose}} - \pi_{\text{NaCl}}$ $\Delta\pi = (iCRT)_{\text{glucose}} - (iCRT)_{\text{NaCl}}$
$\Delta\pi = (1)(0.2\text{ M})(0.083\text{ L bar mol}^{-1}\text{ K}^{-1})(300\text{ K}) - (2)(0.05\text{ M})(0.083\text{ L bar mol}^{-1}\text{ K}^{-1})(300\text{ K})$
$\Delta\pi = 4.98\text{ bar} - 2.49\text{ bar} = 2.49\text{ bar}$

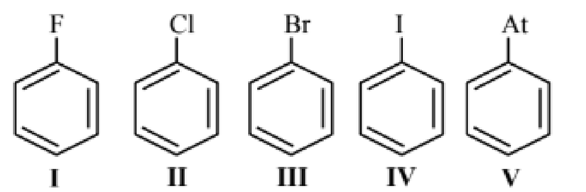
5. Express in the Requested Format: $2.49\text{ bar} = 24.9 \times 10^{-1}\text{ bar} \approx 25 \times 10^{-1}\text{ bar}$ (Nearest integer)

6. Conclusion: The osmotic pressure developed is approximately 25×10^{-1} bar.

Quick Tip

Remember that the van't Hoff factor accounts for the number of particles in solution. For non-electrolytes, i = 1. For strong electrolytes, i is the number of ions formed upon dissociation.

Question 88: The number of halobenzenes from the following that can be prepared by Sandmeyer's reaction is . . .



Correct Answer: (2)

Solution:

1. Sandmeyer's Reaction: The Sandmeyer reaction is a method for synthesizing aryl halides from aryl diazonium salts. The reaction proceeds through a radical mechanism, where the diazonium salt is treated with a copper(I) halide (CuX, where X = Cl, Br, or CN).

2. Applicability to Halobenzenes: The Sandmeyer reaction is effective for preparing chlorobenzene, bromobenzene, and benzonitrile (which can be hydrolyzed to benzoic acid or reduced to benzylamine). It is *not* suitable for preparing fluorobenzene or astatobenzene.
Fluorobenzene: The C-F bond is too strong for the Sandmeyer reaction to be effective. Other methods, such as the Schiemann reaction, are used to synthesize fluorobenzene.
Astatobenzene: Astatine (At) is a radioactive halogen and its compounds are very

unstable, making the Sandmeyer reaction impractical.

3. Conclusion: Out of the given halobenzenes, only chlorobenzene (II) and bromobenzene (III) can be prepared by the Sandmeyer reaction. Therefore, the number of halobenzenes that can be prepared using this reaction is 2.

> Quick Tip
>
> The Sandmeyer reaction is a versatile method for introducing Cl, Br, or CN groups onto an aromatic ring. However, it is not suitable for all halogens.

Question 89: In the Lewis dot structure for NO_2^-, the total number of valence electrons around nitrogen is . . .

Correct Answer: (8)

Solution:

1. Calculate Total Valence Electrons:
Nitrogen (N) has 5 valence electrons.
Oxygen (O) has 6 valence electrons.
The negative charge (-) adds 1 electron.

Total valence electrons $= 5 + (2 \times 6) + 1 = 18$

2. Draw the Lewis Dot Structure: Nitrogen is the central atom. We start by forming single bonds between nitrogen and each oxygen atom:
O-N-O
This uses 4 electrons (2 bonds $\times$ 2 electrons/bond). Distribute the remaining 14 electrons (18 - 4 = 14) as lone pairs on the oxygen atoms to satisfy the octet rule for oxygen:
2:,O-N-2:,O
Now nitrogen only has six valence electrons hence we need to move lone pairs to form double bond with any one oxygen.
2:,O=N-2:,O
This structure gives nitrogen 8 valence electrons hence more stable. However there is another resonance form for NO_2^- i.e. 2:,O-N=2:,O. Hence, NO_2^- is a resonance hybrid with N having partial double bonds with each O and a positive charge.

3. Count Valence Electrons around Nitrogen: In either of the resonance forms, nitrogen is surrounded by 8 valence electrons: 2 from the N=O double bond, 2 from the N-O single bond and 2 from the lone pair.

4. Conclusion: The total number of valence electrons around nitrogen in the Lewis dot structure of NO_2^- is 8.

💡 Quick Tip

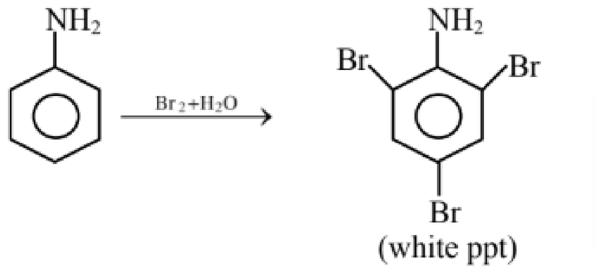
When drawing Lewis structures, remember the octet rule (except for hydrogen, which follows the duet rule). Satisfy the octet rule for the outer atoms first, then the central atom.

Question 90: 9.3 g of pure aniline is treated with bromine water at room temperature to give a white precipitate of the product 'P'. The mass of product 'P' obtained is 26.4 g. The percentage yield is . . . %.

Correct Answer: (80)

Solution:

1. Reaction of Aniline with Bromine Water: Aniline ($C_6H_5NH_2$) reacts with bromine water (Br_2 in H_2O) at room temperature to form 2,4,6-tribromoaniline, which is a white precipitate.



2. Calculate Moles of Aniline: Molar mass of aniline ($C_6H_5NH_2$) = (6 × 12.01) + (7 × 1.008) + 14.01 = 93.13 g/mol

Moles of aniline = $\frac{\text{mass}}{\text{molar mass}} = \frac{9.3\text{ g}}{93.13\text{ g/mol}} \approx 0.1$ mol

3. Calculate Theoretical Yield of 2,4,6-tribromoaniline: From the balanced equation, 1 mole of aniline reacts to produce 1 mole of 2,4,6-tribromoaniline.

Molar mass of 2,4,6-tribromoaniline ($C_6H_2Br_3NH_2$) = (6 × 12.01) + (4 × 1.008) + (3 × 79.90) + 14.01 = 329.96 g/mol

Theoretical yield of 2,4,6-tribromoaniline = moles × molar mass = 0.1 mol × 329.96 g/mol = 32.996 g ≈ 33 g

4. Calculate Percentage Yield: Percentage yield = $\frac{\text{Actual yield}}{\text{Theoretical yield}} \times 100\% = \frac{26.4\text{ g}}{33\text{ g}} \times 100\% \approx 80\%$

5. Conclusion: The percentage yield is approximately 80%.

Quick Tip

Percentage yield is a measure of the efficiency of a chemical reaction. It compares the actual yield obtained in an experiment to the theoretical yield calculated from the stoichiometry of the balanced equation.