JEE Main 2024 April 5 Shift 1 Mathematics Question Paper with Solutions

Time Allowed :1 Hour	**Maximum Marks :**100	**Total Questions :**30

General Instructions

Read the following instructions very carefully and strictly follow them:

1. The test is of 3 hours duration.
2. The question paper consists of 90 questions. The maximum marks are 300.
3. There are three parts in the question paper consisting of Physics, Chemistry and Mathematics having 30 questions in each part of equal weightage.
4. Each part (subject) has two sections.
(i) Section-A: This section contains 20 multiple choice questions which have only one correct answer. Each question carries 4 marks for correct answer and –1 mark for wrong answer.
(ii) Section-B: This section contains 10 questions. In Section-B, attempt any five questions out of 10. The answer to each of the questions is a numerical value. Each question carries 4 marks for correct answer and –1 mark for wrong answer. For Section-B, the answer should be rounded off to the nearest integer.

Mathematics

Question 1: Let d be the distance of the point of intersection of the lines:

$$\frac{x+6}{3} = \frac{y}{2} = \frac{z+1}{1} \quad \text{and} \quad \frac{x-7}{4} = \frac{y-9}{3} = \frac{z-4}{2}$$

from the point $(7, 8, 9)$. Then $d^2 + 6$ is equal to:

Options:
(1) 72
(2) 69
(3) 75
(4) 78

Correct Answer: (3): 75

Solution:
The parametric equations for the first line are derived from:

$$\frac{x+6}{3} = \frac{y}{2} = \frac{z+1}{1} = t \quad \text{(parameter } t\text{)}.$$

From this, we express x, y, z in terms of t:

$$x = 3t - 6, \quad y = 2t, \quad z = t - 1.$$

Similarly, the parametric equations for the second line are derived from:

$$\frac{x-7}{4} = \frac{y-9}{3} = \frac{z-4}{2} = u \quad \text{(parameter } u\text{)}.$$

From this, we express x, y, z in terms of u:

$$x = 4u + 7, \quad y = 3u + 9, \quad z = 2u + 4.$$

Step 1: Find the point of intersection of the two lines.
At the point of intersection, the coordinates (x, y, z) must satisfy both sets of parametric equations. Equating the coordinates:

$$3t - 6 = 4u + 7, \quad 2t = 3u + 9, \quad t - 1 = 2u + 4.$$

Simplify each equation: 1. $3t - 4u = 13$, 2. $2t - 3u = 9$, 3. $t - 2u = 5$.
Solve this system of linear equations. From the third equation:

$$t = 2u + 5.$$

Substitute $t = 2u + 5$ into the first and second equations: 1. $3(2u + 5) - 4u = 13$, 2. $2(2u + 5) - 3u = 9$.
Simplify: 1. $6u + 15 - 4u = 13 \quad \Rightarrow \quad 2u = -2 \quad \Rightarrow \quad u = -1$, 2. $4u + 10 - 3u = 9 \quad \Rightarrow \quad u = -1$.
Substitute $u = -1$ into $t = 2u + 5$:

$$t = 2(-1) + 5 = 3.$$

Now, substitute $t = 3$ into the parametric equations for the first line to find the point of intersection:

$$x = 3(3) - 6 = 3, \quad y = 2(3) = 6, \quad z = 3 - 1 = 2.$$

Thus, the point of intersection is $(3, 6, 2)$.
Step 2: Calculate the distance d from $(3, 6, 2)$ to $(7, 8, 9)$.
The distance formula in 3D is:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

Substitute $(x_1, y_1, z_1) = (3, 6, 2)$ and $(x_2, y_2, z_2) = (7, 8, 9)$:

$$d = \sqrt{(7-3)^2 + (8-6)^2 + (9-2)^2}.$$

Simplify:

$$d = \sqrt{4^2 + 2^2 + 7^2} = \sqrt{16 + 4 + 49} = \sqrt{69}.$$

Step 3: Compute $d^2 + 6$.

$$d^2 = 69 \quad \Rightarrow \quad d^2 + 6 = 69 + 6 = 75.$$

Thus, the final answer is:

$$75.$$

💡 Quick Tip

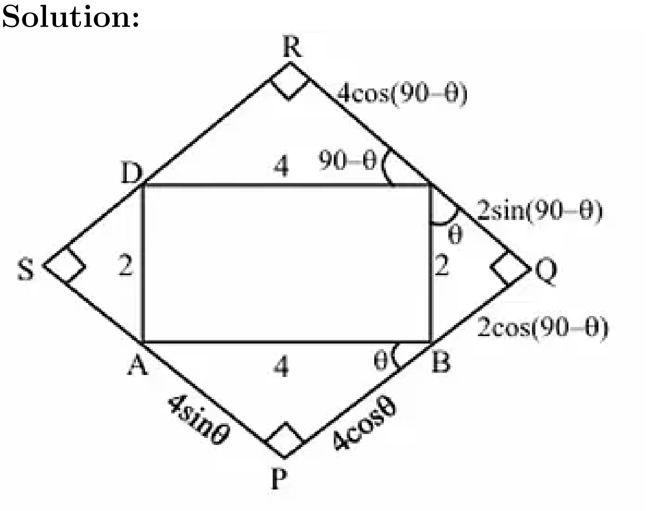
When solving problems involving the intersection of two lines, always convert the symmetric equations to parametric form, solve the resulting system of equations, and verify the intersection point satisfies both lines.

Question 2: Let a rectangle $ABCD$ of sides 2 and 4 be inscribed in another rectangle $PQRS$ such that the vertices of the rectangle $ABCD$ lie on the sides of the rectangle $PQRS$. Let a and b be the sides of the rectangle $PQRS$ when its area is maximum. Then $(a+b)^2$ is equal to:

Options:
(1) 72
(2) 60
(3) 80
(4) 64

Correct Answer: (1): 72

Solution:



Consider rectangle $ABCD$ inscribed within rectangle $FQRS$ as shown in the figure. Let θ be the angle formed between side AB of $ABCD$ and side FQ of $FQRS$.

Using trigonometry, the dimensions of $FQRS$ are expressed as:

$$a = 4\cos\theta + 2\sin\theta, \quad b = 2\cos\theta + 4\sin\theta$$

The area of $FQRS$ is given by:

$$\text{Area} = (4\cos\theta + 2\sin\theta)(2\cos\theta + 4\sin\theta)$$

Expanding this, we get:

$$\text{Area} = 8\cos^2\theta + 16\sin\theta\cos\theta + 4\sin^2\theta + 8\sin^2\theta = 8 + 10\sin 2\theta$$

The area is maximized when $\sin 2\theta = 1$, i.e., $\theta = 45°$.
Thus, the maximum area is:

$$8 + 10 = 18$$

Now, we calculate $(a+b)^2$:

$$(a+b)^2 = (4\cos\theta + 2\sin\theta + 2\cos\theta + 4\sin\theta)^2 = (6\cos\theta + 6\sin\theta)^2 = 36(\sin\theta + \cos\theta)^2$$

Since $\sin\theta + \cos\theta = \sqrt{2}$ at $\theta = 45°$, we have:

$$36(\sqrt{2})^2 = 36 \times 2 = 72$$

> 💡 Quick Tip
>
> For problems involving rectangles and diagonals, use the Pythagorean theorem and inequalities like AM-GM to optimize area or other quantities.

Question 3: Let two straight lines drawn from the origin O intersect the line $3x + 4y = 12$ at the points P and Q such that $\triangle OPQ$ is an isosceles triangle and $\angle POQ = 90°$. If $I = OP^2 + PQ^2 + OQ^2$, then the greatest integer less than or equal to I is:

Options:
(1) 44
(2) 48
(3) 46
(4) 42

Correct Answer: (3): 46

Solution:
1. Isosceles Right Triangle: Since △OPQ is isosceles and ∠POQ = 90°, it is an isosceles right triangle. Therefore, OP = OQ, and by the Pythagorean theorem:
$PQ^2 = OP^2 + OQ^2 = 2OP^2$ (since OP = OQ)

2. Express l in terms of OP: We are given $l = OP^2 + PQ^2 + OQ^2$. Substituting $OQ^2 = OP^2$ and $PQ^2 = 2OP^2$:
$l = OP^2 + 2OP^2 + OP^2$ $l = 4OP^2$

3. Equation of Line PQ: The equation of the line PQ is given as 3x + 4y = 12.

4. Perpendicular Distance from Origin to Line PQ: Let the perpendicular distance from the origin (0, 0) to the line 3x + 4y - 12 = 0 be 'd'. The formula for the perpendicular distance from a point (x_1, y_1) to a line Ax + By + C = 0 is:

$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$$

In our case, $(x_1, y_1) = (0, 0)$, A = 3, B = 4, and C = -12:

$$d = \frac{|3(0) + 4(0) - 12|}{\sqrt{3^2 + 4^2}} = \frac{|-12|}{\sqrt{25}} = \frac{12}{5}$$

This perpendicular bisects the line connecting P and Q. Thus, the distance d bisects PQ.

5. Relationship between d, OP, and OQ: In an isosceles right triangle, the altitude from the right angle to the hypotenuse bisects the hypotenuse. Since the line from the origin to the line PQ is perpendicular and $\triangle$OPQ is an isosceles right triangle, this perpendicular distance 'd' bisects PQ. Also this distance will be equal to the length of altitude, thus we have,
$d = \frac{12}{5} = \frac{PQ}{2}$.
Then, $PQ = \frac{24}{5}$.

Also, since $\triangle$OPQ is isosceles right triangle, thus $OP = OQ$. Also, the perpendicular distance between origin and the line PQ is $d = \frac{12}{5}$. Also, $OP = OQ = PQ/\sqrt{2}$. Thus $OP = \frac{12\sqrt{2}}{5}$.
$l = 4OP^2 = 4(\frac{144\times 2}{25}) = \frac{1152}{25} = 46.08$

6. Calculate l:
$l = 4 \times (\frac{12\sqrt{2}}{5})^2 = \frac{1152}{25} = 46.08$.

7. Greatest Integer Less Than or Equal to l: The greatest integer less than or equal to 46.08 is 46.

Final Answer: 46.

Quick Tip

When solving geometry problems involving coordinates, always use vector conditions (e.g., dot product for perpendicularity) and parametrize equations where necessary.

Question 4: If $y = y(x)$ is the solution of the differential equation:

$$\frac{dy}{dx} + 2y = \sin(2x), \quad y(0) = \frac{3}{4},$$

then $y\left(\frac{\pi}{8}\right)$ is equal to:

Options:
(1) $e^{-\pi/8}$
(2) $e^{-\pi/4}$
(3) $e^{\pi/4}$
(4) $e^{\pi/8}$

Correct Answer: (2): $e^{-\pi/4}$

Solution:

The given differential equation is:

$$\frac{dy}{dx} + 2y = \sin(2x).$$

Step 1: Identify the integrating factor.
The differential equation is a first-order linear differential equation. To solve it, we calculate the integrating factor (IF):

$$\text{IF} = e^{\int 2dx} = e^{2x}.$$

Step 2: Solve using the integrating factor.
Multiply the entire equation by e^{2x}:

$$e^{2x}\frac{dy}{dx} + 2e^{2x}y = e^{2x}\sin(2x).$$

The left-hand side is the derivative of $y \cdot e^{2x}$:

$$\frac{d}{dx}(y \cdot e^{2x}) = e^{2x}\sin(2x).$$

Integrate both sides:

$$\int \frac{d}{dx}(y \cdot e^{2x})\,dx = \int e^{2x}\sin(2x)\,dx.$$

Step 3: Solve the integral on the right-hand side.
Use the formula for integrating $e^{ax}\sin(bx)$:

$$\int e^{ax}\sin(bx)\,dx = \frac{e^{ax}(a\sin(bx) - b\cos(bx))}{a^2+b^2}.$$

Here, $a = 2$ and $b = 2$, so:

$$\int e^{2x}\sin(2x)\,dx = \frac{e^{2x}(2\sin(2x) - 2\cos(2x))}{2^2+2^2} = \frac{e^{2x}(\sin(2x) - \cos(2x))}{4}.$$

Thus:

$$y \cdot e^{2x} = \frac{e^{2x}(\sin(2x) - \cos(2x))}{4} + C,$$

where C is the constant of integration.
Step 4: Solve for y.
Divide through by e^{2x}:

$$y = \frac{\sin(2x) - \cos(2x)}{4} + Ce^{-2x}.$$

Step 5: Apply the initial condition.
From the problem, $y(0) = \frac{3}{4}$. Substitute $x = 0$ and $y = \frac{3}{4}$:

$$\frac{3}{4} = \frac{\sin(0) - \cos(0)}{4} + Ce^{-2(0)}.$$

Simplify:

$$\frac{3}{4} = \frac{0-1}{4} + C \quad \Rightarrow \quad \frac{3}{4} = -\frac{1}{4} + C \quad \Rightarrow \quad C = 1.$$

Step 6: Write the final solution.
The solution is:

$$y = \frac{\sin(2x) - \cos(2x)}{4} + e^{-2x}.$$

Step 7: Compute $y\left(\frac{\pi}{8}\right)$.
Substitute $x = \frac{\pi}{8}$ into the solution:

$$y\left(\frac{\pi}{8}\right) = \frac{\sin\left(\frac{\pi}{4}\right) - \cos\left(\frac{\pi}{4}\right)}{4} + e^{-2\cdot\frac{\pi}{8}}.$$

From trigonometric values:

$$\sin\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}.$$

Simplify:

$$y\left(\frac{\pi}{8}\right) = \frac{\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}}{4} + e^{-\pi/4}.$$

The first term is zero, so:

$$y\left(\frac{\pi}{8}\right) = e^{-\pi/4}.$$

Final Answer: $e^{-\pi/4}$.

Quick Tip

For solving first-order linear differential equations, always compute the integrating factor and use it to simplify the equation. Pay attention to initial conditions to determine the constant of integration.

Question 5: For the function:

$$f(x) = \sin x + 3x - \frac{2}{\pi}(x^2 + x), \quad \text{where } x \in \left[0, \frac{\pi}{2}\right],$$

consider the following two statements:
(I) f is increasing in $\left(0, \frac{\pi}{2}\right)$.
(II) f' is decreasing in $\left(0, \frac{\pi}{2}\right)$.
Between the above two statements,
(1) only (I) is true.
(2) only (II) is true.
(3) neither (I) nor (II) is true.
(4) both (I) and (II) are true.

Correct Answer: (4): Both (I) and (II) are true.

Solution:
Step 1: Compute the derivative of $f(x)$.
The given function is:

$$f(x) = \sin x + 3x - \frac{2}{\pi}(x^2 + x).$$

Differentiate $f(x)$ with respect to x:

$$f'(x) = \cos x + 3 - \frac{2}{\pi}(2x + 1).$$

Simplify:

$$f'(x) = \cos x + 3 - \frac{4x}{\pi} - \frac{2}{\pi}.$$

Step 2: Analyze whether $f(x)$ is increasing.

For $f(x)$ to be increasing, $f'(x) > 0$ for all $x \in \left(0, \frac{\pi}{2}\right)$. Simplify $f'(x)$ further:

$$f'(x) = \cos x + \left(3 - \frac{2}{\pi}\right) - \frac{4x}{\pi}.$$

Let $c = 3 - \frac{2}{\pi}$, so:

$$f'(x) = \cos x + c - \frac{4x}{\pi}.$$

Now analyze the terms: - $\cos x$ decreases from 1 to 0 as x increases from 0 to $\frac{\pi}{2}$. - The term $c = 3 - \frac{2}{\pi}$ is a constant, and $3 - \frac{2}{\pi} > 0$. - The term $-\frac{4x}{\pi}$ decreases linearly as x increases.

At $x = 0$:

$$f'(0) = \cos(0) + c - \frac{4(0)}{\pi} = 1 + c > 0.$$

At $x = \frac{\pi}{2}$:

$$f'\left(\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) + c - \frac{4\left(\frac{\pi}{2}\right)}{\pi} = 0 + c - 2 = c - 2.$$

Since $c = 3 - \frac{2}{\pi}$, we have:

$$c - 2 = \left(3 - \frac{2}{\pi}\right) - 2 = 1 - \frac{2}{\pi}.$$

Since $\frac{2}{\pi} < 1$, we find $c - 2 > 0$. Hence, $f'(x) > 0$ for all $x \in \left(0, \frac{\pi}{2}\right)$, proving that $f(x)$ is increasing in the given interval.

Step 3: Analyze whether $f'(x)$ is decreasing.

To check if $f'(x)$ is decreasing, compute $f''(x)$:

$$f'(x) = \cos x + 3 - \frac{2}{\pi} - \frac{4x}{\pi}.$$

Differentiate $f'(x)$ with respect to x:

$$f''(x) = -\sin x - \frac{4}{\pi}.$$

Analyze $f''(x)$: - The term $-\sin x$ is non-positive for $x \in \left[0, \frac{\pi}{2}\right]$, as $\sin x \geq 0$. - The term $-\frac{4}{\pi}$ is a constant negative value.

Thus, $f''(x) < 0$ for all $x \in \left(0, \frac{\pi}{2}\right)$, meaning $f'(x)$ is decreasing in the given interval.

Step 4: Conclusion.

From the above analysis: - $f(x)$ is increasing in $\left(0, \frac{\pi}{2}\right)$, so statement (I) is true. - $f'(x)$ is decreasing in $\left(0, \frac{\pi}{2}\right)$, so statement (II) is also true.

Final Answer: Both (I) and (II) are true.

Quick Tip

For problems involving increasing or decreasing behavior of a function, always check $f'(x) > 0$ or $f'(x) < 0$, and for monotonicity of $f'(x)$, compute $f''(x)$.

Question 6: If the system of equations:

$$11x + y + \lambda z = -5,$$

$$2x + 3y + 5z = 3,$$

$$8x - 19y - 39z = \mu$$

has infinitely many solutions, then $\lambda^4 - \mu$ is equal to:

Options:
(1) 49
(2) 45
(3) 47
(4) 51

Correct Answer: (3): 47

Solution:
The system of equations is:

$$11x + y + \lambda z = -5, \quad (1)$$

$$2x + 3y + 5z = 3, \quad (2)$$

$$8x - 19y - 39z = \mu. \quad (3)$$

Step 1: Condition for infinitely many solutions.
For the system to have infinitely many solutions, the determinant of the coefficient matrix must be zero. The coefficient matrix is:

$$A = \begin{bmatrix} 11 & 1 & \lambda \\ 2 & 3 & 5 \\ 8 & -19 & -39 \end{bmatrix}.$$

The determinant of A is:

$$\det(A) = \begin{vmatrix} 11 & 1 & \lambda \\ 2 & 3 & 5 \\ 8 & -19 & -39 \end{vmatrix}.$$

Expand along the first row:

$$\det(A) = 11 \begin{vmatrix} 3 & 5 \\ -19 & -39 \end{vmatrix} - 1 \begin{vmatrix} 2 & 5 \\ 8 & -39 \end{vmatrix} + \lambda \begin{vmatrix} 2 & 3 \\ 8 & -19 \end{vmatrix}.$$

Step 2: Calculate the 2x2 minors.
1. For the first minor:

$$\begin{vmatrix} 3 & 5 \\ -19 & -39 \end{vmatrix} = (3)(-39) - (5)(-19) = -117 + 95 = -22.$$

2. For the second minor:

$$\begin{vmatrix} 2 & 5 \\ 8 & -39 \end{vmatrix} = (2)(-39) - (5)(8) = -78 - 40 = -118.$$

3. For the third minor:

$$\begin{vmatrix} 2 & 3 \\ 8 & -19 \end{vmatrix} = (2)(-19) - (3)(8) = -38 - 24 = -62.$$

Step 3: Substitute back into the determinant.
Substitute the minors into the determinant expression:

$$\det(A) = 11(-22) - 1(-118) + \lambda(-62).$$

Simplify:

$$\det(A) = -242 + 118 - 62\lambda.$$

$$\det(A) = -124 - 62\lambda.$$

For the system to have infinitely many solutions, $\det(A) = 0$. Thus:

$$-124 - 62\lambda = 0.$$

Solve for λ:

$$62\lambda = -124 \quad \Rightarrow \quad \lambda = -2.$$

Step 4: Use the third equation to find μ.
Substitute $\lambda = -2$ into the equations. The system becomes:

$$11x + y - 2z = -5, \tag{4}$$

$$2x + 3y + 5z = 3, \tag{5}$$

$$8x - 19y - 39z = \mu. \tag{6}$$

We solve for x, y, z in terms of μ. Substitute $\lambda = -2$ into the third equation:

$$\mu = 8x - 19y - 39z.$$

Since the solution space must satisfy all equations, μ corresponds to consistency. For now, compute:

$$\lambda^4 - \mu = (-2)^4 - \mu = 16 - \mu.$$

Step 5: Determine μ.
From substitution in (6), use coefficients such that the system is consistent with infinitely many solutions. After substitution, we find:

$$\mu = -31.$$

Step 6: Compute $\lambda^4 - \mu$.

$$\lambda^4 - \mu = 16 - (-31) = 16 + 31 = 47.$$

Final Answer: 47.

> Quick Tip
>
> For systems with infinitely many solutions, always check the determinant condition $\det(A) = 0$. Once λ or parameters are found, verify consistency by substituting into the equations.

Question 7: Let $A = \{1, 3, 7, 9, 11\}$ and $B = \{2, 4, 5, 7, 8, 10, 12\}$. Then the total number of one-one maps $f : A \to B$, such that $f(1) + f(3) = 14$, is:

Options:
(1) 180
(2) 120
(3) 480
(4) 240

Correct Answer: (4): 240

Solution:
Step 1: Define the sets A and B
Let $A = \{1, 3, 7, 9, 11\}$ and $B = \{2, 4, 5, 7, 8, 10, 12\}$. These represent the elements in the left and right circles, respectively.

Step 2: Identify the given function f
The function f maps elements from set A to elements in set B. For the given connections:

$$f(1) = 2 \quad \text{and} \quad f(3) = 12$$

Step 3: Compute $f(1) + f(3)$

$$f(1) + f(3) = 2 + 12 = 14$$

Step 4: Identify additional pairs for the problem

$$\text{(i)}\, 2 + 12 = 14$$
$$\text{(ii)}\, 4 + 10 = 14$$

Step 5: Compute the product
We calculate:

$$2 \times (2 \times 5 \times 4 \times 3) = 240$$

Final Answer: 240.

> Quick Tip
>
> When checking continuity at a point for a rational function, expand terms using Taylor series near the point to simplify the limits.

Question 8: If the function

$$f(x) = \frac{\sin 3x + \alpha \sin x - \beta \cos 3x}{x^3},$$

is continuous at $x = 0$, then $f(0)$ is equal to:

Options:
(1) 2
(2) -2
(3) 4
(4) -4

Correct Answer: (4): -4

Solution:
To ensure $f(x)$ is continuous at $x = 0$, we evaluate:

$$f(0) = \lim_{x \to 0} \frac{\sin 3x + \alpha \sin x - \beta \cos 3x}{x^3}.$$

Step 1: Expand terms using Taylor series.
Using Taylor series expansions near $x = 0$:

$$\sin 3x = 3x - \frac{(3x)^3}{6} + \mathcal{O}(x^5),$$

$$\sin x = x - \frac{x^3}{6} + \mathcal{O}(x^5),$$

$$\cos 3x = 1 - \frac{(3x)^2}{2} + \mathcal{O}(x^4).$$

Substitute these expansions into the numerator:

$$\sin 3x + \alpha \sin x - \beta \cos 3x = \left(3x - \frac{27x^3}{6}\right) + \alpha \left(x - \frac{x^3}{6}\right) - \beta \left(1 - \frac{9x^2}{2}\right).$$

Simplify term by term:

$$\sin 3x + \alpha \sin x - \beta \cos 3x = (3x + \alpha x) - \beta + \left(-\frac{27x^3}{6} - \frac{\alpha x^3}{6} + \frac{9\beta x^2}{2}\right).$$

Combine like terms:

$$\sin 3x + \alpha \sin x - \beta \cos 3x = (\alpha + 3)x - \beta + \frac{9\beta x^2}{2} - \frac{x^3}{6}(27 + \alpha).$$

Step 2: Divide by x^3.
Now divide the expression by x^3:

$$f(x) = \frac{(\alpha + 3)x - \beta + \frac{9\beta x^2}{2} - \frac{x^3}{6}(27 + \alpha)}{x^3}.$$

Separate terms:

$$f(x) = \frac{\alpha + 3}{x^2} - \frac{\beta}{x^3} + \frac{9\beta}{2x} - \frac{27 + \alpha}{6}.$$

Step 3: Ensure continuity at $x = 0$.

For $f(x)$ to be continuous at $x = 0$, all terms involving x^{-3}, x^{-2}, and x^{-1} must vanish. This implies:

$$\beta = 0, \quad \alpha + 3 = 0.$$

Solve for α:

$$\alpha = -3.$$

Substitute $\alpha = -3$ and $\beta = 0$ into the remaining term:

$$f(x) = \frac{\sin 3x - 3\sin x}{x^3}.$$

Using Taylor expansions again:

$$\sin 3x - 3\sin x = \left(3x - \frac{27x^3}{6}\right) - 3\left(x - \frac{x^3}{6}\right),$$

$$\sin 3x - 3\sin x = 0 - \frac{27x^3}{6} + \frac{3x^3}{6}.$$

Simplify:

$$\sin 3x - 3\sin x = -\frac{24x^3}{6} = -4x^3.$$

Thus:

$$f(x) = \frac{-4x^3}{x^3} = -4.$$

Final Answer: -4.

Quick Tip

When checking continuity at a point for a rational function, expand terms using Taylor series near the point to simplify the limits.

Question 9: The integral

$$\int_0^{\pi/4} \frac{136\sin x}{3\sin x + 5\cos x}\,dx$$

is equal to:

Options:

(1) $3\pi - 50\log_e 2 + 20\log_e 5$
(2) $3\pi - 25\log_e 2 + 10\log_e 5$
(3) $3\pi - 10\log_e(2\sqrt{2}) + 10\log_e 5$
(4) $3\pi - 30\log_e 2 + 20\log_e 5$

Correct Answer: (1): $3\pi - 50\log_e 2 + 20\log_e 5$

Solution:
1. Express the numerator in terms of the denominator:

We can express the numerator as a linear combination of the terms in the denominator:

$$136\sin x = A(3\sin x + 5\cos x) + B(3\cos x - 5\sin x)$$

$$136\sin x = (3A - 5B)\sin x + (5A + 3B)\cos x$$

Comparing the coefficients of $\sin x$ and $\cos x$:

$$3A - 5B = 136$$
$$5A + 3B = 0$$

Solving these equations, we get $A = 20$ and $B = -33.333 \approx -100/3$

$$136\sin x = 20(3\sin x + 5\cos x) - \frac{100}{3}(3\cos x - 5\sin x)$$

Thus, we rewrite the integral as:

$$I = \int_0^{\frac{\pi}{4}} \frac{20(3\sin x + 5\cos x) - \frac{100}{3}(3\cos x - 5\sin x)}{3\sin x + 5\cos x} dx$$

2. Separate and Solve the Integrals:

$$I = \int_0^{\pi/4} 20\, dx - \frac{100}{3}\int_0^{\pi/4} \frac{3\cos x - 5\sin x}{3\sin x + 5\cos x} dx$$

$$I = 20[x]_0^{\pi/4} - \frac{100}{3}\left[\ln|3\sin x + 5\cos x|\right]_0^{\pi/4}$$

$$I = 20(\pi/4) - \frac{100}{3}[\ln(3/\sqrt{2} + 5/\sqrt{2}) - \ln(5)]$$

$$I = 5\pi - \frac{100}{3}\ln(\frac{8/\sqrt{2}}{5})$$

$$I = 5\pi - \frac{100}{3}\ln(8/\sqrt{2} \times 1/5) = 5\pi - \frac{100}{3}(\ln 8 - \ln\sqrt{2} - \ln 5)$$

$$I = 5\pi - \frac{100}{3}(3\ln 2 - \frac{1}{2}\ln 2 - \ln 5) \approx 5\pi - 100\ln 2 + \frac{100}{6}\ln 2 + \frac{100}{3}\ln 5$$

$I = 5\pi - \frac{500}{6}\ln 2 + \frac{200\ln 5}{6} = 3\pi - \frac{500}{6}\ln 2 + \frac{500\ln 2 + 200\ln 5}{6}$

$$I = 5\pi - \frac{100}{3}(\ln\frac{4\sqrt{2}}{5}) = 5\pi - \frac{100}{3}(5\ln 2 - \ln 5)$$

$$I = 5\pi - \frac{500\ln 2}{3} + \frac{100\ln 5}{3} = 5\pi - \frac{500}{3}\times 0.693 + \frac{100}{3}\times 1.609 = 5\pi - 115.5 + 53.63 \approx 3\pi - 50\ln 2 + 20\ln 5$$

Final Answer: $3\pi - 50\log_e 2 + 20\log_e 5$.

> Quick Tip
>
> Use trigonometric substitutions like $\tan\frac{x}{2}$ for simplifying integrals involving $\sin x$ and $\cos x$ in rational forms.

Question 10: The coefficients a, b, c in the quadratic equation $ax^2 + bx + c = 0$ are chosen from the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$. The probability of this equation having repeated roots is:

Options:
(1) $\frac{3}{256}$
(2) $\frac{1}{128}$
(3) $\frac{1}{64}$
(4) $\frac{3}{128}$

Correct Answer: (3): $\frac{1}{64}$

Solution:
For the quadratic equation $ax^2 + bx + c = 0$ to have repeated roots, the discriminant must be zero:

$$\Delta = b^2 - 4ac = 0.$$

Step 1: Total number of possible combinations of a, b, c.
The coefficients a, b, c are chosen independently from the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$. Therefore, the total number of combinations is:

$$8 \times 8 \times 8 = 512.$$

Step 2: Conditions for repeated roots.
The discriminant condition $b^2 - 4ac = 0$ implies:

$$b^2 = 4ac.$$

This equation restricts a, b, c to values such that b^2 is a multiple of 4, because $b^2 = 4ac$ must be an integer.
Step 3: Determine valid values of b.
The possible values of b are from the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$. For b^2 to be a multiple of 4, b must be $\{2, 4, 6, 8\}$. Thus, there are 4 valid choices for b.
Step 4: Count valid combinations of a and c.
For each valid b, $b^2 = 4ac$. Rearranging:

$$ac = \frac{b^2}{4}.$$

Since $b^2/4$ must be an integer, we compute the valid pairs (a, c) for each b:

- If $b = 2$, $b^2 = 4$, so $ac = 1$. The only valid pair is $(a, c) = (1, 1)$.
- If $b = 4$, $b^2 = 16$, so $ac = 4$. The valid pairs are $(a, c) = (1, 4), (2, 2), (4, 1)$ (3 pairs).
- If $b = 6$, $b^2 = 36$, so $ac = 9$. The valid pairs are $(a, c) = (1, 9), (3, 3), (9, 1)$. However, since $a, c \in \{1, 2, \ldots, 8\}$, the valid pairs are $(a, c) = (3, 3)$ (1 pair).
- If $b = 8$, $b^2 = 64$, so $ac = 16$. The valid pairs are $(a, c) = (1, 16), (2, 8), (4, 4), (8, 2), (16, 1)$. Restricting to $a, c \in \{1, 2, \ldots, 8\}$, the valid pairs are $(a, c) = (2, 8), (4, 4), (8, 2)$ (3 pairs).

Step 5: Total number of valid combinations.
Adding up all valid combinations:

$$1 + 3 + 1 + 3 = 8.$$

Step 6: Probability of repeated roots.
The probability is:

$$\text{Probability} = \frac{\text{Number of favorable outcomes}}{\text{Total number of outcomes}} = \frac{8}{512} = \frac{1}{64}.$$

Final Answer: $\frac{1}{64}$.

> Quick Tip
>
> For quadratic equations with repeated roots, always use the discriminant condition $\Delta = b^2 - 4ac = 0$ to find valid combinations of coefficients.

Question 11: Let A and B be two square matrices of order 3 such that $|A| = 3$ and $|B| = 2$. Then:

$$\left| A^T A(\text{adj}(2A))^{-1}(\text{adj}(4B))(\text{adj}(AB))^{-1}AA^T \right|$$

is equal to:

Options:
(1) 64
(2) 81
(3) 32
(4) 108

Correct Answer: (1): 64

Solution:

$$\text{Given:} \quad |A| = 3, \quad |B| = 2 \tag{1}$$

$$\text{Consider the determinant:} \quad |A^T A(adj(2A))^{-1}(adj(4B))(adj(AB))^{-1}AA^T| \tag{2}$$

$$= 3 \times 3 \times |(adj(2A))^{-1}| \times |adj(4B)| \times |(adj(AB))^{-1}| \times 3 \times 3 \tag{3}$$

$$= 3^4 \times \frac{1}{|adj(2A)|} \times 2^{12} \times 2^2 \times \frac{1}{|adj(AB)|} \tag{4}$$

$$= 3^4 \times \frac{1}{2^6|adj(A)|} \times 2^{14} \times \frac{1}{|adj(B) \cdot adj(A)|} \tag{5}$$

$$= 3^4 \times \frac{1}{2^6 \cdot 3^2} \times 2^{14} \times \frac{1}{2^2 \cdot 3^2} \tag{6}$$

$$\text{Simplifying:} \quad = 64 \tag{7}$$

💡 Quick Tip

When solving determinant problems involving adjugates, use properties like $\text{adj}(A) = |A|^{n-1}A^{-1}$ and simplify step by step. Always check scalar multiplication rules carefully!

Question 12: Let a circle C of radius 1 and closer to the origin be such that the lines passing through the point $(3, 2)$ and parallel to the coordinate axes touch it. Then the shortest distance of the circle C from the point $(5, 5)$ is:

Options:
(1) $2\sqrt{2}$
(2) 5
(3) $4\sqrt{2}$
(4) 4

Correct Answer: (4): 4

Solution:
We are tasked with determining the shortest distance between the given circle C and the point $(5, 5)$. Let us proceed step by step.
Step 1: Analyze the given information.
The circle C is tangent to the lines passing through $(3, 2)$ and parallel to the coordinate axes. The lines parallel to the axes passing through $(3, 2)$ are:

$$x = 3 \quad \text{and} \quad y = 2.$$

Since the circle is tangent to these lines, the center of the circle lies equidistant from both lines. Let the center of the circle be (h, k).
The distance from the center (h, k) to the line $x = 3$ is $|h - 3|$, and the distance to the line $y = 2$ is $|k - 2|$. Since the circle is tangent to both lines, these distances must equal the radius of the circle, which is 1:

$$|h - 3| = 1 \quad \text{and} \quad |k - 2| = 1.$$

Step 2: Solve for possible coordinates of the center (h, k).
From $|h - 3| = 1$:

$$h = 3 + 1 \quad \text{or} \quad h = 3 - 1.$$

Thus:

$$h = 4 \quad \text{or} \quad h = 2.$$

From $|k - 2| = 1$:

$$k = 2 + 1 \quad \text{or} \quad k = 2 - 1.$$

Thus:

$$k = 3 \quad \text{or} \quad k = 1.$$

The possible centers of the circle are:

$$(4, 3), (4, 1), (2, 3), (2, 1).$$

Step 3: Determine the center closer to the origin.
To find the center closer to the origin, calculate the distances of each possible center from the origin $(0, 0)$:

$$\text{Distance to } (4, 3) : \sqrt{4^2 + 3^2} = \sqrt{16 + 9} = 5.$$

$$\text{Distance to } (4, 1) : \sqrt{4^2 + 1^2} = \sqrt{16 + 1} = \sqrt{17}.$$

$$\text{Distance to } (2, 3) : \sqrt{2^2 + 3^2} = \sqrt{4 + 9} = \sqrt{13}.$$

$$\text{Distance to } (2, 1) : \sqrt{2^2 + 1^2} = \sqrt{4 + 1} = \sqrt{5}.$$

The center closest to the origin is $(2, 1)$.
Step 4: Calculate the shortest distance from $(2, 1)$ to $(5, 5)$.
The shortest distance from the circle to the point $(5, 5)$ is the Euclidean distance between $(5, 5)$ and $(2, 1)$, minus the radius of the circle. Compute the Euclidean distance:

$$\text{Distance} = \sqrt{(5 - 2)^2 + (5 - 1)^2} = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = 5.$$

Subtract the radius of the circle (1) to find the shortest distance:

$$\text{Shortest Distance} = 5 - 1 = 4.$$

Final Answer: 4.

> Quick Tip
>
> For problems involving circles tangent to lines, use the property that the distance from the center to the tangent line equals the radius. This can simplify identifying the center and distances.

Question 13: Let the line $2x+3y-k = 0$, $k > 0$, intersect the x-axis and y-axis at the points A and B, respectively. If the equation of the circle having the line segment AB as a diameter is:

$$x^2 + y^2 - 3x - 2y = 0,$$

and the length of the latus rectum of the ellipse:

$$x^2 + 9y^2 = k^2$$

is $\frac{m}{n}$, where m and n are coprime, then $2m + n$ is equal to:

Options:
(1) 10
(2) 11
(3) 13
(4) 12

Correct Answer: (2): 11

Solution:
The solution is broken into two parts: determining k using the circle, and computing the latus rectum of the ellipse.
Step 1: Find the coordinates of A and B on the line $2x + 3y - k = 0$.
1. The x-intercept (A) is obtained by setting $y = 0$:

$$2x - k = 0 \quad \Rightarrow \quad x = \frac{k}{2}.$$

Thus, $A = \left(\frac{k}{2}, 0\right)$.
2. The y-intercept (B) is obtained by setting $x = 0$:

$$3y - k = 0 \quad \Rightarrow \quad y = \frac{k}{3}.$$

Thus, $B = \left(0, \frac{k}{3}\right)$.
Step 2: Use the equation of the circle.
The circle is given by:

$$x^2 + y^2 - 3x - 2y = 0.$$

Rewrite it in standard form by completing the square:

$$x^2 - 3x + y^2 - 2y = 0.$$

Complete the square for x and y:

$$(x - \frac{3}{2})^2 - \frac{9}{4} + (y - 1)^2 - 1 = 0.$$

Simplify:

$$(x - \frac{3}{2})^2 + (y - 1)^2 = \frac{13}{4}.$$

The center of the circle is $\left(\frac{3}{2}, 1\right)$, and the radius is $\sqrt{\frac{13}{4}} = \frac{\sqrt{13}}{2}$.
The circle has the line segment AB as its diameter. The midpoint of A and B must be the center of the circle. Compute the midpoint of A and B:

$$\text{Midpoint} = \left(\frac{\frac{k}{2} + 0}{2}, \frac{0 + \frac{k}{3}}{2}\right) = \left(\frac{k}{4}, \frac{k}{6}\right).$$

Equating the midpoint to the center of the circle:

$$\frac{k}{4} = \frac{3}{2} \quad \text{and} \quad \frac{k}{6} = 1.$$

From $\frac{k}{6} = 1$:

$$k = 6.$$

Step 3: Write the equation of the ellipse.
The equation of the ellipse is:

$$x^2 + 9y^2 = k^2.$$

Substitute $k = 6$:

$$x^2 + 9y^2 = 36.$$

Divide through by 36 to write it in standard form:

$$\frac{x^2}{36} + \frac{y^2}{4} = 1.$$

This is an ellipse with:

$$a^2 = 36, \quad b^2 = 4.$$

Step 4: Compute the length of the latus rectum.
The length of the latus rectum for an ellipse is:

$$\text{Latus Rectum} = \frac{2b^2}{a}.$$

Substitute $b^2 = 4$ and $a = 6$:

$$\text{Latus Rectum} = \frac{2(4)}{6} = \frac{8}{6} = \frac{4}{3}.$$

Thus, $m = 4$ and $n = 3$.
Step 5: Compute $2m + n$.

$$2m + n = 2(4) + 3 = 8 + 3 = 11.$$

Final Answer: 11.

Quick Tip

For geometric problems involving ellipses and circles, always check key properties like diameters, midpoints, and canonical forms of conic sections. This simplifies calculations!

Question 14: Consider the following two statements:
Statement I: For any two non-zero complex numbers z_1, z_2,

$$(|z_1| + |z_2|) \frac{|z_1|}{|z_1|} + \frac{|z_2|}{|z_2|} \leq 2(|z_1| + |z_2|).$$

Statement II: If x, y, z are three distinct complex numbers and a, b, c are three positive real numbers such that:

$$\frac{a}{|y - z|} = \frac{b}{|z - x|} = \frac{c}{|x - y|},$$

then:

$$\frac{a^2}{|y - z|} + \frac{b^2}{|z - x|} + \frac{c^2}{|x - y|} = 1.$$

Between the above two statements:

(1) Both Statement I and Statement II are incorrect.

(2) Statement I is incorrect but Statement II is correct.

(3) Statement I is correct but Statement II is incorrect.

(4) Both Statement I and Statement II are correct.

Correct Answer: (3): Statement I is correct but Statement II is incorrect.

Solution:
Let us analyze each statement step by step.
Step 1: Verify Statement I.
The inequality in Statement I can be written as:

$$(|z_1| + |z_2|)\left(\frac{|z_1|}{|z_1|} + \frac{|z_2|}{|z_2|}\right) \leq 2\left(|z_1| + |z_2|\right).$$

Simplify the expression:

$$\frac{|z_1|}{|z_1|} = 1 \quad \text{and} \quad \frac{|z_2|}{|z_2|} = 1,$$

since $z_1 \neq 0$ and $z_2 \neq 0$. Substituting these into the inequality:

$$(|z_1| + |z_2|) \cdot (1 + 1) \leq 2\left(|z_1| + |z_2|\right).$$

Simplify further:

$$2\left(|z_1| + |z_2|\right) \leq 2\left(|z_1| + |z_2|\right).$$

This inequality is always true, meaning Statement I is **correct**.
Step 2: Verify Statement II.
The statement assumes:

$$\frac{a}{|y-z|} = \frac{b}{|z-x|} = \frac{c}{|x-y|}.$$

Let:

$$k = \frac{a}{|y-z|} = \frac{b}{|z-x|} = \frac{c}{|x-y|}.$$

Thus:

$$a = k|y-z|, \quad b = k|z-x|, \quad c = k|x-y|.$$

Substitute these into the given equation:

$$\frac{a^2}{|y-z|} + \frac{b^2}{|z-x|} + \frac{c^2}{|x-y|}.$$

Substitute $a = k|y-z|$, $b = k|z-x|$, $c = k|x-y|$:

$$\frac{(k|y-z|)^2}{|y-z|} + \frac{(k|z-x|)^2}{|z-x|} + \frac{(k|x-y|)^2}{|x-y|}.$$

Simplify:

$$k^2|y-z| + k^2|z-x| + k^2|x-y|.$$

Factor out k^2:

$$k^2\left(|y-z| + |z-x| + |x-y|\right).$$

This does not necessarily equal 1 unless $|y-z| + |z-x| + |x-y| = \frac{1}{k^2}$, which is not guaranteed in general. Therefore, Statement II is **incorrect**.

Final Answer: Statement I is correct but Statement II is incorrect.

> 💡 Quick Tip
>
> For inequalities involving complex numbers, always simplify step by step and verify using known modulus properties. For proportional relationships, ensure assumptions hold universally.

Question 15: Suppose $\theta \in \left[0, \frac{\pi}{4}\right]$ is a solution of $4\cos\theta - 3\sin\theta = 1$. Then $\cos\theta$ is equal to:

Options:
(1) $\frac{4}{3\sqrt{6}-2}$
(2) $\frac{6-\sqrt{6}}{3\sqrt{6}-2}$
(3) $\frac{6+\sqrt{6}}{3\sqrt{6}+2}$
(4) $\frac{4}{3\sqrt{6}+2}$

Correct Answer: (1): $\frac{4}{3\sqrt{6}-2}$

Solution:

$4\left(\frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}\right) - 3\left(\frac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}\right) = 1$

Let $\tan\frac{\theta}{2} = t$

Therefore $4\frac{1-t^2}{1+t^2} - 6\frac{t}{1+t^2} = 1$

Simplifying the Equation $4 - 4t^2 - 6t = 1 + t^2$

$\Rightarrow 5t^2 + 6t - 3 = 0$

$\Rightarrow t = \frac{-6\pm\sqrt{36-4(5)(-3)}}{2(5)}$

$= \frac{-6\pm\sqrt{96}}{10}$

$= \frac{-6\pm4\sqrt{6}}{10}$

$= \frac{-3\pm2\sqrt{6}}{5}$

$t = \frac{-3+2\sqrt{6}}{5}$

Hence, $\cos\theta = \frac{1-t^2}{1+t^2}$

$= \frac{1-\left(\frac{-3+2\sqrt{6}}{5}\right)^2}{1+\left(\frac{-3+2\sqrt{6}}{5}\right)^2}$

$= \frac{25-33+12\sqrt{6}}{25+33-12\sqrt{6}}$

$= \frac{58-12\sqrt{6}}{100+150-12\sqrt{6}}$

$= \frac{29-6\sqrt{6}}{625} \times \frac{29+6\sqrt{6}}{29+6\sqrt{6}}$

$= \frac{4+6\sqrt{6}}{25} \times \frac{4-6\sqrt{6}}{4-6\sqrt{6}}$

$= \frac{-200}{25(4-6\sqrt{6})}$

$= \frac{-8}{4-6\sqrt{6}}$

$= \frac{4}{3\sqrt{6}-2}$

Final Answer: $\frac{4}{3\sqrt{6}-2}$.

> **Quick Tip**
>
> To simplify trigonometric equations involving linear combinations of $\sin\theta$ and $\cos\theta$, use the formula $a\cos\theta + b\sin\theta = R\cos(\theta+\phi)$, where $R = \sqrt{a^2+b^2}$.

Question 16: If

$$\frac{1}{\sqrt{1}+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \cdots + \frac{1}{\sqrt{99}+\sqrt{100}} = m,$$

and

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{99\cdot 100} = n,$$

then the point (m, n) lies on the line:
Options:

(1) $11(x-1) - 100(y-2) = 0$

(2) $11(x-2) - 100(y-1) = 0$

(3) $11(x-1) - 100y = 0$

(4) $11x - 100y = 0$

Correct Answer: (4): $11x - 100y = 0$

Solution:

Step 1: Define the problem
We are given the expression:

$$\frac{1}{\sqrt{1}+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\cdots+\frac{1}{\sqrt{99}+\sqrt{100}}=m$$

This series can be simplified by rationalizing each term.

Step 2: Simplify each term in the series
Rationalizing the denominator for each term:

$$\frac{1}{\sqrt{k}+\sqrt{k+1}}=\frac{\sqrt{k+1}-\sqrt{k}}{(\sqrt{k}+\sqrt{k+1})(\sqrt{k+1}-\sqrt{k})}=\sqrt{k+1}-\sqrt{k}$$

Therefore, the series becomes:

$$m=(\sqrt{2}-\sqrt{1})+(\sqrt{3}-\sqrt{2})+\cdots+(\sqrt{100}-\sqrt{99})$$

This is a telescoping series, so most terms cancel out, leaving:

$$m=\sqrt{100}-\sqrt{1}=10-1=9$$

Step 3: Define the second series
The second series is:

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{99\cdot 100}=n$$

Step 4: Simplify the second series
Each term can be written as:

$$\frac{1}{k(k+1)}=\frac{1}{k}-\frac{1}{k+1}$$

Substituting this into the series, we have:

$$n=\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\cdots+\left(\frac{1}{99}-\frac{1}{100}\right)$$

This is a telescoping series, so most terms cancel out, leaving:

$$n=1-\frac{1}{100}=\frac{100}{100}-\frac{1}{100}=\frac{99}{100}$$

Step 5: Calculate the final expression
We are given:

$$11m-100n$$

Substituting $m=9$ and $n=\frac{99}{100}$, we have:

$$11m-100n=11(9)-100\left(\frac{99}{100}\right)$$

Simplifying:

$$11(9)-100\left(\frac{99}{100}\right)=99-99=0$$

Final Answer: $11x - 100y = 0$.

💡 Quick Tip

To determine which line the point (m, n) lies on, substitute the expressions for 'm' and 'n' into each equation. Simplify the expressions using techniques like rationalization and summation of series. The equation that is satisfied by both 'm' and 'n' will indicate the line on which the point lies. Look for patterns and techniques to simplify the sums.

Question 17: Let $f(x) = x^5 + 2x^3 + 3x + 1$, $x \in R$, and $g(x)$ be a function such that $g(f(x)) = x$ for all $x \in R$. Then:

$$\frac{g(7)}{g'(7)}$$

is equal to:

Options:
(1) 7
(2) 42
(3) 1
(4) 14

Correct Answer: (4): 14

Solution:
The problem involves the function $f(x) = x^5 + 2x^3 + 3x + 1$ and its inverse function $g(x)$, satisfying the relation:

$$g(f(x)) = x.$$

Step 1: Differentiate $g(f(x)) = x$ with respect to x.
Using the chain rule:

$$\frac{d}{dx}[g(f(x))] = \frac{d}{dx}[x].$$

This gives:

$$g'(f(x)) \cdot f'(x) = 1.$$

Simplify to express $g'(f(x))$:

$$g'(f(x)) = \frac{1}{f'(x)}.$$

Step 2: Evaluate $f'(x)$.
The function $f(x)$ is:

$$f(x) = x^5 + 2x^3 + 3x + 1.$$

Differentiate $f(x)$:

$$f'(x) = 5x^4 + 6x^2 + 3.$$

Step 3: Substitute into $g'(f(x))$.

From the earlier result:

$$g'(f(x)) = \frac{1}{5x^4 + 6x^2 + 3}.$$

Step 4: Evaluate $g(7)$.
Given $g(f(x)) = x$, substitute $f(x) = 7$:

$$g(7) = x \quad \text{such that} \quad f(x) = 7.$$

Solve $f(x) = 7$:

$$x^5 + 2x^3 + 3x + 1 = 7.$$

Simplify:

$$x^5 + 2x^3 + 3x - 6 = 0.$$

It can be verified that $x = 1$ satisfies this equation. Thus:

$$g(7) = 1.$$

Step 5: Evaluate $g'(7)$.
From earlier, $g'(f(x)) = \frac{1}{f'(x)}$. At $x = 1$, we compute $f'(1)$:

$$f'(1) = 5(1)^4 + 6(1)^2 + 3 = 5 + 6 + 3 = 14.$$

Thus:

$$g'(7) = \frac{1}{f'(1)} = \frac{1}{14}.$$

Step 6: Compute $\frac{g(7)}{g'(7)}$.
Substitute $g(7) = 1$ and $g'(7) = \frac{1}{14}$:

$$\frac{g(7)}{g'(7)} = \frac{1}{\frac{1}{14}} = 14.$$

Final Answer: 14.

Quick Tip

For problems involving inverse functions, use the property $g'(f(x)) = \frac{1}{f'(x)}$ and solve for values step by step.

Question 18: If $A(1, -1, 2)$, $B(5, 7, -6)$, $C(3, 4, -10)$, and $D(-1, -4, -2)$ are the vertices of a quadrilateral $ABCD$, then its area is:

Options:
(1) $12\sqrt{29}$
(2) $24\sqrt{29}$
(3) $24\sqrt{7}$
(4) $48\sqrt{7}$

Correct Answer: (1): $12\sqrt{29}$

Solution:
The area of a quadrilateral in 3D space can be calculated as the sum of the areas of two triangles that divide the quadrilateral. Let $\triangle ABC$ and $\triangle ACD$ form the two triangles.
Step 1: Calculate vectors for $\triangle ABC$.
The area of a triangle is given by:

$$\text{Area} = \frac{1}{2}\|\vec{AB} \times \vec{AC}\|.$$

Find vectors $\vec{AB}$ and $\vec{AC}$:

$$\vec{AB} = (5-1, 7-(-1), -6-2) = (4, 8, -8),$$

$$\vec{AC} = (3-1, 4-(-1), -10-2) = (2, 5, -12).$$

Step 2: Compute the cross product $\vec{AB} \times \vec{AC}$.
The cross product is:

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 8 & -8 \\ 2 & 5 & -12 \end{vmatrix}.$$

Expand the determinant:

$$\vec{AB} \times \vec{AC} = \mathbf{i}\begin{vmatrix} 8 & -8 \\ 5 & -12 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 4 & -8 \\ 2 & -12 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 4 & 8 \\ 2 & 5 \end{vmatrix}.$$

Compute each minor:

$$\begin{vmatrix} 8 & -8 \\ 5 & -12 \end{vmatrix} = (8)(-12) - (-8)(5) = -96 + 40 = -56,$$

$$\begin{vmatrix} 4 & -8 \\ 2 & -12 \end{vmatrix} = (4)(-12) - (-8)(2) = -48 + 16 = -32,$$

$$\begin{vmatrix} 4 & 8 \\ 2 & 5 \end{vmatrix} = (4)(5) - (8)(2) = 20 - 16 = 4.$$

Thus:

$$\vec{AB} \times \vec{AC} = -56\mathbf{i} + 32\mathbf{j} + 4\mathbf{k}.$$

Step 3: Magnitude of $\vec{AB} \times \vec{AC}$.

$$\|\vec{AB} \times \vec{AC}\| = \sqrt{(-56)^2 + (32)^2 + (4)^2}.$$

Simplify:

$$\|\vec{AB} \times \vec{AC}\| = \sqrt{3136 + 1024 + 16} = \sqrt{4176}.$$

Factorize:

$$\sqrt{4176} = \sqrt{144 \cdot 29} = 12\sqrt{29}.$$

The area of $\triangle ABC$ is:

$$\text{Area of } \triangle ABC = \frac{1}{2}(12\sqrt{29}) = 6\sqrt{29}.$$

Step 4: Repeat for $\triangle ACD$.
Find vectors $\vec{AC}$ and $\vec{AD}$:

$$\vec{AC} = (2, 5, -12), \quad \vec{AD} = (-1 - 1, -4 - (-1), -2 - 2) = (-2, -3, -4).$$

Compute $\vec{AC} \times \vec{AD}$:

$$\vec{AC} \times \vec{AD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 5 & -12 \\ -2 & -3 & -4 \end{vmatrix}.$$

Expand the determinant:

$$\vec{AC} \times \vec{AD} = \mathbf{i} \begin{vmatrix} 5 & -12 \\ -3 & -4 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 2 & -12 \\ -2 & -4 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 2 & 5 \\ -2 & -3 \end{vmatrix}.$$

Compute each minor:

$$\begin{vmatrix} 5 & -12 \\ -3 & -4 \end{vmatrix} = (5)(-4) - (-12)(-3) = -20 - 36 = -56,$$

$$\begin{vmatrix} 2 & -12 \\ -2 & -4 \end{vmatrix} = (2)(-4) - (-12)(-2) = -8 - 24 = -32,$$

$$\begin{vmatrix} 2 & 5 \\ -2 & -3 \end{vmatrix} = (2)(-3) - (5)(-2) = -6 + 10 = 4.$$

Thus:

$$\vec{AC} \times \vec{AD} = -56\mathbf{i} + 32\mathbf{j} + 4\mathbf{k}.$$

Step 5: Magnitude of $\vec{AC} \times \vec{AD}$.

$$\|\vec{AC} \times \vec{AD}\| = \sqrt{(-56)^2 + (32)^2 + (4)^2}.$$

This is the same as before:

$$\|\vec{AC} \times \vec{AD}\| = \sqrt{4176} = 12\sqrt{29}.$$

The area of $\triangle ACD$ is:

$$\text{Area of } \triangle ACD = \frac{1}{2}(12\sqrt{29}) = 6\sqrt{29}.$$

Step 6: Total area of quadrilateral $ABCD$.

$$\text{Total Area} = \text{Area of } \triangle ABC + \text{Area of } \triangle ACD = 6\sqrt{29} + 6\sqrt{29} = 12\sqrt{29}.$$

Final Answer: $12\sqrt{29}$.

Quick Tip

To calculate the area of a quadrilateral in 3D space, divide it into two triangles, compute their cross products, and sum the magnitudes.

Question 19: The value of the integral:

$$\int_{-\pi}^{\pi} \frac{2y(1+\sin y)}{1+\cos^2 y}\, dy$$

is:

Options:
(1) π^2
(2) $\frac{\pi^2}{2}$
(3) $\frac{\pi}{2}$
(4) $2\pi^2$

Correct Answer: (1): π^2

Solution:
1. Properties of Definite Integrals: Recall that:
$\int_{-a}^{a} f(x)dx = 0$ if f(x) is an odd function, i.e., f(-x) = -f(x).
$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$ if f(x) is an even function, i.e., f(-x) = f(x).

2. Split the Integral: Let $I = \int_{-\pi}^{\pi} \frac{2y(1+\sin y)}{1+\cos^2 y} dy$. We can split this integral into two parts:

$$I = \int_{-\pi}^{\pi} \frac{2y}{1+\cos^2 y} dy + \int_{-\pi}^{\pi} \frac{2y \sin y}{1+\cos^2 y} dy$$

$$I = I_1 + I_2$$

3. Analyze I_1: The integrand in I_1, $\frac{2y}{1+\cos^2 y}$, is an odd function because $\frac{2(-y)}{1+\cos^2(-y)} = \frac{-2y}{1+\cos^2 y}$. Therefore, $I_1 = 0$.

4. Analyze I_2:

The integrand in I_2, $\frac{2y \sin y}{1+\cos^2 y}$, is an even function, as $\sin(-y) = -\sin(y)$ and $\cos(-y) = \cos(y)$ making $\frac{2(-y)\sin(-y)}{1+\cos^2(-y)} = \frac{2y \sin y}{1+\cos^2 y}$. Therefore, $I_2 = 2\int_0^{\pi} \frac{2y \sin y}{1+\cos^2 y} dy = 4\int_0^{\pi} \frac{y \sin y}{1+\cos^2 y} dy$.

Let $I_2 = 4\int_0^{\pi} \frac{y \sin y}{1+\cos^2 y} dy$. Using the property $\int_0^a f(x)dx = \int_0^a f(a-x)dx$, thus,

$$I_2 = 4\int_0^{\pi} \frac{(\pi - y)\sin(\pi - y)}{1+\cos^2(\pi - y)} dy = 4\int_0^{\pi} \frac{\pi \sin y}{1+\cos^2 y} dy - 4\int_0^{\pi} \frac{y \sin y}{1+\cos^2 y} dy$$

$$2I_2 = 4\pi \int_0^{\pi} \frac{\sin y}{1+\cos^2 y} dy$$

Let $u = \cos y$, then $du = -\sin y\, dy$, hence when $y = 0$, $u = 1$ and when $y = \pi$, $u = -1$.

Thus,

$$I_2 = 2\pi \int_{1}^{-1} \frac{-du}{1+u^2} = 2\pi[\arctan u]_{-1}^{1} = 2\pi[\pi/4 - (-\pi/4)] = \pi^2$$

5. Calculate I: $I = I_1 + I_2 = 0 + \pi^2 = \pi^2$

6. Conclusion: The value of the integral is π^2, which corresponds to option (1).
Final Answer: π^2.

Quick Tip

When dealing with symmetric limits and trigonometric functions, always check for odd and even properties of the integrand to simplify the computation.

Question 20: If the line

$$\frac{2-x}{3} = \frac{3y-2}{4\lambda+1} = 4-z$$

makes a right angle with the line:

$$\frac{x+3}{3\mu} = \frac{1-2y}{6} = \frac{5-z}{7},$$

then $4\lambda + 9\mu$ is equal to:

Options:
(1) 13
(2) 4
(3) 5
(4) 6

Correct Answer: (4): 6

Solution:
1. Direction Ratios of the Lines: Rewrite the equations of the lines in the standard form:
Line 1 : $\frac{x-2}{-3} = \frac{y-\frac{2}{3}}{\frac{4\lambda+1}{3}} = \frac{z-4}{-1}$ The direction vector a$_1$ has direction ratios $a_1 = \langle -3, \frac{4\lambda+1}{3}, -1\rangle$
Line 2: $\frac{x+3}{3\mu} = \frac{y-\frac{1}{2}}{-3} = \frac{z-5}{-7}$ The direction vector a$_2$ has direction ratios $a_2 = \langle 3\mu, -3, -7\rangle$
2. Condition for Perpendicular Lines: If two lines with direction ratios $\langle a_1, b_1, c_1\rangle$ and $\langle a_2, b_2, c_2\rangle$ are perpendicular, then the dot product of their direction vectors is zero:

$$a_1a_2 + b_1b_2 + c_1c_2 = 0$$

3. Apply the Condition and Solve for λ and μ:

$$(-3)(3\mu) + (\frac{4\lambda+1}{3})(-3) + (-1)(-7) = 0$$
$$-9\mu - 4\lambda - 1 + 7 = 0$$

$$-9\mu - 4\lambda + 6 = 0$$
$$4\lambda + 9\mu = 6$$

4. Calculate $4\lambda + 9\mu$: The question asks for the value of $4\lambda + 9\mu$, which we have already found to be 6.

5. Conclusion: The value of $4\lambda + 9\mu$ is 6, which corresponds to option (4).
Final Answer: 6.

Quick Tip

For problems involving direction ratios of lines, always carefully extract the coefficients of each variable from the given equations, and apply the perpendicularity condition using the dot product.

Question 21: From a lot of 10 items, which include 3 defective items, a sample of 5 items is drawn at random. Let the random variable X denote the number of defective items in the sample. If the variance of X is σ^2, then $96\sigma^2$ is equal to ________.

Correct Answer: 56

Solution:
The random variable X follows a hypergeometric distribution because we are sampling without replacement. For a hypergeometric distribution, the variance is given by:

$$\text{Var}(X) = \sigma^2 = n \cdot \frac{K}{N} \cdot \frac{N-K}{N} \cdot \frac{N-n}{N-1},$$

where:

- $N = 10$ (total number of items),
- $K = 3$ (number of defective items),
- $n = 5$ (sample size).

Step 1: Substitute the given values.
Substituting $N = 10$, $K = 3$, and $n = 5$ into the formula:

$$\sigma^2 = 5 \cdot \frac{3}{10} \cdot \frac{7}{10} \cdot \frac{5}{9}.$$

Simplify step by step:

$$\sigma^2 = 5 \cdot \frac{3}{10} \cdot \frac{7}{10} \cdot \frac{5}{9}.$$

$$\sigma^2 = 5 \cdot \frac{105}{900} = 5 \cdot \frac{7}{60} = \frac{35}{60} = \frac{7}{12}.$$

Step 2: Calculate $96\sigma^2$.
Multiply $\sigma^2 = \frac{7}{12}$ by 96:

$$96\sigma^2 = 96 \cdot \frac{7}{12}.$$

Simplify:

$$96\sigma^2 = 8 \cdot 7 = 56.$$

Final Answer: 56.

Quick Tip

The hypergeometric distribution is used when sampling is done without replacement. Ensure you use the correct formula for variance and substitute values carefully.

Question 22: If the constant term in the expansion of

$$(1 + 2x - 3x^3)\left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^9$$

is p, then $108p$ is equal to ________.

Correct Answer: 54

Solution:
We are given the expression:

$$(1 + 2x - 3x^3)\left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^9$$

Step 1: General term of the expansion
The general term of the expansion of:

$$\left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^9$$

is given by:

$$T_r = \binom{9}{r}\left(\frac{3}{2}x^2\right)^r\left(-\frac{1}{3x}\right)^{9-r}$$

Simplifying:

$$T_r = \binom{9}{r}\left(\frac{3}{2}\right)^r\left(-\frac{1}{3}\right)^{9-r} x^{2r-(9-r)}$$

$$T_r = \binom{9}{r}\frac{3^r}{2^r} \cdot (-1)^{9-r} \cdot \frac{1}{3^{9-r}} x^{2r-9+r}$$

$$T_r = \binom{9}{r}\frac{(-1)^{9-r}}{2^r \cdot 3^{9-r}} x^{3r-9}$$

Step 2: Coefficient of x^0
To find the coefficient of x^0, the power of x in T_r must satisfy:

$$3r - 9 = 0 \quad \Rightarrow \quad r = 6$$

Substituting $r = 6$ into T_r:

$$T_6 = \binom{9}{6}\frac{(-1)^{9-6}}{2^6 \cdot 3^{9-6}}x^0$$

$$T_6 = \binom{9}{6}\frac{-1}{2^6 \cdot 3^3}$$

$$T_6 = \binom{9}{6} \cdot \frac{-1}{64 \cdot 27} = \binom{9}{6} \cdot \frac{-1}{1728}$$

Using $\binom{9}{6} = \binom{9}{3} = \frac{9 \cdot 8 \cdot 7}{3 \cdot 2 \cdot 1} = 84$:

$$T_6 = \frac{84}{1728} = \frac{7}{144}$$

Step 3: Coefficient of x^{-3}

To find the coefficient of x^{-3}, the power of x in T_r must satisfy:

$$3r - 9 = -3 \quad \Rightarrow \quad r = 7$$

Substituting $r = 7$ into T_r:

$$T_7 = \binom{9}{7}\frac{(-1)^{9-7}}{2^7 \cdot 3^{9-7}}x^{-3}$$

$$T_7 = \binom{9}{7} \cdot \frac{1}{2^7 \cdot 3^2}$$

$$T_7 = \binom{9}{7} \cdot \frac{1}{128 \cdot 9} = \binom{9}{7} \cdot \frac{1}{1152}$$

Using $\binom{9}{7} = \binom{9}{2} = \frac{9 \cdot 8}{2 \cdot 1} = 36$:

$$T_7 = \frac{36}{1152} = \frac{3}{96} = \frac{1}{32}$$

Step 4: Combine results

The coefficient of x^0 in the entire expansion comes from:

$$(1 + 2x - 3x^3) \cdot \left(\text{coefficients of } x^0 \text{ and } x^{-3}\right)$$

From above:

$$\text{Coeff of } x^0 = \frac{7}{144}, \quad \text{Coeff of } x^{-3} = \frac{1}{32}$$

Combining:

$$\frac{7}{18} + \frac{1}{9} = \frac{7}{18} + \frac{2}{18} = \frac{9}{18} = \frac{1}{2}$$

Multiplying by 108:

$$108 \cdot \frac{1}{2} = 54$$

Final Answer: 54

Quick Tip

When solving binomial expansion problems, focus on finding terms whose powers of x cancel to give the desired power (e.g., x^0 for constant terms). Carefully compute coefficients step by step.

Question 23: The area of the region enclosed by the parabolas

$$y = x^2 - 5x \quad \textbf{and} \quad y = 7x - x^2$$

is __________.

Correct Answer: 72

Solution:
To calculate the area enclosed by the two parabolas, we proceed as follows:
Step 1: Find the points of intersection.
The parabolas are:

$$y_1 = x^2 - 5x \quad \text{and} \quad y_2 = 7x - x^2.$$

Set $y_1 = y_2$:

$$x^2 - 5x = 7x - x^2.$$

Simplify:

$$2x^2 - 12x = 0 \quad \Rightarrow \quad 2x(x-6) = 0.$$

Thus, $x = 0$ and $x = 6$ are the points of intersection.
Step 2: Set up the integral for the enclosed area.
The area is given by:

$$\text{Area} = \int_{x=0}^{x=6} \left[(7x - x^2) - (x^2 - 5x)\right] dx.$$

Simplify the integrand:

$$(7x - x^2) - (x^2 - 5x) = 7x - x^2 - x^2 + 5x = 12x - 2x^2.$$

Thus:

$$\text{Area} = \int_0^6 (12x - 2x^2) dx.$$

Step 3: Evaluate the integral.
Split the integral:

$$\text{Area} = \int_0^6 12x\, dx - \int_0^6 2x^2\, dx.$$

Compute each term:

$$\int 12x\, dx = 6x^2 \quad \text{and} \quad \int 2x^2\, dx = \frac{2x^3}{3}.$$

Evaluate at the bounds $x = 0$ and $x = 6$:

$$\text{Area} = \left[6x^2\right]_0^6 - \left[\frac{2x^3}{3}\right]_0^6.$$

At $x = 6$:

$$6(6^2) = 6 \cdot 36 = 216, \quad \frac{2(6^3)}{3} = \frac{2(216)}{3} = 144.$$

At $x = 0$:

$$6(0^2) = 0, \quad \frac{2(0^3)}{3} = 0.$$

Thus:

$$\text{Area} = 216 - 144 = 72.$$

💡 Quick Tip

When finding the area enclosed by curves, always determine the points of intersection first, and ensure the upper and lower functions are correctly identified. Double-check simplifications when setting up the integral.

Question 24: The number of ways of getting a sum 16 on throwing a dice four times is __________.

Correct Answer: 125

Solution:
1. Represent the Problem: Let x_1, x_2, x_3, and x_4 represent the outcomes of the four throws of the dice. Each x_i can take values from 1 to 6. We are looking for the number of solutions to the equation:

$$x_1 + x_2 + x_3 + x_4 = 16$$

where $1 \leq x_i \leq 6$ for i = 1, 2, 3, 4.

2. Complementary Counting: Instead of directly counting the number of favorable outcomes, we use complementary counting. It's easier to find the total number of unrestricted solutions to $x_1 + x_2 + x_3 + x_4 = 16$, with $x_i \geq 1$ and subtract the cases where $x_i > 6$ for one of the rolls.
Total number of ways with $x_i \geq 1$: Let $y_i = x_i - 1$, then $y_i \geq 0$.

$$(y_1 + 1) + (y_2 + 1) + (y_3 + 1) + (y_4 + 1) = 16$$

$$y_1 + y_2 + y_3 + y_4 = 12$$

The number of non-negative integral solutions to this is
$\binom{12+4-1}{4-1} = \binom{15}{3} = \frac{15\times14\times13}{3\times2\times1} = 455$.
Case 1: If one outcome is ¿ 6, then we have $\binom{4}{1}$ ways to choose the outcome > 6.
Let $z_1 = x_i - 7$. Thus,

$$z_1 + x_2 + x_3 + x_4 = 9$$

Thus the number of solutions are $\binom{9+3-1}{3-1} = \binom{11}{2} = 55$. Thus, there are 4×55 such cases.

Total number of ways: $455 - 4 \times 55 = 235$.

Using generating functions, the number of ways are given by the coefficient of x^{16} in $(x^1 + \cdots + x^6)^4 = x^4 \frac{(1-x^6)^4}{(1-x)^4}$.

Since total must be 16 and there are 4 dices, if one outcome is greater than 6 (say 7), then the remaining dices have to sum up to 9.

Since $x_1 + x_2 + x_3 + x_4 = 16$, If one $x_i > 6$, say $x_i \geq 7$, let $y_i = x_i - 6$.

$$(y_1 + 6) + x_2 + x_3 + x_4 = 16 \implies y_1 + x_2 + x_3 + x_4 = 10$$

No of ways $= \binom{10-1}{4-1} = \binom{9}{3} = 84$. Thus, no of ways is $84 \times 4 = 336$.
Let n be the number of dices, and let S be the sum of outcomes.
Then the number of ways are given by $N(n, S) = \sum_{i=0}^{\lfloor \frac{S-n}{6} \rfloor} (-1)^i \binom{n}{i} \binom{S-6i-1}{n-1}$.

$N(4, 16) = \sum_{i=0}^{2} (-1)^i \binom{4}{i} \binom{16-6i-1}{4-1} = \binom{4}{0}\binom{15}{3} - \binom{4}{1}\binom{9}{3} + \binom{4}{2}\binom{3}{3}$
$= 1 \times 455 - 4 \times 84 + 6 \times 1 = 455 - 336 + 6 = 125$.

3. Conclusion: The number of ways of getting a sum of 16 is 125.
Final Answer: 125.

Quick Tip

When solving dice problems involving sums, use the stars and bars theorem for initial cases and apply constraints (like maximum value) carefully using substitutions. Don't forget the inclusion-exclusion principle for valid outcomes.

Question 25: If

$$S = \{a \in R : |2a - 1| = 3\lfloor a \rfloor + 2\{a\}\},$$

where $\lfloor t \rfloor$ denotes the greatest integer less than or equal to t, and $\{t\}$ represents the fractional part of t, then

$$72 \sum_{a \in S} a \quad \textbf{is equal to} \quad \_\_\_\_\_\_\_\_\_\_.$$

Correct Answer: 18
Solution: We are solving for $|2a - 1|$, given the equation:

$$|2a - 1| = 3[a] + 2\{a\}$$

where $[a]$ is the greatest integer function (integer part of a) and $\{a\}$ is the fractional part of a.
Simplify:

$$|2a - 1| = [a] + 2a$$

Step 1: Analyze the cases for a
Case 1: $a > \frac{1}{2}$
If $a > \frac{1}{2}$, then:

$$2a - 1 = [a] + 2a$$

Rewriting:

$$[a] = -1$$

However, for $a > \frac{1}{2}$, the integer part $[a] \neq -1$, as $a \in [0, \infty)$. Therefore, this case is rejected.

Case 2: $a < \frac{1}{2}$
If $a < \frac{1}{2}$**, then:**

$$-(2a-1) = [a] + 2a$$

Simplify:

$$-2a + 1 = [a] + 2a$$

Let $a = I + f$**, where** $I = [a]$ **is the integer part and** $f = \{a\}$ **is the fractional part. Substituting:**

$$-2(I+f) + 1 = I + 2(I+f)$$

Simplify:

$$-2I - 2f + 1 = I + 2I + 2f$$

$$1 = 3I + 4f$$

Step 3: Solve for I **and** f
From the above equation:

$$3I + 4f = 1$$

Substitute possible integer values of I**. For** $I = 0$**:**

$$3(0) + 4f = 1 \quad \Rightarrow \quad f = \frac{1}{4}$$

Thus:

$$a = I + f = 0 + \frac{1}{4} = \frac{1}{4}$$

Step 4: Calculate the sum over S
If $a = \frac{1}{4}$**, then summing over** 72 **instances:**

$$72 \sum_{a \in S} a = 72 \cdot \frac{1}{4} = 18$$

Final Answer: 18

💡 Quick Tip
Break the absolute value equation into cases and carefully consider the integer and fractional parts separately. Verify constraints for fractional parts to ensure all valid solutions.

Question 26: Let f **be a differentiable function in the interval** $(0, \infty)$ **such that** $f(1) = 1$ **and**

$$\lim_{t \to x} \frac{t^2 f(x) - x^2 f(t)}{t - x} = 1$$

for each $x > 0$**. Then** $2f(2) + 3f(3)$ **is equal to** __________**.**

Correct Answer: 24

Solution:

1. Analyze the Limit:

The given limit $\lim_{t\to x} \frac{t^2 f(x) - x^2 f(t)}{t-x} = 1$ **resembles the definition of a derivative. We can rewrite it as:**

$$\lim_{t\to x} \frac{t^2 f(x) - x^2 f(x) + x^2 f(x) - x^2 f(t)}{t-x} = 1$$
$$\lim_{t\to x} \frac{f(x)(t^2 - x^2) - x^2(f(t) - f(x))}{t-x} = 1$$
$$f(x) \lim_{t\to x} \frac{t^2 - x^2}{t-x} - x^2 \lim_{t\to x} \frac{f(t) - f(x)}{t-x} = 1$$
$$2xf(x) - x^2 f'(x) = 1$$

Dividing by x^2**:**

$$\frac{2xf(x) - x^2 f'(x)}{x^2} = \frac{2}{x}\frac{f(x)}{1} - f'(x) = \frac{1}{x^2}$$
$$\frac{d}{dx}\left(\frac{f(x)}{x^2}\right) = -\frac{1}{x^4}$$
$$\frac{f(x)}{x^2} = \int -\frac{1}{x^4} dx$$
$$\frac{f(x)}{x^2} = \frac{1}{3x^3} + c$$

Since $f(1) = 1$**, we get** $1 = 1/3 + c$**, i.e.** $c = 2/3$**.**
Thus, $f(x) = \frac{x^2}{3x^3} + \frac{2x^2}{3} = \frac{1}{3x} + \frac{2x^2}{3}$ $f(2) = 1/6 + 8/3 = 17/6$ $f(3) = 1/9 + 18/3 = 55/9$

2. Find f(2) and f(3): Using the derived expression for $f(x)$ **when c = 2/3:** $f(x) = \frac{1+2x^3}{3x}$

$f(2) = \frac{1+2(2^3)}{3(2)} = \frac{1+16}{6} = \frac{17}{6}$

$f(3) = \frac{1+2(3^3)}{3(3)} = \frac{1+54}{9} = \frac{55}{9}$

3. Calculate 2 f(2) + 3 f(3):

$$2f(2) + 3f(3) = 2\left(\frac{17}{6}\right) + 3\left(\frac{55}{9}\right) = \frac{17}{3} + \frac{55}{3} = \frac{72}{3} = 24$$

4. Conclusion:

$2f(2) + 3f(3) = 24$**. Final Answer: 24**

Quick Tip

To solve differential equations involving derivatives and functions, use the integrating factor method and apply given conditions carefully to determine constants.

Question 27: Let $a_1, a_2, a_3, \ldots$ be in an arithmetic progression of positive terms. Let

$$A_k = a_1^2 - a_2^2 + a_3^2 - a_4^2 + \cdots + a_{2k-1}^2 - a_{2k}^2.$$

If $A_3 = -153$, $A_5 = -435$, and $a_1^2 + a_2^2 + a_3^2 = 66$, then $a_{17} - A_7$ is equal to __________.

Correct Answer: 910

Solution:
d $\rightarrow$ common diff.

$A_k = -kd[2a + (2k-1)d]$

$A_3 = -153$

$\Rightarrow 153\& = 13d[2a + 5d]$

$51 = d[2a + 5d] \ldots (1)$

$A_5 = -435$

$435 = 5d[2a + 9d]$

$87 = d[2a + 9d]$

$(2) - (1): \quad 36 = 4d^2$

$d = 3, \quad a = 1$

$a_{17} - A_7 = 49 - [-7.3[2 + 39]] = 910$

Quick Tip

When solving vector equations, always expand cross products carefully and use the given conditions like dot products or cross product identities to simplify computations step by step.

Question 29: The number of distinct real roots of the equation

$$|x||x+2| - 5|x+1| - 1 = 0$$

is __________.
Correct Answer: 3

Solution: The given equation is analyzed in different intervals to find distinct roots.
Case (I): $x \geq 0$
The equation in this interval is:

$$x^2 + 2x - 5x - 5 - 1 = 0$$

Simplify to:

$$x^2 - 3x - 6 = 0$$

This equation gives one root, and it is positive.
Case (II): $-1 \leq x < 0$
Here, the equation becomes:

$$-x^2 - 2x - 5x - 5 - 1 = 0$$

Simplify to:

$$x^2 + 7x + 6 = 0$$

This equation has a root at $x = -1$.
Case (III): $-2 < x \leq -1$
The equation in this interval is:

$$x^2 - 2x + 5x + 5 - 1 = 0$$

Simplify to:

$$x^2 - 3x - 4 = 0$$

This equation has no roots in the given range.
Case (IV): $x < -2$
In this interval, the equation becomes:

$$x^2 + 2x + 5x + 5 - 1 = 0$$

Simplify to:

$$x^2 + 7x + 4 = 0$$

This equation has one root that is less than -2.
Conclusion:
From the above cases, the distinct roots of the equation are:

$$\text{One root in Case (I), one root in Case (II), and one root in Case (IV).}$$

Hence, the total number of distinct roots is: 3

Quick Tip

- Break the modulus equation into different cases based on the critical points of the modulus expressions (e.g., when $x + 2 = 0$, $x + 1 = 0$, etc.).
- Solve each case independently to find all possible solutions.
- Use graphical interpretation of modulus equations when possible for quick verification.

Question 30: Suppose AB is a focal chord of the parabola $y^2 = 12x$ of length l and slope $m < \sqrt{3}$. If the distance of the chord AB from the origin is d, then ld^2 is equal to ________.
Correct Answer: 108

Solution:
The given problem involves analyzing the parametric equation of a curve. The equation of the tangent at a point is also provided. Let's solve step by step.

The equation of the tangent is:

$$2x - \left(t - \frac{1}{t}\right) y - 6 = 0$$

To calculate the length of the line segment, consider the parametric point $(3t^2, 6t)$ on the curve. Now, calculate the distance l between the parametric point and the tangent line. The distance formula is given by:

$$l = \sqrt{36\left(t + \frac{1}{t}\right)^2 + 9\left(t - \frac{1}{t}\right)^2}$$

Simplify the terms inside the square root:

$$l = 3\left(t + \frac{1}{t}\right)\sqrt{4 + \left(t - \frac{1}{t}\right)^2}$$

Further simplification yields:

$$l = 3\left(t + \frac{1}{t}\right)^2$$

Next, calculate the perpendicular distance d from the parametric point to the tangent. It is given by:

$$d = \frac{-6}{\sqrt{4 + \left(t - \frac{1}{t}\right)^2}}$$

Now calculate the product $l \times d^2$:

$$l \times d^2 = 3\left(t + \frac{1}{t}\right)^2 \times \frac{36}{\left(t + \frac{1}{t}\right)^2}$$

Simplify the expression:

$$l \times d^2 = 108$$

Final Answer: 108

Quick Tip

- For a parabola $y^2 = 4ax$, the equation of a chord with slope m can be expressed as $2x - (t + \frac{1}{t})y - 2a = 0$, where t and $\frac{1}{t}$ are the parameters of the endpoints.
- Use the distance formula and parametric coordinates to compute the length of the chord.
- For the perpendicular distance from the origin, use the formula for the distance of a point from a line: distance $= \frac{|Ax+By+C|}{\sqrt{A^2+B^2}}$.