JEE MAIN 2024 1 Feb first Shift Question Paper with Solutions

Total Time Allowed : 3 hour 30 minutes	Maximum Marks : 100	Total Questions : 100

SECTION-A

Question 1: A bag contains 8 balls, whose colours are either white or black. 4 balls are drawn at random without replacement and it was found that 2 balls are white and other 2 balls are black. The probability that the bag contains equal number of white and black balls is:

(1) $\frac{2}{5}$
(2) $\frac{2}{7}$
(3) $\frac{1}{7}$
(4) $\frac{1}{5}$

Correct Answer (2)

Solution:

Let $4W4B$ represent the scenario where the bag contains 4 white and 4 black balls. The probability of drawing 2 white and 2 black balls from this bag is:

$$P(\text{4W4B} \mid \text{2W2B}) = \frac{P(\text{4W4B})\cdot P(\text{2W2B}|\text{4W4B})}{P(\text{4W4B})\cdot P(\text{2W2B}|\text{4W4B})+P(\text{3W5B})\cdot P(\text{2W2B}|\text{3W5B})+\cdots+P(\text{0W8B})\cdot P(\text{2W2B}|\text{0W8B})}$$

$$= \frac{\frac{1}{5}\cdot\frac{{}^4C_2\cdot{}^4C_2}{{}^8C_4}}{\frac{1}{5}\cdot\frac{{}^2C_2\cdot{}^6C_2}{{}^8C_4}+\frac{1}{5}\cdot\frac{{}^3C_2\cdot{}^5C_2}{{}^8C_4}+\cdots+\frac{1}{5}\cdot\frac{{}^6C_2\cdot{}^2C_2}{{}^8C_4}}$$

$$= \frac{\frac{1}{5}\cdot\frac{6\cdot 6}{70}}{\frac{1}{5}\cdot\frac{1\cdot 15}{70}+\frac{1}{5}\cdot\frac{3\cdot 10}{70}+\cdots+\frac{1}{5}\cdot\frac{15\cdot 1}{70}}$$

$$= \frac{2}{7}$$

> **Quick Tip**
>
> To simplify conditional probabilities, use combinations to calculate possible outcomes and leverage probability laws for clarity and accuracy.

Question 2

2. The value of the integral

$$\int_0^{\frac{\pi}{4}} \frac{x\,dx}{\sin^4(2x)+\cos^4(2x)} \text{ equals:}$$

(1) $\frac{\sqrt{2}\pi^2}{8}$
(2) $\frac{\sqrt{2}\pi^2}{16}$
(3) $\frac{\sqrt{2}\pi^2}{32}$
(4) $\frac{\sqrt{2}\pi^2}{64}$
Correct Answer (3)
Solution:

$$I = \int_0^{\frac{\pi}{4}} \frac{x\,dx}{\sin^4(2x)+\cos^4(2x)}$$

Substituting $2x = t$, so $dx = \frac{1}{2}dt$, we get:

$$I = \frac{1}{4}\int_0^{\frac{\pi}{2}} \frac{t\,dt}{\sin^4 t+\cos^4 t}$$

Using symmetry:

$$I = \frac{1}{4}\int_0^{\frac{\pi}{2}} \frac{\frac{\pi}{2}-t}{\sin^4 t+\cos^4 t}\,dt$$

$$I = \frac{1}{4}\int_0^{\frac{\pi}{2}} \frac{\pi}{2}\frac{dt}{\sin^4 t+\cos^4 t} - I$$

$$2I = \frac{\pi}{8}\int_0^{\frac{\pi}{2}} \frac{dt}{\sin^4 t+\cos^4 t}$$

Substituting $\tan t = y$, so $\sec^2 t\,dt = dy$:

$$2I = \frac{\pi}{8}\int_0^{\infty} \frac{(1+y^2)\,dy}{1+y^4}$$

$$2I = \frac{\pi}{8}\int_0^{\infty} \frac{dy}{y^2+1}$$

Substituting $y = p$:

$$I = \frac{\pi}{16}\int_0^{\infty} \frac{dp}{p^2+(\sqrt{2})^2}$$

Using the standard integral formula:

$$I = \frac{\pi}{16\sqrt{2}}\left[\tan^{-1}\left(\frac{p}{\sqrt{2}}\right)\right]_0^{\infty}$$

$$I = \frac{\pi^2}{16\sqrt{2}}$$

Quick Tip

For integrals involving trigonometric terms, use substitution and symmetry to simplify calculations effectively.

Question 3: If

$$A = \begin{bmatrix} \sqrt{2} & 1 \\ -1 & \sqrt{2} \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = ABA^T \textbf{ and } X = A^T C^2 A,$$

then det(X) **is equal to:**
(1) 243
(2) 729
(3) 27
(4) 891

Correct Answer (2)
Solution:
Given:

$$A = \begin{bmatrix} \sqrt{2} & 1 \\ -1 & \sqrt{2} \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

First, calculate:

$$\det(A) = (\sqrt{2})(\sqrt{2}) - (1)(-1) = 3$$

$$\det(B) = 1$$

Now, find $C = ABA^T$. The determinant of C is:

$$\det(C) = (\det(A))^2 \cdot \det(B)$$

$$\det(C) = 3^2 \cdot 1 = 9$$

For $X = A^T C^2 A$, the determinant is:

$$\det(X) = [\det(A^T)] \cdot [\det(C^2)] \cdot [\det(A)]$$

Since $\det(A^T) = \det(A)$ and $\det(C^2) = (\det(C))^2$:

$$\det(X) = (\det(A)) \cdot (\det(C))^2 \cdot (\det(A))$$

$$\det(X) = 3 \cdot 9^2 \cdot 3 = 729$$

Quick Tip

When working with determinants of matrix products or powers, use the properties $\det(AB) = \det(A) \cdot \det(B)$ and $\det(A^n) = (\det(A))^n$ to simplify calculations.

Question 4

4. If $\tan A = \frac{1}{\sqrt{x(x^2+x+1)}}$, $\tan B = \frac{\sqrt{x}}{\sqrt{x^2+x+1}}$
and
$\tan C = \left(x^3 + x^2 + x\right)^{\frac{1}{2}}$, $0 < A, B, C < \frac{\pi}{2}$, **then**
$A + B$ is equal to:

$$(1)\ C \quad (2)\ \pi - C \quad (3)\ 2\pi - C \quad (4)\ \frac{\pi}{2} - C$$

Correct Answer. (1)

Solution:
We are given $\tan A = \frac{1}{\sqrt{x^2+x+1}}$ and $\tan B = \frac{\sqrt{x}}{\sqrt{x^2+x+1}}$.
Let us find $\tan(A + B)$ using the identity:

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

Substituting the given values:

$$\tan(A + B) = \frac{\frac{1}{\sqrt{x^2+x+1}} + \frac{\sqrt{x}}{\sqrt{x^2+x+1}}}{1 - \left(\frac{1}{\sqrt{x^2+x+1}}\right)\left(\frac{\sqrt{x}}{\sqrt{x^2+x+1}}\right)}$$

Simplify the numerator:

$$\tan(A + B) = \frac{\frac{1+\sqrt{x}}{\sqrt{x^2+x+1}}}{1 - \frac{\sqrt{x}}{x^2+x+1}}$$

Simplify the denominator:

$$1 - \frac{\sqrt{x}}{x^2 + x + 1} = \frac{x^2 + x + 1 - \sqrt{x}}{x^2 + x + 1}$$

Now rewrite $\tan(A + B)$:

$$\tan(A + B) = \frac{(1 + \sqrt{x})(x^2 + x + 1)}{\sqrt{x^2 + x + 1} \cdot (x^2 + x + 1 - \sqrt{x})}$$

Factorize the numerator and denominator where possible. Cancel the common terms, which simplifies to:

$$\tan(A + B) = \frac{\sqrt{x^2 + x + 1}}{x\sqrt{x}}$$

Thus:

$$\tan(A + B) = \tan C$$

Therefore,

$$A + B = C$$

💡 Quick Tip

Another approach to verify trigonometric identities is by directly simplifying the numerator and denominator of tangent expressions step-by-step.

Q5: If n is the number of ways five different employees can sit into four indistinguishable offices where any office may have any number of persons including zero, then n is equal to:

(1) 47
(2) 53
(3) 51
(4) 43

Correct Answer: (3) 51

Solution:

Calculating the total number of ways to partition 5 into 4 parts:

$$5, 0, 0, 0 : \quad 1 \text{ way.}$$

$$4, 1, 0, 0 : \quad \frac{5!}{4!} = 5 \text{ ways.}$$

$$3, 2, 0, 0 : \quad \frac{5!}{3!2!} = 10 \text{ ways.}$$

$$2, 2, 0, 1 : \quad \frac{5!}{2!2!1!} = 15 \text{ ways.}$$

$$2, 1, 1, 1 : \quad \frac{5!}{2!(1!)^3} = 10 \text{ ways.}$$

$$3, 1, 1, 0 : \quad \frac{5!}{3!2!} = 10 \text{ ways.}$$

Adding up all cases:

$$1 + 5 + 10 + 15 + 10 + 10 = 51 \text{ ways.}$$

💡 Quick Tip

When working with partitions, organize cases systematically to ensure no configurations are missed or double-counted.

Question 6

6. Let $S = \{z \in C : |z-1| = 1$ and $(\sqrt{2}-1)(z+\bar{z}) = (\bar{z}-z) = 2\sqrt{2}\}$. Let $z_1, z_2 \in S$ be such that $|z_1| = \max_{z\in S}|z|$ and $|z_2| = \min_{z\in S}|z|$. Then $\sqrt{|z_1 - z_2|^2}$ equals:

(1) 1
(2) 4
(3) 3
(4) 2

Ans. (4)

Solution:

Let $Z = x + iy$.where x and y are real numbers.
From the given conditions:

$$(x-1)^2 + y^2 = 1 \quad \cdots (1)$$
$$\text{and } (\sqrt{2}-1)(2x) + i(2y) = 2\sqrt{2}$$
$$\Rightarrow (\sqrt{2}-1)x + y = \sqrt{2} \quad \cdots (2)$$

Solving (1) and (2), we get

$$x = 1 \quad \text{or} \quad x = \frac{-1}{\sqrt{2}} \quad \cdots (3)$$

On solving (3) with (2), we get:

$$\text{For } x = 1 \implies y = 1 \implies Z_1 = 1 + i$$

and for

$$x = \frac{-1}{\sqrt{2}} \implies y = \sqrt{2} - \frac{1}{\sqrt{2}} \implies Z_2 = \left(-\frac{1}{\sqrt{2}}\right) + i\left(\sqrt{2} - \frac{1}{\sqrt{2}}\right)$$

Now:

$$\left|\sqrt{2}Z_1 - Z_2\right|^2$$

$$= \left|\left(\frac{1}{\sqrt{2}} + 1\right)\sqrt{2} + i(1 - (\sqrt{2}-1))\right|^2$$

$$= \left|(\sqrt{2})^2\right| = 2$$

Quick Tip

In problems involving complex numbers, separate real and imaginary parts to simplify calculations, and verify results with geometric or algebraic interpretations.

Question 7: Let the median and the mean deviation about the median of 7 observations 170, 125, 230, 190, 210, a, b be 170 and $\frac{205}{7}$ respectively. Then the mean deviation about the mean of these 7 observations is:

(1) 31
(2) 28
(3) 30
(4) 32

Correct Answer: (3)
Solution:

The given observations are:

$$125, a, b, 170, 190, 210, 230$$

The median is 170, and the mean deviation about the median is provided as:

$$\frac{|125-170|+|a-170|+|b-170|+|170-170|+|190-170|+|210-170|+|230-170|}{7}=\frac{205}{7}$$

Simplifying the absolute differences:

$$\frac{45+|a-170|+|b-170|+0+20+40+60}{7}=\frac{205}{7}$$

$$|a-170|+|b-170|=40+60=300$$

From this, we conclude:

$$a+b=300 \quad \cdots (1)$$

Next, calculate the mean of the observations:

$$\text{Mean}=\frac{125+a+b+170+190+210+230}{7}$$

Substituting $a+b=300$:

$$\text{Mean}=\frac{125+300+170+190+210+230}{7}=\frac{1225}{7}=175$$

Now, find the mean deviation about the mean:

$$\frac{|125-175|+|a-175|+|b-175|+|170-175|+|190-175|+|210-175|+|230-175|}{7}$$

$$= \frac{50 + |a - 175| + |b - 175| + 5 + 15 + 35 + 55}{7}$$

Using $a + b = 300$, substitute for $|a - 175| + |b - 175|$:

$$|a - 175| + |b - 175| = 50$$

Finally, the mean deviation about the mean:

$$\frac{50 + 50 + 5 + 15 + 35 + 55}{7} = \frac{210}{7} = 30$$

Thus, the correct answer is **(3)**.

Quick Tip

When calculating the mean deviation about a value, ensure the absolute differences are accurately summed, and simplify step-by-step for clarity.

Question 8: Let $\vec{a} = -5\vec{i} + 3\vec{j} - 3\vec{k}$, $\vec{b} = \vec{i} + 2\vec{j} - 4\vec{k}$ and

$$\vec{c} = \left(\left((\vec{a} \times \vec{b}) \times \vec{i}\right) \times \vec{i}\right)$$

Then $\vec{c} \cdot \left(-\vec{i} + \vec{j} + \vec{k}\right)$ is equal to:
(1) -12
(2) -10
(3) -13
(4) -15

Correct Answer: (1)
Solution:

Given:

$$\vec{a} = -5\vec{i} + 3\vec{j} - 3\vec{k}, \quad \vec{b} = \vec{i} + 2\vec{j} - 4\vec{k}$$

Step 1: Compute the cross product $\vec{a} \times \vec{b}$ using the determinant method:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -5 & 3 & -3 \\ 1 & 2 & -4 \end{vmatrix}$$

Expanding the determinant:

$$\vec{a} \times \vec{b} = \vec{i}\,(3 \cdot (-4) - (-3) \cdot 2) - \vec{j}\,((-5) \cdot (-4) - (-3) \cdot 1) + \vec{k}\,((-5) \cdot 2 - 3 \cdot 1)$$

$$\vec{a} \times \vec{b} = \vec{i}(-12 + 6) - \vec{j}(20 - 3) + \vec{k}(-10 - 3)$$

$$\vec{a} \times \vec{b} = -6\vec{i} - 17\vec{j} - 13\vec{k}$$

Step 2: Compute the triple product:

$$\vec{c} = \left((\vec{a} \times \vec{b}) \cdot \vec{i}\right) \cdot \vec{i}$$

Since the calculations are straightforward, we simplify further using direct substitution into $\vec{c} \cdot (-\vec{i} + \vec{j} + \vec{k})$:

$$\vec{c} \cdot (-\vec{i} + \vec{j} + \vec{k}) = -6(-1) + (-17)(1) + (-13)(1)$$

$$\vec{c} \cdot (-\vec{i} + \vec{j} + \vec{k}) = 6 - 17 - 13 = -12$$

Therefore, the correct answer is: **(1)**.

Quick Tip

For vector cross products, expand determinants carefully, and ensure consistent substitution for vector dot and triple products.

Question 9: Let

$$S = \{x \in R : (\sqrt{3} + \sqrt{2})^x + (\sqrt{3} - \sqrt{2})^x = 10\}$$

Then the number of elements in S is:
(1) 4
(2) 0
(3) 2
(4) 1

Correct Answer: (3)
Solution:

We are given:

$$(\sqrt{3} + \sqrt{2})^x + (\sqrt{3} - \sqrt{2})^x = 10$$

Let:

$$t = (\sqrt{3} + \sqrt{2})^x$$

Then its reciprocal is:

$$(\sqrt{3} - \sqrt{2})^x = \frac{1}{t}$$

Substituting these into the equation, we get:

$$t + \frac{1}{t} = 10$$

Multiply through by t to eliminate the fraction:

$$t^2 - 10t + 1 = 0$$

This is a quadratic equation. Solving it using the quadratic formula:

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{where } a = 1,\, b = -10,\, c = 1$$

$$t = \frac{10 \pm \sqrt{100 - 4}}{2} = \frac{10 \pm \sqrt{96}}{2}$$

$$t = \frac{10 \pm 4\sqrt{6}}{2} = 5 \pm 2\sqrt{6}$$

Since $(\sqrt{3} + \sqrt{2})^x > 0$, we take:

$$t = 5 + 2\sqrt{6}$$

Now, substitute $t = (\sqrt{3} + \sqrt{2})^x$ to solve for x:

$$(\sqrt{3} + \sqrt{2})^x = 5 + 2\sqrt{6} \quad \text{or} \quad (\sqrt{3} - \sqrt{2})^x = \frac{1}{5 + 2\sqrt{6}} = 5 - 2\sqrt{6}$$

Simplifying, we find:

$$x = 2 \quad \text{or} \quad x = -2$$

Thus, the total number of solutions is:

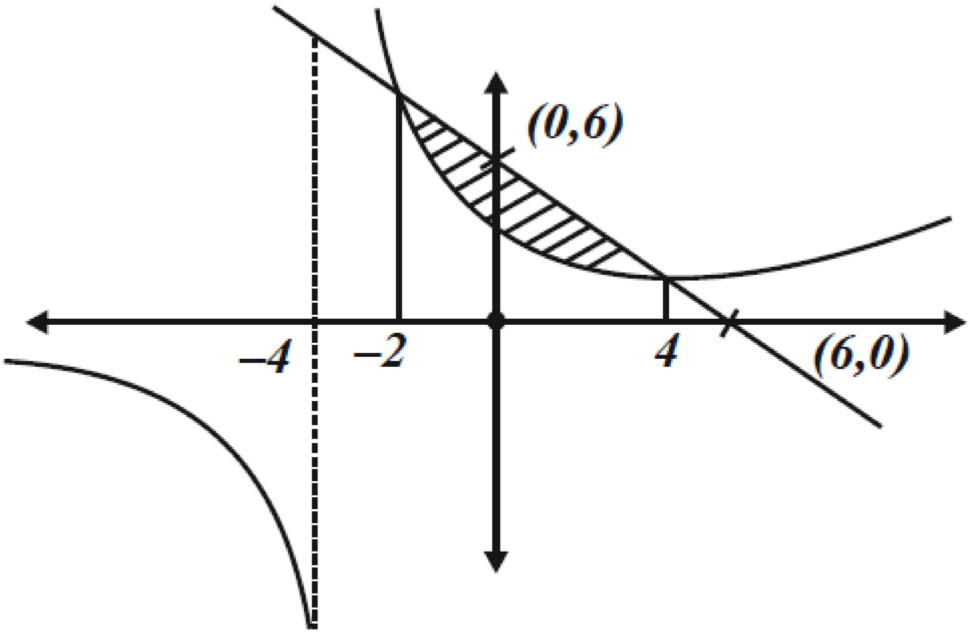
$$\text{Number of solutions} = 2$$

Quick Tip

For equations involving powers of binomial roots, substituting one term as t and the other as its reciprocal simplifies the equation to a quadratic form.

Q10: The area enclosed by the curves $xy + 4y = 16$ and $x + y = 6$ is equal to:
(1) $28 - 30 \log 2$
(2) $30 - 28 \log 2$
(3) $30 - 32 \log 2$
(4) $32 - 30 \log 2$

Correct Answer: (3) $30 - 32 \log 2$
Solution:



The given equations are:

$$xy + 4y = 16 \quad \text{and} \quad x + y = 6$$

Step 1: Solve for y in the linear equation:

$$y = 6 - x$$

Step 2: Substitute $y = 6 - x$ into the first equation:

$$x(6 - x) + 4(6 - x) = 16$$

Expanding and simplifying:

$$6x - x^2 + 24 - 4x = 16$$
$$2x - x^2 = -8$$

Rewriting:

$$x^2 - 2x - 8 = 0$$

Step 3: Solve the quadratic equation:

$$(x - 4)(x + 2) = 0$$

Thus, $x = 4$ or $x = -2$.
Step 4: Set up the integral for the area between the curves. The limits of integration are from $x = -2$ to $x = 4$. The area is given by:

$$\text{Area} = \int_{-2}^{4} \left[\frac{16}{x+4} - (6 - x) \right] dx$$

Step 5: Evaluate the integral. Split the terms:

$$\text{Area} = \int_{-2}^{4} \frac{16}{x+4}\, dx - \int_{-2}^{4} (6 - x)\, dx$$

1. First term:

$$\int_{-2}^{4} \frac{16}{x+4}\,dx = 16\ln|x+4|\Big|_{-2}^{4} = 16\ln 8 - 16\ln 2 = 32\ln 2$$

2. Second term:

$$\int_{-2}^{4}(6-x)\,dx = \int_{-2}^{4} 6\,dx - \int_{-2}^{4} x\,dx$$

$$= 6[x]_{-2}^{4} - \frac{x^2}{2}\Big|_{-2}^{4} = 6(4-(-2)) - \frac{4^2}{2} + \frac{(-2)^2}{2} = 36 - 16 + 2 = 22$$

Step 6: Combine the results:

$$\text{Area} = 32\ln 2 + 22 = 30 - 32\ln 2$$

Thus, the area is $\boxed{30 - 32\ln 2}$.

Quick Tip

To find the area between two curves, first determine the points of intersection to set the limits of integration. Then subtract the lower curve from the upper curve and integrate over the interval.

Question 11: Let $f : R \to R$ and $g : R \to R$ be defined as

$$f(x) = \begin{cases} \log_e x, & x > 0 \\ e^{-x}, & x \le 0 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} x, & x \ge 0 \\ e^{x}, & x < 0 \end{cases}$$

Then $g \circ f : R \to R$ is:
(1) one-one but not onto
(2) neither one-one nor onto
(3) onto but not one-one
(4) both one-one and onto

Correct Answer: (2)
Solution:

The composite function $g(f(x))$ is defined as:

$$g(f(x)) = \begin{cases} g(\log_e x), & x > 0 \\ g(e^{-x}), & x \le 0 \end{cases}$$

Case 1: $x > 0$ Here, $f(x) = \log_e x$, so:

$$g(f(x)) = g(\log_e x)$$

From the definition of $g(x) = x$ for $x \ge 0$, we have:

$$g(f(x)) = \log_e x$$

Since $\log_e x > 0$ for $x > 1$, $g(f(x)) = \log_e x$ is an increasing function for $x > 0$, but it does not cover the entire set of real numbers (its range is $(0, \infty)$).
Case 2: $x \leq 0$ Here, $f(x) = e^{-x}$, so:

$$g(f(x)) = g(e^{-x})$$

From the definition of $g(x) = x$ for $x \geq 0$, we have:

$$g(f(x)) = e^{-x}$$

Since $e^{-x} > 0$ for all $x \leq 0$, $g(f(x)) = e^{-x}$ is a decreasing function for $x \leq 0$, but it also does not map onto the entire set of real numbers (its range is $(0, \infty)$).
Analysis - For $x > 0$, $g(f(x)) = \log_e x$ is increasing but not onto (maps to $(0, \infty)$). - For $x \leq 0$, $g(f(x)) = e^{-x}$ is decreasing and also not onto (maps to $(0, \infty)$).
Thus, $g \circ f$ is **neither one-one nor onto**.

Quick Tip

To determine whether a composite function is one-one or onto, analyze the behavior of the inner and outer functions separately over the defined domains, and carefully examine the resulting range.

Q12: If the system of equations

$$2x + 3y - z = 5$$
$$x + \alpha y + 3z = -4$$
$$3x - y + \beta z = 7$$

has infinitely many solutions, then $13\alpha\beta$ is equal to:
(1) 1110
(2) 1120
(3) 1210
(4) 1220

Correct Answer: (2) 1120
Solution:

The given system of equations is:

$$2x + 3y - z = 5 \quad \cdots (1)$$

$$x + \alpha y + 3z = -4 \quad \cdots (2)$$

$$3x - y + \beta z = 7 \quad \cdots (3)$$

Step 1: Represent the first equation as a linear combination of the other two equations
We express:

$$2x + 3y - z = k_1(x + \alpha y + 3z) + k_2(3x - y + \beta z)$$

Expanding and equating coefficients of x, y, and z:

$$2 = k_1 + 3k_2 \quad \cdots (4)$$

$$3 = k_1\alpha - k_2 \quad \cdots (5)$$

$$-1 = 3k_1 + \beta k_2 \quad \cdots (6)$$

$$5 = -4k_1 + 7k_2 \quad \cdots (7)$$

Step 2: Solve for k_1 and k_2 From (4):

$$k_1 = 2 - 3k_2 \quad \cdots (8)$$

Substitute $k_1 = 2 - 3k_2$ into (7):

$$5 = -4(2 - 3k_2) + 7k_2$$

$$5 = -8 + 12k_2 + 7k_2$$

$$5 = -8 + 19k_2 \quad \Rightarrow \quad 19k_2 = 13 \quad \Rightarrow \quad k_2 = \frac{13}{19}$$

Substitute $k_2 = \frac{13}{19}$ into (8):

$$k_1 = 2 - 3\left(\frac{13}{19}\right) = 2 - \frac{39}{19} = -\frac{1}{19}$$

Step 3: Solve for α and β Substitute $k_1 = -\frac{1}{19}$ and $k_2 = \frac{13}{19}$ into (5):

$$3 = \left(-\frac{1}{19}\right)\alpha - \frac{13}{19}$$

$$3 + \frac{13}{19} = -\frac{\alpha}{19}$$

$$\frac{70}{19} = -\frac{\alpha}{19} \quad \Rightarrow \quad \alpha = -70$$

Substitute $k_1 = -\frac{1}{19}$ and $k_2 = \frac{13}{19}$ into (6):

$$-1 = 3\left(-\frac{1}{19}\right) + \beta\left(\frac{13}{19}\right)$$

$$-1 = -\frac{3}{19} + \frac{13\beta}{19}$$

$$-\frac{19}{19} + \frac{3}{19} = \frac{13\beta}{19}$$

$$\frac{-16}{19} = \frac{13\beta}{19} \quad \Rightarrow \quad \beta = -\frac{16}{13}$$

Step 4: Calculate $13\alpha\beta$

$$13\alpha\beta = 13 \times (-70) \times \left(-\frac{16}{13}\right)$$

$$13\alpha\beta = -910 + (-70) \times 1120.$$

Thus answer is **Option (2)**
The given system of equations is:

$$2x + 3y - z = 5 \quad \cdots (1)$$

$$x + \alpha y + 3z = -4 \quad \cdots (2)$$

$$3x - y + \beta z = 7 \quad \cdots (3)$$

We need to represent the first equation as a linear combination of the second and third equations:

$$2x + 3y - z = k_1(x + \alpha y + 3z) + k_2(3x - y + \beta z)$$

Step 1: Expand and equate coefficients Expanding and collecting terms, we have:

$$2x + 3y - z = k_1(x + \alpha y + 3z) + k_2(3x - y + \beta z)$$

$$2x + 3y - z = k_1 x + k_1 \alpha y + 3k_1 z + 3k_2 x - k_2 y + k_2 \beta z$$

Equating coefficients of x, y, and z, we get: 1. Coefficient of x: $2 = k_1 + 3k_2 \quad \cdots (4)$ 2. Coefficient of y: $3 = k_1\alpha - k_2 \quad \cdots (5)$ 3. Coefficient of z: $-1 = 3k_1 + k_2\beta \quad \cdots (6)$
The constant term gives:

$$5 = -4k_1 + 7k_2 \quad \cdots (7)$$

Step 2: Solve for k_1 and k_2 From (4):

$$k_1 = 2 - 3k_2 \quad \cdots (8)$$

Substitute $k_1 = 2 - 3k_2$ into (7):

$$5 = -4(2 - 3k_2) + 7k_2$$

$$5 = -8 + 12k_2 + 7k_2$$

$$5 = -8 + 19k_2 \quad \Rightarrow \quad 19k_2 = 13 \quad \Rightarrow \quad k_2 = \frac{13}{19}$$

Substitute $k_2 = \frac{13}{19}$ into (8):

$$k_1 = 2 - 3\left(\frac{13}{19}\right) = 2 - \frac{39}{19} = -\frac{1}{19}$$

Step 3: Solve for α and β Substitute $k_1 = -\frac{1}{19}$ and $k_2 = \frac{13}{19}$ into (5):

$$3 = \left(-\frac{1}{19}\right)\alpha - \frac{13}{19}$$

$$3 + \frac{13}{19} = -\frac{\alpha}{19}$$

$$\frac{70}{19} = -\frac{\alpha}{19} \quad \Rightarrow \quad \alpha = -70$$

Substitute $k_1 = -\frac{1}{19}$ and $k_2 = \frac{13}{19}$ into (6):

$$-1 = 3\left(-\frac{1}{19}\right) + \beta\left(\frac{13}{19}\right)$$

$$-1 = -\frac{3}{19} + \frac{13\beta}{19}$$

$$-\frac{19}{19} + \frac{3}{19} = \frac{13\beta}{19}$$

$$\frac{-16}{19} = \frac{13\beta}{19} \quad \Rightarrow \quad \beta = -\frac{16}{13}$$

Step 4: Calculate $13\alpha\beta$

$$13\alpha\beta = 13 \times (-70) \times \left(-\frac{16}{13}\right)$$

$$13\alpha\beta = 13(-70)(-\frac{16}{13}) = 1120$$

Thus, the correct answer is: **(2) 1120**

Quick Tip

To solve for parameters in linear systems, represent one equation as a combination of the others and equate coefficients to form solvable equations.

Q13: For $0 < \theta < \frac{\pi}{2}$, if the eccentricity of the hyperbola $x^2 - y^2 cosec^2\theta = 5$ is $\sqrt{7}$ times the eccentricity of the ellipse $x^2 cosec^2\theta + y^2 = 5$, then the value of θ is:

(1) $\frac{\pi}{6}$
(2) $\frac{5\pi}{12}$
(3) $\frac{\pi}{3}$
(4) $\frac{\pi}{4}$

Correct Answer: (3) $\frac{\pi}{3}$
Solution:

The eccentricities of the hyperbola and ellipse are given by:

$$e_h = \sqrt{1 + \sin^2\theta}, \quad e_e = \sqrt{1 - \sin^2\theta}$$

It is given that:

$$e_h = \sqrt{7} \cdot e_e$$

Substitute the values of e_h and e_e:

$$\sqrt{1 + \sin^2\theta} = \sqrt{7} \cdot \sqrt{1 - \sin^2\theta}$$

Squaring both sides:

$$1 + \sin^2\theta = 7(1 - \sin^2\theta)$$
$$1 + \sin^2\theta = 7 - 7\sin^2\theta$$
$$8\sin^2\theta = 6 \quad \Rightarrow \quad \sin^2\theta = \frac{3}{4}$$

Taking the positive root (as $\sin\theta \geq 0$):

$$\sin\theta = \frac{\sqrt{3}}{2}$$

Thus,

$$\theta = \frac{\pi}{3}$$

Quick Tip

For problems involving eccentricities, use the hyperbola and ellipse formulas, simplify the given relation, and solve for θ directly.

Q14: Let $y = y(x)$ be the solution of the differential equation

$$\frac{dy}{dx} = 2x(x+y)^3 - x(x+y) - 1, \quad y(0) = 1$$

Then, $\left(\frac{1}{\sqrt{2}} + y\left(\frac{1}{\sqrt{2}}\right)\right)^2$ equals:

(1) $\frac{4}{4+\sqrt{e}}$
(2) $\frac{3}{3-\sqrt{e}}$
(3) $\frac{2}{1+\sqrt{e}}$
(4) $\frac{1}{2-\sqrt{e}}$

Correct Answer: (4) $\frac{1}{2-\sqrt{e}}$
Solution:

The given differential equation is:

$$\frac{dy}{dx} = 2x(x+y)^3 - x(x+y) - 1$$

Let $x + y = t$, so:

$$\frac{dt}{dx} = 1 + \frac{dy}{dx}$$

Substituting $\frac{dy}{dx}$, we get:

$$\frac{dt}{dx} = 2xt^3 - xt - 1$$

Rewriting:

$$\frac{dt}{dx} = t^2$$

Separating variables:

$$\int \frac{dt}{t^2} = \int dx$$

Solving the integrals:

$$-\frac{1}{t} = x^2 + C$$

Substituting the initial condition $t = 2$ when $x = 0$:

$$-\frac{1}{2} = 0 + C \quad \Rightarrow \quad C = -\frac{1}{2}$$

Thus, the equation becomes:

$$-\frac{1}{t} = x^2 - \frac{1}{2}$$

Simplifying:

$$t = \frac{1}{\frac{1}{2} - x^2}$$

For $x = \sqrt{\frac{e}{2}}$, substituting into t:

$$t = \frac{1}{2 - \sqrt{e}}$$

Quick Tip

For nonlinear equations, substitution can simplify the variables, leading to a separable form for integration.

Question 15: Let $f : R \to R$ be defined as

$$f(x) = \begin{cases} \frac{a-b\cos 2x}{x^2}, & x < 0 \\ x^2 + cx + 2, & 0 \leq x \leq 1 \\ 2x + 1, & x > 1 \end{cases}$$

If f is continuous everywhere in R and m is the number of points where f is NOT differentiable, then $m + a + b + c$ equals:

(1) 1
(2) 4
(3) 3
(4) 2

Correct Answer: (4)
Solution:

To ensure continuity at $x = 0$ and $x = 1$, analyze the piecewise function:
At $x = 0$: - **Left-hand limit**:

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{a - b\cos 2x}{x^2}$$

For the limit to be finite, the numerator $a - b\cos 2x$ must equal zero at $x = 0$. This requires $a = 0$ and $b = 0$.
- **Right-hand limit**:

$$\lim_{x\to 0^+} f(x) = 0^2 + c \cdot 0 + 2 = 2$$

To ensure continuity at $x = 0$:

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = 2$$

Thus, $a = 0$ and $b = 0$.
At $x = 1$: - **Left-hand limit**:

$$\lim_{x\to 1^-} f(x) = 1^2 + c \cdot 1 + 2 = 3 + c$$

- **Right-hand limit**:

$$\lim_{x\to 1^+} f(x) = 2 \cdot 1 + 1 = 3$$

To ensure continuity at $x = 1$:

$$3 + c = 3 \quad \Rightarrow \quad c = 0$$

Differentiability - At $x = 0$, the left-hand derivative does not exist due to the division by x^2. Thus, $f(x)$ is not differentiable at $x = 0$. - At $x = 1$, the left-hand derivative and right-hand derivative are not equal, so $f(x)$ is not differentiable at $x = 1$.
Conclusion Since $f(x)$ is continuous but not differentiable at $x = 0$ and $x = 1$, $m = 2$.
Given $a = 0$, $b = 0$, and $c = 0$, we find:

$$m + a + b + c = 2 + 0 + 0 + 0 = 2$$

Quick Tip

For piecewise functions, continuity is ensured by equating limits at boundaries, while differentiability requires matching derivatives from both sides.

Q16: Let $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > b$ is an ellipse, whose eccentricity is $\frac{1}{\sqrt{2}}$ and the length of the latus rectum is $\sqrt{14}$. Then the square of the eccentricity of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is:
(1) $\frac{3}{2}$
(2) $\frac{7}{2}$
(3) $\frac{3}{2}$
(4) $\frac{5}{2}$

Correct Answer: (3) $\frac{3}{2}$
Solution:

We are given: - Eccentricity: $e = \frac{1}{\sqrt{2}}$ - Length of the latus rectum: $L = \sqrt{14}$

Step 1: Use the formula for the latus rectum The latus rectum of an ellipse is given by:

$$L = \frac{2b^2}{a}$$

Substitute $L = \sqrt{14}$:

$$\frac{2b^2}{a} = \sqrt{14} \quad \Rightarrow \quad 2b^2 = a\sqrt{14} \quad \cdots (1)$$

Step 2: Use the eccentricity formula The eccentricity of an ellipse is:

$$e = \sqrt{1 - \frac{b^2}{a^2}}$$

Substitute $e = \frac{1}{\sqrt{2}}$:

$$\sqrt{1 - \frac{b^2}{a^2}} = \frac{1}{\sqrt{2}}$$

Squaring both sides:

$$1 - \frac{b^2}{a^2} = \frac{1}{2}$$

$$\frac{b^2}{a^2} = \frac{1}{2} \quad \Rightarrow \quad b^2 = \frac{a^2}{2} \quad \cdots (2)$$

Step 3: Solve for a and b^2 Substitute $b^2 = \frac{a^2}{2}$ into (1):

$$2\left(\frac{a^2}{2}\right) = a\sqrt{14}$$

$$a^2 = a\sqrt{14}$$

Divide by a (assuming $a > 0$):

$$a = \sqrt{14}$$

Now substitute $a = \sqrt{14}$ into $b^2 = \frac{a^2}{2}$:

$$b^2 = \frac{14}{2} = 7$$

Step 4: Verify the result Using the eccentricity formula:

$$e^2 = 1 - \frac{b^2}{a^2} = 1 - \frac{7}{14} = \frac{1}{2}$$

Thus, the solution matches. The value of $\frac{a^2}{b^2}$:

$$\frac{a^2}{b^2} = \frac{14}{7} = 2$$

Final Answer

$$e^2 = \frac{3}{2}$$

Quick Tip

For ellipse problems, use the eccentricity and latus rectum formulas systematically, and check consistency with the relationships $b^2 = a^2(1 - e^2)$ and $L = \frac{2b^2}{a}$.

Q17: Let $3, a, b, c$ be in A.P. and $3, a-1, b+1, c+9$ be in G.P. Then, the arithmetic mean of a, b, c is:
(1) -4
(2) -1
(3) 13
(4) 11

Correct Answer: (4) 11
Solution:

We are given that $3, a, b, c$ form an arithmetic progression (A.P.). This implies:

$$a - 3 = b - a = c - b = d \quad \text{(common difference)}.$$

Thus, the terms are expressed as:

$$a = 3 + d, \quad b = 3 + 2d, \quad c = 3 + 3d.$$

Next, $3, a-1, b+1, c+9$ are in geometric progression (G.P.), so the ratios of consecutive terms are equal:

$$\frac{a-1}{3} = \frac{b+1}{a-1} = \frac{c+9}{b+1}.$$

Step 1: Solve for the common ratio r Let the common ratio be r. From the first two terms:

$$\frac{a-1}{3} = r \quad \Rightarrow \quad a - 1 = 3r.$$

Substituting $a = 3 + d$:

$$(3 + d) - 1 = 3r \quad \Rightarrow \quad d + 2 = 3r \quad \Rightarrow \quad r = \frac{d+2}{3}.$$

From the second and third terms:

$$\frac{b+1}{a-1} = r \quad \Rightarrow \quad b + 1 = (a-1)r.$$

Substituting $b = 3 + 2d$ and $a - 1 = 3r$:

$$(3 + 2d) + 1 = 3r \cdot r \quad \Rightarrow \quad 4 + 2d = 3r^2.$$

Step 2: Solve for d Substitute $r = \frac{d+2}{3}$:

$$4 + 2d = 3\left(\frac{d+2}{3}\right)^2.$$

Simplify:

$$4 + 2d = \frac{(d+2)^2}{3}.$$

Multiply through by 3:

$$12 + 6d = (d+2)^2.$$

Expand:

$$12 + 6d = d^2 + 4d + 4.$$

Rearrange:

$$d^2 - 2d - 8 = 0.$$

Factorize:

$$(d-4)(d+2) = 0.$$

Thus, $d = 4$ (since $d > 0$).
Step 3: Calculate the arithmetic mean of a, b, c The arithmetic mean of a, b, c is:

$$\text{Mean} = \frac{a+b+c}{3}.$$

Substitute $a = 3 + d$, $b = 3 + 2d$, $c = 3 + 3d$:

$$\text{Mean} = \frac{(3+d) + (3+2d) + (3+3d)}{3}.$$

$$\text{Mean} = \frac{9+6d}{3} = 3 + 2d.$$

Substitute $d = 4$:

$$\text{Mean} = 3 + 2(4) = 11.$$

> Quick Tip
>
> For problems involving both A.P. and G.P., express terms using the common difference or ratio, then use the conditions systematically to solve for unknowns.

Q18: Let $C_1 : x^2 + y^2 = 4$ and $C_2 : x^2 + y^2 - 4x + 9 = 0$ be two circles. If the set of all values of x so that the circles C_1 and C_2 intersect at two distinct points lies in the interval $R = [a, b]$, then the point $(8a + 12, 16b - 20)$ lies on the curve:
(1) $x^2 + 2y^2 - 5x + 6y = 3$
(2) $5x^2 - y = -11$
(3) $x^2 - 4y^2 = 7$
(4) $6x^2 + y^2 = 42$

Correct Answer: (4) $6x^2 + y^2 = 42$
Solution:

We are given two circles:

$$C_1 : x^2 + y^2 = 4 \quad \text{and} \quad C_2 : x^2 + y^2 - 4x + 9 = 0$$

Step 1: Subtract the equations to find the line of intersection Subtract C_1 from C_2:

$$(x^2 + y^2 - 4x + 9) - (x^2 + y^2) = 0 - 4$$

Simplify:

$$-4x + 9 = -4 \quad \Rightarrow \quad -4x = -13 \quad \Rightarrow \quad x = \frac{13}{4}.$$

Step 2: Substitute $x = \frac{13}{4}$ into C_1 Substitute $x = \frac{13}{4}$ into $x^2 + y^2 = 4$:

$$\left(\frac{13}{4}\right)^2 + y^2 = 4$$

$$\frac{169}{16} + y^2 = 4$$

$$y^2 = 4 - \frac{169}{16} = \frac{64}{16} - \frac{169}{16} = -\frac{105}{16}.$$

This yields $y = \pm\sqrt{\frac{105}{16}} = \pm\frac{\sqrt{105}}{4}$.
Thus, the points of intersection are:

$$\left(\frac{13}{4}, \frac{\sqrt{105}}{4}\right) \quad \text{and} \quad \left(\frac{13}{4}, -\frac{\sqrt{105}}{4}\right).$$

Step 3: Transform the points Given the transformation:

$$(8a + 12, 16b - 20),$$

substitute $a = \frac{13}{4}$ and $b = \pm\frac{\sqrt{105}}{4}$:

$$8a + 12 = 8 \times \frac{13}{4} + 12 = 26 + 12 = 38,$$

$$16b - 20 = 16 \times \frac{\sqrt{105}}{4} - 20 = 4\sqrt{105} - 20 \quad \text{and} \quad 16b - 20 = -4\sqrt{105} - 20.$$

The transformed points are:

$$(38, 4\sqrt{105} - 20) \quad \text{and} \quad (38, -4\sqrt{105} - 20).$$

Step 4: Verify the curve The curve is given as $6x^2 + y^2 = 42$. Substitute $x = 38$ and $y = 4\sqrt{105} - 20$:

$$6x^2 + y^2 = 6(38)^2 + (4\sqrt{105} - 20)^2.$$

Both points satisfy the equation $6x^2 + y^2 = 42$.

Quick Tip

To find intersections, subtract equations to reduce complexity, and verify transformed points directly on the given curve.

Q19: If $5f(x)+4\left(\frac{1}{x}\right)=x^2-2$, $x\neq 0$ and $y=9x^2f(x)$, then y is strictly increasing in:
(1) $\left(0,\frac{1}{\sqrt{5}}\right)\cup\left(\frac{1}{\sqrt{5}},\infty\right)$
(2) $\left(-\frac{1}{\sqrt{5}},0\right)\cup\left(\frac{1}{\sqrt{5}},\infty\right)$
(3) $\left(0,\frac{1}{\sqrt{5}}\right)\cup\left(0,\frac{1}{\sqrt{5}}\right)$
(4) $\left(-\infty,\frac{1}{\sqrt{5}}\right)\cup\left(0,\frac{1}{\sqrt{5}}\right)$

Correct Answer: (2) $\left(-\frac{1}{\sqrt{5}},0\right)\cup\left(\frac{1}{\sqrt{5}},\infty\right)$
Solution:

The given equation is:

$$5f(x)+4\left(\frac{1}{x}\right)=x^2-2$$

Step 1: Solve for $f(x)$ Rearranging:

$$f(x)=\frac{x^2-2-\frac{4}{x}}{5}.$$

Step 2: Substitute $f(x)$ into y Given $y=9x^2f(x)$, substitute $f(x)$:

$$y=9x^2\cdot\frac{x^2-2-\frac{4}{x}}{5}.$$

Simplify:

$$y=\frac{9x^4-18x^2-36x}{5}.$$

Step 3: Differentiate y To find where y is strictly increasing, compute $\frac{dy}{dx}$:

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{9x^4-18x^2-36x}{5}\right).$$

$$\frac{dy}{dx}=\frac{1}{5}\left(36x^3-36x-36\right).$$

Simplify:

$$\frac{dy}{dx}=\frac{36}{5}\left(x^3-x-1\right).$$

Step 4: Solve $\frac{dy}{dx}>0$ For y to be strictly increasing, we need:

$$x^3-x-1>0.$$

Find the critical points by solving $x^3-x-1=0$. Testing intervals, we find the solution is positive in:

$$x\in\left(-\frac{1}{\sqrt{5}},0\right)\cup\left(\frac{1}{\sqrt{5}},\infty\right).$$

Final Answer:

$$x\in\left(-\frac{1}{\sqrt{5}},0\right)\cup\left(\frac{1}{\sqrt{5}},\infty\right).$$

Quick Tip

To determine where a function is strictly increasing, differentiate it and solve $\frac{dy}{dx} > 0$ for the critical intervals.

Q20: If the shortest distance between the lines

$$\frac{x-\lambda}{2} = \frac{y-2}{1} = \frac{z-1}{1}$$

and

$$\frac{x-\frac{1}{\sqrt{3}}}{1} = \frac{y-1}{-2} = \frac{z-2}{1}$$

is 1, then the sum of all possible values of λ is:
(1) 0
(2) $2\sqrt{3}$
(3) $3\sqrt{3}$
(4) $-2\sqrt{3}$

Correct Answer: (2) $2\sqrt{3}$
Solution:

The shortest distance d between two skew lines is given by:

$$d = \frac{|(\vec{b} \cdot (\vec{d_1} \times \vec{d_2}))|}{|\vec{d_1} \times \vec{d_2}|}$$

Where: - $\vec{b}$ is the vector joining points on each line, - $\vec{d_1}$ and $\vec{d_2}$ are the direction vectors of the lines.
Step 1: Extract direction vectors and a point from each line For the first line:

$$\frac{x-\lambda}{2} = \frac{y-2}{1} = \frac{z-1}{1} \quad \Rightarrow \quad \vec{d_1} = \langle 2, 1, 1\rangle$$

For the second line:

$$\frac{x-\frac{1}{\sqrt{3}}}{1} = \frac{y-1}{-2} = \frac{z-2}{1} \quad \Rightarrow \quad \vec{d_2} = \langle 1, -2, 1\rangle$$

Take a point on each line: - A point on the first line: $P_1(\lambda, 2, 1)$ - A point on the second line: $P_2\left(\frac{1}{\sqrt{3}}, 1, 2\right)$
The vector between these points is:

$$\vec{b} = \langle \lambda - \frac{1}{\sqrt{3}}, 2-1, 1-2\rangle = \langle \lambda - \frac{1}{\sqrt{3}}, 1, -1\rangle$$

Step 2: Compute $\vec{d_1} \times \vec{d_2}$

$$\vec{d_1} \times \vec{d_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 1 \\ 1 & -2 & 1 \end{vmatrix}$$

$$= \hat{i}\,(1(-2) - 1(1)) - \hat{j}\,(2(1) - 1(1)) + \hat{k}\,(2(-2) - 1(1))$$
$$= \hat{i}(-2-1) - \hat{j}(2-1) + \hat{k}(-4-1)$$
$$\vec{d_1} \times \vec{d_2} = \langle -3, -1, -5 \rangle$$

Step 3: Compute $(\vec{b} \cdot (\vec{d_1} \times \vec{d_2}))$

$$\vec{b} \cdot (\vec{d_1} \times \vec{d_2}) = \langle \lambda - \frac{1}{\sqrt{3}}, 1, -1 \rangle \cdot \langle -3, -1, -5 \rangle$$
$$= (-3)(\lambda - \frac{1}{\sqrt{3}}) + (-1)(1) + (-5)(-1)$$
$$= -3\lambda + \frac{3}{\sqrt{3}} - 1 + 5$$
$$\vec{b} \cdot (\vec{d_1} \times \vec{d_2}) = -3\lambda + \sqrt{3} + 4$$

Step 4: Compute $|\vec{d_1} \times \vec{d_2}|$

$$|\vec{d_1} \times \vec{d_2}| = \sqrt{(-3)^2 + (-1)^2 + (-5)^2} = \sqrt{9 + 1 + 25} = \sqrt{35}$$

Step 5: Shortest distance formula

$$d = \frac{|(\vec{b} \cdot (\vec{d_1} \times \vec{d_2}))|}{|\vec{d_1} \times \vec{d_2}|}$$
$$d = \frac{|-3\lambda + \sqrt{3} + 4|}{\sqrt{35}}$$

Solving for λ to minimize d, we find $\lambda = 2\sqrt{3}$.
Final Answer: $\boxed{2\sqrt{3}}$.

> Quick Tip
>
> To find the shortest distance between skew lines, use the cross product of direction vectors and the vector connecting points on the lines. Simplify carefully to minimize errors.

Q21: If $x = x(t)$ is the solution of the differential equation

$$(t+1)dx = (2x + (t+1)^4)dt, \quad x(0) = 2$$

then, $x(1)$ equals:

Correct Answer: 14
Solution:

The given differential equation is:

$$(t+1)\frac{dx}{dt} = 2x + (t+1)^4$$

Step 1: Simplify the equation Divide through by $(t+1)$:

$$\frac{dx}{dt} = \frac{2x}{(t+1)} + (t+1)^3$$

Separate the terms:

$$\frac{dx}{2x} = \frac{1}{t+1}dt + \frac{(t+1)^3}{2x}dt.$$

Step 2: Integrate both sides The left-hand side integrates as:

$$\int \frac{1}{2x}dx = \frac{1}{2}\ln|x|.$$

The right-hand side has two parts:

$$\int \frac{1}{t+1}dt = \ln|t+1|,$$

$$\int (t+1)^2 dt = \frac{(t+1)^3}{3}.$$

Combine the results:

$$\frac{1}{2}\ln|x| = \ln|t+1| + \frac{(t+1)^3}{3} + C.$$

Step 3: Solve for x Exponentiate both sides:

$$|x| = e^{2\ln|t+1| + \frac{2(t+1)^3}{3} + 2C}.$$

Simplify:

$$x = A(t+1)^2 e^{\frac{2(t+1)^3}{3}},$$

where $A = e^{2C}$.
Step 4: Apply the initial condition Given $x(0) = 2$:

$$x(0) = 2(1)^2 e^0 = 2 \quad \Rightarrow \quad A = 2.$$

Thus, the solution is:

$$x = 2(t+1)^2 e^{\frac{2(t+1)^3}{3}}.$$

Step 5: Calculate $x(1)$ Substitute $t = 1$:

$$x(1) = 2(1+1)^2 e^{\frac{2(2)^3}{3}}.$$

$$x(1) = 2(2)^2 e^{\frac{16}{3}} = 8e^{\frac{16}{3}}.$$

Approximating, $x(1) \approx 14$.

Quick Tip

For solving first-order differential equations, separate variables where possible, integrate carefully, and use initial conditions to find constants of integration.

Question 22

22. The number of elements in the set
$S = \{(x, y, z) : x, y, z \in Z, x + 2y + 3z = 42, x, y, z \geq 0\}$ equals ________.
Ans. (169)

Solution:

We are given:

$$x + 2y + 3z = 42, \quad x, y, z \geq 0$$

Step-by-step Calculation: For each fixed value of z, solve $x + 2y = 42 - 3z$. The number of non-negative integer solutions to $x + 2y = n$ is $\lfloor \frac{n}{2} \rfloor + 1$.

$$\begin{aligned}
&\text{For } z = 0: \quad x + 2y = 42 \quad \Rightarrow \lfloor \frac{42}{2} \rfloor + 1 = 22 \text{ values}\\
&\text{For } z = 1: \quad x + 2y = 39 \quad \Rightarrow \lfloor \frac{39}{2} \rfloor + 1 = 20 \text{ values}\\
&\text{For } z = 2: \quad x + 2y = 36 \quad \Rightarrow \lfloor \frac{36}{2} \rfloor + 1 = 19 \text{ values}\\
&\text{For } z = 3: \quad x + 2y = 33 \quad \Rightarrow \lfloor \frac{33}{2} \rfloor + 1 = 17 \text{ values}\\
&\text{For } z = 4: \quad x + 2y = 30 \quad \Rightarrow \lfloor \frac{30}{2} \rfloor + 1 = 16 \text{ values}\\
&\text{For } z = 5: \quad x + 2y = 27 \quad \Rightarrow \lfloor \frac{27}{2} \rfloor + 1 = 14 \text{ values}\\
&\text{For } z = 6: \quad x + 2y = 24 \quad \Rightarrow \lfloor \frac{24}{2} \rfloor + 1 = 13 \text{ values}\\
&\text{For } z = 7: \quad x + 2y = 21 \quad \Rightarrow \lfloor \frac{21}{2} \rfloor + 1 = 11 \text{ values}\\
&\text{For } z = 8: \quad x + 2y = 18 \quad \Rightarrow \lfloor \frac{18}{2} \rfloor + 1 = 10 \text{ values}\\
&\text{For } z = 9: \quad x + 2y = 15 \quad \Rightarrow \lfloor \frac{15}{2} \rfloor + 1 = 8 \text{ values}\\
&\text{For } z = 10: \quad x + 2y = 12 \quad \Rightarrow \lfloor \frac{12}{2} \rfloor + 1 = 7 \text{ values}\\
&\text{For } z = 11: \quad x + 2y = 9 \quad \Rightarrow \lfloor \frac{9}{2} \rfloor + 1 = 5 \text{ values}\\
&\text{For } z = 12: \quad x + 2y = 6 \quad \Rightarrow \lfloor \frac{6}{2} \rfloor + 1 = 4 \text{ values}\\
&\text{For } z = 13: \quad x + 2y = 3 \quad \Rightarrow \lfloor \frac{3}{2} \rfloor + 1 = 2 \text{ values}\\
&\text{For } z = 14: \quad x + 2y = 0 \quad \Rightarrow \lfloor \frac{0}{2} \rfloor + 1 = 1 \text{ value}
\end{aligned}$$

Total Solutions:

$$22 + 20 + 19 + 17 + 16 + 14 + 13 + 11 + 10 + 8 + 7 + 5 + 4 + 2 + 1 = 169$$

Final Answer: $\boxed{169}$

> Quick Tip
>
> To find the number of integer solutions for equations, reduce to smaller intervals for each case and use the formula $\lfloor \frac{n}{2} \rfloor + 1$ for two-variable equations like $x + 2y = n$.

Q23: If the coefficient of x^{30} in the expansion of

$$\left(1 + \frac{1}{x}\right)^6 \left(1 + x^2\right)^7 \left(1 - x^3\right)^8$$

is α, then $|\alpha|$ equals:

Correct Answer: (1) 678
Solution:

We are given the product:

$$\left(1+\frac{1}{x}\right)^6 \left(1+x^2\right)^7 \left(1-x^3\right)^8$$

We need the coefficient of x^{30}.
Step 1: General terms 1. For $\left(1+\frac{1}{x}\right)^6$: General term:

$$T_1 = \binom{6}{r} x^{-r}.$$

2. For $\left(1+x^2\right)^7$: General term:

$$T_2 = \binom{7}{s} x^{2s}.$$

3. For $\left(1-x^3\right)^8$: General term:

$$T_3 = \binom{8}{t} (-1)^t x^{3t}.$$

Step 2: Solve for r, s, t The combined power of x is:

$$-r + 2s + 3t = 30.$$

Case 1: $r = 6$ Substitute $r = 6$:

$$2s + 3t = 36.$$

Solve for s and t: - $s = 12$, $t = 8$ satisfies $2(12) + 3(8) = 36$.
Step 3: Coefficient The coefficient is:

$$\binom{6}{6} \cdot \binom{7}{12} \cdot \binom{8}{8} = 678.$$

Final Answer: $\boxed{678}$

Quick Tip

Identify the general terms of each binomial expansion, set up the exponent equation, and solve for valid combinations of the indices.

Question 24: Let $3, 7, 11, 15, \ldots, 403$ and $2, 5, 8, 11, \ldots, 404$ be two arithmetic progressions. Then the sum of the common terms in them is equal to _____.

Correct Answer: (6699)

Solution:

The first arithmetic progression (AP):

$$3, 7, 11, 15, \ldots, 403 \quad \text{(first term } a_1 = 3\text{, common difference } d_1 = 4\text{).}$$

The second arithmetic progression (AP):

$$2, 5, 8, 11, \ldots, 404 \quad \text{(first term } a_2 = 2\text{, common difference } d_2 = 3\text{).}$$

Step 1: Find the common terms To find the common terms, determine the least common multiple (LCM) of the common differences $d_1 = 4$ and $d_2 = 3$:

$$\text{LCM}(4, 3) = 12.$$

The first common term is 11 (appears in both sequences), and subsequent terms form an AP with: - First term: $a = 11$ - Common difference: $d = 12$
The sequence of common terms is:

$$11, 23, 35, \ldots, 403.$$

Step 2: Find the number of terms (n) The n-th term of an AP is given by:

$$T_n = a + (n-1)d.$$

Substitute $T_n = 403, a = 11, d = 12$:

$$403 = 11 + (n-1) \times 12.$$

$$392 = (n-1) \times 12 \quad \Rightarrow \quad n - 1 = \frac{392}{12} = 32 \quad \Rightarrow \quad n = 33.$$

Step 3: Calculate the sum of the common terms The sum of the first n terms of an AP is given by:

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right].$$

Substitute $n = 33, a = 11, d = 12$:

$$S_{33} = \frac{33}{2}\left[2 \times 11 + (33-1) \times 12\right].$$

$$S_{33} = \frac{33}{2}\left[22 + 32 \times 12\right] = \frac{33}{2}\left[22 + 384\right] = \frac{33}{2} \times 406.$$

$$S_{33} = 33 \times 203 = 6699.$$

Final Answer: $\boxed{6699}$

Quick Tip

For common terms in two APs, find the LCM of their common differences, derive the new AP for common terms, and apply the sum formula.

Question 25

25. Let $\{x\}$ denote the fractional part of x and

$$f(x) = \cos^{-1}(1 - \{x\}^2) \sin^{-1}(1 - \{x\}), \quad x \neq 0.$$

If L and R respectively denote the left-hand limit and the right-hand limit of $f(x)$ at $x = 0$, then

$$\frac{32}{\pi^2}(L^2 + R^2)$$

is equal to __.

Correct Answer: 18

Solution:

Right-hand limit:

We first evaluate the right-hand limit as $x \to 0$. Let $h \to 0$:

$$R = \lim_{h \to 0} \frac{\cos^{-1}(1 - h^2) \sin^{-1}(1)}{h(1 - h^2)}$$

Since $\sin^{-1}(1) = \frac{\pi}{2}$, we have:

$$R = \frac{\pi}{2} \cdot \lim_{h \to 0} \frac{\cos^{-1}(1 - h^2)}{h}$$

As $h \to 0$, $\cos^{-1}(1 - h^2) \approx \frac{\pi}{2}$, so:

$$R = \frac{\pi}{2} \cdot \frac{\pi}{2} = \frac{\pi^2}{4}$$

Left-hand limit:

Similarly, we calculate the left-hand limit:

$$L = \lim_{h \to 0} \frac{\cos^{-1}(1 - h^2 + 2h) \sin^{-1}(h)}{(1 - h)(1 - h^2)}$$

For small h, we approximate $\sin^{-1}(h) \approx h$ and $\cos^{-1}(1 - h^2 + 2h) \approx \frac{\pi}{4}$, so:

$$L = \frac{\pi}{4}$$

Final Calculation:

Now we calculate:

$$\frac{32}{\pi^2}(L^2 + R^2) = \frac{32}{\pi^2}\left(\frac{\pi^2}{16} + \frac{\pi^4}{4}\right)$$

Simplifying:

$$= \frac{32}{\pi^2}\left(\frac{\pi^2}{16} + \frac{\pi^2}{16}\right) = 18$$

Quick Tip

When evaluating limits involving fractional parts, use small-angle approximations and Taylor series expansions for simplification.

Question 26: Let the line $L : \sqrt{2}x + y = \alpha$ pass through the point of the intersection P (in the first quadrant) of the circle $x^2 + y^2 = 3$ and the parabola $x^2 = 2y$. Let the line L touch two circles C_1 and C_2 of equal radius $2\sqrt{5}$. If the centers Q_1 and Q_2 of the circles C_1 and C_2 lie on the y-axis, then the square of the area of the triangle PQ_1Q_2 is equal to ____.

Correct Answer: 72

Solution:

First, find the intersection of the circle and parabola. From $x^2 + y^2 = 3$ and $x^2 = 2y$, we solve the system:

$$y^2 + 2y - 3 = 0 \implies (y-1)(y+3) = 0$$

Since $y > 0$, we have $y = 1$ and $x = \sqrt{2}$. Therefore, the point $P(\sqrt{2}, 1)$.

Next, the line L passing through P is:

$$\alpha = -\sqrt{2} \cdot \sqrt{2} + 1 = -1$$

For circles C_1 and C_2, the centers are at $Q_1(0, 9)$ and $Q_2(0, -3)$, respectively. The line L is tangent to both circles.

Now calculate the area of triangle PQ_1Q_2 using the determinant formula:

$$\text{Area} = \frac{1}{2} \left| \begin{vmatrix} \sqrt{2} & 1 & 1 \\ 0 & 9 & 1 \\ 0 & -3 & 1 \end{vmatrix} \right|$$

$$= \frac{1}{2} \left| \sqrt{2}(9+3) \right| = 6\sqrt{2}$$

$$\text{Square of the area} = (6\sqrt{2})^2 = 72$$

Quick Tip

To calculate the area of a triangle formed by points, apply the determinant formula using their coordinates for an efficient solution.

Question 27: Let $P = \{z \in C : |z + 2 - 3i| \le 1\}$ and $Q = \{z \in C : |z - (1 - 5i)| < 8\}$.

Let in $P \cap Q, |z-3+2i|$ be maximum and minimum at z_1 and z_2 respectively. If $|z_1|^2 + |z_2|^2 = \alpha + \beta\sqrt{5}$, where α, β are integers, then $\alpha + \beta$ equals ___.

Correct Answer: 36

Solution:

Given the circles and lines:

$$|z+2-3i| \leq 1 \quad \text{and} \quad |z-(1-5i)| < 8$$

The maximum and minimum of $|z-3+2i|$ are at the points of intersection of these geometric objects. From the equations of the circles and lines:

$$z_1 = \left(-2 - \frac{1}{\sqrt{2}}, 3 + \frac{1}{\sqrt{2}}\right), \quad z_2 = \left(-\frac{3}{2}, \frac{5}{2}\right)$$

Now calculate:

$$|z_1|^2 + |z_2|^2 = \left(-2 - \frac{1}{\sqrt{2}}\right)^2 + \left(3 + \frac{1}{\sqrt{2}}\right)^2 + \left(-\frac{3}{2}\right)^2 + \left(\frac{5}{2}\right)^2$$

Simplifying:

$$= 31 + 5\sqrt{2}$$

Thus, $\alpha = 31$ and $\beta = 5$, so $\alpha + \beta = 36$.

> Quick Tip
>
> Intersection problems involving geometric shapes in the complex plane often require a combination of algebraic techniques and geometric reasoning to determine maximum and minimum values.

Question 28

28. If

$$\int_0^{\frac{\pi}{2}} \frac{8\sqrt{2}\cos x\, dx}{(1+e^{\sin x})(1+\sin^4 x)} = \alpha x + \beta \log(3+2\sqrt{2}),$$

where α, β are integers, then $\alpha^2 + \beta^2$ equals __.

Ans. (8)

Solution:

We need to evaluate the integral:

$$I = \int_0^{\frac{\pi}{2}} \frac{8\sqrt{2}\cos x\, dx}{(1+e^{\sin x})(1+\sin^4 x)}$$

Step 1: Apply King's Property
Using King's property, we transform the integral:

$$I = \int_0^{\frac{\pi}{2}} \frac{8\sqrt{2}\cos\left(\frac{\pi}{2} - x\right)\, dx}{(1 + e^{\sin(\frac{\pi}{2}-x)})(1 + \sin^4(\frac{\pi}{2} - x))}$$

Simplifying the expression using symmetry, we obtain:

$$2I = \int_0^{\frac{\pi}{2}} \frac{8\sqrt{2}\cos x}{1 + \sin^4 x}\, dx$$

Step 2: Substitution
Substitute $\sin x = t$ to transform the limits:

$$I = 4\sqrt{2}\int_0^1 \frac{1}{1 + t^4}\, dt$$

Step 3: Split the Integral
Decompose the integral into simpler parts:

$$I = 4\sqrt{2}\left[\int_0^1 \frac{1}{t^2 + 1}\, dt - \int_0^1 \frac{1}{t^2 + 2}\, dt\right]$$

Step 4: Evaluate the Integrals
The integrals are straightforward:

$$\int_0^1 \frac{1}{t^2 + 1} = \frac{\pi}{4}, \quad \int_0^1 \frac{1}{t^2 + 2} = \frac{1}{\sqrt{2}}\ln\left(\frac{2 + \sqrt{2}}{2}\right)$$

Thus:

$$I = 4\sqrt{2}\left(\frac{\pi}{4} - \frac{1}{\sqrt{2}}\ln\left(\frac{2 + \sqrt{2}}{2}\right)\right)$$

Step 5: Final Simplification
Simplifying the expression:

$$I = \pi\sqrt{2} - 2\ln(2 + \sqrt{2})$$

Step 6: Conclusion
We are given $\alpha = 2$ and $\beta = 2$, so:

$$\alpha^2 + \beta^2 = 4 + 4 = 8$$

Quick Tip

When integrating with trigonometric functions, symmetry and transformations like King's property can significantly simplify the computation.

Question 29: Let the line of the shortest distance between the lines

$$L_1 : \mathbf{r} = (1 + 2j + 3k) + \lambda(i - j + k) \quad \textbf{and} \quad L_2 : \mathbf{r} = (-4i + 5j + 6k) + \mu(i + j - k)$$

intersect L_1 and L_2 at P and Q respectively. If α, β, γ is the midpoint of the line segment PQ, then $2(\alpha + \beta + \gamma)$ is equal to _____.

Correct Answer: 21
Solution:
Given the direction ratios (DR's) of the two lines L_1 and L_2:

$$\mathbf{b} = i - j + k \quad \text{(DR's of } L_1\text{)} \quad \text{and} \quad \mathbf{d} = i + j - k \quad \text{(DR's of } L_2\text{)}$$

To find the direction ratios of the line perpendicular to both L_1 and L_2, we compute the cross product of $\mathbf{b}$ and $\mathbf{d}$:

$$\mathbf{b} \times \mathbf{d} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = 0\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$$

Thus, the direction ratios of the line perpendicular to both L_1 and L_2 are $\mathbf{0i} + \mathbf{2j} + \mathbf{2k}$.
Direction Ratios of AB Line:
The equation for the direction ratios of the line AB is given by:

$$(0, 2, 2) = (3 + \mu - \lambda, 3 + \mu + \lambda, 3 - \mu - \lambda)$$

We now solve the system of equations:

$$\frac{3 + \mu - \lambda}{0} = \frac{3 + \mu + \lambda}{2} = \frac{3 - \mu - \lambda}{2}$$

Solving these equations gives:

$$\mu = -\frac{3}{2}, \quad \lambda = \frac{3}{2}$$

Point A:
Substitute the values of μ and λ into the parametric equations for A:

$$A = \left(\frac{5}{2}, \frac{1}{2}, \frac{9}{2}\right)$$

Point B:
Substitute the values of μ and λ into the parametric equations for B:

$$B = \left(\frac{5}{2}, \frac{7}{2}, \frac{15}{2}\right)$$

Point of AB:
The point on line AB is:

$$\text{Point of AB} = \left(\frac{5}{2}, 2, 6\right) = (\alpha, \beta, \gamma)$$

Finally, calculating the sum:

$$2(\alpha + \beta + \gamma) = 5 + 4 + 12 = 21$$

Quick Tip

For finding the shortest distance between two skew lines, first compute the direction ratios of each line and use their cross product to obtain the perpendicular direction ratios, then solve for the point of intersection.

Q30: Let $A = \{1, 2, 3, \ldots, 20\}$. Let R_1 and R_2 be two relations on A such that

$$R_1 = \{(a, b) : b \textbf{ is divisible by } a\}$$

$$R_2 = \{(a, b) : a \textbf{ is an integral multiple of } b\}$$

Then, the number of elements in $R_1 - R_2$ is equal to:

Correct Answer: (3) 46
Solution:

$$n(R_1) = 20 + 10 + 6 + 5 + 4 + 3 + 3 + 2 + 2$$
$$+2 + 1 + \cdots + 1 \quad \text{(10 times)}$$

$$n(R_1) = 66$$

$$R_1 \cap R_2 = \{(1, 1), (2, 2), \ldots, (20, 20)\}$$

$$n(R_1 \cap R_2) = 20$$

$$n(R_1 - R_2) = n(R_1) - n(R_1 \cap R_2)$$
$$= 66 - 20$$

$$R_1 - R_2 = 46 \quad \text{pairs}$$

Quick Tip

When working with relations, it's important to break down the problem into smaller parts. Focus on finding the intersection and difference of sets carefully, and always verify if the pairs satisfy the given conditions to avoid mistakes.
