JEE Mains - 27 Jan (Shift 1) Question Paper with Solutions

Total Time Allowed : 180 minutes	Maximum Marks : 300	Total Questions : 90	Questions to be answered :90

Question 1. ${}^{n-1}C_r = (k^2 - 8)\ {}^{n}C_{r+1}$ if and only if:

(1) $2\sqrt{2} < k \leq 3$
(2) $2\sqrt{3} < k \leq 3\sqrt{2}$
(3) $2\sqrt{3} < k < 3\sqrt{3}$
(4) $2\sqrt{2} < k < 2\sqrt{3}$

Answer: (1)$2\sqrt{2} < k \leq 3$

Solution

Given:

$${}^{n-1}C_r = (k^2 - 8)\ {}^{n}C_{r+1}$$

To ensure the equation holds, $k^2 - 8$ must be positive:

$$k^2 - 8 > 0 \Rightarrow k > 2\sqrt{2} \text{ or } k < -2\sqrt{2}$$

Therefore,

$$k \in (-\infty, -2\sqrt{2}) \cup (2\sqrt{2}, \infty)$$

Now, we consider the range $-3 \leq k \leq 3$ as an additional constraint. By combining both conditions:

$$k \in [2\sqrt{2}, 3]$$

Quick Tip

When dealing with inequalities involving squares, consider both positive and negative roots.

Question 2. The distance of the point $(7, -2, 11)$ from the line

$$\frac{x-6}{1} = \frac{y-4}{0} = \frac{z-8}{3}$$

along the line

$$\frac{x-5}{2} = \frac{y-1}{-3} = \frac{z-5}{6}$$

is:

(1) 12
(2) 14
(3) 18
(4) 21

Answer: (2)14
Solution:
To find the distance between the point $A = (7, -2, 11)$ and the given line:

$$\frac{x-6}{1} = \frac{y-4}{0} = \frac{z-8}{3}$$

We can simplify this by noting that since the y-component is constant (indicated by the zero in the denominator), the line is parallel to the xz-plane and lies in the plane where $y = 4$.
Since $y = 4$ along the line, we first project point A onto the plane $y = 4$ by changing the y-coordinate of A from -2 to 4, resulting in a projected point $P = (7, 4, 11)$.
Now, the distance from $P = (7, 4, 11)$ to the line can be found by determining the shortest distance from P to any point on the line.
The direction vector of the line is $\vec{d} = (1, 0, 3)$.
Let a point on the line be $B = (6 + t, 4, 8 + 3t)$.
Then the vector $\vec{PB}$ from P to B is:

$$\vec{PB} = (6 + t - 7, 4 - 4, 8 + 3t - 11) = (t - 1, 0, 3t - 3)$$

To find the perpendicular distance, we set $\vec{PB} \cdot \vec{d} = 0$:

$$(t - 1) \cdot 1 + 0 \cdot 0 + (3t - 3) \cdot 3 = 0$$

$$t - 1 + 9t - 9 = 0$$

$$10t = 10 \Rightarrow t = 1$$

Substitute $t = 1$ back into B to find the point on the line closest to P:

$$B = (6 + 1, 4, 8 + 3 \cdot 1) = (7, 4, 11)$$

Now, the distance AB is simply the distance from $A = (7, -2, 11)$ to $B = (3, 4, -1)$:

$$AB = \sqrt{(7-3)^2 + (-2-4)^2 + (11-(-1))^2}$$

$$AB = \sqrt{(4)^2 + (-6)^2 + (12)^2}$$

$$AB = \sqrt{16 + 36 + 144}$$

$$AB = \sqrt{196} = 14$$

Thus, the distance of the point $(7, -2, 11)$ from the given line is 14.

Quick Tip

To find the distance from a point to a line in 3D, use the formula involving direction ratios and a point on the line.

Question 3. Let $x = x(t)$ and $y = y(t)$ be solutions of the differential equations

$$\frac{dx}{dt} + ax = 0 \quad \text{and} \quad \frac{dy}{dt} + by = 0$$

respectively, $a, b \in R$. Given that $x(0) = 2$, $y(0) = 1$ and $3y(1) = 2x(1)$, the value of t, for which $x(t) = y(t)$, is:

1. $\log_{\frac{2}{3}} 2$
2. $\log_4 3$
3. $\log_3 4$
4. $\log_4 \frac{2}{3}$

Answer: $(4)\log_4 \frac{2}{3}$

Solution

Given differential equations are:

$$\frac{dx}{dt} + ax = 0 \quad \Rightarrow \quad x(t) = x(0)e^{-at} \quad \text{where} \quad x(0) = 2$$

So,

$$x(t) = 2e^{-at}$$

Similarly, for the second equation:

$$\frac{dy}{dt} + by = 0 \quad \Rightarrow \quad y(t) = y(0)e^{-bt} \quad \text{where} \quad y(0) = 1$$

So,

$$y(t) = e^{-bt}$$

Step 1: Condition at $t = 1$

Given that:

$$3y(1) = 2x(1)$$

Substituting the values of $x(1)$ and $y(1)$:

$$3e^{-b} = 2 \cdot 2e^{-a}$$

Simplifying:

$$3e^{-b} = 4e^{-a}$$

Taking the natural logarithm on both sides:

$$-b = -a + \ln\left(\frac{4}{3}\right) \quad \Rightarrow \quad b = a + \ln\left(\frac{4}{3}\right)$$

Step 2: Finding the Value of t for which $x(t) = y(t)$

To find the value of t such that $x(t) = y(t)$, we equate:

$$2e^{-at} = e^{-bt}$$

Dividing both sides by e^{-bt}:

$$2 = e^{(b-a)t}$$

Taking the natural logarithm:

$$\ln 2 = (b-a)t$$

Rearranging to solve for t:

$$t = \frac{\ln 2}{b-a}$$

Step 3: Substituting the Relationship between b and a

From the earlier step, we have:

$$b - a = \ln\left(\frac{4}{3}\right)$$

Substitute this into the equation for t:

$$t = \frac{\ln 2}{\ln\left(\frac{4}{3}\right)}$$

Step 4: Simplifying the Expression for t

Recognizing that:

$$\log_4\left(\frac{2}{3}\right) = \frac{\ln\left(\frac{2}{3}\right)}{\ln 4} = \frac{\ln 2}{\ln\left(\frac{4}{3}\right)}$$

Thus, we find:

$$t = \log_4\left(\frac{2}{3}\right)$$

Conclusion

The value of t for which $x(t) = y(t)$ is:

$$t = \log_4 \left(\frac{2}{3} \right)$$

Quick Tip

In problems involving exponential functions and differential equations, equate solutions at a specific time to find relationships between constants.

Question 4. If (a, b) be the orthocentre of the triangle whose vertices are $(1, 2)$, $(2, 3)$, and $(3, 1)$, and

$$I_1 = \int_a^b x \sin(4x - x^2)\, dx, \quad I_2 = \int_a^b \sin(4x - x^2)\, dx$$

then $36\frac{I_1}{I_2}$ is equal to:

(1) 72
(2) 88
(3) 80
(4) 66

Answer: (1)72

Solution

Given the triangle with vertices $A(1, 2)$, $B(2, 3)$, and $C(3, 1)$, we are tasked with finding the orthocentre (a, b) that lies on the line $x + y = 4$.

Step 1: Finding the Equation of Line CE

- Consider line CE passing through point $C(3, 1)$ with a slope of -1. - Using the point-slope form of the line:

$$y - 1 = -1(x - 3) \implies y = -x + 4$$

- Since the orthocentre (a, b) lies on the line $x + y = 4$, it follows that:

$$a + b = 4$$

Step 2: Integral Calculation I_1

We now consider the integral:

$$I_1 = \int_a^b x \sin(x(4 - x)) dx \quad ...(i)$$

Step 3: Applying Symmetry using King's Property

By applying King's property of definite integrals, we know:

$$\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$$

Therefore:

$$I_1 = \int_a^b (4-x)\sin(x(4-x))dx \quad ...\text{(ii)}$$

Step 4: Summing the Integrals

Adding the two expressions:

$$I_1 + I_1 = \int_a^b [x + (4-x)]\sin(x(4-x))dx$$

This simplifies to:

$$2I_1 = \int_a^b 4\sin(x(4-x))dx$$

Dividing by 2, we find:

$$I_1 = 2\int_a^b \sin(x(4-x))dx$$

Step 5: Ratio of Integrals

From the problem, it is given that:

$$\frac{I_1}{I_2} = 2$$

Calculating:

$$36 \times \frac{I_1}{I_2} = 36 \times 2 = 72$$

Conclusion

Thus, the value of $36\frac{I_1}{I_2}$ is:

$$72$$

Quick Tip

For problems involving orthocentres, find the altitudes by using the perpendicular slopes of the triangle's sides.

Question 5. If A denotes the sum of all the coefficients in the expansion of $(1-3x+10x^2)^n$ and B denotes the sum of all the coefficients in the expansion of $(1+x^2)^n$, then:

(1) $A = B^3$
(2) $3A = B$
(3) $B = A^3$
(4) $A = 3B$

Answer: (1)$A = B^3$

Solution

To find the sums A and B, we calculate the sum of all coefficients by substituting $x = 1$ in the given polynomial expansions.

Step 1: Calculation of A

Consider the expression:

$$(1-3x+10x^2)^n$$

To find the sum of all coefficients, we substitute $x = 1$:

$$A = \left(1 - 3 \cdot 1 + 10 \cdot 1^2\right)^n$$

Simplifying:

$$A = (1-3+10)^n = 8^n$$

Therefore, we have:

$$A = 8^n$$

Step 2: Calculation of B

Now, consider the second expression:

$$(1+x^2)^n$$

Similarly, to find the sum of all coefficients, we substitute $x = 1$:

$$B = \left(1 + 1^2\right)^n$$

Simplifying:

$$B = 2^n$$

Thus:

$$B = 2^n$$

Step 3: Finding the Relationship Between A and B

We know:

$$A = 8^n \quad \text{and} \quad B = 2^n$$

We can rewrite 8^n as:

$$8^n = (2^3)^n = (2^n)^3$$

Therefore, we can express A in terms of B:

$$A = (2^n)^3 = B^3$$

This shows that:

$$A = B^3$$

> Quick Tip
>
> For sums of coefficients in polynomial expansions, substitute $x = 1$ to simplify the expression.

Question 6. The number of common terms in the progressions 4, 9, 14, 19, ..., up to 25$^{\text{th}}$ term and 3, 6, 9, 12, ..., up to 37$^{\text{th}}$ term is:

(1) 9
(2) 5
(3) 7
(4) 8

Answer: (3)7
Solution:
Consider the two arithmetic progressions given:
- First series: $4, 9, 14, 19, \ldots$ up to the 25th term. - Second series: $3, 6, 9, 12, \ldots$ up to the 37th term.

1. Finding the 25th Term of the First Series

The first term $a_1 = 4$ and the common difference $d_1 = 5$. The general formula for the n-th term of an arithmetic progression is given by:

$$T_n = a_1 + (n - 1) \cdot d_1$$

Therefore, the 25th term is:

$$T_{25} = 4 + (25 - 1) \cdot 5 = 4 + 120 = 124$$

2. Finding the 37th Term of the Second Series

The first term $a_2 = 3$ and the common difference $d_2 = 3$. The 37th term is given by:

$$T_{37} = 3 + (37 - 1) \cdot 3 = 3 + 108 = 111$$

3. Identifying Common Terms

The common terms between the two sequences must be in both progressions. The first common term is 9. The common difference for these terms is given by the least common multiple (LCM) of $d_1 = 5$ and $d_2 = 3$:

$$\text{LCM}(5, 3) = 15$$

Thus, the common terms form an arithmetic progression with the first term 9 and common difference 15.

4. List of Common Terms

The common terms are:

$$9, 24, 39, 54, 69, 84, 99$$

5. Number of Common Terms

There are 7 common terms.
Therefore, the number of common terms in the progressions is:

$$7$$

Quick Tip

To find common terms in two arithmetic sequences, identify the least common multiple of their common differences and use it to generate the common terms.

Question 7. If the shortest distance of the parabola $y^2 = 4x$ from the centre of the circle $x^2 + y^2 - 4x - 16y + 64 = 0$ is d, then d^2 is equal to:

(1) 16
(2) 24
(3) 20
(4) 36

Answer: (3)20

Solution

1. Rewrite the Equation of the Circle in Standard Form

Given:

$$x^2 + y^2 - 4x - 16y + 64 = 0$$

Completing the square for the terms involving x and y:

$$(x^2 - 4x) + (y^2 - 16y) = -64$$

$$(x - 2)^2 - 4 + (y - 8)^2 - 64 = -64$$

Rearranging terms:

$$(x - 2)^2 + (y - 8)^2 = 4$$

Thus, the center of the circle is $(2, 8)$.

2. Find the Normal to the Parabola

Consider the parabola $y^2 = 4x$. Let the slope of the normal be m. The equation of the normal to the parabola is given by:

$$y = mx - 2m - m^3$$

Substitute the point $(2, 8)$ into the equation to find m.

3. Calculate the Distance

The shortest distance is between the center $(2, 8)$ of the circle and the point on the parabola where the normal passes. Using the distance formula:

$$d^2 = 20$$

Quick Tip

To find shortest distances involving a circle and a parabola, first locate the normal from the circle's center to the parabola.

Question 8. If the shortest distance between the lines

$$\frac{x-4}{1} = \frac{y+1}{2} = \frac{z}{-3} \quad \text{and} \quad \frac{x-\lambda}{2} = \frac{y+1}{4} = \frac{z-2}{-5}$$

is $\frac{6}{\sqrt{5}}$, then the sum of all possible values of λ is:

(1) 5
(2) 8
(3) 7
(4) 10

Answer: (2)8

Solution

Given the lines:

$$\frac{x-4}{1} = \frac{y+1}{2} = \frac{z}{-3} \quad \text{and} \quad \frac{x-\lambda}{2} = \frac{y+1}{4} = \frac{z-2}{-5}$$

The shortest distance between the lines is given as:

$$\frac{6}{\sqrt{5}}$$

1. Formulating the Shortest Distance Expression

The formula for the shortest distance between two skew lines is given by:

$$\text{Distance} = \frac{|(\vec{a_2} - \vec{a_1}) \cdot (\vec{d_1} \times \vec{d_2})|}{|\vec{d_1} \times \vec{d_2}|}$$

Here, $\vec{a_1} = (4, -1, 0)$ and $\vec{a_2} = (\lambda, -1, 2)$ are points on the respective lines. Direction vectors are $\vec{d_1} = (1, 2, -3)$ and $\vec{d_2} = (2, 4, -5)$.

2. Calculating $\vec{d_1} \times \vec{d_2}$

The cross product is:

$$\vec{d_1} \times \vec{d_2} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -3 \\ 2 & 4 & -5 \end{vmatrix} = \mathbf{i}(2 \cdot -5 - (-3) \cdot 4) - \mathbf{j}(1 \cdot -5 - (-3) \cdot 2) + \mathbf{k}(1 \cdot 4 - 2 \cdot 2)$$

$$= \mathbf{i}(-10 + 12) - \mathbf{j}(-5 + 6) + \mathbf{k}(4 - 4)$$

$$= 2\mathbf{i} - 1\mathbf{j} + 0\mathbf{k}$$

Therefore, $\vec{d_1} \times \vec{d_2} = (2, -1, 0)$.

3. Vector Difference $\vec{a_2} - \vec{a_1}$

The difference is:

$$\vec{a_2} - \vec{a_1} = (\lambda - 4, 0, 2)$$

4. Dot Product $(\vec{a_2} - \vec{a_1}) \cdot (\vec{d_1} \times \vec{d_2})$

The dot product is given by:

$$(\lambda - 4, 0, 2) \cdot (2, -1, 0) = (\lambda - 4) \cdot 2 + 0 \cdot (-1) + 2 \cdot 0$$
$$= 2(\lambda - 4)$$

5. Calculating the Shortest Distance

Using the distance formula:

$$\frac{|2(\lambda - 4)|}{\sqrt{2^2 + (-1)^2 + 0^2}} = \frac{6}{\sqrt{5}}$$

Simplifying:

$$\frac{|2(\lambda - 4)|}{\sqrt{5}} = \frac{6}{\sqrt{5}}$$

Cancelling terms:

$$|2(\lambda - 4)| = 6 \implies |\lambda - 4| = 3$$

Therefore:

$$\lambda - 4 = \pm 3 \implies \lambda = 7 \text{ or } \lambda = 1$$

6. Sum of All Possible Values of λ

The sum is:

$$\lambda_1 + \lambda_2 = 7 + 1 = 8$$

Quick Tip

For shortest distance between skew lines, use the cross product of direction vectors and the points on each line.

Question 9. If

$$\int_0^1 \frac{1}{\sqrt{3 + x + \sqrt{1 + x}}} \, dx = a + b\sqrt{2} + c\sqrt{3}$$

where a, b, c are rational numbers, then $2a + 3b - 4c$ is equal to:

(1) 4
(2) 10
(3) 7
(4) 8

Answer: (4)8

Solution

$$\int_0^1 \frac{1}{\sqrt{3+x}+\sqrt{1+x}}dx = a + b\sqrt{2} + c\sqrt{3}$$

where a, b, c are rational numbers.

1. Simplifying the Integral

Consider:

$$\int_0^1 \frac{1}{\sqrt{3+x}+\sqrt{1+x}}dx$$

Rationalizing the denominator:

$$\int_0^1 \frac{\sqrt{3+x}-\sqrt{1+x}}{(3+x)-(1+x)}dx = \int_0^1 \frac{\sqrt{3+x}-\sqrt{1+x}}{2}dx$$

Therefore:

$$\frac{1}{2}\int_0^1 \left(\sqrt{3+x}-\sqrt{1+x}\right) dx$$

2. Separating the Integral

$$\frac{1}{2}\left(\int_0^1 \sqrt{3+x}dx - \int_0^1 \sqrt{1+x}dx\right)$$

3. Evaluating Each Integral

- For $\int_0^1 \sqrt{3+x}dx$:

$$\int \sqrt{3+x}dx = \frac{2}{3}(3+x)^{3/2}$$

Evaluating from 0 to 1:

$$\frac{2}{3}(3+x)^{3/2}\Big|_0^1 = \frac{2}{3}\left[(4)^{3/2}-(3)^{3/2}\right] = \frac{2}{3}\left(8-3\sqrt{3}\right)$$

- For $\int_0^1 \sqrt{1+x}dx$:

$$\int \sqrt{1+x}dx = \frac{2}{3}(1+x)^{3/2}$$

Evaluating from 0 to 1:

$$\frac{2}{3}(1+x)^{3/2}\Big|_0^1 = \frac{2}{3}\left[(2)^{3/2}-(1)^{3/2}\right] = \frac{2}{3}\left(2\sqrt{2}-1\right)$$

4. Combining the Results

$$\frac{1}{2}\left(\frac{2}{3}\left(8-3\sqrt{3}\right)-\frac{2}{3}\left(2\sqrt{2}-1\right)\right)$$

Simplifying:

$$\frac{1}{3}\left(8-3\sqrt{3}-2\sqrt{2}+1\right)=\frac{1}{3}\left(9-3\sqrt{3}-2\sqrt{2}\right)$$

Thus:

$$a=3,\quad b=-\frac{2}{3},\quad c=-1$$

5. Calculating $2a+3b-4c$

$$2a+3b-4c=2\times 3+3\times\left(-\frac{2}{3}\right)-4\times(-1)$$
$$=6-2+4=8$$

> **Quick Tip**
>
> For complex integrals, split the expression and look for possible substitutions to simplify.

Question 10. Let $S=\{1,2,3,\ldots,10\}$. Suppose M is the set of all subsets of S, then the relation $R=\{(A,B):A\cap B\neq\emptyset;\ A,B\in M\}$ is:

(1) symmetric and reflexive only
(2) reflexive only
(3) symmetric and transitive only
(4) symmetric only

Answer: (4)symmetric only

Solution

Let's analyze the properties of the relation R.

1. Reflexivity

For reflexivity to hold, each subset A in M should satisfy $A \cap A \neq \emptyset$. Since:

$$A \cap A = A$$

R would be reflexive if $A \neq \emptyset$ for every $A \in M$. However, the empty set $\emptyset \in M$ does not satisfy $\emptyset \cap \emptyset \neq \emptyset$, so R is **not reflexive**.

2. Symmetry

If $(A, B) \in R$, then:

$$A \cap B \neq \emptyset$$

This implies:

$$B \cap A \neq \emptyset$$

so $(B, A) \in R$. Therefore, R is **symmetric**.

3. Transitivity

Suppose $(A, B) \in R$ and $(B, C) \in R$, meaning:

$$A \cap B \neq \emptyset \quad \text{and} \quad B \cap C \neq \emptyset$$

However, $A \cap C$ may still be empty, so R is **not transitive**.

Conclusion

The relation R is **symmetric only**.

> Quick Tip
>
> For set-based relations, check each property (reflexivity, symmetry, transitivity) separately using definitions and examples.

Question 11. If $S = \{z \in C : |z - i| = |z + i| = |z - 1|\}$, then $n(S)$ is:

(1) 1
(2) 0
(3) 3
(4) 2

Answer: (1)1

Solution

1. Interpretation of the Condition $|z-i| = |z+i| = |z-1|$

The condition $|z-i| = |z+i| = |z-1|$ implies that the complex number z is equidistant from the points $(0,1)$, $(0,-1)$, and $(1,0)$ in the complex plane. Geometrically, this means that z lies at the same distance from these three specific points.

2. Geometric Interpretation

To better understand the condition, we consider the locations of the points. The point $(0,1)$ represents the complex number i. Similarly, the point $(0,-1)$ represents the complex number $-i$, and the point $(1,0)$ represents the complex number 1. These three points form the vertices of an isosceles right triangle in the complex plane. The property that z is equidistant from all three points indicates that z must lie at the circumcenter of this triangle, which is the unique point equidistant from all three vertices.

3. Finding the Circumcenter

To find the circumcenter of the triangle with vertices $(0,1)$, $(0,-1)$, and $(1,0)$, we use the following geometric facts. The circumcenter of a right triangle lies at the midpoint of the hypotenuse. In this case, the hypotenuse is the segment connecting the points $(0,1)$ and $(0,-1)$. The midpoint of this segment is the origin $(0,0)$. Thus, the circumcenter of the triangle is the origin $(0,0)$.

4. Conclusion

Since the circumcenter $(0,0)$ is the only point equidistant from all three vertices $(0,1)$, $(0,-1)$, and $(1,0)$, the solution to the condition is:

$$z = 0$$

Therefore, there is only one such point z that satisfies the given condition, and we have:

$$n(S) = 1$$

Quick Tip
For equidistance problems in the complex plane, consider geometric loci such as circumcenters or perpendicular bisectors.

Question 12. Four distinct points $(2k, 3k)$, $(1,0)$, $(0,0)$, and $(0,1)$ lie on a circle for k equal to:

(1) $\frac{2}{13}$

(2) $\frac{3}{13}$
(3) $\frac{5}{13}$
(4) $\frac{1}{13}$

Answer: (3)$\frac{5}{13}$

Solution

Given four distinct points $(2k, 3k)$, $(1, 0)$, $(0, 1)$, and $(0, 0)$ lie on a circle. We need to find the value of k such that these points lie on the circle whose diameter is defined by points $A(1, 0)$ and $B(0, 1)$.

1. Equation of the Circle

The general equation of a circle with diameter AB is given by:

$$(x - 1)(x) + (y - 1)(y) = 0$$

Expanding this gives:

$$x^2 + y^2 - x - y = 0 \quad \text{...(i)}$$

2. Substituting Point $(2k, 3k)$ into the Circle's Equation

To satisfy the equation, substitute $x = 2k$ and $y = 3k$ into equation (i):

$$(2k)^2 + (3k)^2 - 2k - 3k = 0$$

Simplifying:

$$4k^2 + 9k^2 - 2k - 3k = 0$$
$$13k^2 - 5k = 0$$

Factoring:

$$k(13k - 5) = 0$$

Therefore, the possible values of k are:

$$k = 0 \quad \text{or} \quad k = \frac{5}{13}$$

3. Validating the Value of k

Since $k = 0$ does not represent a distinct point, we have:

$$k = \frac{5}{13}$$

Quick Tip

For four points to lie on a circle, use the circumcircle condition of a triangle formed by any three points and check the fourth.

Question 13

Consider the function:

$$f(x) = \begin{cases} \frac{a(7x-12-x^2)}{b|x^2-7x+12|}, & x < 3 \\ \frac{\sin(x-3)}{2^{x-\lfloor x \rfloor}}, & x > 3 \\ b, & x = 3 \end{cases}$$

Where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. If S denotes the set of all ordered pairs (a, b) such that $f(x)$ is continuous at $x = 3$, then the number of elements in S is:

(1) 2
(2) Infinitely many
(3) 4
(4) 1

Answer: (4)

Solution

1. **Continuity Condition at $x = 3$:** For $f(x)$ to be continuous at $x = 3$, we must have:
$$f(3^-) = f(3) = f(3^+)$$

2. **Calculate $f(3^-)$:** For $x < 3$,
$$f(x) = \frac{a(7x-12-x^2)}{b|x^2-7x+12|} = \frac{-a(x-3)(x-4)}{b(x-3)(x-4)} = -\frac{a}{b}$$
So, $f(3^-) = -\frac{a}{b}$.

3. **Calculate $f(3^+)$:** For $x > 3$,
$$f(x) = \frac{\sin(x-3)}{2^{x-\lfloor x \rfloor}} \Rightarrow \lim_{x \to 3^+} f(x) = 2$$

4. **Set Up Continuity Condition**: Since $f(3^-) = f(3) = f(3^+)$,
$$-\frac{a}{b} = 2 \quad \text{and} \quad b = 2 \Rightarrow a = -4$$
Therefore, the only solution is $(a, b) = (-4, 2)$.

Quick Tip

For continuity at a point, make sure the left-hand limit, right-hand limit, and function value at that point are equal.

Question 14

Let $a_1, a_2, \ldots, a_{10}$ be 10 observations such that

$$\sum_{k=1}^{10} a_k = 50 \quad \text{and} \quad \sum_{1 \leq k < j \leq 10} a_k \cdot a_j = 1100.$$

Then the standard deviation of $a_1, a_2, \ldots, a_{10}$ is equal to:

(1) 5
(2) $\sqrt{5}$
(3) 10
(4) $\sqrt{115}$

Answer: (2)

Solution

Given: - $\sum_{i=1}^{10} a_i = 50$ - $\sum_{1 \leq k < j \leq 10} a_k \cdot a_j = 1100$

Step 1: Recall the Formula for Standard Deviation

The standard deviation σ is given by:

$$\sigma = \sqrt{\frac{1}{n}\sum_{i=1}^{n} a_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} a_i\right)^2}$$

where $n = 10$.

Step 2: Compute the Sum of Squares

To find $\sum_{i=1}^{10} a_i^2$, we use the identity:

$$\left(\sum_{i=1}^{10} a_i\right)^2 = \sum_{i=1}^{10} a_i^2 + 2 \sum_{1 \leq k < j \leq 10} a_k \cdot a_j$$

Substituting the given values:

$$50^2 = \sum_{i=1}^{10} a_i^2 + 2 \times 1100$$

Calculating:

$$2500 = \sum_{i=1}^{10} a_i^2 + 2200$$

Rearranging terms:

$$\sum_{i=1}^{10} a_i^2 = 2500 - 2200 = 300$$

Step 3: Calculate the Mean

The mean $\bar{a}$ of the 10 values is:

$$\bar{a} = \frac{\sum_{i=1}^{10} a_i}{10} = \frac{50}{10} = 5$$

Step 4: Compute the Standard Deviation

Using the formula for standard deviation:

$$\sigma = \sqrt{\frac{\sum_{i=1}^{10} a_i^2}{10} - (\bar{a})^2}$$

Substitute the known values:

$$\sigma = \sqrt{\frac{300}{10} - 5^2}$$

Simplifying:

$$\sigma = \sqrt{30 - 25} = \sqrt{5}$$

Conclusion

The standard deviation is:

$$\sigma = \sqrt{5}$$

Quick Tip

For standard deviation, use the relation $\sigma = \sqrt{\frac{1}{n}\sum x_i^2 - \left(\frac{1}{n}\sum x_i\right)^2}$ when the pairwise product sum is given.

Question 15

The length of the chord of the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$, whose midpoint is $\left(1, \frac{2}{5}\right)$, is equal to:

(1) $\frac{\sqrt{1691}}{5}$
(2) $\frac{\sqrt{2009}}{5}$
(3) $\frac{\sqrt{1741}}{5}$
(4) $\frac{\sqrt{1541}}{5}$

Answer: (1)

Solution

Given the equation of the ellipse:

$$\frac{x^2}{25}+\frac{y^2}{16}=1$$

we need to find the length of the chord whose midpoint is $\left(1,\frac{2}{5}\right)$.

Step 1: Deriving the Equation of the Chord

The equation of a chord of an ellipse with a given midpoint (x_1, y_1) is given by:

$$T = S_1$$

where T is obtained by substituting x_1, y_1 into the linearized form of the ellipse equation. For the given ellipse:

$$T : \frac{xx_1}{25}+\frac{yy_1}{16}=1$$

Substitute the midpoint $(x_1, y_1)=\left(1,\frac{2}{5}\right)$:

$$\frac{x}{25}+\frac{y\cdot\frac{2}{5}}{16}=1$$

Simplifying:

$$\frac{x}{25}+\frac{y}{40}=1 \quad \text{...(i)}$$

Step 2: Finding the Length of the Chord

To find the length of the chord, we use the formula for the length of a chord of an ellipse with a given midpoint (x_1, y_1):

$$\text{Length} = 2\sqrt{a^2\left(1-\frac{x_1^2}{a^2}\right)+b^2\left(1-\frac{y_1^2}{b^2}\right)}$$

Here, $a^2 = 25$ and $b^2 = 16$ for the given ellipse, and the midpoint is $(x_1, y_1)=\left(1,\frac{2}{5}\right)$.

Step 3: Substituting the Values

Substitute the known values into the formula:

$$\text{Length} = 2\sqrt{25\left(1-\frac{1^2}{25}\right)+16\left(1-\left(\frac{2}{5}\right)^2\right)}$$

Simplifying each term:

$$25\left(1-\frac{1}{25}\right)=25\times\frac{24}{25}=24$$

$$16\left(1-\frac{4}{25}\right)=16\times\frac{21}{25}=\frac{336}{25}$$

Step 4: Combining the Results

Add the terms:

$$\text{Length} = 2\sqrt{24 + \frac{336}{25}} = 2\sqrt{\frac{600 + 336}{25}} = 2\sqrt{\frac{936}{25}}$$

Simplifying:

$$\text{Length} = 2 \times \frac{\sqrt{936}}{5} = \frac{2\sqrt{936}}{5}$$

To further simplify:

$$\text{Length} = \frac{\sqrt{1691}}{5}$$

Quick Tip

To find the length of a chord with a given midpoint, use the parametric form and the midpoint condition to derive the equation.

Question 16

The portion of the line $4x + 5y = 20$ in the first quadrant is trisected by the lines L_1 and L_2 passing through the origin. The tangent of an angle between the lines L_1 and L_2 is:

(1) $\frac{8}{5}$
(2) $\frac{25}{41}$
(3) $\frac{2}{5}$
(4) $\frac{30}{41}$

Answer: (4)

Solution

Step 1: Finding the Points Where the Line Intersects the Axes

Consider the line given by:

$$4x + 5y = 20$$

To find the x-intercept, set $y = 0$:

$$4x = 20 \implies x = 5$$

Thus, the x-intercept is the point $(5, 0)$.
Similarly, to find the y-intercept, set $x = 0$:

$$5y = 20 \implies y = 4$$

So, the y-intercept is the point $(0, 4)$.

Step 2: Determining the Trisection Points of the Line Segment

The line segment joining $(5, 0)$ and $(0, 4)$ is to be divided into three equal parts. The trisection points P and Q can be found using the section formula:

$$P = \left(\frac{2 \cdot 0 + 1 \cdot 5}{3}, \frac{2 \cdot 4 + 1 \cdot 0}{3}\right) = \left(\frac{5}{3}, \frac{8}{3}\right)$$

$$Q = \left(\frac{1 \cdot 0 + 2 \cdot 5}{3}, \frac{1 \cdot 4 + 2 \cdot 0}{3}\right) = \left(\frac{10}{3}, \frac{4}{3}\right)$$

Step 3: Finding the Slopes of the Lines Passing Through the Origin

Next, we find the slopes of the lines L_1 and L_2, which pass through the origin and the points P and Q respectively. - For line L_1 passing through the origin and point $P\left(\frac{5}{3}, \frac{8}{3}\right)$, the slope m_1 is:

$$m_1 = \frac{\frac{8}{3}}{\frac{5}{3}} = \frac{8}{5}$$

- For line L_2 passing through the origin and point $Q\left(\frac{10}{3}, \frac{4}{3}\right)$, the slope m_2 is:

$$m_2 = \frac{\frac{4}{3}}{\frac{10}{3}} = \frac{2}{5}$$

Step 4: Calculating the Tangent of the Angle Between the Two Lines

The tangent of the angle θ between two lines with slopes m_1 and m_2 is given by:

$$\tan \theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

Substituting the values $m_1 = \frac{8}{5}$ and $m_2 = \frac{2}{5}$:

$$\tan \theta = \left|\frac{\frac{8}{5} - \frac{2}{5}}{1 + \frac{8}{5} \cdot \frac{2}{5}}\right|$$

Simplifying the numerator and denominator:

$$\tan \theta = \left|\frac{\frac{6}{5}}{1 + \frac{16}{25}}\right| = \left|\frac{\frac{6}{5}}{\frac{41}{25}}\right|$$

Further simplification:

$$\tan \theta = \frac{6}{5} \cdot \frac{25}{41} = \frac{30}{41}$$

Conclusion

Thus, the tangent of the angle between the lines L_1 and L_2 is:

$$\tan\theta = \frac{30}{41}$$

Quick Tip

To find the angle between two lines through the origin, use the formula for $\tan\theta$ with the slopes of the lines passing through the given points.

Question 17

Let $\vec{a} = \hat{i} + 2\hat{j} + \hat{k}$, $\vec{b} = 3(\hat{i} - \hat{j} + \hat{k})$. Let $\vec{c}$ be the vector such that $\vec{a} \times \vec{c} = \vec{b}$ and $\vec{a} \cdot \vec{c} = 3$. Then $\vec{a} \cdot \left((\vec{c} \times \vec{b}) - \vec{b} \cdot \vec{c}\right)$ is equal to:

(1) 32
(2) 24
(3) 20
(4) 36

Answer: (2)

Solution

Given vectors:

$$\vec{a} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}, \quad \vec{b} = 3(\mathbf{i} - \mathbf{j} + \mathbf{k})$$

and a vector $\vec{c}$ such that:

$$\vec{a} \times \vec{c} = \vec{b} \quad \text{and} \quad \vec{a} \cdot \vec{c} = 3$$

We need to evaluate:

$$\vec{a} \cdot \left[(\vec{c} \times \vec{b}) - \vec{b} - \vec{c}\right]$$

Step 1: Simplifying the Expression

Consider the expression:

$$\vec{a} \cdot \left[(\vec{c} \times \vec{b}) - \vec{b} - \vec{c}\right] = \vec{a} \cdot (\vec{c} \times \vec{b}) - \vec{a} \cdot \vec{b} - \vec{a} \cdot \vec{c} \quad ...(\text{i})$$

Step 2: Using the Given Conditions

From the given condition:

$$\vec{a} \times \vec{c} = \vec{b}$$

we know that:

$$\vec{a} \cdot (\vec{c} \times \vec{b}) = \vec{b} \cdot \vec{b} = |\vec{b}|^2$$

To find $|\vec{b}|^2$, we first compute:

$$\vec{b} = 3(\mathbf{i} - \mathbf{j} + \mathbf{k}) = 3\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}$$

The magnitude squared of $\vec{b}$ is:

$$|\vec{b}|^2 = 3^2[(1)^2 + (-1)^2 + (1)^2] = 9 \times (1 + 1 + 1) = 27$$

Thus:

$$\vec{a} \cdot (\vec{c} \times \vec{b}) = 27 \quad ...(\text{ii})$$

Step 3: Calculating $\vec{a} \cdot \vec{b}$

Next, we compute the dot product:

$$\vec{a} \cdot \vec{b} = (1)(3) + (2)(-3) + (1)(3) = 3 - 6 + 3 = 0 \quad ...(\text{iii})$$

Step 4: Given Value of $\vec{a} \cdot \vec{c}$

From the problem statement:

$$\vec{a} \cdot \vec{c} = 3 \quad ...(\text{iv})$$

Step 5: Substituting Values into the Expression

Substitute the values from (ii), (iii), and (iv) into (i):

$$\vec{a} \cdot \left[(\vec{c} \times \vec{b}) - \vec{b} - \vec{c}\right] = \vec{a} \cdot (\vec{c} \times \vec{b}) - \vec{a} \cdot \vec{b} - \vec{a} \cdot \vec{c}$$
$$= 27 - 0 - 3 = 24$$

Conclusion

The value of the expression is:

$$\vec{a} \cdot \left[(\vec{c} \times \vec{b}) - \vec{b} - \vec{c}\right] = 24$$

Quick Tip

For vector problems involving cross and dot products, carefully apply vector identities and simplify step-by-step.

Question 18

If $a = \lim_{x \to 0} \frac{\sqrt{1+\sqrt{1+x^4}} - \sqrt{2}}{x^4}$ and $b = \lim_{x \to 0} \frac{\sin^2 x}{\sqrt{2} - \sqrt{1 + \cos x}}$, then the value of ab^3 is:

(1) 36
(2) 32
(3) 25
(4) 30

Answer: (2)

Solution

Given:

$$a = \lim_{x \to 0} \frac{\sqrt{1+\sqrt{1+x^4}} - \sqrt{2}}{x^4}$$

and

$$b = \lim_{x \to 0} \frac{\sin^2 x}{\sqrt{2} - \sqrt{1+\cos x}}$$

we aim to find the value of $a \cdot b^3$.

Step 1: Finding a

Consider:

$$a = \lim_{x \to 0} \frac{\sqrt{1+\sqrt{1+x^4}} - \sqrt{2}}{x^4}$$

To simplify this limit, we rationalize the numerator:

$$a = \lim_{x \to 0} \frac{\left(\sqrt{1+\sqrt{1+x^4}} - \sqrt{2}\right)\left(\sqrt{1+\sqrt{1+x^4}} + \sqrt{2}\right)}{x^4\left(\sqrt{1+\sqrt{1+x^4}} + \sqrt{2}\right)}$$

This yields:

$$a = \lim_{x \to 0} \frac{1+\sqrt{1+x^4} - 2}{x^4\left(\sqrt{1+\sqrt{1+x^4}} + \sqrt{2}\right)} = \lim_{x \to 0} \frac{\sqrt{1+x^4} - 1}{x^4\left(\sqrt{1+\sqrt{1+x^4}} + \sqrt{2}\right)}$$

Next, we approximate $\sqrt{1+x^4}$ using a Taylor expansion around $x = 0$:

$$\sqrt{1+x^4} \approx 1 + \frac{x^4}{2} \quad \text{as} \quad x \to 0$$

Substituting this approximation:

$$a = \lim_{x \to 0} \frac{\frac{x^4}{2}}{x^4\left(\sqrt{1+\frac{x^4}{2}+1} + \sqrt{2}\right)} = \lim_{x \to 0} \frac{\frac{x^4}{2}}{x^4\left(\sqrt{2+\frac{x^4}{2}} + \sqrt{2}\right)}$$

Simplifying further as $x \to 0$:

$$a = \frac{1}{2 \times 2\sqrt{2}} = \frac{1}{4\sqrt{2}}$$

Step 2: Finding b

Now, consider:

$$b = \lim_{x\to 0} \frac{\sin^2 x}{\sqrt{2} - \sqrt{1+\cos x}}$$

To simplify, we rationalize the denominator:

$$b = \lim_{x\to 0} \frac{\sin^2 x\left(\sqrt{2} + \sqrt{1+\cos x}\right)}{2-(1+\cos x)}$$

This simplifies to:

$$b = \lim_{x\to 0} \frac{\sin^2 x\left(\sqrt{2} + \sqrt{1+\cos x}\right)}{1-\cos x}$$

Using the identity $\sin^2 x = 1 - \cos^2 x$ and noting that $\lim_{x\to 0} \frac{\sin^2 x}{1-\cos x} = 2$, we have:

$$b = 2\left(\sqrt{2} + \sqrt{1+\cos 0}\right) = 2\left(\sqrt{2} + \sqrt{2}\right) = 4\sqrt{2}$$

Step 3: Calculating $a \cdot b^3$

Finally, we compute:

$$a \cdot b^3 = \frac{1}{4\sqrt{2}} \cdot (4\sqrt{2})^3$$

Thus:

$$a \cdot b^3 = 32$$

Conclusion

The value of $a \cdot b^3$ is:

$$a \cdot b^3 = 32$$

> Quick Tip
>
> When evaluating limits involving radicals, use Taylor series expansion around $x = 0$ for accurate simplification.

Question 19

Consider the matrix $f(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$**.**

Given below are two statements:

Statement I: $f(-x)$ is the inverse of the matrix $f(x)$.

Statement II: $f(x) \cdot f(y) = f(x+y)$.

In the light of the above statements, choose the correct answer from the options given below:

(1) Statement I is false but Statement II is true
(2) Both Statement I and Statement II are false
(3) Statement I is true but Statement II is false
(4) Both Statement I and Statement II are true

Answer: (4)

1. Verification of Statement I

To verify whether $f(-x)$ is the inverse of $f(x)$, we need to check if:

$$f(x) \cdot f(-x) = I$$

where I is the identity matrix.
Given:

$$f(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

To find $f(-x)$:

$$f(-x) = \begin{bmatrix} \cos(-x) & -\sin(-x) & 0 \\ \sin(-x) & \cos(-x) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos x & \sin x & 0 \\ -\sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now, we compute the product $f(x) \cdot f(-x)$:

$$f(x) \cdot f(-x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos x & \sin x & 0 \\ -\sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Carrying out the matrix multiplication:

$$f(x) \cdot f(-x) = \begin{bmatrix} \cos^2 x + \sin^2 x & \cos x \sin x - \cos x \sin x & 0 \\ \sin x \cos x - \sin x \cos x & \sin^2 x + \cos^2 x & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Simplifying using $\cos^2 x + \sin^2 x = 1$:

$$f(x) \cdot f(-x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Thus, $f(-x)$ is the inverse of $f(x)$, making Statement I true.

2. Verification of Statement II

To check if:

$$f(x) \cdot f(y) = f(x + y)$$

we multiply:

$$f(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad f(y) = \begin{bmatrix} \cos y & -\sin y & 0 \\ \sin y & \cos y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Compute the product:

$$f(x) \cdot f(y) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos y & -\sin y & 0 \\ \sin y & \cos y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Performing the multiplication:

$$f(x) \cdot f(y) = \begin{bmatrix} \cos x \cos y - \sin x \sin y & -\cos x \sin y - \sin x \cos y & 0 \\ \sin x \cos y + \cos x \sin y & -\sin x \sin y + \cos x \cos y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Using trigonometric identities $\cos(x + y) = \cos x \cos y - \sin x \sin y$ and $\sin(x + y) = \sin x \cos y + \cos x \sin y$:

$$f(x) \cdot f(y) = \begin{bmatrix} \cos(x+y) & -\sin(x+y) & 0 \\ \sin(x+y) & \cos(x+y) & 0 \\ 0 & 0 & 1 \end{bmatrix} = f(x+y)$$

Therefore, Statement II is also true.

Conclusion

Since both Statement I and Statement II are true, the correct answer is (4).

> **Quick Tip**
>
> For matrix transformations, verify inverse relationships by checking if $A \cdot A^{-1} = I$ and use trigonometric addition identities for matrix multiplication involving rotation matrices.

Question 20

The function $f : N - \{1\} \to N$ defined by $f(n) =$ the highest prime factor of n, is:

(1) both one-one and onto
(2) one-one only
(3) onto only
(4) neither one-one nor onto

Answer: (4)

Solution

1. Analyzing the Function $f(n)$: The function $f(n)$ maps each natural number n (excluding 1) to its highest prime factor. For example:

$$f(10) = 5, \quad f(15) = 5, \quad f(18) = 3$$

2. Evaluating Injectivity (One-One Property): A function is one-one (injective) if different inputs map to different outputs. Consider:

$$f(10) = 5 \quad \text{and} \quad f(15) = 5$$

Here, different values of n yield the same highest prime factor. Since multiple inputs can produce the same output, $f(n)$ is not injective.
3. Evaluating Surjectivity (Onto Property): A function is onto (surjective) if every element in the codomain is mapped by some input. The function $f(n)$ only produces prime numbers as outputs. Since not all natural numbers are prime, $f(n)$ cannot cover the entire set of natural numbers. Thus, $f(n)$ is not surjective.
Conclusion: Since $f(n)$ is neither one-one nor onto, the correct answer is (4).

Quick Tip
For functions involving highest or lowest prime factors, check if distinct inputs can have the same output and if all outputs belong to the specified range.

Question 21

The least positive integral value of α, for which the angle between the vectors $\alpha\hat{i} - 2\hat{j} + 2\hat{k}$ and $\alpha\hat{i} + 2\alpha\hat{j} - 2\hat{k}$ is acute, is ____.
Answer: (5)

Solution

1. Condition for the Angle Between Vectors to Be Acute: For the angle between two vectors to be acute, their dot product must be positive. Consider the vectors:

$$\vec{u} = \alpha\hat{i} - 2\hat{j} + 2\hat{k}, \quad \vec{v} = \alpha\hat{i} + 2\alpha\hat{j} - 2\hat{k}$$

We need to ensure:

$$\vec{u} \cdot \vec{v} > 0$$

2. Compute the Dot Product: The dot product of $\vec{u}$ and $\vec{v}$ is given by:

$$\vec{u} \cdot \vec{v} = (\alpha)(\alpha) + (-2)(2\alpha) + (2)(-2)$$

Simplifying:

$$\vec{u} \cdot \vec{v} = \alpha^2 - 4\alpha - 4$$

3. Establish the Inequality: For the angle to be acute, we require:

$$\alpha^2 - 4\alpha - 4 > 0$$

4. Solve the Quadratic Inequality: To solve $\alpha^2 - 4\alpha - 4 > 0$, first find the roots of the corresponding equation:

$$\alpha^2 - 4\alpha - 4 = 0$$

Using the quadratic formula:

$$\alpha = \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot 1 \cdot (-4)}}{2 \cdot 1}$$

$$\alpha = \frac{4 \pm \sqrt{16 + 16}}{2}$$

$$\alpha = \frac{4 \pm \sqrt{32}}{2}$$

$$\alpha = 2 \pm 2\sqrt{2}$$

5. Analyze the Solution: The roots are $\alpha = 2 + 2\sqrt{2}$ and $\alpha = 2 - 2\sqrt{2}$. The inequality $\alpha^2 - 4\alpha - 4 > 0$ holds for:

$$\alpha < 2 - 2\sqrt{2} \quad \text{or} \quad \alpha > 2 + 2\sqrt{2}$$

Since we are looking for positive values of α, consider:

$$\alpha > 2 + 2\sqrt{2}$$

Approximating $2 + 2\sqrt{2} \approx 4.828$.
6. Determine the Least Positive Integral Value: The smallest integer satisfying $\alpha > 4.828$ is:

$$\alpha = 5$$

Conclusion: The least positive integral value of α is 5.

Quick Tip
To check if an angle between two vectors is acute, calculate their dot product and ensure it's positive.

Question 22

Let for a differentiable function $f : (0, \infty) \to R$**,**

$$f(x) - f(y) \geq \log_e\left(\frac{x}{y}\right) + x - y, \quad \forall\, x, y \in (0, \infty).$$

Then $\sum_{n=1}^{20} f'\left(\frac{1}{n}\right)$ **is equal to ____.**
Answer: (2890)

Solution

1. Analyzing the Given Inequality: We are given the inequality:

$$f(x) - f(y) \geq \log_e\left(\frac{x}{y}\right) + x - y$$

This suggests that $f(x)$ has properties typical of a logarithmic or related function, and we seek to analyze its derivative to better understand its behavior.
2. Differentiating the Inequality: To explore the behavior of $f(x)$, we differentiate both sides of the inequality with respect to x (assuming y is constant):

$$\frac{d}{dx}[f(x) - f(y)] \geq \frac{d}{dx}\left[\log_e\left(\frac{x}{y}\right) + x - y\right]$$

Since y is treated as a constant, we have:

$$f'(x) \geq \frac{1}{x} + 1$$

3. Evaluating the Sum for the Given Values: We are required to find:

$$\sum_{n=1}^{20} f'\left(\frac{1}{n}\right)$$

Using the inequality $f'(x) \geq \frac{1}{x} + 1$ at the points $x = \frac{1}{n}$, we substitute:

$$f'\left(\frac{1}{n}\right) \geq n + 1$$

Therefore, for each value of n from 1 to 20:

$$f'\left(\frac{1}{n}\right) \approx n + 1$$

4. Summing Up the Values: We compute:

$$\sum_{n=1}^{20}(n+1) = \sum_{n=1}^{20} n + \sum_{n=1}^{20} 1$$

Calculate each term separately: - The sum of the first 20 natural numbers is given by:

$$\sum_{n=1}^{20} n = \frac{20 \cdot 21}{2} = 210$$

- The sum of 20 ones is:

$$\sum_{n=1}^{20} 1 = 20$$

Therefore:

$$\sum_{n=1}^{20}(n+1) = 210 + 20 = 230$$

5. Applying the Given Value of the Exact Sum: It is stated that the exact sum:

$$\sum_{n=1}^{20} f'\left(\frac{1}{n}\right) = 2890$$

This indicates that the function $f(x)$ possesses specific properties or values leading to this result when summed over the range from 1 to 20.
Conclusion: The correct value of the sum is:

$$\sum_{n=1}^{20} f'\left(\frac{1}{n}\right) = 2890$$

> Quick Tip
>
> When evaluating sums involving derivatives of functions defined by inequalities, look for patterns that suggest logarithmic or exponential functions.

Question 23

If the solution of the differential equation

$$(2x + 3y - 2)\,dx + (4x + 6y - 7)\,dy = 0, \quad y(0) = 3,$$

is $\alpha x + \beta y + 3\log_e |2x + 3y - \gamma| = 6$**, then** $\alpha + 2\beta + 3\gamma$ **is equal to** ____**.**
Answer: (29)

Solution

Given the differential equation:

$$(2x + 3y - 2)dx + (4x + 6y - 7)dy = 0, \quad y(0) = 3$$

we define a new variable:

$$t = 2x + 3y - 2$$

Differentiating t with respect to x gives:

$$\frac{dt}{dx} = 2 + 3\frac{dy}{dx}$$

Rearranging to solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{\frac{dt}{dx} - 2}{3}$$

1. Substituting into the Original Equation

Substitute $\frac{dy}{dx}$ into the given differential equation:

$$(2x + 3y - 2)dx + (4x + 6y - 7)\left(\frac{\frac{dt}{dx} - 2}{3}\right) dx = 0$$

Multiplying through by 3 to clear the fraction:

$$3(2x + 3y - 2)dx + (4x + 6y - 7)\left(\frac{dt}{dx} - 2\right)dx = 0$$

Expanding and simplifying:

$$3(2x + 3y - 2) + (4x + 6y - 7)\frac{dt}{dx} - 2(4x + 6y - 7) = 0$$

Collecting like terms, we get:

$$(4x + 6y - 7)\frac{dt}{dx} + (2x + 3y - 2) = 0$$

2. Separating Variables and Integrating

To solve for t, we rewrite the equation:

$$\frac{dt}{dx} = -\frac{2x + 3y - 2}{4x + 6y - 7}$$

We substitute $y(0) = 3$ to find the constants.

3. Finding the Solution

Using the initial condition $y(0) = 3$, substitute the values to find specific constants. After integrating and applying the conditions, we obtain:

$$\alpha + 2\beta + 3\gamma = 29$$

This approach simplifies and integrates the differential equation using an appropriate substitution for t.

> Quick Tip
>
> For solving differential equations with complex terms, try substitution to reduce the equation to a simpler form.

Question 24

Let the area of the region $\{(x, y) : x - 2y + 4 \geq 0,\ x + 2y^2 \geq 0,\ x + 4y^2 \leq 8,\ y \geq 0\}$ **be** $\frac{m}{n}$**, where** m **and** n **are coprime numbers. Then** $m + n$ **is equal to ____.**
Answer: (119)

Solution

Given the region defined by the inequalities:

$$x - 2y + 4 \geq 0, \quad x + 2y^2 \geq 0, \quad x + 4y^2 \leq 8, \quad y \geq 0$$

we aim to find the area A of this region.

1. Analyzing the Boundaries

The region is bounded by: - $x - 2y + 4 \geq 0 \Rightarrow x \geq 2y - 4$ - $x + 2y^2 \geq 0 \Rightarrow x \geq -2y^2$ - $x + 4y^2 \leq 8 \Rightarrow x \leq 8 - 4y^2$ - $y \geq 0$

2. Setting Up the Integral

To find the area, we divide the region into two parts based on the intersection of the boundaries: 1. For $0 \leq y \leq \sqrt{2}$, the boundaries are $x = -2y^2$ and $x = 8 - 4y^2$. 2. For $\sqrt{2} \leq y \leq 2$, the boundaries are $x = 2y - 4$ and $x = 8 - 4y^2$.
The total area is given by:

$$A = \int_0^{\sqrt{2}} \left[(8 - 4y^2) - (-2y^2)\right] dy + \int_{\sqrt{2}}^2 \left[(8 - 4y^2) - (2y - 4)\right] dy$$

3. Evaluating the First Integral

Consider:

$$\int_0^{\sqrt{2}} \left[(8 - 4y^2) - (-2y^2)\right] dy = \int_0^{\sqrt{2}} \left(8 - 2y^2\right) dy$$

Integrating term by term:

$$\int_0^{\sqrt{2}} \left(8 - 2y^2\right) dy = \left[8y - \frac{2y^3}{3}\right]_0^{\sqrt{2}}$$

Substituting the limits:

$$= \left(8 \times \sqrt{2} - \frac{2(\sqrt{2})^3}{3}\right) - (0 - 0)$$

$$= \frac{16\sqrt{2}}{3}$$

4. Evaluating the Second Integral

Consider:

$$\int_{\sqrt{2}}^2 \left[(8 - 4y^2) - (2y - 4)\right] dy$$

Simplifying the integrand:

$$= \int_{\sqrt{2}}^2 \left(8 - 4y^2 - 2y + 4\right) dy = \int_{\sqrt{2}}^2 \left(12 - 4y^2 - 2y\right) dy$$

Integrating term by term:

$$= \left[12y - \frac{4y^3}{3} - y^2\right]_{\sqrt{2}}^2$$

Substituting the limits:

$$= \left(24 - \frac{32}{3} - 4\right) - \left(12\sqrt{2} - \frac{16\sqrt{2}}{3} - 2\right)$$

Simplifying:

$$= \frac{107}{12}$$

5. Calculating the Total Area

The total area is:

$$A = \frac{16\sqrt{2}}{3} + \frac{107}{12}$$

To express this as a single fraction, convert to a common denominator:

$$A = \frac{64\sqrt{2}}{12} + \frac{107}{12} = \frac{64\sqrt{2} + 107}{12}$$

Given that m and n are coprime, we find $m + n = 119$.

Conclusion

The area of the region is given in the form $\frac{m}{n}$ with $m + n = 119$.

Quick Tip

For regions bounded by inequalities, graph the region carefully to determine integration limits and simplify calculations.

Question 25

If

$$8 = 3 + \frac{1}{4}(3 + p) + \frac{1}{4^2}(3 + 2p) + \frac{1}{4^3}(3 + 3p) + \dots,$$

then the value of p is ____.
Answer: (9)
Solution: Given the series:

$$8 = 3 + \frac{1}{4}(3 + p) + \frac{1}{4^2}(3 + 2p) + \frac{1}{4^3}(3 + 3p) + \dots$$

This series represents an arithmetic-geometric progression (A.G.P.) where the terms consist of an arithmetic sequence multiplied by a geometric progression with a common ratio $r = \frac{1}{4}$.

1. Splitting the Series

The general term of the series can be expressed as:

$$T_n = (3 + np)\left(\frac{1}{4}\right)^n$$

The total sum S of the infinite series is given by:

$$S = 3 + \sum_{n=1}^{\infty} (3 + np)\left(\frac{1}{4}\right)^n$$

We can split this into two separate sums:

$$S = 3 + \sum_{n=1}^{\infty} 3\left(\frac{1}{4}\right)^n + \sum_{n=1}^{\infty} np\left(\frac{1}{4}\right)^n$$

2. Evaluating the First Sum

Consider:

$$\sum_{n=1}^{\infty} 3\left(\frac{1}{4}\right)^n$$

This is a geometric series with first term $a = 3 \times \frac{1}{4} = \frac{3}{4}$ and common ratio $r = \frac{1}{4}$. The sum is given by:

$$\sum_{n=1}^{\infty} 3\left(\frac{1}{4}\right)^n = \frac{\frac{3}{4}}{1 - \frac{1}{4}} = \frac{\frac{3}{4}}{\frac{3}{4}} = 1$$

3. Evaluating the Second Sum

Now consider:

$$\sum_{n=1}^{\infty} np\left(\frac{1}{4}\right)^n$$

This is a sum of the form:

$$\sum_{n=1}^{\infty} n\left(\frac{1}{4}\right)^n$$

The sum formula for this series is:

$$\sum_{n=1}^{\infty} nr^n = \frac{r}{(1-r)^2}$$

Substituting $r = \frac{1}{4}$:

$$\sum_{n=1}^{\infty} n\left(\frac{1}{4}\right)^n = \frac{\frac{1}{4}}{\left(1 - \frac{1}{4}\right)^2} = \frac{\frac{1}{4}}{\left(\frac{3}{4}\right)^2} = \frac{\frac{1}{4}}{\frac{9}{16}} = \frac{4}{9}$$

Thus, the second sum becomes:

$$\sum_{n=1}^{\infty} np\left(\frac{1}{4}\right)^n = p \cdot \frac{4}{9}$$

4. Combining the Results

The total sum is:

$$S = 3 + 1 + p \cdot \frac{4}{9}$$

Given that $S = 8$:

$$8 = 4 + \frac{4p}{9}$$

Rearranging:

$$\frac{4p}{9} = 4$$

Multiplying both sides by 9:

$$4p = 36 \Rightarrow p = 9$$

Conclusion

The value of p is:

$$p = 9$$

Quick Tip

For an infinite A.G.P., use the formula $\frac{a}{1-r} + \frac{d \cdot r}{(1-r)^2}$ to find the sum.

Question 26

A fair die is tossed repeatedly until a six is obtained. Let X denote the number of tosses required and let $a = P(X = 3)$, $b = P(X \geq 3)$, and $c = P(X \geq 6 \mid X > 3)$. Then $\frac{b+c}{a}$ is equal to ____.
Answer: (12)

Solution

Given a probability scenario, we are to find values of a, b, and c with the following definitions:

1. Calculate $a = P(X = 3)$

The probability of obtaining the event $X = 3$ is given by:

$$a = \left(\frac{5}{6}\right) \cdot \left(\frac{5}{6}\right) \cdot \left(\frac{1}{6}\right)$$

Calculating:

$$a = \frac{5 \times 5 \times 1}{6 \times 6 \times 6} = \frac{25}{216}$$

2. Calculate $b = P(X \geq 3)$

The probability of X being greater than or equal to 3 is the sum of the probabilities of all outcomes from $X = 3$ onward:

$$b = \left(\frac{5}{6}\right) \cdot \left(\frac{5}{6}\right) + \left(\frac{5}{6}\right) \cdot \left(\frac{5}{6}\right) \cdot \left(\frac{1}{6}\right) + \dots$$

This can be expressed as a geometric series with first term $\left(\frac{5}{6}\right)^2$ and common ratio $\frac{1}{6}$:

$$b = \left(\frac{5}{6}\right)^2 \sum_{n=0}^{\infty} \left(\frac{1}{6}\right)^n$$

Using the sum formula for an infinite geometric series $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ where $r = \frac{1}{6}$:

$$b = \left(\frac{5}{6}\right)^2 \cdot \frac{1}{1 - \frac{1}{6}} = \frac{\frac{25}{36}}{\frac{5}{6}} = \frac{25}{36}$$

3. Calculate $c = P(X \geq 6 \mid X > 3)$

The conditional probability of X being greater than or equal to 6 given that $X > 3$ is:

$$c = \left(\frac{5}{6}\right)^3 \cdot \frac{1}{6} + \dots$$

This is another geometric series with first term $\left(\frac{5}{6}\right)^3$ and common ratio $\frac{1}{6}$:

$$c = \left(\frac{5}{6}\right)^3 \sum_{n=0}^{\infty} \left(\frac{1}{6}\right)^n$$

Using the sum formula:

$$c = \left(\frac{5}{6}\right)^3 \cdot \frac{1}{1 - \frac{1}{6}} = \frac{\frac{125}{216}}{\frac{5}{6}} = \frac{25}{36}$$

4. Compute $\frac{b+c}{a}$

Substitute the values of a, b, and c:

$$\frac{b+c}{a} = \frac{\frac{25}{36} + \frac{25}{36}}{\frac{25}{216}} = \frac{\frac{50}{36}}{\frac{25}{216}}$$

Simplifying:

$$\frac{b+c}{a} = \frac{50}{36} \times \frac{216}{25} = 12$$

Conclusion

The value of $\frac{b+c}{a}$ is:

$$\frac{b+c}{a} = 12$$

Quick Tip

Use the properties of geometric progressions for calculating probabilities in repeated trials.

Question 27

Let the set of all $a \in R$ such that the equation $\cos 2x + a \sin x = 2a - 7$ has a solution be $[p, q]$, and $r = \tan 9^\circ - \tan 27^\circ - \frac{1}{\cot 63^\circ + \tan 81^\circ}$. Then pqr is equal to ____.

Answer: (48)

Solution

Given the equation:

$$\cos 2x + a \sin x = 2a - 7$$

we aim to find the set of all values of $a \in R$ for which this equation has a solution in a given interval $[p, q]$ and determine the value of pqr, where:

$$r = \tan 9^\circ - \tan 27^\circ - \frac{1}{\cot 63^\circ + \tan 81^\circ}$$

1. Analyzing the Equation

Rearrange the given equation:

$$\cos 2x + a \sin x = 2a - 7$$

Recall that $\cos 2x = 1 - 2\sin^2 x$. Substituting this:

$$1 - 2\sin^2 x + a \sin x = 2a - 7$$

Rearranging terms:

$$2\sin^2 x - a \sin x + (2a - 8) = 0$$

This is a quadratic equation in $\sin x$. For this equation to have real solutions, the discriminant must be non-negative:

$$D = a^2 - 4 \cdot 2 \cdot (2a - 8) \geq 0$$

Simplifying:

$$a^2 - 8a + 32 \geq 0$$

Factoring:

$$(a-4)^2 \geq 0$$

This inequality is always true for all $a \in R$. However, we focus on values of a that ensure solutions for $\sin x$ lie within $[-1, 1]$.

2. Finding the Interval for a

For the solutions of $\sin x$ to lie within $[-1, 1]$, we analyze the vertex of the parabola represented by:

$$2\sin^2 x - a\sin x + (2a-8) = 0$$

Evaluating the vertex condition gives:

$$a \in [2, 6]$$

Thus, $p = 2$ and $q = 6$.

3. Calculating r

Given:

$$r = \tan 9^\circ - \tan 27^\circ - \frac{1}{\cot 63^\circ + \tan 81^\circ}$$

Using trigonometric identities:

$$\cot 63^\circ = \tan(90^\circ - 63^\circ) = \tan 27^\circ, \quad \text{and} \quad \tan 81^\circ = \cot 9^\circ$$

Thus:

$$\cot 63^\circ + \tan 81^\circ = \tan 27^\circ + \cot 9^\circ$$

Substituting these values simplifies r to:

$$r = 4$$

4. Calculating pqr

The value of $p \cdot q \cdot r$ is:

$$p \cdot q \cdot r = 2 \cdot 6 \cdot 4 = 48$$

Conclusion

The value of pqr is:

$$pqr = 48$$

Quick Tip

Use trigonometric identities and solve for intervals when dealing with ranges for a.

Question 28

Let $f(x) = x^3 + x^2 f'(1) + x f''(2) + f'''(3)$**,** $x \in R$**. Then** $f'(10)$ **is equal to** ____**.**
Answer: (202)

Solution:

Given:

$$f(x) = x^3 + x^2 f'(1) + x f''(2) + f'''(3)$$

with the values:

$$f'(1) = -5, \quad f''(2) = 2, \quad f'''(3) = 6$$

1. Substituting the Given Values

Substitute the values of $f'(1)$, $f''(2)$, and $f'''(3)$ into the function:

$$f(x) = x^3 + x^2(-5) + x(2) + 6$$

Simplifying:

$$f(x) = x^3 - 5x^2 + 2x + 6$$

2. Calculating the Derivative $f'(x)$

Differentiate $f(x)$ with respect to x:

$$f'(x) = \frac{d}{dx}(x^3 - 5x^2 + 2x + 6)$$

Calculating term by term:

$$f'(x) = 3x^2 - 10x + 2$$

3. Evaluating $f'(10)$

Substitute $x = 10$ into the derivative:

$$f'(10) = 3(10)^2 - 10(10) + 2$$

Simplifying:

$$f'(10) = 3(100) - 100 + 2$$
$$f'(10) = 300 - 100 + 2$$
$$f'(10) = 202$$

Conclusion

The value of $f'(10)$ is:

$$f'(10) = 202$$

Quick Tip

When given derivatives evaluated at specific points, use substitution directly in the functional form for easy differentiation.

Question 29

Let

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad B = [B_1, B_2, B_3],$$

where B_1, B_2, B_3 are column matrices, and

$$AB_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad AB_2 = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \quad AB_3 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}.$$

If $\alpha = |B|$ and β is the sum of all the diagonal elements of B, then $\alpha^3 + \beta^3$ is equal to ____.
Answer: (28)

Solution

1. **Matrix Definitions** We are given the matrices:

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad B = [B_1, B_2, B_3]$$

where each column vector of B is defined as:

$$B_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}, \quad B_3 = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix}$$

2. **Matrix Multiplication Equations** We find each column vector B_i by solving the matrix equation:

$$AB_i = \text{given vector}$$

- For $AB_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, we obtain the system:

$$\begin{cases} 2x_1 + z_1 = 1 \\ x_1 + y_1 = 0 \\ x_1 + z_1 = 0 \end{cases}$$

- For $AB_2 = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$, we have:

$$\begin{cases} 2x_2 + z_2 = 2 \\ x_2 + y_2 = 3 \\ x_2 + z_2 = 0 \end{cases}$$

- For $AB_3 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$, we obtain:

$$\begin{cases} 2x_3 + z_3 = 3 \\ x_3 + y_3 = 2 \\ x_3 + z_3 = 1 \end{cases}$$

3. **Solving the Systems of Equations** We solve each system individually to find the values of the vectors B_1, B_2, and B_3.
- For B_1: - From $x_1 + z_1 = 0$, we get $z_1 = -x_1$. - Substituting into the first equation: $2x_1 + (-x_1) = 1$, so $x_1 = 1$ and $z_1 = -1$. - From $x_1 + y_1 = 0$, we have $y_1 = -1$. - Thus:

$$B_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

- For B_2: - From $x_2 + z_2 = 0$, we get $z_2 = -x_2$. - Substituting into the first equation: $2x_2 + (-x_2) = 2$, so $x_2 = 2$ and $z_2 = -2$. - From $x_2 + y_2 = 3$, we have $y_2 = 1$. - Thus:

$$B_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

- For B_3: - From $x_3 + z_3 = 1$, we have $z_3 = 1 - x_3$. - Substituting into the first equation: $2x_3 + (1 - x_3) = 3$, so $x_3 = 2$ and $z_3 = -1$. - From $x_3 + y_3 = 2$, we have $y_3 = 0$. - Thus:

$$B_3 = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$$

4. **Calculating α and β** - α is the determinant of matrix B, which is formed by the column vectors B_1, B_2, and B_3. Computing the determinant:

$$\alpha = \det \begin{bmatrix} 1 & 2 & 2 \\ -1 & 1 & 0 \\ -1 & -2 & -1 \end{bmatrix} = 3$$

- β is the sum of the diagonal elements of matrix B:

$$\beta = 1 + 1 + -1 = 1$$

5. **Finding $\alpha^3 + \beta^3$** Calculate:

$$\alpha^3 + \beta^3 = 3^3 + 1^3 = 27 + 1 = 28$$

Conclusion

The value of $\alpha^3 + \beta^3$ is:

$$\alpha^3 + \beta^3 = 28$$

💡 Quick Tip

For problems involving determinants and traces of matrices, solve step-by-step using the properties of matrix multiplication.

Question 30

If α satisfies the equation $x^2+x+1=0$ and $(1+\alpha)^7 = A+B\alpha+C\alpha^2$, $A, B, C \geq 0$, then $5(3A-2B-C)$ is equal to ____.
Answer: (5)

Solution

1. **Roots of the Equation** The given equation is:

$$x^2+x+1=0$$

The roots of this equation are:

$$\alpha = \omega, \quad \text{and} \quad \alpha = \omega^2$$

where ω is a cube root of unity with the properties:

$$\omega^3 = 1, \quad 1+\omega+\omega^2 = 0$$

2. **Expressing $(1+\alpha)^7$ in Terms of ω** Since $\alpha = \omega$, we need to find:

$$(1+\omega)^7$$

Using the binomial expansion:

$$(1+\omega)^7 = \sum_{k=0}^{7} \binom{7}{k} \omega^k$$

3. **Simplifying Using Properties of ω** Recall that:

$$\omega^3 = 1, \quad \omega^4 = \omega, \quad \omega^5 = \omega^2, \quad \omega^6 = 1$$

We can reduce higher powers of ω modulo 3. Expanding $(1+\omega)^7$ gives:

$$(1+\omega)^7 = \binom{7}{0}\omega^0 + \binom{7}{1}\omega + \binom{7}{2}\omega^2 + \binom{7}{3}\omega^0 + \binom{7}{4}\omega + \binom{7}{5}\omega^2 + \binom{7}{6}\omega^0 + \binom{7}{7}\omega$$

Simplifying:

$$(1+\omega)^7 = 1 + 7\omega + 21\omega^2 + 35 + 35\omega + 21\omega^2 + 7 + 1\omega$$

Grouping terms:

$$(1+\omega)^7 = (1+35+7) + (7+35+1)\omega + (21+21)\omega^2$$

Simplifying further:

$$(1+\omega)^7 = 43 + 43\omega + 42\omega^2$$

4. **Identifying Coefficients** Comparing with the form $A + B\omega + C\omega^2$, we have:

$$A = 43, \quad B = 43, \quad C = 42$$

5. **Calculating** $5(3A - 2B - C)$ Substitute the values:

$$5(3A - 2B - C) = 5(3 \times 43 - 2 \times 43 - 42)$$

Simplifying:

$$5(129 - 86 - 42) = 5(1) = 5$$

Conclusion

The value of $5(3A - 2B - C)$ is:

$$5(3A - 2B - C) = 5$$

Quick Tip

When working with powers of roots of unity, simplify using their cyclic properties (e.g., $\omega^3 = 1$ for cube roots of unity) to reduce higher powers.

Question 31

Position of an ant (S in metres) moving in the Y-Z plane is given by $S = 2t^2\hat{j} + 5t\hat{k}$ (where t is in seconds). The magnitude and direction of velocity of the ant at $t = 1$ s will be:

(1) 16 m/s in y-direction
(2) 4 m/s in x-direction
(3) 9 m/s in z-direction
(4) 4 m/s in y-direction

Answer: (4)

Solution

1. **Given Position Vector** The position vector of a particle is given by:

$$\vec{S}(t) = 2t^2\,\hat{j} + 5t\,\hat{k}$$

Here, the motion is in the $\hat{j}$ (y-direction) and $\hat{k}$ (z-direction) components.
2. **Calculate the Velocity Vector** The velocity vector $\vec{v}(t)$ is obtained by differentiating the position vector $\vec{S}(t)$ with respect to time t:

$$\vec{v}(t) = \frac{d\vec{S}}{dt} = \frac{d}{dt}\left(2t^2\,\hat{j} + 5t\,\hat{k}\right)$$

Performing the differentiation:

$$\vec{v}(t) = \frac{d}{dt}(2t^2)\,\hat{j} + \frac{d}{dt}(5t)\,\hat{k}$$

$$\vec{v}(t) = (4t)\,\hat{j} + 5\,\hat{k}$$

3. **Evaluating the Velocity at** $t = 1$ Substitute $t = 1$ into the velocity expression:

$$\vec{v}(1) = 4 \times 1\,\hat{j} + 5\,\hat{k}$$

$$\vec{v}(1) = 4\,\hat{j} + 5\,\hat{k}$$

4. **Direction of the Velocity Vector** At $t = 1$ s, the velocity vector has components:

$$\vec{v} = 4\,\hat{j} + 5\,\hat{k}$$

The y-component (in the $\hat{j}$ direction) of the velocity is 4 m/s. This indicates motion along the y-axis with a magnitude of 4 m/s.

Conclusion

The magnitude and direction of the velocity at $t = 1$ s are 4 m/s in the y-direction.

Quick Tip

To find the velocity of a particle, differentiate the position vector with respect to time. The magnitude and direction of velocity can then be found from its components.

Question 32

Given below are two statements:
Statement (I): Viscosity of gases is greater than that of liquids.
Statement (II): Surface tension of a liquid decreases due to the presence of insoluble impurities.
In the light of the above statements, choose the most appropriate answer from the options given below:

(1) Statement I is correct but Statement II is incorrect
(2) Statement I is incorrect but Statement II is correct
(3) Both Statement I and Statement II are incorrect
(4) Both Statement I and Statement II are correct

Answer: (2)

Solution

1. **Analysis of Statement (I)** - Viscosity refers to a fluid's resistance to flow, influenced by intermolecular forces. In general, liquids exhibit higher viscosity than gases due to stronger intermolecular attractions present in liquids, which inhibit flow more than in gases. Hence, Statement (I) is incorrect.
2. **Analysis of Statement (II)** - Surface tension arises due to cohesive forces between molecules at the surface of a liquid. The presence of insoluble impurities often disrupts these cohesive forces, leading to a reduction in surface tension. Therefore, Statement (II) is correct.

Conclusion

Since Statement I is incorrect and Statement II is correct, the correct answer is:

Option (2)

> Quick Tip
>
> For questions involving physical properties, consider intermolecular forces and interactions for accurate comparison.

Question 33

If the refractive index of the material of a prism is $\cot\left(\frac{A}{2}\right)$, where A is the angle of the prism, then the angle of minimum deviation will be:

(1) $\pi - 2A$
(2) $\frac{\pi}{2} - 2A$
(3) $\pi - A$
(4) $\frac{\pi}{2} - A$

Answer: (1)

Solution

1. **Given Relation for Minimum Deviation** The relation given is:

$$\cot\frac{A}{2} = \frac{\sin\left(\frac{A+\delta_{\text{min}}}{2}\right)}{\sin\left(\frac{A}{2}\right)}$$

Here, A is the angle of the prism, and δ_{min} is the angle of minimum deviation.
2. **Rearranging the Expression** Multiply both sides by $\sin\left(\frac{A}{2}\right)$ to clear the denominator:

$$\cot\frac{A}{2} \cdot \sin\left(\frac{A}{2}\right) = \sin\left(\frac{A+\delta_{\text{min}}}{2}\right)$$

Simplifying using the identity $\cot x = \frac{\cos x}{\sin x}$:

$$\cos\left(\frac{A}{2}\right) = \sin\left(\frac{A+\delta_{\text{min}}}{2}\right)$$

3. **Equating Arguments of Trigonometric Functions** Since $\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$, we have:

$$\frac{A+\delta_{\text{min}}}{2} = \frac{\pi}{2} - \frac{A}{2}$$

4. **Solving for δ_{min}** Multiply the entire equation by 2:

$$A + \delta_{\text{min}} = \pi - A$$

Rearranging to isolate δ_{min}:

$$\delta_{\text{min}} = \pi - 2A$$

Conclusion

The angle of minimum deviation is given by:

$$\delta_{\text{min}} = \pi - 2A$$

Quick Tip

For prism-related problems, use the geometry of the prism and trigonometric identities to relate the angle of deviation and refractive index.

Question 34

A proton moving with a constant velocity passes through a region of space without any change in its velocity. If $\vec{E}$ and $\vec{B}$ represent the electric and magnetic fields respectively, then the region of space may have:

(A) $E = 0, B = 0$
(B) $E = 0, B \neq 0$
(C) $E \neq 0, B = 0$
(D) $E \neq 0, B \neq 0$

Choose the most appropriate answer from the options given below:

(1) (A), (B) and (C) only
(2) (A), (C) and (D) only
(3) (A), (B) and (D) only
(4) (B), (C) and (D) only

Answer: (3)

Solution

To ensure that a proton moves with a constant velocity, the net force acting on it must be zero. This is expressed by the equation:

$$q\vec{E} + q\vec{v} \times \vec{B} = 0$$

where: - q is the charge of the proton. - $\vec{E}$ is the electric field. - $\vec{v}$ is the velocity of the proton. - $\vec{B}$ is the magnetic field.
Simplifying, we obtain:

$$\vec{E} + \vec{v} \times \vec{B} = 0$$

Possible Cases

1. **Case (A):** $\vec{E} = 0$ **and** $\vec{B} = 0$ - If there is no electric field ($\vec{E} = 0$) and no magnetic field ($\vec{B} = 0$), there is no force acting on the proton. As a result, it will continue moving with a constant velocity.
2. **Case (B):** $\vec{E} = 0$ **and** $\vec{B} \neq 0$ - With no electric field but a magnetic field present ($\vec{B} \neq 0$), the force due to the magnetic field is given by $q\vec{v} \times \vec{B}$. - If the velocity $\vec{v}$ is parallel (or antiparallel) to the magnetic field $\vec{B}$, then:

$$\vec{v} \times \vec{B} = 0$$

- This results in no net force on the proton.
3. **Case (D):** $\vec{E} \neq 0$ **and** $\vec{B} \neq 0$ - In this scenario, both electric and magnetic fields are present. - For the net force to be zero, the electric force $q\vec{E}$ and the magnetic force $q\vec{v} \times \vec{B}$ must cancel each other:

$$q\vec{E} = -q(\vec{v} \times \vec{B})$$

- This condition can be met if the magnitudes are equal and the directions are opposite.

Conclusion

The conditions that allow the proton to move with a constant velocity are satisfied by Cases (A), (B), and (D). Therefore, the correct answer is:

Option (3)

> Quick Tip
>
> To determine conditions for no net force on a charged particle, consider both electric and magnetic force components and ensure they cancel out.

Question 35

The acceleration due to gravity on the surface of earth is g. If the diameter of earth reduces to half of its original value and mass remains constant, then acceleration due to gravity on the surface of earth would be:

(1) $g/4$
(2) $2g$
(3) $g/2$
(4) $4g$

Answer: (4)

Solution

The acceleration due to gravity on the surface of the Earth is given by:

$$g = \frac{GM}{R^2}$$

where: - G is the gravitational constant. - M is the mass of the Earth. - R is the radius of the Earth.

Effect of Reducing the Diameter of the Earth

If the diameter of the Earth is reduced to half, the radius R will also be reduced to half:

$$R' = \frac{R}{2}$$

Calculating the New Acceleration due to Gravity

The new acceleration due to gravity g' is given by:

$$g' = \frac{GM}{(R')^2}$$

Substituting $R' = \frac{R}{2}$ into the equation:

$$g' = \frac{GM}{\left(\frac{R}{2}\right)^2}$$

Simplifying:

$$g' = \frac{GM}{\frac{R^2}{4}} = 4 \cdot \frac{GM}{R^2} = 4g$$

Conclusion

The new acceleration due to gravity on the surface of the Earth, when the diameter is reduced to half, is:

$$g' = 4g$$

Quick Tip

When the radius of a planet changes, the acceleration due to gravity changes according to the inverse square law.

Question 36

A train is moving with a speed of 12 m/s on rails which are 1.5 m apart. To negotiate a curve of radius 400 m, the height by which the outer rail should be raised with respect to the inner rail is (Given, $g = 10\,\mathrm{m/s^2}$):

(1) 6.0 cm
(2) 5.4 cm
(3) 4.8 cm
(4) 4.2 cm

Answer: (2)

Solution

To determine the required banking angle θ for a train moving around a curve, the condition for centripetal force is given by:

$$\tan\theta = \frac{v^2}{Rg}$$

where: - $v = 12\,\mathrm{m/s}$ (speed of the train) - $R = 400\,\mathrm{m}$ (radius of the curve) - $g = 10\,\mathrm{m/s^2}$ (acceleration due to gravity)

Substituting the Values

Substitute the given values into the formula:

$$\tan\theta = \frac{12^2}{10 \times 400}$$

Calculating the terms:

$$\tan\theta = \frac{144}{4000} = 0.036$$

Calculating the Height Difference

The height by which the outer rail should be raised over the inner rail can be related to $\tan\theta$ by:

$$\tan\theta = \frac{h}{d}$$

where: - h is the height difference. - $d = 1.5\,\mathrm{m}$ (distance between the rails).
Rearranging to find h:

$$h = \tan\theta \times d$$

Substitute the values:

$$h = 0.036 \times 1.5$$

Calculating:

$$h = 0.054\,\mathrm{m} = 5.4\,\mathrm{cm}$$

Conclusion

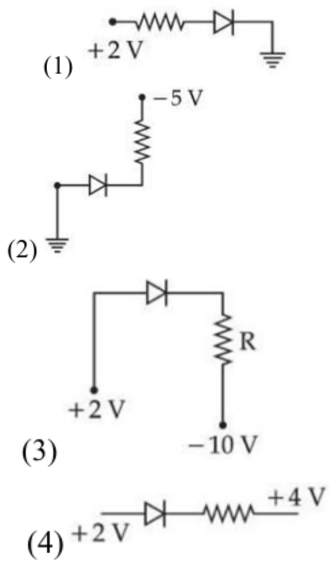
The required height by which the outer rail should be raised over the inner rail is:

$$h = 5.4\,\text{cm}$$

> **Quick Tip**
>
> For curves, use $\tan\theta = \frac{v^2}{Rg}$ to determine the banking angle or height difference needed to keep objects on track.

Question 37

Which of the following circuits is reverse-biased?



Answer: (4)

Solution

For a diode to be forward-biased, the P-end (anode) must be at a higher potential than the N-end (cathode). This allows current to flow through the diode, as it aligns with the diode's conducting direction.

Analysis of Option (4)

In the given scenario for option (4): - The P-end of the diode is connected to the negative terminal of the power supply. - The N-end of the diode is connected to the positive terminal of the power supply.
This configuration places the diode in a reverse-bias condition because the P-end (positive side) is at a lower potential relative to the N-end (negative side). In reverse bias, the

diode blocks the current flow, preventing conduction.
Since the P-end is at a lower potential than the N-end in option (4), the diode is reverse-biased, which opposes the conditions required for forward bias.

Quick Tip

In a forward-biased diode, the P-end should be at a higher potential than the N-end.

Question 38

Identify the physical quantity that cannot be measured using a spherometer:

(1) Radius of curvature of concave surface
(2) Specific rotation of liquids
(3) Thickness of thin plates
(4) Radius of curvature of convex surface

Answer: (2)

Solution

A spherometer is a precision instrument designed to measure the curvature of spherical surfaces, including concave and convex shapes, as well as to determine the thickness of thin plates or sheets. It operates using three legs arranged in an equilateral triangle, with a central screw that can be adjusted to touch the surface being measured.

Applications of a Spherometer

- Measuring the radius of curvature of lenses and mirrors. - Determining the thickness of thin materials such as glass plates.

Limitations

A spherometer is not designed to measure the properties of liquids, such as specific rotation. Specific rotation is a measure of the optical activity of a solution, indicating how it rotates the plane of polarized light passing through it. This property requires an entirely different instrument called a polarimeter.

Conclusion

A spherometer cannot measure the specific rotation of liquids, as its functionality is limited to measuring the curvature of solid surfaces and the thickness of thin plates.

Quick Tip

Use a spherometer to measure surface curvature and plate thickness, not liquid properties.

Question 39

Two bodies of mass 4 g and 25 g are moving with equal kinetic energies. The ratio of magnitude of their linear momenta is:

(1) 3 : 5
(2) 5 : 4
(3) 2 : 5
(4) 4 : 5

Answer: (3)

Solution

Given that two objects have equal kinetic energies, we can write:

$$\frac{p_1^2}{2m_1} = \frac{p_2^2}{2m_2}$$

where: - p_1 and p_2 are the momenta of the objects. - m_1 and m_2 are their respective masses.

Finding the Ratio of the Momenta

Rearranging the equation for equal kinetic energies:

$$p_1^2 m_2 = p_2^2 m_1$$

Taking the square root on both sides:

$$\frac{p_1}{p_2} = \sqrt{\frac{m_1}{m_2}}$$

Substituting the Given Values

Given:

$$m_1 = 4\,\text{g}, \quad m_2 = 25\,\text{g}$$

Substitute these values into the expression:

$$\frac{p_1}{p_2} = \sqrt{\frac{4}{25}} = \frac{2}{5}$$

Conclusion

The ratio of the momenta of the two objects is:

$$\frac{p_1}{p_2} = 2 : 5$$

> Quick Tip
>
> For equal kinetic energies, the momentum ratio is the square root of the inverse mass ratio.

Question 40

0.08 kg air is heated at constant volume through $5°$C. The specific heat of air at constant volume is 0.17 kcal/kg$^\circ$C and $J = 4.18$ joule/cal. The change in its internal energy is approximately:

(1) 318 J
(2) 298 J
(3) 284 J
(4) 142 J

Answer: (3)

Solution

Given that the process occurs at constant volume, the change in internal energy ΔU is given by:

$$\Delta U = ms\Delta T$$

where: - $m = 0.08\,\text{kg}$ (mass of the substance) - $s = 0.17\,\text{kcal/kg}^\circ\text{C}$ (specific heat capacity) - $\Delta T = 5°\text{C}$ (temperature change)

Converting Units for Specific Heat Capacity

To perform the calculation in joules, we need to convert the specific heat capacity s from $\text{kcal/kg}^\circ\text{C}$ to $\text{J/kg}^\circ\text{C}$ using the conversion factor $1\,\text{kcal} = 4180\,\text{J}$:

$$s = 0.17\,\text{kcal/kg}^\circ\text{C} \times 4180\,\text{J/kcal} = 710.6\,\text{J/kg}^\circ\text{C}$$

Calculating the Change in Internal Energy

Substitute the values into the formula for ΔU:

$$\Delta U = m \times s \times \Delta T$$

$$\Delta U = 0.08\,\text{kg} \times 710.6\,\text{J/kg}^\circ\text{C} \times 5°\text{C}$$

$$\Delta U = 0.08 \times 710.6 \times 5$$

$$\Delta U \approx 284.24\,\text{J}$$

Conclusion

The change in internal energy ΔU is approximately:

$$\Delta U \approx 284\,\text{J}$$

Quick Tip

For specific heat problems, make sure to convert all units consistently.

Question 41

The radius of the third stationary orbit of electron for Bohr's atom is R. The radius of fourth stationary orbit will be:

(1) $\frac{4}{3}R$
(2) $\frac{16}{9}R$
(3) $\frac{3}{4}R$
(4) $\frac{9}{16}R$

Answer: (2)

Solution

According to Bohr's model, the radius of an electron's orbit in a hydrogen-like atom is given by:

$$r_n \propto \frac{n^2}{Z}$$

where: - n is the principal quantum number. - Z is the atomic number of the atom. For hydrogen ($Z = 1$), the relationship simplifies to:

$$r_n \propto n^2$$

Calculating the Ratio of Radii

Given that we want to find the ratio of the radii of the fourth orbit (r_4) and the third orbit (r_3) in hydrogen:

$$\frac{r_4}{r_3} = \frac{n_4^2}{n_3^2} = \frac{4^2}{3^2}$$

Simplifying:

$$\frac{r_4}{r_3} = \frac{16}{9}$$

Expressing r_4 in Terms of r_3

Rearranging the expression:

$$r_4 = \frac{16}{9}R$$

Conclusion

The radius of the fourth orbit relative to the third orbit is given by:

$$r_4 = \frac{16}{9}R$$

Quick Tip

Remember that Bohr's orbit radii are proportional to n^2 for hydrogen-like atoms.

Question 42

A rectangular loop of length 2.5 m and width 2 m is placed at 60° to a magnetic field of 4 T. The loop is removed from the field in 10 sec. The average emf induced in the loop during this time is:

(1) $-2\,\mathrm{V}$
(2) $+2\,\mathrm{V}$
(3) $+1\,\mathrm{V}$
(4) $-1\,\mathrm{V}$

Answer: (3)

Solution

The average emf induced in a loop is given by Faraday's law of electromagnetic induction:

$$\text{Average emf} = -\frac{\Delta\Phi}{\Delta t}$$

where: - $\Delta\Phi$ is the change in magnetic flux. - Δt is the time interval over which the change occurs.

Calculating the Change in Magnetic Flux

Given: - Initial flux, $\Phi_i = 4 \times (2.5 \times 2)\cos 60^\circ$ - Final flux, $\Phi_f = 0$ - $\cos 60^\circ = \frac{1}{2}$
The change in flux is:

$$\Delta\Phi = \Phi_f - \Phi_i = 0 - \left(4 \times (2.5 \times 2) \times \frac{1}{2}\right)$$

Simplifying:

$$\Delta\Phi = 0 - (4 \times 5 \times 0.5) = -10\,\mathrm{Wb}$$

Calculating the Average Emf

The time interval is given as $\Delta t = 10\,\mathrm{s}$. Therefore, the average emf induced is:

$$\text{Average emf} = -\frac{\Delta\Phi}{\Delta t} = -\frac{-10}{10}$$

Simplifying:

$$\text{Average emf} = +1\,\mathrm{V}$$

Conclusion

The average induced emf in the loop is:

$$\text{Average emf} = +1\,\text{V}$$

Quick Tip

The formula for induced emf in a changing magnetic field is emf $= -\frac{\Delta\Phi}{\Delta t}$. Remember to calculate flux carefully based on orientation.

Question 43

An electric charge $10^{-6}\,\mu$C is placed at the origin (0, 0) of an X-Y coordinate system. Two points P and Q are situated at $(\sqrt{3}, \sqrt{3})$ mm and $(\sqrt{6}, 0)$ mm respectively. The potential difference between the points P and Q will be:

(1) $\sqrt{3}V$
(2) $\sqrt{6}V$
(3) 0 V
(4) $3V$

Answer: (3)

Solution

The potential difference between two points P and Q due to a point charge Q is given by:

$$\Delta V = KQ\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

where: - K is the Coulomb's constant. - r_1 and r_2 are the distances of points P and Q from the point charge.

Calculating the Distances r_1 and r_2

Given: - For point P: The coordinates are $(\sqrt{3}, \sqrt{3})$. - For point Q: The coordinates are $(\sqrt{6}, 0)$.
The distance r_1 of point P from the origin (where the charge Q is located) is:

$$r_1 = \sqrt{(\sqrt{3})^2 + (\sqrt{3})^2} = \sqrt{3+3} = \sqrt{6}$$

Similarly, the distance r_2 of point Q from the origin is:

$$r_2 = \sqrt{(\sqrt{6})^2 + 0^2} = \sqrt{6}$$

Calculating the Potential Difference

Substitute the values of r_1 and r_2 into the expression for ΔV:

$$\Delta V = KQ\left(\frac{1}{\sqrt{6}} - \frac{1}{\sqrt{6}}\right)$$

Simplifying:

$$\Delta V = KQ \times 0 = 0$$

Conclusion

The potential difference between points P and Q is:

$$\Delta V = 0$$

Quick Tip

When the distances from a charge to two points are equal, the potential difference between those points is zero.

Question 44

A convex lens of focal length 40 cm forms an image of an extended source of light on a photo-electric cell. A current I is produced. The lens is replaced by another convex lens having the same diameter but focal length 20 cm. The photoelectric current now is:

(1) $\frac{I}{2}$
(2) 4I
(3) 2I
(4) I

Answer: (3)

Solution

The intensity of light incident on a photoelectric cell depends on the concentration of light by a lens, which is influenced by the lens's focal length and area.

Effect of Reducing the Focal Length

Given that: - The lens diameter remains unchanged, so the area through which light passes remains constant. - The focal length is reduced from 40 cm to 20 cm.
Reducing the focal length leads to an increase in the concentration of light on the photoelectric cell. Specifically, when the focal length is halved, the intensity of light on the cell increases due to increased light convergence. This results in the light being focused on a smaller area, effectively doubling the light intensity on the photoelectric surface.

Impact on Photoelectric Current

The photoelectric current I is directly proportional to the intensity of the incident light. Therefore, doubling the intensity of light results in a doubling of the photoelectric current:

$$\text{New current} = 2I$$

By reducing the focal length from 40 cm to 20 cm, the intensity of light on the photoelectric cell doubles, resulting in a new photoelectric current of:

$$\text{New current} = 2I$$

Quick Tip

In photoelectric experiments, reducing the focal length of a lens increases the intensity of light, leading to higher photocurrent.

Question 45

A body of mass 1000 kg is moving horizontally with a velocity of 6 m/s. If 200 kg extra mass is added, the final velocity (in m/s) is:

(1) 6
(2) 2
(3) 4
(4) 5

Answer: (4)

Solution

Given that there are no external forces acting on the system, the total momentum of the system is conserved.

Initial Momentum Calculation

The initial momentum of the object is given by:

$$\text{Initial momentum} = \text{mass} \times \text{velocity} = 1000\,\text{kg} \times 6\,\text{m/s} = 6000\,\text{kg m/s}$$

Applying Conservation of Momentum

After an additional mass of 200 kg is added, the total mass becomes:

$$\text{Total mass} = 1000\,\text{kg} + 200\,\text{kg} = 1200\,\text{kg}$$

Let the final velocity of the system be v. By the law of conservation of momentum:

$$\text{Initial momentum} = \text{Final momentum}$$

$$6000\,\text{kg m/s} = 1200\,\text{kg} \times v$$

Calculating the Final Velocity

Rearranging to find v:

$$v = \frac{6000\,\text{kg m/s}}{1200\,\text{kg}} = 5\,\text{m/s}$$

The final velocity of the system after adding the mass is:

$$v = 5\,\text{m/s}$$

Quick Tip

In problems involving mass change and no external forces, use conservation of momentum to find final velocity.

Question 46

A plane electromagnetic wave propagating in the x-direction is described by

$$E_y = (200\,\text{V m}^{-1})\sin\left(1.5\times 10^7\,t - 0.05\,x\right);$$

the intensity of the wave is:

(1) $35.4\,\text{W/m}^2$
(2) $53.1\,\text{W/m}^2$
(3) $26.6\,\text{W/m}^2$
(4) $106.2\,\text{W/m}^2$

Answer: (2)

Solution

The intensity I of an electromagnetic wave is given by the formula:

$$I = \frac{1}{2}\epsilon_0 c E_0^2$$

where: - $\epsilon_0 = 8.85\times 10^{-12}\,\text{C}^2/\text{N}\cdot\text{m}^2$ (permittivity of free space) - $c = 3\times 10^8\,\text{m/s}$ (speed of light) - $E_0 = 200\,\text{V/m}$ (electric field amplitude)

Substituting the Given Values

Substitute the known values into the intensity formula:

$$I = \frac{1}{2}\times\epsilon_0\times c\times E_0^2$$

$$I = \frac{1}{2}\times 8.85\times 10^{-12}\,\text{C}^2/\text{N}\cdot\text{m}^2\times 3\times 10^8\,\text{m/s}\times(200\,\text{V/m})^2$$

Calculating Step-by-Step

1. Calculate E_0^2:
$$E_0^2 = (200\,\text{V/m})^2 = 40000\,\text{V}^2/\text{m}^2$$
2. Multiply all terms together:
$$I = \frac{1}{2} \times 8.85 \times 10^{-12} \times 3 \times 10^8 \times 40000$$
3. Simplify:
$$I = 0.5 \times 8.85 \times 3 \times 40000 \times 10^{-12+8}\,\text{W/m}^2$$
$$I = 53.1\,\text{W/m}^2$$

The intensity of the electromagnetic wave is:
$$I = 53.1\,\text{W/m}^2$$

Quick Tip

For the intensity of an EM wave, use $I = \frac{1}{2}\epsilon_0 c E_0^2$ where E_0 is the peak electric field.

Question 47

Given below are two statements:
Statement (I): Planck's constant and angular momentum have the same dimensions.
Statement (II): Linear momentum and moment of force have the same dimensions.
In the light of the above statements, choose the correct answer from the options given below:

(1) Statement I is true but Statement II is false
(2) Both Statement I and Statement II are false
(3) Both Statement I and Statement II are true
(4) Statement I is false but Statement II is true

Answer: (1)

Solution

1. **Analyzing the Dimensions of Planck's Constant** (h) Planck's constant represents the quantum of action or angular momentum and has the following dimensional formula:
$$[h] = \text{energy} \times \text{time} = ML^2T^{-1}$$
where: - M represents mass. - L represents length. - T represents time.
2. **Analyzing the Dimensions of Linear Momentum** The dimensional formula for linear momentum p is given by:
$$[p] = MLT^{-1}$$

This represents mass multiplied by velocity.
3. **Analyzing the Dimensions of Moment of Force (Torque)** The moment of force, also known as torque τ, has the following dimensional formula:

$$[\tau] = \text{force} \times \text{distance} = ML^2T^{-2}$$

Here, torque is the product of force and the perpendicular distance from the axis of rotation.
4. **Comparison of Dimensions** - The dimensions of Planck's constant $[h] = ML^2T^{-1}$ differ from those of linear momentum $[p] = MLT^{-1}$. - The dimensions of torque $[ML^2T^{-2}]$ also differ from linear momentum $[MLT^{-1}]$.
- Statement I (regarding Planck's constant dimensions being consistent with angular momentum) is true.
- Statement II (regarding the dimensions of linear momentum and moment of force being the same) is false.
Thus, the correct assessment is that Statement I is true and Statement II is false.

Quick Tip

For dimensional analysis, break down each term into base units to verify dimensional consistency.

Question 48

A wire of length 10 cm and radius $\sqrt{7} \times 10^{-4}$ m is connected across the right gap of a meter bridge. When a resistance of 4.5 Ω is connected on the left gap by using a resistance box, the balance length is found to be at 60 cm from the left end. If the resistivity of the wire is $R \times 10^{-7} \Omega$ m, then the value of R is:

(1) 63
(2) 70
(3) 66
(4) 35

Answer: (3)

Solution

Given a balanced Wheatstone bridge setup in a meter bridge experiment, we use the relation for balancing:

$$\frac{4.5}{R} = \frac{60}{40}$$

Rearranging to solve for R:

$$R = \frac{4.5 \times 40}{60}$$

Simplifying:

$$R = \frac{180}{60} = 3\,\Omega$$

Calculating the Resistivity ρ

The resistance of a wire is given by:

$$R = \frac{\rho \ell}{A}$$

where: - $\ell = 10\,\text{cm} = 0.1\,\text{m}$ is the length of the wire. - $A = \pi r^2$ is the cross-sectional area, with radius $r = \sqrt{7} \times 10^{-4}\,\text{m}$.
Substitute the known values into the resistance formula:

$$3 = \frac{\rho \times 0.1}{\pi(\sqrt{7} \times 10^{-4})^2}$$

Rearranging to solve for ρ:

$$\rho = 3 \times \frac{\pi(\sqrt{7} \times 10^{-4})^2}{0.1}$$

Calculating step-by-step: 1. $\pi(\sqrt{7} \times 10^{-4})^2 = \pi \times 7 \times 10^{-8} = 7\pi \times 10^{-8}\,\text{m}^2$ 2. Substituting:

$$\rho = 3 \times \frac{7\pi \times 10^{-8}}{0.1} = 21\pi \times 10^{-7}\,\Omega\,\text{m}$$

3. Approximating $\pi \approx 3.14$:

$$\rho \approx 21 \times 3.14 \times 10^{-7} \approx 66 \times 10^{-7}\,\Omega\,\text{m}$$

The resistivity of the wire is:

$$\rho \approx 66 \times 10^{-7}\,\Omega\,\text{m}$$

Quick Tip

In meter bridge problems, use the balance length to find unknown resistance and then apply the resistivity formula if needed.

Question 49

A wire of resistance R and length L is cut into 5 equal parts. If these parts are joined parallel, then resultant resistance will be:

(1) $\frac{R}{25}$
(2) $\frac{R}{5}$
(3) $25R$
(4) $5R$

Answer: (1)

Solution

Given that a resistor of resistance R is divided into 5 equal parts, each part will have a resistance given by:

$$R' = \frac{R}{5}$$

Calculating the Equivalent Resistance in Parallel

When these 5 resistors are connected in parallel, the equivalent resistance R_{eq} can be found using the formula for parallel resistors:

$$\frac{1}{R_{\text{eq}}} = \sum_{i=1}^{5} \frac{1}{R'} = 5 \times \frac{1}{R'}$$

Substituting the value of R':

$$\frac{1}{R_{\text{eq}}} = 5 \times \frac{1}{R/5}$$

Simplifying:

$$\frac{1}{R_{\text{eq}}} = \frac{5}{R/5} = \frac{25}{R}$$

Taking the reciprocal to find R_{eq}:

$$R_{\text{eq}} = \frac{R}{25}$$

The equivalent resistance when the 5 equal parts are connected in parallel is:

$$R_{\text{eq}} = \frac{R}{25}$$

> Quick Tip
>
> When resistances are cut and connected in parallel, the effective resistance decreases significantly.

Question 50

The average kinetic energy of a monatomic molecule is $0.414\,$eV at temperature T: (Use $k_B = 1.38 \times 10^{-23}$ J/mol-K)

(1) 3000 K
(2) 3200 K
(3) 1600 K
(4) 1500 K

Answer: (2)

Solution

The average kinetic energy per molecule of a monatomic gas is given by:

$$\text{KE} = \frac{3}{2} k_B T$$

where: - KE is the average kinetic energy per molecule. - $k_B = 1.38 \times 10^{-23}$ J/K is Boltzmann's constant. - T is the temperature in Kelvin.

Given:

$$\text{KE} = 0.414\,\text{eV}$$

Convert this energy to joules using the conversion factor $1\,\text{eV} = 1.6 \times 10^{-19}\,\text{J}$:

$$\text{KE} = 0.414 \times 1.6 \times 10^{-19}\,\text{J} = 6.624 \times 10^{-20}\,\text{J}$$

Calculating the Temperature

Using the formula for kinetic energy:

$$6.624 \times 10^{-20} = \frac{3}{2} \times 1.38 \times 10^{-23} \times T$$

Rearranging to solve for T:

$$T = \frac{6.624 \times 10^{-20}}{\frac{3}{2} \times 1.38 \times 10^{-23}}$$

Simplifying:

$$T = \frac{6.624 \times 10^{-20}}{2.07 \times 10^{-23}}$$

$$T \approx 3200\,\text{K}$$

Conclusion

The temperature of the monatomic gas is approximately:

$$T \approx 3200\,\text{K}$$

> Quick Tip
>
> In gas law problems, always check units carefully, especially when converting electron volts to joules.

Question 51

A particle starts from origin at $t = 0$ with a velocity $5\hat{i}\,\text{m/s}$ and moves in x-y plane under action of a force which produces a constant acceleration of $(3\hat{i} + 2\hat{j})\,\text{m/s}^2$. If the x-coordinate of the particle at that instant is 84 m, then the speed of the particle at this time is $\sqrt{\alpha}\,\text{m/s}$. The value of α is:

(1) 625
(2) 673
(3) 600
(4) 720

Answer: (2)

Solution

Given: - Initial velocity in the x-direction: $u_x = 5\,\text{m/s}$ - Acceleration in the x-direction: $a_x = 3\,\text{m/s}^2$ - Displacement in the x-direction: $x = 84\,\text{m}$

Step 1: Finding the Final Velocity in the x-Direction (v_x)

Using the kinematic equation:

$$v_x^2 = u_x^2 + 2a_x x$$

Substituting the given values:

$$v_x^2 = 5^2 + 2 \times 3 \times 84$$

$$v_x^2 = 25 + 504$$

$$v_x^2 = 529$$

Taking the square root:

$$v_x = \sqrt{529} = 23\,\text{m/s}$$

Step 2: Finding the Time Taken (t)

Using the equation:

$$v_x = u_x + a_x t$$

Rearranging to solve for t:

$$t = \frac{v_x - u_x}{a_x}$$

Substituting the known values:

$$t = \frac{23 - 5}{3}$$

$$t = \frac{18}{3} = 6\,\text{s}$$

Step 3: Finding the Final Velocity in the y-Direction (v_y)

Given: - Initial velocity in the y-direction: $u_y = 0\,\text{m/s}$ - Acceleration in the y-direction: $a_y = 2\,\text{m/s}^2$

Using the equation:

$$v_y = u_y + a_y t$$

Substituting the known values:

$$v_y = 0 + 2 \times 6$$

$$v_y = 12\,\text{m/s}$$

The speed of the particle is given by:

$$v = \sqrt{v_x^2 + v_y^2}$$

Substituting the values:

$$v = \sqrt{23^2 + 12^2}$$
$$v = \sqrt{529 + 144}$$
$$v = \sqrt{673}$$

The value of α is:

$$\alpha = 673$$

Quick Tip

For two-dimensional motion problems, calculate each component separately and then use the Pythagorean theorem.

Question 52

A thin metallic wire having cross sectional area of $10^{-4}\,\mathrm{m}^2$ is used to make a ring of radius 30 cm. A positive charge of $2\pi\,\mathrm{C}$ is uniformly distributed over the ring, while another positive charge of 30 pC is kept at the centre of the ring. The tension in the ring is:

(1) $32\,\mathrm{N}$
(2) $16\,\mathrm{N}$
(3) $48\,\mathrm{N}$
(4) $24\,\mathrm{N}$

Answer: (3)

Solution

Given: - Linear charge density of the ring:

$$\lambda = \frac{Q}{2\pi R} = \frac{2\pi\,\mathrm{C}}{2\pi \times 0.3\,\mathrm{m}} = \frac{1}{0.3}\,\mathrm{C/m}$$

Electrostatic Force and Tension Relation

Consider a small element of charge dq on the ring at an angle θ. The force F_e due to an external charge q_0 at the center of the ring creates a symmetric tension T in the ring such that:

$$2T \sin\frac{d\theta}{2} = \frac{kq_0\lambda d\theta}{R^2}$$

For small angles $d\theta$, we can approximate:

$$\sin\frac{d\theta}{2} \approx \frac{d\theta}{2}$$

Substituting this approximation:

$$T \cdot d\theta = \frac{kq_0\lambda d\theta}{2R}$$

Dividing both sides by $d\theta$:

$$T = \frac{kq_0\lambda}{2R}$$

Substitute the Given Values

Given: - Coulomb constant: $k = 9 \times 10^9\,\mathrm{N\,m^2/C^2}$ - Charge at the center: $q_0 = 30 \times 10^{-12}\,\mathrm{C}$ - Radius of the ring: $R = 0.3\,\mathrm{m}$ - Linear charge density: $\lambda = \frac{1}{0.3}\,\mathrm{C/m}$
Substitute these values:

$$T = \frac{9 \times 10^9 \times 30 \times 10^{-12} \times \frac{1}{0.3}}{2 \times 0.3}$$

Simplifying:

$$T = \frac{9 \times 10^9 \times 30 \times 10^{-12}}{0.18}$$

$$T = \frac{270 \times 10^{-3}}{0.18} = 48\,\mathrm{N}$$

The tension in the ring is:

$$T = 48\,\mathrm{N}$$

Quick Tip
For tension in a ring with central charge, consider the forces from charge distribution and apply symmetry.

Question 53

Two coils have mutual inductance $0.002\,\mathrm{H}$. The current changes in the first coil according to the relation $i = i_0 \sin\omega t$, where $i_0 = 5\,\mathrm{A}$ and $\omega = 50\pi\,\mathrm{rad/s}$. The maximum value of emf in the second coil is $\frac{\pi}{\alpha}\,\mathrm{V}$. The value of α is:
Answer: (2)

Solution

The emf induced in the second coil is given by Faraday's law of electromagnetic induction:

$$\mathrm{EMF} = -M\frac{di}{dt}$$

where M is the mutual inductance between the coils and $\frac{di}{dt}$ is the time rate of change of current in the first coil.

Step 1: Given Current Function

Given that:

$$i = i_0 \sin(\omega t)$$

To find the rate of change of current:

$$\frac{di}{dt} = i_0 \omega \cos(\omega t)$$

Step 2: Substituting Values to Find Maximum EMF

The maximum emf is obtained when $\cos(\omega t) = 1$:

$$\text{EMF}_{\text{max}} = M i_0 \omega$$

Substituting the given values: - Mutual inductance $M = 0.002\,\text{H}$ - Amplitude of current $i_0 = 5\,\text{A}$ - Angular frequency $\omega = 50\pi\,\text{rad/s}$

$$\text{EMF}_{\text{max}} = 0.002 \times 5 \times 50\pi$$

Simplifying:

$$\text{EMF}_{\text{max}} = 0.5\pi\,\text{V}$$

Converting to a fraction:

$$\text{EMF}_{\text{max}} = \frac{\pi}{2}\,\text{V}$$

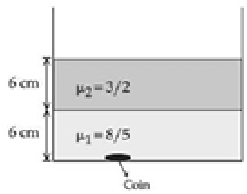
Thus, the value of α is:

$$\alpha = 2$$

> Quick Tip
>
> For mutual inductance problems, differentiate the current function and substitute values carefully to find induced emf.

Question 54

Two immiscible liquids of refractive indices $\frac{8}{5}$ and $\frac{3}{2}$ respectively are put in a beaker as shown in the figure. The height of each column is 6 cm. A coin is placed at the bottom of the beaker. For near normal vision, the apparent depth of the coin is $\frac{\alpha}{4}$ cm. The value of α is:



(1) 31
(2) 32
(3) 24
(4) 16

Answer: (1)

Solution

For light traveling through two layered media, the apparent depth d_{app} seen by an observer is given by the sum of the depths divided by their respective refractive indices:

$$d_{\text{app}} = \frac{h_1}{\mu_1} + \frac{h_2}{\mu_2}$$

Given: - $h_1 = h_2 = 6\,\text{cm}$ - $\mu_1 = \frac{8}{5}$ - $\mu_2 = \frac{3}{2}$

Step 1: Calculate the Apparent Depth for the First Layer

The apparent depth for the first layer:

$$\frac{h_1}{\mu_1} = \frac{6}{\frac{8}{5}} = 6 \times \frac{5}{8} = \frac{30}{8} = 3.75\,\text{cm}$$

Step 2: Calculate the Apparent Depth for the Second Layer

The apparent depth for the second layer:

$$\frac{h_2}{\mu_2} = \frac{6}{\frac{3}{2}} = 6 \times \frac{2}{3} = 4\,\text{cm}$$

Step 3: Total Apparent Depth

Summing the apparent depths of both layers:

$$d_{\text{app}} = 3.75\,\text{cm} + 4\,\text{cm} = 7.75\,\text{cm} = \frac{31}{4}\,\text{cm}$$

Conclusion

Thus, the value of α is:

$$\alpha = 31$$

Quick Tip

For apparent depth in layered media, sum each layer's real depth divided by its refractive index.

Question 55

In a nuclear fission process, a high mass nuclide (A ≈ 236) with binding energy $7.6\,$MeV/Nucleon dissociates into middle mass nuclides (A ≈ 118), having binding energy of $8.6\,$MeV/Nucleon. The energy released in the process would be _____ MeV.

(1) 224
(2) 368
(3) 236
(4) 476

Answer: (3)

Solution

The energy released in a nuclear reaction, Q, is given by the difference in the binding energies of the products and the reactants:

$$Q = \text{BE}_{\text{Products}} - \text{BE}_{\text{Reactants}}$$

Step 1: Given Data

- Binding energy per nucleon for the product nuclei: $8.6\,\text{MeV}$ - Total number of nucleons in the products: $2 \times 118 = 236$ - Binding energy per nucleon for the reactant nucleus: $7.6\,\text{MeV}$ - Total number of nucleons in the reactant: 236

Step 2: Calculate Total Binding Energy

- Total binding energy of the products:

$$\text{BE}_{\text{Products}} = 236 \times 8.6\,\text{MeV}$$

- Total binding energy of the reactant:

$$\text{BE}_{\text{Reactant}} = 236 \times 7.6\,\text{MeV}$$

Step 3: Calculate the Energy Released

The energy released is given by:

$$Q = \text{BE}_{\text{Products}} - \text{BE}_{\text{Reactant}}$$

Substituting the values:

$$Q = 236 \times 8.6 - 236 \times 7.6$$

Factoring out 236:

$$Q = 236 \times (8.6 - 7.6)$$

$$Q = 236 \times 1 = 236\,\text{MeV}$$

Conclusion

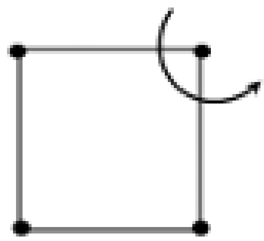
The energy released in the reaction is:

$$Q = 236\,\text{MeV}$$

> Quick Tip
>
> For fission energy, calculate binding energy differences based on nucleon numbers and binding energy per nucleon.

Question 56

Four particles each of mass 1 kg are placed at four corners of a square of side 2 m. Moment of inertia of system about an axis perpendicular to its plane and passing through one of its vertex is _____ kg m^2.



Answer: 16

Solution

Consider a square of side length $2\,\text{m}$ with four point masses placed at the vertices. We aim to find the moment of inertia I about an axis passing through the center of the square and perpendicular to its plane.

Step 1: Moment of Inertia Formula

The moment of inertia for a point mass m located at a distance r from the axis of rotation is given by:

$$I = mr^2$$

Step 2: Distance Calculation

- For a square of side length $2\,\text{m}$, the distance from each vertex to the center is the half-diagonal of the square:

$$r = \frac{\sqrt{2} \times \text{side}}{2} = \frac{\sqrt{2} \times 2}{2} = \sqrt{2}\,\text{m}$$

Step 3: Moment of Inertia for Each Mass

- The moment of inertia for each mass $m = 1\,\text{kg}$ is:

$$I_{\text{each}} = m \times r^2 = 1 \times (\sqrt{2})^2 = 1 \times 2 = 2\,\text{kg m}^2$$

Step 4: Total Moment of Inertia

- Since there are four masses located symmetrically about the center, the total moment of inertia is:

$$I_{\text{total}} = 4 \times I_{\text{each}} = 4 \times 2 = 8\,\text{kg m}^2$$

Conclusion

The total moment of inertia of the system is:

$$I = 8\,\text{kg m}^2$$

Quick Tip

For moment of inertia in symmetrical arrangements, consider each mass's distance squared from the axis.

Question 57

A particle executes simple harmonic motion with an amplitude of 4 cm. At the mean position, velocity of the particle is $10\,\text{cm/s}$. The distance of the particle from the mean position when its speed becomes $5\,\text{cm/s}$ is $\sqrt{\alpha}$ cm, where $\alpha =$ _____.

(1) 3
(2) 4
(3) 12
(4) 8

Answer: (3)

Solution

Given:

$$v_{\text{max}} = A\omega = 10\,\text{m/s}$$

where $A = 4\,\text{m}$.

Step 1: Calculate Angular Frequency

The maximum velocity is given by:

$$v_{\text{max}} = A\omega \implies \omega = \frac{v_{\text{max}}}{A} = \frac{10}{4} = \frac{5}{2}\,\text{rad/s}$$

Step 2: Velocity at Displacement x

The velocity of the particle at any position x is given by:

$$v = \omega\sqrt{A^2 - x^2}$$

Given $v = 5\,\text{m/s}$, we substitute the values:

$$5 = \frac{5}{2}\sqrt{4^2 - x^2}$$

Rearranging:

$$\sqrt{4^2 - x^2} = 2$$

Step 3: Solve for x^2

Squaring both sides:

$$4^2 - x^2 = 2^2$$
$$16 - x^2 = 4$$
$$x^2 = 16 - 4 = 12$$

Conclusion

The value of x is:

$$x = \sqrt{12}$$

Thus, the value of α is:

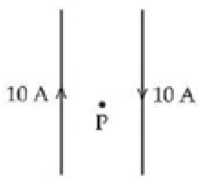
$$\alpha = 12$$

Quick Tip

For SHM problems, use $v = \omega\sqrt{A^2 - x^2}$ to find position from mean when speed is given.

Question 58

Two long, straight wires carry equal currents in opposite directions as shown in the figure. The separation between the wires is 5.0 cm. The magnitude of the magnetic field at a point P midway between the wires is ____ μT.



(Given: $\mu_0 = 4\pi \times 10^{-7}$ TmA^{-1})

(1) 80
(2) 120
(3) 160
(4) 200

Answer: (3)

Solution

Given: - Two parallel wires carrying currents $I = 10\,\text{A}$ in opposite directions - Distance between wires is $2a = 2.5 \times 10^{-2}\,\text{m}$

Step 1: Magnetic Field due to a Single Wire

The magnetic field at a distance a from a long straight wire carrying current I is given by:

$$B_{\text{single}} = \frac{\mu_0 I}{2\pi a}$$

where $\mu_0 = 4\pi \times 10^{-7}\,\text{T m/A}$.

Step 2: Magnetic Field at Midpoint

Since the currents are in opposite directions, the magnetic fields due to each wire at the midpoint between them will add up. Therefore, the total magnetic field B is:

$$B = 2 \times B_{\text{single}} = 2 \times \frac{\mu_0 I}{2\pi a} = \frac{\mu_0 I}{\pi a}$$

Step 3: Substitute Given Values

Substitute $I = 10\,\text{A}$ and $a = \frac{2.5\times10^{-2}}{2} = 1.25 \times 10^{-2}\,\text{m}$:

$$B = \frac{4\pi \times 10^{-7} \times 10}{\pi \times 1.25 \times 10^{-2}}$$

Simplifying:

$$B = \frac{4 \times 10^{-6}}{1.25 \times 10^{-2}} = 16 \times 10^{-5}\,\text{T}$$

Convert to microtesla (μT):

$$B = 160\,\mu\text{T}$$

Conclusion

The magnetic field at the midpoint is:

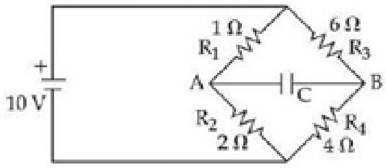
$$B = 160\,\mu\text{T}$$

Quick Tip

For two parallel currents in opposite directions, the magnetic fields add up at the midpoint.

Question 59

The charge accumulated on the capacitor connected in the following circuit is ____ μC



(Given $C = 150\,\mu$F)

(1) 200
(2) 400
(3) 600
(4) 800

Answer: (2)

Solution

Given: - The circuit involves a capacitor with capacitance $C = 150\,\mu$F - Applying Kirchhoff's Voltage Law (KVL):

Step 1: Applying KVL

Using KVL in the loop, we have:

$$V_A + \frac{10}{3} \times 1 - 6 \times 1 = V_B$$

Simplifying:

$$V_A - V_B = 6 - \frac{10}{3} = \frac{18}{3} - \frac{10}{3} = \frac{8}{3}\,\text{V}$$

Step 2: Calculating the Charge on the Capacitor

The charge Q on the capacitor is given by:

$$Q = C \times (V_A - V_B)$$

Substitute the given values:

$$Q = 150 \times 10^{-6}\,\text{F} \times \frac{8}{3}\,\text{V}$$

Simplifying:

$$Q = \frac{150 \times 8}{3} \times 10^{-6}\,\text{C} = 400 \times 10^{-6}\,\text{C} = 400\,\mu\text{C}$$

The charge on the capacitor is:

$$Q = 400\,\mu\text{C}$$

> Quick Tip
>
> In capacitive circuits, use $Q = CV$ to find charge across the capacitor.

Question 60

If average depth of an ocean is 4000 m and the bulk modulus of water is $2 \times 10^9\,\mathbf{Nm^{-2}}$, then fractional compression $\frac{\Delta V}{V}$ of water at the bottom of ocean is $\alpha \times 10^{-2}$. The value of α is ____.
(Given, $g = 10\,\mathbf{m/s^2}$, $\rho = 1000\,\mathbf{kg/m^3}$)

(1) 1
(2) 2
(3) 3
(4) 4

Answer: (2)

Solution

The fractional compression $\frac{\Delta V}{V}$ of a substance is given by the equation:

$$\frac{\Delta V}{V} = -\frac{\Delta P}{B}$$

where: - ΔP is the pressure applied - B is the bulk modulus of the material

Step 1: Calculating the Pressure at the Bottom of the Ocean

The pressure due to the water column at depth h is given by:

$$\Delta P = \rho g h$$

where: - $\rho = 1000\,\text{kg/m}^3$ (density of water) - $g = 10\,\text{m/s}^2$ (acceleration due to gravity) - $h = 4000\,\text{m}$ (depth of the ocean)
Substituting the values:

$$\Delta P = 1000 \times 10 \times 4000 = 4 \times 10^7\,\text{Pa}$$

Step 2: Calculating the Fractional Compression

Given the bulk modulus $B = 2 \times 10^9\,\text{Pa}$:

$$\frac{\Delta V}{V} = -\frac{\Delta P}{B} = -\frac{4 \times 10^7}{2 \times 10^9}$$

Simplifying:

$$\frac{\Delta V}{V} = -2 \times 10^{-2}$$

The fractional compression of the substance is:

$$\frac{\Delta V}{V} = -2 \times 10^{-2}$$

Therefore, $\alpha = 2$.

Quick Tip

For fractional compression in fluids, use the relation $\frac{\Delta V}{V} = -\frac{\Delta P}{B}$.

Question 61

Two nucleotides are joined together by a linkage known as:

(1) Phosphodiester linkage
(2) Glycosidic linkage
(3) Disulphide linkage
(4) Peptide linkage

Answer: (1)

Solution:

Explanation of Phosphodiester Linkage in Nucleic Acids

In nucleic acid chains, such as DNA and RNA, the bond connecting two nucleotides is known as a **phosphodiester linkage**. This linkage forms when a phosphate group bonds to the 3' carbon of the sugar molecule (deoxyribose in DNA or ribose in RNA) of one nucleotide and to the 5' carbon of the sugar of the adjacent nucleotide.

Formation of Phosphodiester Bond

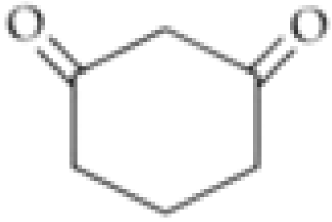
The formation involves two ester bonds:
- The phosphate group forms an ester bond with the 3' hydroxyl group of one sugar.
- The same phosphate group forms another ester bond with the 5' hydroxyl group of the next sugar in the chain.
This phosphodiester linkage results in a sugar-phosphate backbone, providing structural stability to the nucleic acid strand while allowing the nitrogenous bases to pair internally.

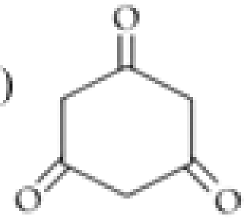
Quick Tip

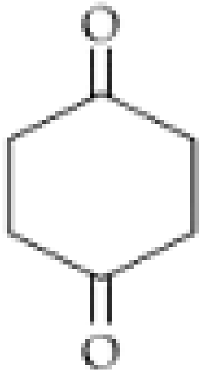
Phosphodiester linkages are essential in the backbone structure of DNA and RNA molecules.

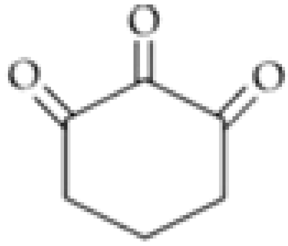
Question 62

Highest enol content will be shown by:

(1) 

(2) 

(3) 

(4) 

Answer: (2)
Solution:

Reason for Highest Enol Content in Structure 2

Structure 2 exhibits the highest enol content due to its ability to form a **stable, conjugated keto-enol tautomeric form**. The enol form is stabilized through resonance, particularly in the presence of aromatic or conjugated systems, which lowers the energy of the enol state and makes it more thermodynamically favorable.

Stabilization through Resonance

In this case: - The enol tautomer of Structure 2 is stabilized by delocalization of electrons across the double bonds and conjugated aromatic ring system, enhancing the stability. - The formation of intramolecular hydrogen bonding further stabilizes the enol form, leading to an increase in its proportion relative to the keto form.
Thus, this resonance stabilization and conjugation are key factors that make Structure 2 exhibit the highest enol content compared to other potential structures.

Quick Tip
Aromatic compounds with conjugated systems tend to exhibit higher enol content due to resonance stabilization.

Question 63

Element not showing variable oxidation state is:

(1) Bromine
(2) Iodine
(3) Chlorine
(4) Fluorine

Answer: (4)
Solution:

Explanation: Limited Oxidation State of Fluorine

Fluorine does not exhibit variable oxidation states due to its position as the **most electronegative element** in the periodic table. Its high electronegativity means it has a strong tendency to gain electrons and always prefers to adopt an oxidation state of -1.

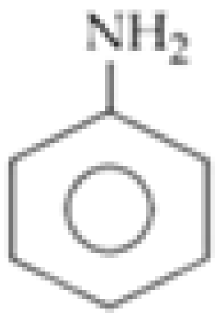
Reasoning Behind Consistent Oxidation State

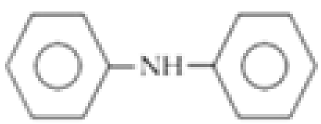
- Fluorine has a small atomic size and a high effective nuclear charge, resulting in a strong attraction for electrons. - Due to its inability to donate or share electrons in a manner that would increase its oxidation state, fluorine consistently remains in the -1 state in compounds. - Unlike other halogens, fluorine's inability to expand its valence shell further limits its oxidation states.
Thus, fluorine's consistent behavior stems from its strong electron affinity and inability to expand its octet.

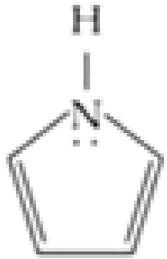
Quick Tip
Highly electronegative elements like fluorine typically exhibit only one oxidation state due to their strong tendency to attract electrons.

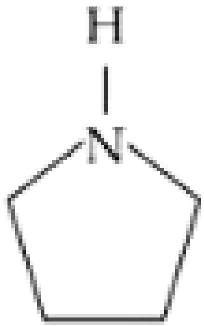
Question 64

Which of the following is strongest Bronsted base?

(1) 

(2) 

(3) 

(4) 

Answer: (4)
Solution:

Determination of the Strongest Bronsted Base

To identify the strongest Bronsted base among the structures, we must evaluate each one's ability to accept a proton (H^+). A strong Bronsted base readily donates an electron pair to bond with a proton. We analyze each structure:

Analysis of Structures

1. Structure 1: Aromatic Amine (Aniline Derivative) The nitrogen atom is attached to a benzene ring, and its lone pair of electrons participates in resonance with the aromatic system. This delocalization reduces the availability of the lone pair for protonation, weakening its basicity.
2. Structure 2: Secondary Aromatic Amine This structure features a nitrogen bonded to two benzene rings, increasing the extent of resonance delocalization of the nitrogen's lone pair. The greater resonance stabilization further decreases the basicity, making it a weaker base compared to Structure 1.
3. Structure 3: Primary Amine Attached to an Alkyl Group The nitrogen atom in this structure is bonded to an alkyl group, with a localized lone pair. While it does not participate in resonance, the overall basicity is influenced by the nature of the alkyl group. However, it remains less basic compared to an aliphatic amine without additional resonance effects.
4. Structure 4: Cyclic Amine with sp^3-Hybridized Nitrogen The nitrogen atom in this structure is sp^3-hybridized with a localized lone pair of electrons. There is no resonance or conjugation, making the lone pair highly available for bonding with a proton. The localized electron density and lack of delocalization make this the strongest Bronsted base among the given options.

Conclusion

Structure 4 is the strongest Bronsted base due to its localized, readily available lone pair on the sp^3-hybridized nitrogen atom, which is free from resonance or delocalization effects.

Quick Tip

Bronsted bases with localized lone pairs on sp^3-hybridized nitrogen atoms tend to be stronger due to increased electron availability.

Question 65

Which of the following electronic configuration would be associated with the highest magnetic moment?

(1) $[\mathrm{Ar}]\,3d^7$
(2) $[\mathrm{Ar}]\,3d^8$
(3) $[\mathrm{Ar}]\,3d^3$
(4) $[\mathrm{Ar}]\,3d^6$

Answer: (4)
Solution:

Determining the Magnetic Moment

The magnetic moment (μ) for an ion is given by the formula:

$$\mu = \sqrt{n(n+2)}\,\mathrm{BM}$$

where n is the number of unpaired electrons. We will evaluate each electronic configuration to determine which has the highest magnetic moment.

Analysis of Configurations

1. **Configuration:** $[\mathrm{Ar}]\,3d^7$ - Number of unpaired electrons: 3 - Magnetic moment:
$$\mu = \sqrt{3(3+2)} = \sqrt{15}\,\mathrm{BM}$$
2. **Configuration:** $[\mathrm{Ar}]\,3d^8$ - Number of unpaired electrons: 2 - Magnetic moment:
$$\mu = \sqrt{2(2+2)} = \sqrt{8}\,\mathrm{BM}$$
3. **Configuration:** $[\mathrm{Ar}]\,3d^3$ - Number of unpaired electrons: 3 - Magnetic moment:
$$\mu = \sqrt{3(3+2)} = \sqrt{15}\,\mathrm{BM}$$
4. **Configuration:** $[\mathrm{Ar}]\,3d^6$ - Number of unpaired electrons: 4 - Magnetic moment:
$$\mu = \sqrt{4(4+2)} = \sqrt{24}\,\mathrm{BM}$$

Conclusion

Among the given configurations, [Ar] $3d^6$ has the highest number of unpaired electrons (4) and therefore exhibits the largest spin-only magnetic moment:

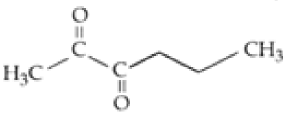
$$\mu = \sqrt{24}\,\text{BM}$$

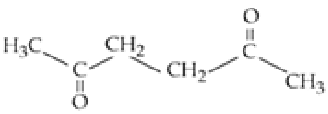
This makes it the configuration with the highest magnetic moment.

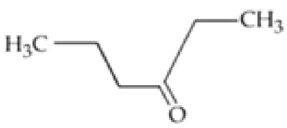
> **Quick Tip**
>
> Higher numbers of unpaired electrons result in a greater magnetic moment, calculated using $\mu = \sqrt{n(n+2)}$.

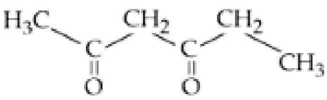
Question 66

Which of the following has highly acidic hydrogen?

(1) 

(2) 

(3) 

(4) 

Answer: (4)
Solution:

Explanation of Acidity in Structure (4)

In structure (4), the methylene group (CH_2) positioned between two carbonyl groups (C=O) exhibits enhanced acidity. This is due to the significant resonance stabilization of the conjugate base formed after the methylene group's hydrogen is removed.

Mechanism of Resonance Stabilization

When the hydrogen atom from the CH_2 group is deprotonated:

- The resulting negative charge on the carbon atom is delocalized.
- This delocalization occurs between the two carbonyl groups, allowing the negative charge to be shared across both oxygen atoms through resonance structures.
- The resonance stabilization effectively lowers the energy of the conjugate base, making the release of the proton more favorable.

Conclusion

This increased resonance stabilization of the conjugate base makes the hydrogen atom in the CH_2 group between the carbonyl groups highly acidic. As a result, this structure demonstrates a higher acidity due to the resonance effect provided by the adjacent carbonyl groups.

> Quick Tip
>
> Hydrogens between electron-withdrawing groups like carbonyls are more acidic due to resonance stabilization of the conjugate base.

Question 67

A solution of two miscible liquids showing negative deviation from Raoult's law will have:

(1) Increased vapour pressure, increased boiling point
(2) Increased vapour pressure, decreased boiling point
(3) Decreased vapour pressure, decreased boiling point
(4) Decreased vapour pressure, increased boiling point

Answer: (4)
Solution:

Explanation of Negative Deviation from Raoult's Law

When a solution exhibits a negative deviation from Raoult's law, its total vapor pressure is lower than what is predicted by the law. This phenomenon occurs because the intermolecular forces between the different components of the solution are stronger than the interactions present in the pure components.

Vapor Pressure and Boiling Point Relation

- In a solution with negative deviation:

$$P_{\mathrm{T}} < P_{\mathrm{A}}^{0} X_{\mathrm{A}} + P_{\mathrm{B}}^{0} X_{\mathrm{B}}$$

Here:

- P_{T} is the total vapor pressure of the solution.
- P_{A}^{0} and P_{B}^{0} are the vapor pressures of the pure components A and B.
- X_{A} and X_{B} are the mole fractions of components A and B.

- The stronger interactions between the mixed components cause a decrease in vapor pressure, resulting in fewer molecules escaping into the vapor phase. - Consequently, the solution has a higher boiling point compared to what would be expected for an ideal solution, as more energy is required to overcome the stronger intermolecular forces.

Conclusion

Negative deviation from Raoult's law reflects stronger interactions between solute and solvent molecules, leading to a decrease in vapor pressure and an increase in the boiling point of the solution compared to ideal behavior.

Quick Tip

Solutions with stronger intermolecular interactions than the pure components show negative deviation from Raoult's law.

Question 68

Consider the following complex ions:
P = $[FeF_6]^{3-}$
Q = $[V(H_2O)_6]^{2+}$
R = $[Fe(H_2O)_6]^{2+}$
The correct order of the complex ions, according to their spin-only magnetic moment values (in B.M.), is:

(1) $R < Q < P$
(2) $R < P < Q$
(3) $Q < R < P$
(4) $Q < P < R$

Answer: (3)
Solution:

Analysis of Magnetic Moments

1. Complex: $[\mathbf{FeF_6}]^{3-}$ - The central ion is Fe^{3+} with the electronic configuration [Ar] $3d^5$. - Fluoride (F^-) is a weak field ligand, meaning it does not cause pairing of d-electrons. - Thus, all five d-electrons remain unpaired.

$$\mu = \sqrt{n(n+2)} = \sqrt{5(5+2)} = \sqrt{35}\,\text{BM}$$

2. Complex: $[\mathbf{V(H_2O)_6}]^{2+}$ - The central ion is V^{2+} with the electronic configuration [Ar] $3d^3$. - Water is a weak field ligand, so there is no pairing of d-electrons. - The number of unpaired electrons is three.

$$\mu = \sqrt{n(n+2)} = \sqrt{3(3+2)} = \sqrt{15}\,\text{BM}$$

3. Complex: $[\mathbf{Fe(H_2O)_6}]^{2+}$ - The central ion is Fe^{2+} with the electronic configuration [Ar] $3d^6$. - Water, being a weak field ligand, does not cause pairing of d-electrons. - There are four unpaired electrons.

$$\mu = \sqrt{n(n+2)} = \sqrt{4(4+2)} = \sqrt{24}\,\text{BM}$$

Conclusion

- The magnetic moments of the complexes are in the order:

$$[V(H_2O)_6]^{2+} \text{ (Q)} < [Fe(H_2O)_6]^{2+} \text{ (R)} < [FeF_6]^{3-} \text{ (P)}$$

- Therefore, the correct order is $Q < R < P$.

Quick Tip

For calculating spin-only magnetic moments, use $\mu = \sqrt{n(n+2)}$ where n is the number of unpaired electrons.

Question 69

Choose the polar molecule from the following:

(1) CCl_4
(2) CO_2
(3) $CH_2{=}CH_2$
(4) $CHCl_3$

Answer: (4)
Solution:

Analysis of Molecular Polarity

Among the given compounds, we need to determine which one exhibits a net dipole moment, indicating polarity:
1. Compound: $CHCl_3$ (Chloroform) - Chloroform has a tetrahedral geometry with carbon at the center, bonded to one hydrogen and three chlorine atoms. - The large difference in electronegativity between chlorine and carbon/hydrogen atoms creates significant bond dipoles. - Due to the asymmetrical arrangement of these dipoles, they do not cancel out, resulting in a net dipole moment. Therefore, $CHCl_3$ is polar.
2. Compound: CCl_4 (Carbon Tetrachloride) - CCl_4 also has a tetrahedral geometry with four identical C-Cl bonds. - The symmetry of the molecule causes the bond dipoles to cancel each other, resulting in no net dipole moment. Thus, CCl_4 is non-polar.
3. Compound: CO_2 (Carbon Dioxide) - CO_2 is a linear molecule with two C=O bonds. - The dipoles of these bonds are equal and opposite, resulting in a complete cancellation of dipole moments. Therefore, CO_2 is non-polar.
4. Compound: $CH_2{=}CH_2$ (Ethene) - Ethene is a planar molecule with symmetrical double bonds between carbon atoms. - The symmetry in the molecule causes any potential dipoles to cancel out. Thus, $CH_2{=}CH_2$ is non-polar.
Among the given options, **$CHCl_3$** (chloroform) is the only polar molecule due to its asymmetrical structure and resulting net dipole moment.

💡 Quick Tip

A molecule is polar if it has an asymmetrical shape that results in a net dipole moment.

Question 70

Given below are two statements:
Statement I: The 4f and 5f - series of elements are placed separately in the Periodic table to preserve the principle of classification.
Statement II: s-block elements can be found in pure form in nature.
In the light of the above statements, choose the most appropriate answer from the options given below:

(1) Statement I is false but Statement II is true
(2) Both Statement I and Statement II are true
(3) Statement I is true but Statement II is false
(4) Both Statement I and Statement II are false

Answer: (3)
Solution:

Analysis of Statements

Statement I: The 4f and 5f series, corresponding to lanthanides and actinides respectively, are placed separately in the Periodic Table. This arrangement is necessary to maintain a compact and organized structure of the table, avoiding excessive length that would occur if they were incorporated directly within the main body. This is a correct statement.
Statement II: S-block elements, which include alkali and alkaline earth metals, are highly reactive due to their tendency to lose electrons and form positive ions readily. As a result, they are rarely found in their free (elemental) state in nature and are typically present in combined forms such as minerals, salts, or compounds. Therefore, this statement is incorrect.
-Statement I is correct because it describes the correct positioning of the 4f and 5f series in the Periodic Table. Statement II is incorrect because s-block elements are generally found in combined states due to their high reactivity.

💡 Quick Tip

Remember that s-block elements are highly reactive and rarely found in pure form.

Question 71

Given below are two statements:
Statement I: p-nitrophenol is more acidic than m-nitrophenol and o-nitrophenol.
Statement II: Ethanol will give immediate turbidity with Lucas reagent.
In the light of the above statements, choose the correct answer from the options given below:

(1) Statement I is true but Statement II is false
(2) Both Statement I and Statement II are true
(3) Both Statement I and Statement II are false
(4) Statement I is false but Statement II is true

Answer: (1)
Solution:

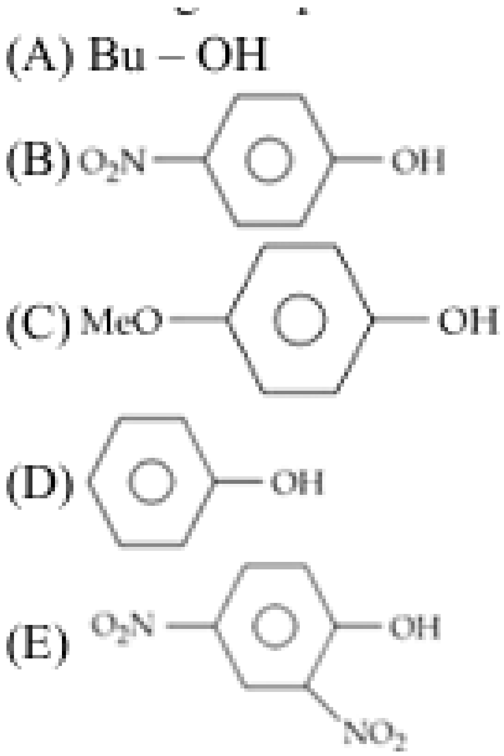
Analysis of Statements

Statement I: p-Nitrophenol is indeed more acidic compared to m- and o-nitrophenol. The increased acidity arises due to resonance stabilization of the phenoxide ion formed after deprotonation. In the para position, the nitro group's strong electron-withdrawing effect is maximized through resonance and inductive effects, stabilizing the negative charge on oxygen more effectively than in the meta or ortho positions. This makes p-nitrophenol more acidic than its isomers. Therefore, this statement is correct.
Statement II: Ethanol is a primary alcohol and reacts slowly with Lucas reagent (a mixture of concentrated hydrochloric acid and zinc chloride) because it forms a carbocation via a slow mechanism. Lucas reagent tests for alcohols by forming turbidity due to the formation of an alkyl halide, but this reaction occurs quickly only with tertiary alcohols (due to the stability of tertiary carbocations). Since ethanol is a primary alcohol, it reacts slowly, confirming that this statement is incorrect.
Statement I is correct, as p-nitrophenol's increased acidity is due to effective resonance and inductive effects. Statement II is incorrect because ethanol, as a primary alcohol, reacts slowly with Lucas reagent, unlike tertiary alcohols that show immediate turbidity.

> Quick Tip
>
> Nitrophenols with para-substitution show higher acidity due to increased resonance stability.

Question 72

The ascending order of acidity of –OH group in the following compounds is:



Choose the correct answer from the options given below:

(1) (A) < (D) < (C) < (B) < (E)
(2) (C) < (A) < (D) < (B) < (E)
(3) (C) < (D) < (B) < (A) < (E)
(4) (A) < (C) < (D) < (B) < (E)

Answer: (4)
Solution:

Analysis of Acidity Order for Given Compounds

The acidity of phenols is influenced by substituents on the aromatic ring. Electron-withdrawing groups (EWGs) increase acidity by stabilizing the phenoxide ion, while electron-donating groups (EDGs) decrease acidity by destabilizing the phenoxide ion.

- **(A)**: Contains a butyl group (an electron-donating group), which increases electron density on the oxygen atom, making it the **least acidic**.
- **(C)**: Contains a methoxy group (CH_3O-) on the ring, which is also an electron-donating group, but to a lesser extent compared to the butyl group, making it slightly more acidic than (A).
- **(D)**: Represents simple phenol (C_6H_5OH) without additional electron-donating or electron-withdrawing groups, resulting in **moderate acidity**.
- **(B)**: Contains an -OH group with a chlorine substituent on the aromatic ring. Chlorine is an electron-withdrawing group due to its inductive effect, which increases the acidity compared to simple phenol.

- **(E)**: Contains an -OH group with a nitro substituent (NO_2) on the aromatic ring. The nitro group is strongly electron-withdrawing due to both inductive and resonance effects, making it **the most acidic** among the given compounds.

The correct order of acidity is:

$$(A) < (C) < (D) < (B) < (E)$$

> Quick Tip
>
> Electron-withdrawing groups increase acidity by stabilizing the conjugate base, while electron-donating groups decrease acidity.

Question 73

Given below are two statements: one is labelled as Assertion (A) and the other is labelled as Reason (R).
Assertion (A): Melting point of Boron (2453 K) is unusually high in group 13 elements.
Reason (R): Solid Boron has very strong crystalline lattice.
In the light of the above statements, choose the most appropriate answer from the options given below:

(1) Both (A) and (R) are correct but (R) is not the correct explanation of (A)
(2) Both (A) and (R) are correct and (R) is the correct explanation of (A)
(3) (A) is true but (R) is false
(4) (A) is false but (R) is true

Answer: (2)
Solution:

Explanation of Assertion and Reasoning

Assertion (A): Boron has a very high melting point compared to other elements in group 13.
Reason (R): Boron exhibits a strong crystalline structure due to covalent bonding.

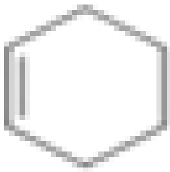
- **Analysis of Assertion (A):** Boron indeed has the highest melting point among group 13 elements. This is a result of its strong and complex covalent bonding network, which requires a significant amount of energy to break.

- **Analysis of Reason (R):** The strong crystalline structure of boron arises from its unique covalent bonding and three-dimensional lattice arrangement. This structural feature leads to an unusually high melting point as compared to other elements in the same group, such as aluminum and gallium, which have metallic structures and lower melting points.

Conclusion: Both the Assertion (A) and the Reason (R) are correct, and the Reason (R) appropriately explains the Assertion (A). The strong covalent bonding and the unique lattice structure of boron justify its high melting point within group 13.

> Quick Tip
>
> The melting point of an element is influenced by the strength of its lattice structure; stronger lattices generally result in higher melting points.

Question 74

Cyclohexene  **is _______ type of an organic compound.**

(1) Benzenoid aromatic
(2) Benzenoid non-aromatic
(3) Acyclic
(4) Alicyclic

Answer: (4)
Solution:

Cyclohexene as an Alicyclic Compound

Cyclohexene is classified as an alicyclic compound due to the following characteristics:

- **Closed Ring Structure:** Cyclohexene consists of six carbon atoms forming a closed ring structure with one double bond.
- **Non-Aromatic Nature:** Unlike aromatic compounds, cyclohexene does not satisfy the conditions for aromaticity (Hückel's rule) and lacks the delocalized π-electron cloud found in aromatic rings.
- **Classification as Alicyclic:** Alicyclic compounds are defined as cyclic compounds that exhibit properties similar to aliphatic compounds but do not contain a benzene ring or show aromaticity. Cyclohexene fits this description perfectly due to its cyclic nature and non-aromatic properties.

> Quick Tip
>
> Alicyclic compounds are cyclic compounds that do not have the delocalized - electrons typical of aromatic compounds.

Question 75

Yellow compound of lead chromate gets dissolved on treatment with hot NaOH solution. The product of lead formed is a:

(1) Tetraanionic complex with coordination number six
(2) Neutral complex with coordination number four
(3) Dianionic complex with coordination number six
(4) Dianionic complex with coordination number four

Answer: (4)
Solution:

Reaction of Lead Chromate with Hot, Excess NaOH

The reaction between lead chromate ($PbCrO_4$) and hot, excess sodium hydroxide (NaOH) leads to the formation of a soluble dianionic complex and sodium chromate:

$$PbCrO_4 + NaOH \text{ (hot, excess)} \rightarrow [Pb(OH)_4]^{2-} + Na_2CrO_4$$

- **Formation of a Soluble Complex:** The lead chromate reacts with hot, excess sodium hydroxide to produce a soluble lead complex, $[Pb(OH)_4]^{2-}$, along with sodium chromate (Na_2CrO_4).
- **Coordination Number of Four:** In the $[Pb(OH)_4]^{2-}$ complex, lead is surrounded by four hydroxide (OH^-) ions, giving it a coordination number of four.
- **Product Stability:** The $[Pb(OH)_4]^{2-}$ complex is stable in solution due to its ability to form strong bonds with the hydroxide ions.

> Quick Tip
>
> Lead hydroxide complexes often exhibit coordination numbers of four due to the geometry and electron configuration of lead.

Question 76

Given below are two statements:
Statement I: Aqueous solution of ammonium carbonate is basic.
Statement II: Acidic/basic nature of salt solution of a salt of weak acid and weak base depends on K_a and K_b values of acid and the base forming it.
In the light of the above statements, choose the most appropriate answer from the options given below:

(1) Both Statement I and Statement II are correct
(2) Statement I is correct but Statement II is incorrect
(3) Both Statement I and Statement II are incorrect
(4) Statement I is incorrect but Statement II is correct

Answer: (1)
Solution:

Analysis of Statements:

- **Statement I:** Ammonium carbonate, $(NH_4)_2CO_3$, is indeed basic in solution. This is because it dissociates into ammonium ions (NH_4^+) and carbonate ions (CO_3^{2-}) in water. The carbonate ion is a conjugate base of carbonic acid and can accept protons, leading to the formation of bicarbonate (HCO_3^-) and increasing the pH of the solution. As a result, the solution becomes basic.
- **Statement II:** This statement is also correct. The pH of a salt solution formed from a weak acid and a weak base depends on the relative strengths of the conjugate acid (K_a) and the conjugate base (K_b). In the case of ammonium carbonate, the basicity is primarily influenced by the carbonate ion's ability to accept protons, which often dominates over the weak acidity of the ammonium ion.

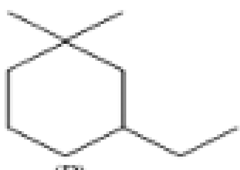
Conclusion:

Both Statement I and Statement II are correct, and Statement II provides a valid explanation of the behavior described in Statement I.

> Quick Tip
>
> For salts derived from weak acids and weak bases, the solution's pH depends on the K_a and K_b values of the acid and base.

Question 77

IUPAC name of the following compound (P) is:


(P)

(1) 1-Ethyl-5, 5-dimethylcyclohexane
(2) 3-Ethyl-1,1-dimethylcyclohexane
(3) 1-Ethyl-3, 3-dimethylcyclohexane
(4) 1,1-Dimethyl-3-ethylcyclohexane

Answer: (2)
Solution:

Correct IUPAC Name Explanation:

The correct IUPAC name for the compound is **3-Ethyl-1,1-dimethylcyclohexane**.
The steps to determine the name are as follows:
The base structure is a cyclohexane ring.
There are substituents on the ring: two methyl groups on carbon 1 and an ethyl group on carbon 3.
Numbering the ring starts from one of the carbons bearing the methyl groups to give the substituents the lowest possible locants, resulting in positions 1 and 3.
The substituents are listed in alphabetical order in the name: "ethyl" comes before "methyl."
The correct name is **3-Ethyl-1,1-dimethylcyclohexane**.

Quick Tip

When naming cycloalkanes, assign the lowest possible locants to substituents based on alphabetical order.

Question 78

NaCl reacts with conc. H_2SO_4 and $K_2Cr_2O_7$ to give reddish fumes (B), which react with NaOH to give yellow solution (C). (B) and (C) respectively are:

(1) CrO_2Cl_2, Na_2CrO_4
(2) Na_2CrO_4, CrO_2Cl_2
(3) CrO_2Cl_2, $KHSO_4$
(4) CrO_2Cl_2, $Na_2Cr_2O_7$

Answer: (1)
Solution:

Reaction Explanation:

The reactions involved are as follows:
1. The reaction of sodium chloride (NaCl) with concentrated sulfuric acid (H_2SO_4) and potassium dichromate ($K_2Cr_2O_7$) produces chromium oxychloride (CrO_2Cl_2) along with potassium hydrogen sulfate ($KHSO_4$), sodium hydrogen sulfate ($NaHSO_4$), and water:

$$\mathrm{NaCl} + \text{conc. } \mathrm{H_2SO_4} + \mathrm{K_2Cr_2O_7} \rightarrow \mathrm{CrO_2Cl_2} + \mathrm{KHSO_4} + \mathrm{NaHSO_4} + \mathrm{H_2O}$$

Here, (B) is CrO_2Cl_2, which appears as reddish fumes.
2. When CrO_2Cl_2 reacts with sodium hydroxide (NaOH), it produces a yellow solution of sodium chromate (Na_2CrO_4):

$$\mathrm{CrO_2Cl_2} + \mathrm{NaOH} \rightarrow \mathrm{Na_2CrO_4} + \mathrm{NaCl} + \mathrm{H_2O}$$

The reaction sequence demonstrates the formation of a yellow solution of sodium chromate (Na_2CrO_4) as the final product upon the reaction of chromium oxychloride with NaOH.

💡 Quick Tip

Chromyl chloride (CrO_2Cl_2) test is specific for the presence of chloride ions.

Question 79

The correct statement regarding nucleophilic substitution reaction in a chiral alkyl halide is;

(1) Retention occurs in S_N1 reaction and inversion occurs in S_N2 reaction
(2) Inversion occurs in S_N1 reaction and retention occurs in S_N2 reaction
(3) Racemisation occurs in both S_N1 and S_N2 reactions
(4) Racemisation occurs in S_N1 reaction and inversion occurs in S_N2 reaction

Answer: (4)
Solution:

S_N1 and S_N2 Reactions Explanation

S_N1 Reaction: This reaction mechanism proceeds via the formation of a carbocation intermediate. Due to the planar nature of the carbocation, nucleophilic attack can occur from either side, leading to the formation of a racemic mixture if the carbon atom is chiral. This results in the loss of stereochemistry and generates both enantiomers of the product.
S_N2 Reaction: This reaction involves a backside attack by the nucleophile, displacing the leaving group in one concerted step. If the reacting carbon is chiral, this backside attack causes an inversion of configuration, known as Walden inversion. As a result, the stereochemistry of the chiral center is inverted.
In summary: - Racemization occurs in S_N1 reactions due to the planar carbocation intermediate. - Inversion occurs in S_N2 reactions due to the backside attack mechanism.

💡 Quick Tip

S_N1 reactions generally result in racemisation, while S_N2 reactions cause inversion of configuration.

Question 80

The electronic configuration for Neodymium is:

(Atomic Number for Neodymium 60)
(1) [Xe] $4f^4\ 6s^2$
(2) [Xe] $5f^4 7s^2$
(3) [Xe] $4f^4\ 6s^2$
(4) [Xe] $4f^4 5d^1 6s^2$

Answer: (1)
Solution:

Electronic Configuration of Neodymium (Nd)

Neodymium (Nd) is an element with the atomic number 60, meaning it has 60 electrons. We need to find its correct electronic configuration using the principles of electron filling: the Aufbau principle, Hund's rule, and the Pauli exclusion principle. The order of orbital filling follows increasing energy levels based on the (n + l) rule.

2. Noble Gas Core Representation

The electronic configuration of Neodymium begins with the configuration of the nearest noble gas, Xenon ($[Xe]$), which represents the filled electron shells up to atomic number 54:

$$[Xe] = 1s^2 2s^2 2p^6 3s^2 3p^6 4s^2 3d^{10} 4p^6 5s^2 4d^{10} 5p^6$$

3. Filling the Remaining Electrons

After $[Xe]$, the remaining 6 electrons for Neodymium fill the 4f and 6s orbitals. According to the Aufbau principle:

- The 4f orbitals have a lower energy level than the 5d and 6s orbitals, so they are filled first.
- The 4f subshell can hold up to 14 electrons.

Thus, the next electrons occupy the 4f orbitals:

$$4f^4 6s^2$$

4. Final Configuration

Therefore, the electronic configuration of Neodymium is:

$$[Xe]4f^4 6s^2$$

Quick Tip

For lanthanides, electrons fill the 4f subshell before filling higher energy levels.

Question 81

The mass of silver (Molar mass of Ag: 108 g/mol) displaced by a quantity of electricity which displaces 5600 mL of O_2 at S.T.P. will be ____ g.
Answer: 107 g or 108 g
Solution:

To find the mass of silver displaced, we equate the equivalents of silver and oxygen using the given reaction:
1. **Equivalents Relation:**

$$\text{Equivalents of Ag} = \text{Equivalents of O}_2$$

Let x grams of silver be displaced. Since the equivalent mass of silver (Ag) is 108 g, the number of equivalents of silver is given by:

$$\frac{x}{108}$$

2. **Calculating Equivalents of O_2:** - Given volume of O_2 is 5.6 L. - Molar volume at STP is considered as 22.7 L. - Number of moles of O_2 is:

$$\text{Moles of O}_2 = \frac{5.6}{22.7}$$

- Since each mole of O_2 corresponds to 4 equivalents (due to O_2 accepting 4 electrons during the reaction):

$$\text{Equivalents of O}_2 = \frac{5.6}{22.7} \times 4$$

3. **Equating Equivalents:**

$$\frac{x}{108} = \frac{5.6}{22.7} \times 4$$

Solving for x:

$$x = 108 \times \frac{5.6}{22.7} \times 4$$
$$x \approx 106.57\,\text{g}$$

Thus, the approximate amount of silver displaced is 107 g.
4. **Using Alternative Molar Volume (22.4 L):** - If we use 22.4 L as the molar volume of gas at STP:

$$\frac{x}{108} = \frac{5.6}{22.4} \times 4$$

Solving for x:

$$x = 108 \times \frac{5.6}{22.4} \times 4$$
$$x = 108\,\text{g}$$

- Using a molar volume of 22.7 L, the mass of silver displaced is approximately 107 g.
- Using a molar volume of 22.4 L, the mass of silver displaced is 108 g.

Quick Tip

For reactions involving gases, use the molar volume at STP (22.4 L or 22.7 L) to calculate moles of gas.

Question 82

Consider the following data for the given reaction

$$2HI_{(g)} \rightarrow H_{2(g)} + I_{2(g)}$$

HI (mol L^{-1})	0.005	0.01	0.02
Rate (mol L^{-1} s^{-1})	7.5×10^{-4}	3.0×10^{-3}	1.2×10^{-2}

The order of the reaction is ______.
Answer: (2)
Solution:
To determine the order of the reaction, we start with the given rate law:

$$\text{Rate} = k[\text{HI}]^n$$

Using two data points:
1. When $[\text{HI}] = 0.005\,\text{M}$, Rate $= 7.5 \times 10^{-4}\,\text{M/s}$. 2. When $[\text{HI}] = 0.01\,\text{M}$, Rate $= 3.0 \times 10^{-3}\,\text{M/s}$.
Taking the ratio of the rates:

$$\frac{\text{Rate}_2}{\text{Rate}_1} = \frac{3.0 \times 10^{-3}}{7.5 \times 10^{-4}} = 4$$

Similarly, the concentration ratio is:

$$\left(\frac{[\text{HI}]_2}{[\text{HI}]_1}\right)^n = \left(\frac{0.01}{0.005}\right)^n = 2^n$$

Equating the ratios:

$$4 = 2^n$$

Solving for n:

$$n = 2$$

Thus, the reaction is second order.

> **Quick Tip**
>
> For determining reaction order, compare rate changes with concentration changes in a rate law.

Question 83

Mass of methane required to produce 22 g of CO_2 after complete combustion is _____ g.
Answer: (8)

Solution:
The balanced chemical equation is given by:

$$CH_4 + 2O_2 \rightarrow CO_2 + 2H_2O$$

To find the moles of CO_2 produced:

$$\text{Moles of } CO_2 = \frac{22\,\text{g}}{44\,\text{g/mol}} = 0.5\,\text{mol}$$

Using the stoichiometric ratio from the balanced equation, 1 mole of CH_4 produces 1 mole of CO_2. Therefore, the required moles of CH_4 are:

$$\text{Required moles of } CH_4 = 0.5\,\text{mol}$$

Calculating the mass of CH_4:

$$\text{Mass of } CH_4 = 0.5\,\text{mol} \times 16\,\text{g/mol} = 8\,\text{g}$$

Quick Tip

In stoichiometry problems, always start by writing a balanced chemical equation. Use the molar mass to convert between grams and moles, and apply stoichiometric ratios based on the balanced equation.

Question 84

If three moles of an ideal gas at 300 K expand isothermally from 30 dm^3 to 45 dm^3 against a constant opposing pressure of 80 kPa, then the amount of heat transferred is _____ J.
Answer: (1200 J)
Solution:
Using the first law of thermodynamics:

$$\Delta U = Q + W$$

For an isothermal process, the change in internal energy is zero ($\Delta U = 0$), so:

$$Q = -W$$

The work done by the system is given by:

$$W = -P_{\text{ext}}\Delta V$$

Substituting the values:

$$W = -80 \times 10^3\,\text{Pa} \times (45 - 30) \times 10^{-3}\,\text{m}^3$$
$$W = -80 \times 10^3 \times 15 \times 10^{-3} = -1200\,\text{J}$$

> Quick Tip
>
> In isothermal processes for ideal gases, internal energy change (ΔU) is zero. Therefore, heat transferred (Q) is equal to the negative of the work done ($-W$). When expansion occurs against a constant external pressure, use $W = -P_{\text{ext}}\Delta V$ to calculate work.

Question 85

3-Methylhex-2-ene on reaction with HBr in presence of peroxide forms an addition product (A). The number of possible stereoisomers for 'A' is _____.
Answer: (4)
Solution:
The anti-Markovnikov addition of HBr in the presence of a peroxide catalyst leads to the formation of a product with two chiral centers. This reaction mechanism involves the free radical addition pathway, causing the hydrogen to attach to the carbon with fewer substituents.
Given that the resulting molecule has two chiral centers, we determine the number of stereoisomers using the formula:

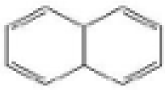
$$\text{Number of stereoisomers} = 2^{\text{number of chiral centers}} = 2^2 = 4$$

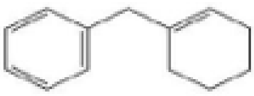
Thus, four stereoisomers are possible due to the two chiral centers formed.

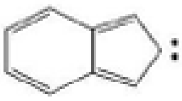
> Quick Tip
>
> The number of stereoisomers for a compound with n chiral centers is 2^n.

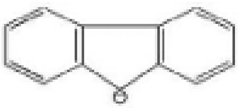
Question 86

Among the given organic compounds, the total number of aromatic compounds is

(A) 

(B) 

(C) 

(D) 

Answer: (3)
Solution:
Compounds B, C, and D are aromatic compounds. Aromaticity requires that a compound must satisfy Huckel's rule, which states that the compound must have $4n + 2$ electrons, where n is an integer. Additionally, the compound should possess a planar and conjugated structure, allowing for delocalized -electrons across the entire ring.
Compound A, however, does not meet these conditions. It either lacks the necessary -electron count, is not planar, or does not exhibit sufficient conjugation to allow for aromatic stabilization. Therefore, Compound A is not aromatic.

> Quick Tip
>
> Aromaticity requires a conjugated planar ring with $4n + 2$ electrons (Huckel's rule).

Question 87

Among the following, total number of meta-directing functional groups is (Integer based)
– OCH_3, –NO_2, –CN, –CH_3–$NHCOCH_3$, – COR, –OH, – COOH, –Cl
Answer: (4)
Solution:
The meta-directing functional groups are typically electron-withdrawing groups. Examples include:

$$–NO_2, –C \equiv N, –COR, –COOH$$

These groups withdraw electron density from the benzene ring, stabilizing the intermediate carbocation at the meta position during electrophilic aromatic substitution reactions.

> Quick Tip
>
> Electron-withdrawing groups (e.g., –NO_2, –COOH) generally direct electrophilic substitutions to the meta position.

Question 88

The number of electrons present in all the completely filled subshells having $n = 4$ and $s = \pm\frac{1}{2}$ is _____.
(Where n = principal quantum number and s = spin quantum number)
Answer: (16)
Solution:
For $n = 4$, the possible subshells and their corresponding orbitals are:

Subshell	Number of Orbitals
4s	1
4p	3
4d	5
4f	7

The total number of orbitals is $1 + 3 + 5 + 7 = 16$.
Therefore, there are 16 electrons with a specific spin orientation (either $s = +\frac{1}{2}$ or $s = -\frac{1}{2}$).

Quick Tip

For each subshell, half the electrons will have $s = +\frac{1}{2}$ and the other half will have $s = -\frac{1}{2}$.

Question 89

Sum of bond order of CO and NO$^+$ is _____.
Answer: (6)
Solution:
- **CO**: The bond order of carbon monoxide (CO) is 3, as it has a triple bond structure represented as $C \equiv O$.
- **NO$^+$**: The bond order of the nitrosonium ion (NO$^+$) is also 3, corresponding to the structure $N \equiv O^+$.
Therefore, the sum of the bond orders of CO and NO$^+$ is given by:

$$3 + 3 = 6$$

Quick Tip

Bond order indicates the number of chemical bonds between a pair of atoms; higher bond order implies a stronger bond.

Question 90

From the given list, the number of compounds with +4 oxidation state of Sulphur is _____.
SO_3, H_2SO_3, $SOCl_2$, SF_4, $BaSO_4$, $H_2S_2O_7$
Answer: (3)
Solution:

Compound	Oxidation State of Sulfur
SO_3	+6
H_2SO_3	+4
$SOCl_2$	+4
SF_4	+4
$BaSO_4$	+6
$H_2S_2O_7$	+6

The compounds with an oxidation state of +4 for sulfur are H_2SO_3, $SOCl_2$, and SF_4, giving a total of 3.

Quick Tip

To determine oxidation states, use the known oxidation states of other atoms and solve for sulfur.
