JEE Main 2022 Question Paper Jun 24 Shift 1 with Solutions

Time Allowed :3 Hour	**Maximum Marks :**300	**Total Questions :**90

General Instructions

Read the following instructions very carefully and strictly follow them:

1. The test is of 3 hours duration.
2. The question paper consists of 90 questions. The maximum marks are 300.
3. There are three parts in the question paper consisting of Mathematics, Physics and Chemistry having 30 questions in each part of equal weightage.
4. Each part (subject) has two sections.
(i) Section-A: This section contains 20 multiple choice questions which have only one correct answer. Each question carries 4 marks for correct answer and –1 mark for wrong answer.
(ii) Section-B: This section contains 10 questions. The answer to each of the questions is a numerical value. Each question carries 4 marks for correct answer and –1 mark for wrong answer. For Section-B, the answer should be rounded off to the nearest integer.

Mathematics

1. Let A $= \{zC : 1|z-(1+i)|2\}$ **and B** $= \{zA : |z-(1-i)| = 1\}$**. Then, B:**

(A) is an empty set
(B) contains exactly two elements
(C) contains exactly three elements
(D) is an infinite set

Correct Answer: (D) is an infinite set

Solution:

Step 1: Understanding the Question
We are given two sets of complex numbers, A and B.
Set A is an annulus (a ring-shaped region) in the complex plane centered at $1+i$ with inner radius 1 and outer radius 2.
Set B consists of all complex numbers z that are in set A and also lie on a circle defined by $|z-(1-i)| = 1$.
We need to determine the number of elements in set B.

Step 2: Geometric Interpretation

Let $z = x + iy$.

Set A is defined by $1 \le |(x-1) + i(y-1)| \le 2$. This represents the region between two concentric circles centered at $C_A = (1, 1)$ with radii $r_1 = 1$ and $r_2 = 2$, inclusive of the boundaries.

Set B contains points that are in A and also satisfy $|z - (1 - i)| = 1$. This equation, $|(x-1) + i(y+1)| = 1$, represents a circle centered at $C_B = (1, -1)$ with radius $r_B = 1$.

So, B is the set of points forming the intersection of the annulus A and the circle defined by $|z - (1 - i)| = 1$.

Step 3: Analyzing the Intersection

Let's analyze the position of the circle of B relative to the annulus of A.

The center of the annulus is $C_A = (1, 1)$.

The center of the circle for B is $C_B = (1, -1)$.

The distance between the centers is $d = \sqrt{(1-1)^2 + (1-(-1))^2} = \sqrt{0^2 + 2^2} = 2$.

Let's consider a point z on the circle of B, i.e., $|z - (1 - i)| = 1$. We need to find the range of values for $|z - (1 + i)|$ for such a point z.

Using the triangle inequality, $|z_1 - z_2| \le |z_1| + |z_2|$ and $||z_1| - |z_2|| \le |z_1 - z_2|$.

Let $w = z - (1 - i)$. Then $|w| = 1$. We want to find the range of $|z - (1 + i)| = |(z - (1 - i)) - (1 + i) + (1 - i)| = |w - 2i|$.

The quantity $|w - 2i|$ represents the distance from a point w on the unit circle centered at the origin to the point $2i$.

The minimum distance is $|2i| - 1 = 2 - 1 = 1$ (when $w = i$).

The maximum distance is $|2i| + 1 = 2 + 1 = 3$ (when $w = -i$).

So, for any point z on the circle $|z - (1 - i)| = 1$, the value of $|z - (1 + i)|$ lies in the interval $[1, 3]$.

The condition for a point z to be in set A is $1 \le |z - (1 + i)| \le 2$.

Since the range of $|z - (1 + i)|$ for points on the circle of B is $[1, 3]$, and the range required for A is $[1, 2]$, there is an intersection.

The intersection consists of points z on the circle $|z - (1 - i)| = 1$ for which $1 \le |z - (1 + i)| \le 2$. This is a continuous arc of the circle. An arc of a circle contains infinitely many points.

Step 4: Final Answer

The set B is the intersection of a circle and a closed annulus. This intersection is a non-empty arc of the circle. Therefore, B contains infinitely many points.

💡 Quick Tip
Visualizing complex number problems geometrically can simplify them greatly. An annulus is the region between two circles, and $\lvert z - c\rvert = r$ is a circle. Finding the intersection helps determine the nature of the resulting set.

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2. The remainder when 3^{2022} is divided by 5 is :

(A) 1
(B) 2
(C) 3
(D) 4

Correct Answer: (D) 4

Solution:

Step 1: Understanding the Question
We need to find the remainder when 3^{2022} is divided by 5. This is a problem of modular arithmetic, which can be expressed as finding the value of $3^{2022} \pmod 5$.

Step 2: Finding the Cyclicity of Powers of 3 Modulo 5
Let's compute the first few powers of 3 and find their remainders when divided by 5.

$$3^1 \equiv 3 \pmod 5$$
$$3^2 = 9 \equiv 4 \pmod 5$$
$$3^3 = 27 \equiv 2 \pmod 5$$
$$3^4 = 81 \equiv 1 \pmod 5$$
$$3^5 = 243 \equiv 3 \pmod 5$$

The remainders repeat in a cycle of length 4: (3, 4, 2, 1).

Step 3: Using the Cyclicity
Since the cycle length is 4, we need to find the remainder of the exponent 2022 when divided by 4.

$$2022 \div 4$$

We can write $2022 = 4 \times 505 + 2$.
So, the remainder is 2.
This means that 3^{2022} will have the same remainder as the second term in the cycle.

$$3^{2022} \equiv 3^{4\times 505+2} \equiv (3^4)^{505} \cdot 3^2 \pmod 5$$

Since $3^4 \equiv 1 \pmod 5$, this becomes:

$$\equiv (1)^{505} \cdot 3^2 \pmod 5$$
$$\equiv 1 \cdot 9 \pmod 5$$
$$\equiv 9 \pmod 5$$
$$\equiv 4 \pmod 5$$

Step 4: Final Answer
The remainder when 3^{2022} is divided by 5 is 4.

> 💡 Quick Tip
>
> For problems involving remainders of large powers ($a^b \pmod{m}$), always look for the cyclicity of remainders of powers of a. Once you find a power k such that $a^k \equiv 1 \pmod{m}$, the problem simplifies to finding $b \pmod{k}$. This is an application of Euler's totient theorem.

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3. The surface area of a balloon of spherical shape being inflated, increases at a constant rate. If initially, the radius of balloon is 3 units and after 5 seconds, it becomes 7 units, then its radius after 9 seconds is :

(A) 9
(B) 10
(C) 11
(D) 12

Correct Answer: (A) 9

Solution:

Step 1: Understanding the Question
We are given that the surface area of a spherical balloon increases at a constant rate. We are given the radius at two different times and asked to find the radius at a third time.

Step 2: Key Formula and Approach
Let S be the surface area and r be the radius of the spherical balloon.
The formula for the surface area of a sphere is $S = 4\pi r^2$.
The problem states that the rate of increase of surface area is constant, which means $\frac{dS}{dt} = k$, where k is a constant.
Integrating this with respect to time t, we get $S(t) = kt + C$, where C is the constant of integration. This means the surface area is a linear function of time.

Step 3: Detailed Explanation
Let's denote the radius at time t as $r(t)$ and the surface area as $S(t)$.
Given: Initial radius, $r(0) = 3$ units.
Radius after 5 seconds, $r(5) = 7$ units.

Now, let's calculate the surface area at these times.
Initial surface area, $S(0) = 4\pi(r(0))^2 = 4\pi(3^2) = 36\pi$.

Surface area after 5 seconds, $S(5) = 4\pi(r(5))^2 = 4\pi(7^2) = 196\pi$.

Since $S(t)$ is a linear function of time, we can write $S(t) = S(0) + kt$.
Using the information at $t = 5$:

$$\begin{aligned} S(5) &= S(0) + k(5) \\ 196\pi &= 36\pi + 5k \\ 160\pi &= 5k \\ k &= \frac{160\pi}{5} = 32\pi \end{aligned}$$

So, the constant rate of increase of surface area is 32π units^2/sec.
The equation for the surface area at any time t is $S(t) = 36\pi + 32\pi t$.

We need to find the radius after 9 seconds, i.e., $r(9)$. First, let's find the surface area at $t = 9$.

$$S(9) = 36\pi + 32\pi(9) = 36\pi + 288\pi = 324\pi$$

Now we can find the radius $r(9)$ from this surface area.

$$\begin{aligned} S(9) &= 4\pi(r(9))^2 \\ 324\pi &= 4\pi(r(9))^2 \\ (r(9))^2 &= \frac{324}{4} = 81 \\ r(9) &= \sqrt{81} = 9 \end{aligned}$$

(Since radius must be positive)

Step 4: Final Answer
The radius of the balloon after 9 seconds is 9 units.
Alternatively, we noticed that $r^2(t)$ is a linear function of t. $r^2(t) = at + b$. $r^2(0) = 9$, $r^2(5) = 49$. $b = 9$. $49 = 5a + 9 \implies 5a = 40 \implies a = 8$. So, $r^2(t) = 8t + 9$. At $t = 9$, $r^2(9) = 8(9) + 9 = 72 + 9 = 81$, so $r(9) = 9$.

> Quick Tip
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> In problems involving rates, always translate the given statements into mathematical equations. "Increases at a constant rate" implies the derivative is a constant. Integrating this gives a linear relationship, which is often easier to work with.

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4. Bag A contains 2 white, 1 black and 3 red balls and bag B contains 3 black, 2 red and n white balls. One bag is chosen at random and 2 balls drawn from it at random, are found to be 1 red and 1 black. If the probability that both balls come from Bag A is $\frac{6}{11}$, then n is equal to

(A) 13
(B) 6
(C) 4
(D) 3

Correct Answer: (C) 4

Solution:

Step 1: Understanding the Question
This is a conditional probability problem that can be solved using Bayes' theorem. We are given information about the outcome of an experiment and asked to find an unknown parameter based on a conditional probability.

Step 2: Defining Events and Probabilities
Let E_A be the event that Bag A is chosen.
Let E_B be the event that Bag B is chosen.
Let D be the event that 2 balls drawn are 1 red and 1 black.

Bag A contains: 2W, 1B, 3R. Total = 6 balls.
Bag B contains: nW, 3B, 2R. Total = $n + 5$ balls.

Since a bag is chosen at random, we have:

$$P(E_A) = P(E_B) = \frac{1}{2}$$

Now, we calculate the conditional probabilities of drawing 1 red and 1 black ball from each bag.
Probability of drawing 1R and 1B from Bag A:

$$P(D|E_A) = \frac{\binom{3}{1} \times \binom{1}{1}}{\binom{6}{2}} = \frac{3 \times 1}{\frac{6\times5}{2}} = \frac{3}{15} = \frac{1}{5}$$

Probability of drawing 1R and 1B from Bag B:

$$P(D|E_B) = \frac{\binom{2}{1} \times \binom{3}{1}}{\binom{n+5}{2}} = \frac{2 \times 3}{\frac{(n+5)(n+4)}{2}} = \frac{12}{(n+5)(n+4)}$$

Step 3: Applying Bayes' Theorem
We are given the probability that the balls came from Bag A, given that they were 1 red and 1 black. This is $P(E_A|D) = \frac{6}{11}$.
Bayes' theorem states:

$$P(E_A|D) = \frac{P(D|E_A)P(E_A)}{P(D|E_A)P(E_A) + P(D|E_B)P(E_B)}$$

Substitute the known values into the formula:

$$\frac{6}{11} = \frac{(\frac{1}{5})(\frac{1}{2})}{(\frac{1}{5})(\frac{1}{2}) + (\frac{12}{(n+5)(n+4)})(\frac{1}{2})}$$

We can cancel the $\frac{1}{2}$ term from the numerator and the denominator.

$$\frac{6}{11} = \frac{\frac{1}{5}}{\frac{1}{5} + \frac{12}{(n+5)(n+4)}}$$

$$\frac{6}{11} = \frac{\frac{1}{5}}{\frac{(n+5)(n+4)+60}{5(n+5)(n+4)}}$$

$$\frac{6}{11} = \frac{1}{5} \times \frac{5(n+5)(n+4)}{(n+5)(n+4)+60}$$

$$\frac{6}{11} = \frac{(n+5)(n+4)}{(n+5)(n+4)+60}$$

Let $X = (n+5)(n+4)$. The equation becomes:

$$\frac{6}{11} = \frac{X}{X+60}$$

Cross-multiply:

$$6(X+60) = 11X$$
$$6X + 360 = 11X$$
$$5X = 360$$
$$X = 72$$

Step 4: Solving for n
Now substitute back $X = (n+5)(n+4)$.

$$(n+5)(n+4) = 72$$

This is a quadratic equation: $n^2 + 9n + 20 = 72$, which simplifies to $n^2 + 9n - 52 = 0$.
We can factor this equation. We need two numbers that multiply to -52 and add to 9. These are 13 and -4.

$$(n+13)(n-4) = 0$$

The possible values for n are $n = -13$ or $n = 4$.
Since n represents the number of balls, it cannot be negative. Therefore, $n = 4$.

Quick Tip

Bayes' theorem problems can be simplified by clearly defining events and calculating the component probabilities first. When the prior probabilities (like choosing a bag) are equal, they often cancel out, simplifying the algebra.

5. Let $x^2 + y^2 + Ax + By + C = 0$ be a circle passing through (0, 6) and touching the parabola $y = x^2$ at (2, 4). Then A + C is equal to ________.

(A) 16
(B) 88/5
(C) 72
(D) -8

Correct Answer: (A) 16

Solution:

Step 1: Understanding the Question
We are given the general equation of a circle and two conditions it must satisfy. First, it passes through a specific point. Second, it is tangent to a given parabola at another specific point. We need to find the value of $A + C$.

Step 2: Using the given conditions to form equations
The equation of the circle is $x^2 + y^2 + Ax + By + C = 0$.
Condition 1: The circle passes through (0, 6).
Substituting $x = 0, y = 6$ into the circle's equation:

$$0^2 + 6^2 + A(0) + B(6) + C = 0 \implies 36 + 6B + C = 0 \quad (1)$$

Condition 2: The circle touches the parabola $y = x^2$ at (2, 4). This implies two things:
a) The point (2, 4) lies on the circle.
Substituting $x = 2, y = 4$ into the circle's equation:

$$2^2 + 4^2 + A(2) + B(4) + C = 0 \implies 4 + 16 + 2A + 4B + C = 0 \implies 2A + 4B + C = -20 \quad (2)$$

b) The circle and parabola have a common tangent at (2, 4). This means their slopes are equal at that point.
First, find the slope of the tangent to the parabola $y = x^2$.

$$\frac{dy}{dx} = 2x$$

At (2, 4), the slope is $m_p = 2(2) = 4$.
Next, find the slope of the tangent to the circle by implicit differentiation.

$$2x + 2y\frac{dy}{dx} + A + B\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(2y + B) = -(2x + A) \implies \frac{dy}{dx} = -\frac{2x + A}{2y + B}$$

At (2, 4), the slope is $m_c = -\frac{2(2)+A}{2(4)+B} = -\frac{4+A}{8+B}$.
Equating the slopes: $m_p = m_c$.

$$4 = -\frac{4+A}{8+B} \implies 4(8+B) = -4-A \implies 32+4B = -4-A \implies A+4B = -36 \quad (3)$$

Step 3: Solving the system of linear equations
We have a system of three equations with three unknowns (A, B, C):
(1) $6B + C = -36$
(2) $2A + 4B + C = -20$
(3) $A + 4B = -36$
From (3), we can express A in terms of B: $A = -36 - 4B$.
Substitute this into (2):

$$2(-36-4B) + 4B + C = -20$$
$$-72 - 8B + 4B + C = -20$$
$$-4B + C = 52 \quad (4)$$

Now we have a system of two equations for B and C:
(1) $6B + C = -36$
(4) $-4B + C = 52$
Subtract equation (4) from equation (1):

$$(6B + C) - (-4B + C) = -36 - 52$$
$$10B = -88 \implies B = -\frac{88}{10} = -\frac{44}{5}$$

Now find C using equation (4):

$$C = 52 + 4B = 52 + 4(-\frac{44}{5}) = 52 - \frac{176}{5} = \frac{260-176}{5} = \frac{84}{5}$$

Finally, find A using equation (3):

$$A = -36 - 4B = -36 - 4(-\frac{44}{5}) = -36 + \frac{176}{5} = \frac{-180+176}{5} = -\frac{4}{5}$$

Step 4: Final Answer
We need to find the value of $A + C$.

$$A + C = -\frac{4}{5} + \frac{84}{5} = \frac{80}{5} = 16$$

Quick Tip

When a curve "touches" another curve at a point, it means they share that point and have a common tangent (and thus the same slope) at that point. This provides two separate conditions to use in solving the problem.

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6. The number of values of α for which the system of equations :
$x + y + z = \alpha$

$\alpha x + 2\alpha y + 3z = -1$
$x + 3\alpha y + 5z = 4$
is inconsistent, is

(A) 0
(B) 1
(C) 2
(D) 3

Correct Answer: (B) 1

Solution:

Step 1: Understanding the Condition for Inconsistency
A system of linear equations $AX = B$ is inconsistent (has no solution) if the determinant of the coefficient matrix, D, is zero, and at least one of the determinants D_x, D_y, D_z is non-zero.

Step 2: Calculate the Determinant of the Coefficient Matrix (D)
The given system of equations is:
$x + y + z = \alpha$
$\alpha x + 2\alpha y + 3z = -1$
$x + 3\alpha y + 5z = 4$
The coefficient matrix is $\begin{pmatrix} 1 & 1 & 1 \\ \alpha & 2\alpha & 3 \\ 1 & 3\alpha & 5 \end{pmatrix}$.
The determinant D is:

$$D = \begin{vmatrix} 1 & 1 & 1 \\ \alpha & 2\alpha & 3 \\ 1 & 3\alpha & 5 \end{vmatrix}$$

$$D = 1(2\alpha \cdot 5 - 3 \cdot 3\alpha) - 1(\alpha \cdot 5 - 3 \cdot 1) + 1(\alpha \cdot 3\alpha - 2\alpha \cdot 1)$$

$$D = (10\alpha - 9\alpha) - (5\alpha - 3) + (3\alpha^2 - 2\alpha)$$

$$D = \alpha - 5\alpha + 3 + 3\alpha^2 - 2\alpha$$

$$D = 3\alpha^2 - 6\alpha + 3$$

$$D = 3(\alpha^2 - 2\alpha + 1) = 3(\alpha - 1)^2$$

Step 3: Find the Value(s) of α for which D=0
For the system to be inconsistent or have infinitely many solutions, we must have $D = 0$.

$$3(\alpha - 1)^2 = 0 \implies \alpha - 1 = 0 \implies \alpha = 1$$

So, $\alpha = 1$ is the only possible value for which the system is not unique.

Step 4: Check for Inconsistency at $\alpha = 1$
Now we must check if the system is inconsistent or has infinite solutions for $\alpha = 1$. We do this by checking D_x, D_y, or D_z. If any of these are non-zero, the system is inconsistent.
Let's substitute $\alpha = 1$ into the system:
$x + y + z = 1$
$x + 2y + 3z = -1$
$x + 3y + 5z = 4$
Let's calculate D_x:

$$D_x = \begin{vmatrix} 1 & 1 & 1 \\ -1 & 2 & 3 \\ 4 & 3 & 5 \end{vmatrix}$$

$$D_x = 1(2 \cdot 5 - 3 \cdot 3) - 1(-1 \cdot 5 - 3 \cdot 4) + 1(-1 \cdot 3 - 2 \cdot 4)$$

$$D_x = 1(10 - 9) - 1(-5 - 12) + 1(-3 - 8)$$

$$D_x = 1 - (-17) + (-11) = 1 + 17 - 11 = 7$$

Since $D = 0$ and $D_x = 7 \neq 0$ for $\alpha = 1$, the system is inconsistent.

Step 5: Final Answer
The only value of α for which the system is inconsistent is $\alpha = 1$. Therefore, the number of such values is 1.

Quick Tip

For a system of 3 linear equations in 3 variables:
- If $D \neq 0$, there is a unique solution.
- If $D = 0$ and at least one of D_x, D_y, D_z is non-zero, there is no solution (inconsistent).
- If $D = 0$ and $D_x = D_y = D_z = 0$, there are infinitely many solutions.
Always remember to check the second condition after finding $D = 0$.

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7. If the sum of the squares of the reciprocals of the roots α and β of the equation $3x^2 + \lambda x - 1 = 0$ is 15, then $6(\alpha^3 + \beta^3)^2$ is equal to :

(A) 18
(B) 24
(C) 36
(D) 96

Correct Answer: (B) 24

Solution:

Step 1: Understanding the Question and Using Vieta's Formulas
We are given a quadratic equation and a condition on its roots α and β. We need to find the value of an expression involving the roots.
For the quadratic equation $3x^2 + \lambda x - 1 = 0$:
Sum of roots: $\alpha + \beta = -\frac{\lambda}{3}$
Product of roots: $\alpha\beta = -\frac{1}{3}$

Step 2: Using the Given Condition to find λ^2
The given condition is the sum of the squares of the reciprocals of the roots is 15.

$$\frac{1}{\alpha^2} + \frac{1}{\beta^2} = 15$$

Let's express this in terms of $\alpha + \beta$ and $\alpha\beta$.

$$\frac{\alpha^2 + \beta^2}{(\alpha\beta)^2} = 15$$
$$\frac{(\alpha + \beta)^2 - 2\alpha\beta}{(\alpha\beta)^2} = 15$$

Now, substitute the values from Vieta's formulas:

$$\frac{(-\lambda/3)^2 - 2(-1/3)}{(-1/3)^2} = 15$$
$$\frac{\lambda^2/9 + 2/3}{1/9} = 15$$

Multiply the numerator and denominator by 9 to clear the fractions:

$$\lambda^2 + 6 = 15$$
$$\lambda^2 = 9$$

Step 3: Calculating the Required Expression
We need to find the value of $6(\alpha^3 + \beta^3)^2$.
First, let's find an expression for $\alpha^3 + \beta^3$.

$$\alpha^3 + \beta^3 = (\alpha + \beta)(\alpha^2 - \alpha\beta + \beta^2) = (\alpha + \beta)((\alpha + \beta)^2 - 3\alpha\beta)$$

Substitute the values in terms of λ:

$$\alpha^3 + \beta^3 = (-\frac{\lambda}{3})\left((-\frac{\lambda}{3})^2 - 3(-\frac{1}{3})\right)$$
$$\alpha^3 + \beta^3 = (-\frac{\lambda}{3})\left(\frac{\lambda^2}{9} + 1\right)$$

Now, we need $(\alpha^3 + \beta^3)^2$:

$$(\alpha^3 + \beta^3)^2 = \left((-\frac{\lambda}{3})\left(\frac{\lambda^2}{9} + 1\right)\right)^2 = \frac{\lambda^2}{9}\left(\frac{\lambda^2}{9} + 1\right)^2$$

Substitute the value $\lambda^2 = 9$:

$$(\alpha^3 + \beta^3)^2 = \frac{9}{9}\left(\frac{9}{9} + 1\right)^2 = 1 \cdot (1+1)^2 = 1 \cdot 2^2 = 4$$

Step 4: Final Answer
Finally, calculate the required value:

$$6(\alpha^3 + \beta^3)^2 = 6 \times 4 = 24$$

> **Quick Tip**
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> Remember the standard algebraic identities involving roots of polynomials, such as $\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$ and $\alpha^3 + \beta^3 = (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)$. These are essential for solving such problems quickly. Notice that we only needed λ^2, not λ itself.

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8. The set of all values of k for which $(\tan^{-1} x)^3 + (\cot^{-1} x)^3 = k\pi^3, x \in \mathbb{R}$, is the interval :

(A) $[\frac{1}{32}, \frac{7}{8})$

(B) $[\frac{1}{24}, \frac{13}{16})$

(C) $[\frac{1}{48}, \frac{13}{16})$

(D) $[\frac{1}{32}, \frac{9}{8})$

Correct Answer: (A) $[\frac{1}{32}, \frac{7}{8})$

Solution:

Step 1: Using Trigonometric Identities
We know the fundamental identity for inverse trigonometric functions:

$$\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2} \quad \text{for all } x \in \mathbb{R}$$

Let $a = \tan^{-1} x$ and $b = \cot^{-1} x$. The given equation becomes $a^3 + b^3 = k\pi^3$, with the constraint $a + b = \frac{\pi}{2}$.
We use the algebraic identity for the sum of cubes:

$$a^3 + b^3 = (a+b)^3 - 3ab(a+b)$$

Substitute $a + b = \frac{\pi}{2}$:

$$a^3 + b^3 = (\frac{\pi}{2})^3 - 3ab(\frac{\pi}{2}) = \frac{\pi^3}{8} - \frac{3\pi}{2}ab$$

Step 2: Finding the Range of ab
We need to find the range of the expression $a^3 + b^3$. This depends on the range of the product ab.
Let $P = ab = (\tan^{-1} x)(\cot^{-1} x)$.
Substitute $\cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x$:

$$P(a) = a(\frac{\pi}{2} - a) = \frac{\pi}{2}a - a^2$$

where $a = \tan^{-1} x$. The range of $\tan^{-1} x$ for $x \in \mathbb{R}$ is $a \in (-\frac{\pi}{2}, \frac{\pi}{2})$.
The function $P(a)$ is a downward-opening parabola. Its maximum value occurs at the vertex.
Vertex of $P(a)$ is at $a = -\frac{(\pi/2)}{2(-1)} = \frac{\pi}{4}$.
The maximum value is $P(\frac{\pi}{4}) = \frac{\pi}{2}(\frac{\pi}{4}) - (\frac{\pi}{4})^2 = \frac{\pi^2}{8} - \frac{\pi^2}{16} = \frac{\pi^2}{16}$.
The minimum value is approached at the boundary of the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ that is farthest from the vertex $\frac{\pi}{4}$. The point $-\frac{\pi}{2}$ is farther.
Let's find the limit as $a \to -\frac{\pi}{2}$:

$$\lim_{a \to (-\pi/2)^+} P(a) = \frac{\pi}{2}(-\frac{\pi}{2}) - (-\frac{\pi}{2})^2 = -\frac{\pi^2}{4} - \frac{\pi^2}{4} = -\frac{\pi^2}{2}$$

So, the range of ab is $(-\frac{\pi^2}{2}, \frac{\pi^2}{16}]$.

Step 3: Finding the Range of $a^3 + b^3$
Let $E = a^3 + b^3 = \frac{\pi^3}{8} - \frac{3\pi}{2}(ab)$.
Since the coefficient of ab is negative, the expression E will be minimized when ab is maximized, and vice versa.
Minimum value of E (occurs when $ab = \frac{\pi^2}{16}$):

$$E_{min} = \frac{\pi^3}{8} - \frac{3\pi}{2}(\frac{\pi^2}{16}) = \frac{\pi^3}{8} - \frac{3\pi^3}{32} = \frac{4\pi^3 - 3\pi^3}{32} = \frac{\pi^3}{32}$$

Maximum value of E (is approached as $ab \to -\frac{\pi^2}{2}$):

$$E_{max} = \frac{\pi^3}{8} - \frac{3\pi}{2}(-\frac{\pi^2}{2}) = \frac{\pi^3}{8} + \frac{3\pi^3}{4} = \frac{\pi^3 + 6\pi^3}{8} = \frac{7\pi^3}{8}$$

The range of $(\tan^{-1} x)^3 + (\cot^{-1} x)^3$ is $[\frac{\pi^3}{32}, \frac{7\pi^3}{8})$.

Step 4: Finding the Range of k
We are given that the expression equals $k\pi^3$.

$$\frac{\pi^3}{32} \le k\pi^3 < \frac{7\pi^3}{8}$$

Dividing by π^3 (which is positive), we get the interval for k:

$$\frac{1}{32} \le k < \frac{7}{8}$$

The set of all values of k is the interval $[\frac{1}{32}, \frac{7}{8})$.

Quick Tip

To find the range of an expression, first simplify it using known identities. Then, express it as a function of a single variable whose range is known (in this case, $a = \tan^{-1} x$). Analyzing this new function (often a quadratic or simple monotonic function) over its domain will give the required range.

---

9. Let S = $\{\sqrt{n} : 1 \quad n \quad 50$ and n is odd$\}$. Let a S and A = $\begin{pmatrix} 1 & 0 & a \\ -1 & 1 & 0 \\ -a & 0 & 1 \end{pmatrix}$. If $\sum_{a \in S} \det(\textbf{adj } A) = 100\lambda$, then λ is equal to :

(A) 218
(B) 221
(C) 663
(D) 1717

Correct Answer: (B) 221

Solution:

Step 1: Understanding the Properties of Adjoint and Determinant
We are asked to compute a sum involving the determinant of the adjoint of a matrix.
A key property is that for an $n \times n$ matrix A, $\det(\text{adj } A) = (\det A)^{n-1}$.
In this case, A is a 3×3 matrix, so $n = 3$. Therefore, $\det(\text{adj } A) = (\det A)^{3-1} = (\det A)^2$.

Step 2: Calculate the Determinant of A
The matrix is A = $\begin{pmatrix} 1 & 0 & a \\ -1 & 1 & 0 \\ -a & 0 & 1 \end{pmatrix}$.
Let's compute its determinant:

$$\det A = 1 \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} - 0 \begin{vmatrix} -1 & 0 \\ -a & 1 \end{vmatrix} + a \begin{vmatrix} -1 & 1 \\ -a & 0 \end{vmatrix}$$

$$\det A = 1(1 \cdot 1 - 0 \cdot 0) - 0 + a(-1 \cdot 0 - 1 \cdot (-a))$$

$$\det A = 1(1) + a(a) = 1 + a^2$$

Therefore, $\det(\text{adj } A) = (\det A)^2 = (1 + a^2)^2$.

Step 3: Evaluate the Summation
We need to calculate $\sum_{a \in S} (1 + a^2)^2$.
The set S is defined as S = $\{\sqrt{n} : 1 \quad n \quad 50$ and n is odd$\}$.
So, the values of n are 1, 3, 5, ..., 49.

The values of a are $\sqrt{1}, \sqrt{3}, \sqrt{5}, ..., \sqrt{49}$.
When we square a, we get $a^2 = n$, where n is an odd integer from 1 to 49.
The sum becomes:

$$\sum_{n\in\{1,3,5,...,49\}} (1+n)^2$$

The terms in the sum are $(1+1)^2, (1+3)^2, (1+5)^2, ..., (1+49)^2$.
This is $2^2 + 4^2 + 6^2 + ... + 50^2$.
We can factor out 2^2 from each term:

$$\sum = (2\cdot 1)^2 + (2\cdot 2)^2 + (2\cdot 3)^2 + ... + (2\cdot 25)^2$$

$$\sum = 4(1^2) + 4(2^2) + 4(3^2) + ... + 4(25^2)$$

$$\sum = 4\sum_{k=1}^{25} k^2$$

We use the formula for the sum of the first N squares: $\sum_{k=1}^{N} k^2 = \frac{N(N+1)(2N+1)}{6}$.
Here, N=25.

$$\sum = 4\left(\frac{25(25+1)(2\cdot 25+1)}{6}\right) = 4\left(\frac{25\cdot 26\cdot 51}{6}\right)$$

$$\sum = 4\left(\frac{33150}{6}\right) = 4(5525) = 22100$$

Alternatively, simplify before multiplying:

$$\sum = 4\cdot\frac{25\cdot 26\cdot 51}{6} = \frac{2\cdot 2\cdot 25\cdot(2\cdot 13)\cdot(3\cdot 17)}{2\cdot 3} = 2\cdot 25\cdot 2\cdot 13\cdot 17 = 100\cdot 221$$

So, the sum is 22100.

Step 4: Find λ
We are given that the sum is equal to 100λ.

$$22100 = 100\lambda$$
$$\lambda = \frac{22100}{100} = 221$$

Quick Tip

When dealing with summations, always look for patterns. In this case, the sum was over squares of even numbers, which could be simplified by factoring out a common term (2^2) to reduce it to a standard sum-of-squares formula. Knowing the formula $\det(\text{adj } A) = (\det A)^{n-1}$ is crucial.

10. For the function $f(x) = 4\log_e(x-1) - 2x^2 + 4x + 5, x > 1$, which one of the following is NOT correct ?

(A) f is increasing in (1, 2) and decreasing in (2, ∞)
(B) f(x) = -1 has exactly two solutions
(C) f' (e) -f"(2) ¡ 0
(D) f(x) = 0 has a root in the interval (e, e + 1)

Correct Answer: (C) f' (e) -f"(2) ¡ 0

Solution:

Step 1: Find the first and second derivatives of the function.
Given $f(x) = 4\ln(x-1) - 2x^2 + 4x + 5$ for $x > 1$.
The first derivative is:

$$f'(x) = \frac{4}{x-1} - 4x + 4 = \frac{4 - (4x-4)(x-1)}{x-1}$$
$$f'(x) = \frac{4 - (4x^2 - 8x + 4)}{x-1} = \frac{-4x^2 + 8x}{x-1} = \frac{-4x(x-2)}{x-1}$$

The second derivative is:

$$f''(x) = \frac{d}{dx}\left(\frac{4}{x-1} - 4x + 4\right) = -\frac{4}{(x-1)^2} - 4$$

Step 2: Analyze each statement.
(A) f is increasing in (1, 2) and decreasing in (2, ∞).
To check for intervals of increase/decrease, we analyze the sign of $f'(x)$.
The denominator $(x-1)$ is positive for $x > 1$.
The term $-4x$ is negative for $x > 1$.
The sign of $f'(x)$ is determined by the sign of $-(x-2) = 2 - x$.
- If $1 < x < 2$, then $x - 2$ is negative, so $f'(x) = \frac{(-)(+)}{+} > 0$. Thus, f is increasing.
- If $x > 2$, then $x - 2$ is positive, so $f'(x) = \frac{(-)(+)}{+} < 0$. Thus, f is decreasing.
Statement (A) is correct.

(B) f(x) = -1 has exactly two solutions.
From (A), we know that f(x) has a local maximum at $x = 2$.
Let's find the behavior at the boundaries and the maximum value.

$$\lim_{x\to 1^+} f(x) = 4(-\infty) - 2 + 4 + 5 = -\infty$$
$$f(2) = 4\ln(1) - 2(4) + 4(2) + 5 = 0 - 8 + 8 + 5 = 5$$
$$\lim_{x\to\infty} f(x) = -\infty \quad (\text{since } -2x^2 \text{ dominates})$$

The function increases from $-\infty$ to a maximum of 5 at $x = 2$, and then decreases to $-\infty$. A horizontal line at $y = -1$ will intersect the graph exactly twice.

Statement (B) is correct.

(C) f'(e) - f"(2) ¡ 0.
First, calculate $f'(e)$. Here $e \approx 2.718$.

$$f'(e) = \frac{-4e(e-2)}{e-1}$$

Since $e > 2$, $e - 2 > 0$ and $e - 1 > 0$. So, $f'(e)$ is negative.
Next, calculate $f''(2)$.

$$f''(2) = -\frac{4}{(2-1)^2} - 4 = -\frac{4}{1} - 4 = -8$$

Now evaluate the expression:

$$f'(e) - f''(2) = (\text{a negative number}) - (-8) = (\text{a negative number}) + 8$$

Let's find the approximate value of $f'(e)$:

$$f'(e) = \frac{-4(2.718)(0.718)}{1.718} \approx -4.54$$
$$f'(e) - f''(2) \approx -4.54 - (-8) = 3.46$$

Since $3.46 > 0$, the statement $f'(e) - f''(2) < 0$ is NOT correct.

(D) f(x) = 0 has a root in the interval (e, e + 1).
We use the Intermediate Value Theorem. $e \approx 2.718$, so the interval is approx $(2.718, 3.718)$.
Let's evaluate f at $x = e$ and some point greater than e.

$$f(e) = 4\ln(e-1) - 2e^2 + 4e + 5$$

Since $1 < e - 1 < 2$, $0 < \ln(e-1) < \ln(e) = 1$. Let's use $e \approx 2.7$. $f(2.7) \approx 4\ln(1.7) - 2(2.7)^2 + 4(2.7) + 5 \approx 4(0.53) - 14.58 + 10.8 + 5 \approx 2.12 - 14.58 + 15.8 = 3.34 > 0$. Now let's check a point near the end of the interval, say $x = 3.5$.

$$f(3.5) = 4\ln(2.5) - 2(3.5)^2 + 4(3.5) + 5 = 4\ln(2.5) - 2(12.25) + 14 + 5 = 4\ln(2.5) - 24.5 + 19 = 4\ln(2.5) - 5.5$$

$\ln(2.5) \approx 0.916$. So $f(3.5) \approx 4(0.916) - 5.5 = 3.664 - 5.5 = -1.836 < 0$.
Since $f(e) > 0$ and $f(3.5) < 0$, by the Intermediate Value Theorem, there must be a root in $(e, 3.5)$, which is within $(e, e + 1)$.
Statement (D) is correct.

Step 3: Final Answer
The only statement that is not correct is (C).

💡 Quick Tip

When analyzing a function, calculating the first and second derivatives is the standard approach. Use $f'(x)$ to determine intervals of increase/decrease and locate local extrema. Use this information to sketch the graph's behavior, which helps in checking statements about roots of equations like $f(x) = c$.

---

11. If the tangent at the point (x_1, y_1) on the curve $y = x^3 + 3x^2 + 5$ passes through the origin, then (x_1, y_1) does NOT lie on the curve :

(A) $x^2 + \frac{y^2}{81} = 2$
(B) $\frac{y^2}{9} - x^2 = 8$
(C) $y = 4x^2 + 5$
(D) $\frac{x}{3} - y^2 = 2$

Correct Answer: (D) $\frac{x}{3} - y^2 = 2$

Solution:

Step 1: Find the equation of the tangent line.
The curve is given by $y = x^3 + 3x^2 + 5$.
The slope of the tangent at any point (x, y) is given by the derivative:

$$\frac{dy}{dx} = 3x^2 + 6x$$

At the point (x_1, y_1), the slope of the tangent is $m = 3x_1^2 + 6x_1$.
The equation of the tangent line at (x_1, y_1) is given by the point-slope form:

$$y - y_1 = m(x - x_1)$$
$$y - y_1 = (3x_1^2 + 6x_1)(x - x_1)$$

Step 2: Use the condition that the tangent passes through the origin.
We are given that the tangent line passes through the origin (0, 0). Substitute $x = 0$ and $y = 0$ into the tangent equation:

$$0 - y_1 = (3x_1^2 + 6x_1)(0 - x_1)$$
$$-y_1 = -(3x_1^2 + 6x_1)x_1$$
$$y_1 = 3x_1^3 + 6x_1^2 \quad (1)$$

Step 3: Find the coordinates of the point (x_1, y_1).
The point (x_1, y_1) also lies on the original curve. So, it must satisfy the curve's equation:

$$y_1 = x_1^3 + 3x_1^2 + 5 \quad (2)$$

Now we have two expressions for y_1. We can equate them to solve for x_1:

$$3x_1^3 + 6x_1^2 = x_1^3 + 3x_1^2 + 5$$
$$2x_1^3 + 3x_1^2 - 5 = 0$$

By the Rational Root Theorem, we can test integer factors of -5. Let's test $x_1 = 1$:

$$2(1)^3 + 3(1)^2 - 5 = 2 + 3 - 5 = 0$$

So, $x_1 = 1$ is a root.
We can perform polynomial division by $(x_1 - 1)$ to find other roots:

$$(2x_1^3 + 3x_1^2 - 5) \div (x_1 - 1) = 2x_1^2 + 5x_1 + 5$$

So, the equation is $(x_1 - 1)(2x_1^2 + 5x_1 + 5) = 0$.
For the quadratic factor $2x_1^2 + 5x_1 + 5$, the discriminant is $\Delta = b^2 - 4ac = 5^2 - 4(2)(5) = 25 - 40 = -15$.
Since $\Delta < 0$, there are no other real roots for x_1.
The only real solution is $x_1 = 1$.
Now find y_1 using equation (2):

$$y_1 = (1)^3 + 3(1)^2 + 5 = 1 + 3 + 5 = 9$$

So, the point of tangency is $(x_1, y_1) = (1, 9)$.

Step 4: Check which curve the point (1, 9) does NOT lie on.
We test the point (1, 9) in each of the given options.
(A) $x^2 + \frac{y^2}{81} = 2 \implies (1)^2 + \frac{9^2}{81} = 1 + \frac{81}{81} = 1 + 1 = 2$. The point lies on this curve.
(B) $\frac{y^2}{9} - x^2 = 8 \implies \frac{9^2}{9} - (1)^2 = \frac{81}{9} - 1 = 9 - 1 = 8$. The point lies on this curve.
(C) $y = 4x^2 + 5 \implies 9 = 4(1)^2 + 5 = 4 + 5 = 9$. The point lies on this curve.
(D) $\frac{x}{3} - y^2 = 2 \implies \frac{1}{3} - 9^2 = \frac{1}{3} - 81 \neq 2$. The point does NOT lie on this curve.

Quick Tip

This problem combines concepts of derivatives (to find the tangent) and coordinate geometry. The key is to set up a system of equations. One equation comes from the point being on the curve, and the other from the geometric condition imposed on its tangent line.

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12. The sum of absolute maximum and absolute minimum values of the function $f(x) = |2x^2 + 3x - 2| + \sin x \cos x$ in the interval [0, 1] is :

(A) $3 + \frac{\sin(1)\cos^2(\frac{1}{2})}{2}$
(B) $3 + \frac{1}{2}(1 + 2\cos(1))\sin(1)$
(C) $5 + \frac{1}{2}(\sin(1) + \sin(2))$

(D) $2 + \sin(\frac{1}{2}) \cos(\frac{1}{2})$

Correct Answer: (B) $3 + \frac{1}{2}(1 + 2\cos(1))\sin(1)$

Solution:

Step 1: Analyze the function and remove the absolute value.
First, analyze the quadratic part $g(x) = 2x^2 + 3x - 2$.

The roots are given by $x = \frac{-3\pm\sqrt{9-4(2)(-2)}}{4} = \frac{-3\pm5}{4}$, which are $x = \frac{1}{2}$ and $x = -2$.

In the interval $[0, 1]$, the quadratic changes sign at $x = \frac{1}{2}$.
- For $x \in [0, \frac{1}{2}]$, $2x^2 + 3x - 2 \leq 0$, so $|2x^2 + 3x - 2| = -(2x^2 + 3x - 2)$.
- For $x \in [\frac{1}{2}, 1]$, $2x^2 + 3x - 2 \geq 0$, so $|2x^2 + 3x - 2| = 2x^2 + 3x - 2$.
Also, note that $\sin x \cos x = \frac{1}{2}\sin(2x)$.
So, the function is piecewise:

$$f(x) = \begin{cases} -2x^2 - 3x + 2 + \frac{1}{2}\sin(2x) & \text{if } 0 \leq x \leq 1/2 \\ 2x^2 + 3x - 2 + \frac{1}{2}\sin(2x) & \text{if } 1/2 \leq x \leq 1 \end{cases}$$

Step 2: Find critical points.
The absolute maximum and minimum values can occur at the endpoints of the interval ([0, 1]) or at critical points inside the interval. A critical point is where $f'(x) = 0$ or $f'(x)$ is undefined. Let's find the derivative:

$$f'(x) = \begin{cases} -4x - 3 + \cos(2x) & \text{if } 0 < x < 1/2 \\ 4x + 3 + \cos(2x) & \text{if } 1/2 < x < 1 \end{cases}$$

The derivative is undefined at $x = 1/2$ because the left and right derivatives are not equal. So, $x = 1/2$ is a critical point.
Let's check for $f'(x) = 0$.
- For $0 < x < 1/2$: $-4x - 3 \in (-5, -3)$ and $\cos(2x) \in (\cos(1), 1]$. Since $\cos(1) > 0.5$, $f'(x)$ is always negative. No critical points here.
- For $1/2 < x < 1$: $4x + 3 \in (5, 7)$ and $\cos(2x) \in (\cos(2), \cos(1))$. Since $-1 \leq \cos(2x) \leq 1$, $f'(x)$ is always positive. No critical points here.
The only critical point to check is $x = 1/2$. The candidates for max/min are $x = 0, x = 1/2, x = 1$.

Step 3: Evaluate the function at the candidate points.

$$f(0) = |0 + 0 - 2| + \sin(0)\cos(0) = 2 + 0 = 2$$

$$f(1/2) = |2(\frac{1}{4}) + 3(\frac{1}{2}) - 2| + \sin(\frac{1}{2})\cos(\frac{1}{2}) = |\frac{1}{2} + \frac{3}{2} - 2| + \frac{1}{2}\sin(1) = |0| + \frac{1}{2}\sin(1) = \frac{1}{2}\sin(1)$$

$$f(1) = |2(1)^2 + 3(1) - 2| + \sin(1)\cos(1) = |3| + \frac{1}{2}\sin(2) = 3 + \frac{1}{2}\sin(2)$$

Step 4: Determine Absolute Max/Min and find their sum.
Comparing the values:
- $f(0) = 2$.
- $f(1/2) = \frac{1}{2}\sin(1)$. Since $0 < 1 < \pi/2$, $0 < \sin(1) < 1$, so $0 < f(1/2) < 1/2$.
- $f(1) = 3 + \frac{1}{2}\sin(2)$. Since $\pi/2 < 2 < \pi$, $\sin(2) > 0$. So $f(1) > 3$.
The absolute minimum is $f(1/2) = \frac{1}{2}\sin(1)$.
The absolute maximum is $f(1) = 3 + \frac{1}{2}\sin(2)$.
Sum of absolute maximum and minimum values is:

$$\text{Sum} = (3 + \frac{1}{2}\sin(2)) + (\frac{1}{2}\sin(1)) = 3 + \frac{1}{2}(\sin(2) + \sin(1))$$

Using the identity $\sin(2) = 2\sin(1)\cos(1)$:

$$\text{Sum} = 3 + \frac{1}{2}(2\sin(1)\cos(1) + \sin(1))$$
$$\text{Sum} = 3 + \frac{1}{2}\sin(1)(2\cos(1) + 1)$$

This matches option (B).

Quick Tip

To find absolute extrema on a closed interval, always check the function values at the endpoints and at all critical points within the interval. For functions with absolute values, critical points often arise where the argument of the absolute value is zero.

---

13. If $\{a_i\}_{i=1}^{n}$, where n is an even integer, is an arithmetic progression with common difference 1, and $\sum_{i=1}^{n} a_i = 192$, $\sum_{i=1}^{n/2} a_{2i} = 120$, then n is equal to :

(A) 48
(B) 96
(C) 92
(D) 104

Correct Answer: (B) 96

Solution:

Step 1: Set up equations based on the given information.
Let the arithmetic progression be $a_1, a_1 + 1, a_1 + 2, ...$ The first term is a_1 and the common difference is $d = 1$.
We are given the sum of the first n terms:

$$\sum_{i=1}^{n} a_i = 192$$

Using the formula for the sum of an AP, $S_n = \frac{n}{2}[2a_1 + (n-1)d]$:

$$\frac{n}{2}[2a_1 + (n-1)(1)] = 192 \implies n(2a_1 + n - 1) = 384 \quad (1)$$

We are also given the sum of the terms with even indices up to n: $a_2 + a_4 + ... + a_n$.
This is a new AP with first term $A_1 = a_2 = a_1 + 1$, common difference $D = a_4 - a_2 = (a_1 + 3) - (a_1 + 1) = 2$, and number of terms $k = n/2$.
The sum is given as:

$$\sum_{i=1}^{n/2} a_{2i} = 120$$

Using the sum formula for this new AP:

$$\frac{n/2}{2}[2(a_1 + 1) + (\frac{n}{2} - 1)(2)] = 120$$
$$\frac{n}{4}[2a_1 + 2 + n - 2] = 120$$
$$\frac{n}{4}[2a_1 + n] = 120 \implies n(2a_1 + n) = 480 \quad (2)$$

Step 2: Solve the system of two equations.
We have the system:
(1) $n(2a_1 + n - 1) = 384 \implies 2a_1 n + n^2 - n = 384$
(2) $n(2a_1 + n) = 480 \implies 2a_1 n + n^2 = 480$
Substitute the value of $2a_1 n + n^2$ from equation (2) into equation (1):

$$(480) - n = 384$$
$$n = 480 - 384$$
$$n = 96$$

Step 3: Final Answer
The value of n is 96. Since 96 is an even integer, it is a valid solution.

Quick Tip

When dealing with sums of AP terms with specific indices (e.g., only even or odd terms), recognize that this subsequence is also an AP. Identify its first term, common difference, and number of terms to apply the standard sum formula. This often leads to a simpler system of equations.

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14. If $x = x(y)$ is the solution of the differential equation $y\frac{dx}{dy} = 2x + y^3(y+1)e^y, x(1) = 0$; then $x(e)$ is equal to :

(A) $e^3(e^e - 1)$
(B) $e^e(e^3 - 1)$
(C) $e^2(e^e + 1)$
(D) $e^e(e^2 - 1)$

Correct Answer: (A) $e^3(e^e - 1)$

Solution:

Step 1: Rearrange the equation into the standard linear form.
The given differential equation is $y\frac{dx}{dy} = 2x + y^3(y+1)e^y$.
We are solving for x as a function of y, so we rearrange it into the linear form $\frac{dx}{dy} + P(y)x = Q(y)$.
Divide by y (assuming $y \neq 0$):

$$\frac{dx}{dy} = \frac{2}{y}x + y^2(y+1)e^y$$
$$\frac{dx}{dy} - \frac{2}{y}x = y^2(y+1)e^y$$

This is a linear differential equation with $P(y) = -\frac{2}{y}$ and $Q(y) = y^2(y+1)e^y$.

Step 2: Find the integrating factor (I.F.).
The integrating factor is given by $I.F. = e^{\int P(y)dy}$.

$$I.F. = e^{\int -\frac{2}{y}dy} = e^{-2\ln|y|} = e^{\ln(y^{-2})} = y^{-2} = \frac{1}{y^2}$$

Step 3: Solve the differential equation.
The solution is given by $x \cdot (I.F.) = \int Q(y) \cdot (I.F.)dy + C$.

$$x \cdot \frac{1}{y^2} = \int (y^2(y+1)e^y) \cdot (\frac{1}{y^2})dy + C$$
$$\frac{x}{y^2} = \int (y+1)e^y dy + C$$

To evaluate the integral, we can use integration by parts or recognize it as the derivative of a product.
Note that $\frac{d}{dy}(ye^y) = 1 \cdot e^y + y \cdot e^y = (y+1)e^y$.
So, $\int (y+1)e^y dy = ye^y$.
The solution becomes:

$$\frac{x}{y^2} = ye^y + C$$

Step 4: Apply the initial condition to find C.
We are given $x(1) = 0$, which means $x = 0$ when $y = 1$.

$$\frac{0}{1^2} = 1 \cdot e^1 + C$$

$$0 = e + C \implies C = -e$$

The particular solution is $\frac{x}{y^2} = ye^y - e$, or $x(y) = y^2(ye^y - e)$.

Step 5: Find the value of $x(e)$.
We need to find the value of x when $y = e$.

$$x(e) = e^2(e \cdot e^e - e)$$
$$x(e) = e^2 \cdot e(e^e - 1)$$
$$x(e) = e^3(e^e - 1)$$

> Quick Tip
>
> Always try to recognize if the differential equation fits a standard form, like linear, separable, or exact. For linear equations of the form $y' + P(x)y = Q(x)$, the integrating factor method is the standard approach. Remember to solve for the constant C using the initial condition before finding the final value.

---

15. Let $\lambda x - 2y = \mu$ be a tangent to the hyperbola $a^2x^2 - y^2 = b^2$. Then $(\frac{\lambda}{a})^2 - (\frac{\mu}{b})^2$ is equal to :

(A) -2
(B) -4
(C) 2
(D) 4

Correct Answer: (D) 4

Solution:

Step 1: Write the hyperbola and the tangent line in standard forms.
The equation of the hyperbola is $a^2x^2 - y^2 = b^2$.
To put it in the standard form $\frac{x^2}{A^2} - \frac{y^2}{B^2} = 1$, we divide by b^2:

$$\frac{a^2x^2}{b^2} - \frac{y^2}{b^2} = 1 \implies \frac{x^2}{(b/a)^2} - \frac{y^2}{b^2} = 1$$

So, for our hyperbola, $A^2 = (b/a)^2$ and $B^2 = b^2$.

The equation of the line is $\lambda x - 2y = \mu$.
To put it in the slope-intercept form $y = mx + c$, we rearrange it:

$$2y = \lambda x - \mu \implies y = \frac{\lambda}{2}x - \frac{\mu}{2}$$

So, the slope is $m = \frac{\lambda}{2}$ and the y-intercept is $c = -\frac{\mu}{2}$.

Step 2: Apply the condition of tangency.
The condition for a line $y = mx + c$ to be a tangent to the hyperbola $\frac{x^2}{A^2} - \frac{y^2}{B^2} = 1$ is:

$$c^2 = A^2m^2 - B^2$$

Substitute the values we found for A^2, B^2, m, and c:

$$(-\frac{\mu}{2})^2 = \left(\frac{b^2}{a^2}\right)\left(\frac{\lambda}{2}\right)^2 - b^2$$
$$\frac{\mu^2}{4} = \frac{b^2}{a^2}\frac{\lambda^2}{4} - b^2$$

Step 3: Simplify the equation to find the required expression.
Multiply the entire equation by 4 to clear the denominators:

$$\mu^2 = \frac{b^2\lambda^2}{a^2} - 4b^2$$

We need to find the value of $(\frac{\lambda}{a})^2 - (\frac{\mu}{b})^2$. Let's rearrange our equation to match this form.

$$\mu^2 + 4b^2 = \frac{b^2\lambda^2}{a^2}$$

Divide the entire equation by b^2 (assuming $b \neq 0$, which is true for a hyperbola):

$$\frac{\mu^2}{b^2} + 4 = \frac{\lambda^2}{a^2}$$

Rearrange the terms to get the desired expression:

$$\frac{\lambda^2}{a^2} - \frac{\mu^2}{b^2} = 4$$
$$(\frac{\lambda}{a})^2 - (\frac{\mu}{b})^2 = 4$$

Step 4: Final Answer
The value of the expression is 4.

Quick Tip

Memorize the conditions of tangency for standard conic sections. For a hyperbola $\frac{x^2}{A^2} - \frac{y^2}{B^2} = 1$, the condition for $y = mx + c$ to be a tangent is $c^2 = A^2m^2 - B^2$. For an ellipse $\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$, it is $c^2 = A^2m^2 + B^2$. For a parabola $y^2 = 4ax$, it is $c = a/m$.

16. Let $\hat{a}, \hat{b}$ be unit vectors. If $\vec{c}$ be a vector such that the angle between $\hat{a}$ and $\vec{c}$ is $\frac{\pi}{12}$, and $\hat{b} = \vec{c} + 2(\vec{c} \times \hat{a})$, then $|6\vec{c}|^2$ is equal to :

(A) $6(3 - \sqrt{3})$
(B) $3 + \sqrt{3}$
(C) $6(3 + \sqrt{3})$
(D) $6(\sqrt{3} + 1)$

Correct Answer: (C) $6(3 + \sqrt{3})$

Solution:

Step 1: Use the given vector equation to find relationships.
We are given $\hat{b} = \vec{c} + 2(\vec{c} \times \hat{a})$.
Also given: $\hat{a}, \hat{b}$ are unit vectors, so $|\hat{a}| = 1, |\hat{b}| = 1$.
The angle θ between $\hat{a}$ and $\vec{c}$ is $\frac{\pi}{12}$.
A useful technique is to take the dot product of the equation with $\hat{a}$.

$$\hat{b} \cdot \hat{a} = (\vec{c} + 2(\vec{c} \times \hat{a})) \cdot \hat{a}$$
$$\hat{b} \cdot \hat{a} = \vec{c} \cdot \hat{a} + 2(\vec{c} \times \hat{a}) \cdot \hat{a}$$

The second term on the right is a scalar triple product involving two identical vectors ($\hat{a}$), so it is zero. $[\vec{c}, \hat{a}, \hat{a}] = 0$.

$$\hat{b} \cdot \hat{a} = \vec{c} \cdot \hat{a}$$

Step 2: Use magnitudes to find an equation for $|\vec{c}|$.
Take the magnitude squared of both sides of the original equation:

$$|\hat{b}|^2 = |\vec{c} + 2(\vec{c} \times \hat{a})|^2$$

Since $\hat{b}$ is a unit vector, $|\hat{b}|^2 = 1$.

$$1 = (\vec{c} + 2(\vec{c} \times \hat{a})) \cdot (\vec{c} + 2(\vec{c} \times \hat{a}))$$
$$1 = \vec{c} \cdot \vec{c} + 2(\vec{c} \cdot (\vec{c} \times \hat{a})) + 2((\vec{c} \times \hat{a}) \cdot \vec{c}) + 4|\vec{c} \times \hat{a}|^2$$

The middle two terms are scalar triple products $[\vec{c}, \vec{c}, \hat{a}]$ which are zero.

$$1 = |\vec{c}|^2 + 4|\vec{c} \times \hat{a}|^2$$

Recall the definition of the cross product magnitude: $|\vec{c} \times \hat{a}| = |\vec{c}||\hat{a}| \sin\theta$.

$$1 = |\vec{c}|^2 + 4(|\vec{c}||\hat{a}| \sin(\frac{\pi}{12}))^2$$
$$1 = |\vec{c}|^2 + 4|\vec{c}|^2(1)^2 \sin^2(\frac{\pi}{12})$$
$$1 = |\vec{c}|^2(1 + 4\sin^2(\frac{\pi}{12}))$$

Step 3: Calculate $\sin^2(\frac{\pi}{12})$ and solve for $|\vec{c}|^2$.
We use the half-angle identity $\cos(2\theta) = 1 - 2\sin^2\theta$.
Let $\theta = \frac{\pi}{12}$, so $2\theta = \frac{\pi}{6}$.

$$\cos(\frac{\pi}{6}) = 1 - 2\sin^2(\frac{\pi}{12})$$
$$\frac{\sqrt{3}}{2} = 1 - 2\sin^2(\frac{\pi}{12})$$
$$2\sin^2(\frac{\pi}{12}) = 1 - \frac{\sqrt{3}}{2} = \frac{2-\sqrt{3}}{2}$$
$$\sin^2(\frac{\pi}{12}) = \frac{2-\sqrt{3}}{4}$$

Substitute this back into our equation for $|\vec{c}|^2$:

$$1 = |\vec{c}|^2(1 + 4(\frac{2-\sqrt{3}}{4}))$$
$$1 = |\vec{c}|^2(1 + 2 - \sqrt{3}) = |\vec{c}|^2(3 - \sqrt{3})$$
$$|\vec{c}|^2 = \frac{1}{3-\sqrt{3}}$$

Step 4: Calculate the final expression.
We need to find $|6\vec{c}|^2$.

$$|6\vec{c}|^2 = 36|\vec{c}|^2 = 36\left(\frac{1}{3-\sqrt{3}}\right)$$

Rationalize the denominator:

$$= \frac{36}{3-\sqrt{3}} \times \frac{3+\sqrt{3}}{3+\sqrt{3}} = \frac{36(3+\sqrt{3})}{3^2 - (\sqrt{3})^2} = \frac{36(3+\sqrt{3})}{9-3} = \frac{36(3+\sqrt{3})}{6}$$
$$= 6(3+\sqrt{3})$$

Quick Tip

In vector problems involving equations, taking dot products or cross products with one of the vectors, or taking the magnitude of both sides, are common and powerful techniques to reveal relationships between the vectors. Remember that $\vec{A} \cdot (\vec{B} \times \vec{C})$ is a scalar triple product which is zero if any two vectors are the same.

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17. If a random variable X follows the Binomial distribution B(33, p) such that 3P(X=0)=P(X=1), then the value of $\frac{P(X=15)}{P(X=18)} - \frac{P(X=16)}{P(X=17)}$ is equal to :

(A) 1320

(B) 1088

(C) $\frac{120}{1331}$

(D) $\frac{1088}{1089}$

Correct Answer: (A) 1320

Solution:

Step 1: Find the value of p using the given condition.
For a Binomial distribution B(n, p), the probability mass function is $P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$.
Here, n=33. Let $q = 1 - p$.
Given: $3P(X = 0) = P(X = 1)$.

$$3\binom{33}{0} p^0 q^{33-0} = \binom{33}{1} p^1 q^{33-1}$$
$$3 \cdot 1 \cdot 1 \cdot q^{33} = 33 \cdot p \cdot q^{32}$$

Assuming $q \neq 0$ (i.e., $p \neq 1$), we can divide by q^{32}:

$$3q = 33p$$
$$3(1 - p) = 33p$$
$$3 - 3p = 33p$$
$$3 = 36p \implies p = \frac{3}{36} = \frac{1}{12}$$

Then $q = 1 - p = 1 - \frac{1}{12} = \frac{11}{12}$.

Step 2: Find a general formula for the ratio of consecutive probabilities.
Let's find the ratio $\frac{P(X=k)}{P(X=k+1)}$. This will simplify calculations.

$$\frac{P(X = k)}{P(X = k+1)} = \frac{\binom{n}{k} p^k q^{n-k}}{\binom{n}{k+1} p^{k+1} q^{n-k-1}} = \frac{\frac{n!}{k!(n-k)!}}{\frac{n!}{(k+1)!(n-k-1)!}} \cdot \frac{q}{p}$$
$$= \frac{(k+1)!(n-k-1)!}{k!(n-k)!} \cdot \frac{q}{p} = \frac{k+1}{n-k} \cdot \frac{q}{p}$$

In our case, $n = 33$ and $\frac{q}{p} = \frac{11/12}{1/12} = 11$.

$$\frac{P(X = k)}{P(X = k+1)} = \frac{k+1}{33-k} \cdot 11$$

Step 3: Calculate the terms in the given expression.
First term is $\frac{P(X=16)}{P(X=17)}$. Let $k = 16$.

$$\frac{P(X = 16)}{P(X = 17)} = \frac{16+1}{33-16} \cdot 11 = \frac{17}{17} \cdot 11 = 11$$

Second term is $\frac{P(X=15)}{P(X=18)}$. We can write this as a product of ratios.

$$\frac{P(X=15)}{P(X=18)} = \frac{P(X=15)}{P(X=16)} \cdot \frac{P(X=16)}{P(X=17)} \cdot \frac{P(X=17)}{P(X=18)}$$

We already have $\frac{P(X=16)}{P(X=17)} = 11$.
For $k=15$: $\frac{P(X=15)}{P(X=16)} = \frac{15+1}{33-15} \cdot 11 = \frac{16}{18} \cdot 11 = \frac{8}{9} \cdot 11 = \frac{88}{9}$.
For $k=17$: $\frac{P(X=17)}{P(X=18)} = \frac{17+1}{33-17} \cdot 11 = \frac{18}{16} \cdot 11 = \frac{9}{8} \cdot 11 = \frac{99}{8}$.
Now, multiply these ratios:

$$\frac{P(X=15)}{P(X=18)} = (\frac{88}{9}) \cdot (11) \cdot (\frac{99}{8}) = \frac{88 \cdot 11 \cdot 99}{9 \cdot 8} = 11 \cdot 11 \cdot 11 = 1331$$

Step 4: Evaluate the final expression.
The value we need to find is $\frac{P(X=15)}{P(X=18)} - \frac{P(X=16)}{P(X=17)}$.

$$\text{Value} = 1331 - 11 = 1320$$

Quick Tip

For binomial probability calculations involving ratios, it's highly efficient to first derive a general formula for the ratio of consecutive terms, $\frac{P(X=k+1)}{P(X=k)}$ or its reciprocal. This avoids calculating large factorial and power terms separately and simplifies the algebra significantly.

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18. The domain of the function $f(x) = \frac{\cos^{-1}(\frac{x^2-5x+6}{x^2-9})}{\log_e(x^2-3x+2)}$ is :

(A) $(-\infty, 1) \cup (2, \infty)$
(B) $(2, \infty)$
(C) $[-\frac{1}{2}, 1) \cup (2, \infty)$
(D) $[-\frac{1}{2}, 1) \cup (2, \infty) - \{\frac{3+\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2}\}$

Correct Answer: (Dropped) The question was found to be erroneous and was dropped from the exam. None of the options are correct.

Solution:
To find the domain of the function $f(x)$, we need to satisfy three conditions.

Condition 1: The argument of the inverse cosine function must be in the interval $[-1, 1]$.

$$-1 \leq \frac{x^2-5x+6}{x^2-9} \leq 1$$

Factorizing the expressions:

$$-1 \leq \frac{(x-2)(x-3)}{(x-3)(x+3)} \leq 1$$

For $x \neq 3$, this simplifies to:

$$-1 \leq \frac{x-2}{x+3} \leq 1$$

This gives two separate inequalities:
a) $\frac{x-2}{x+3} \leq 1 \implies \frac{x-2}{x+3} - 1 \leq 0 \implies \frac{x-2-(x+3)}{x+3} \leq 0 \implies \frac{-5}{x+3} \leq 0$. This implies $x+3 > 0$, so $x > -3$.
b) $\frac{x-2}{x+3} \geq -1 \implies \frac{x-2}{x+3} + 1 \geq 0 \implies \frac{x-2+x+3}{x+3} \geq 0 \implies \frac{2x+1}{x+3} \geq 0$. This holds for $x \geq -1/2$ or $x < -3$.
Combining (a) and (b), we need $x > -3$ AND ($x \geq -1/2$ or $x < -3$). The intersection is $x \geq -1/2$.
So, from this condition, we have $x \in [-1/2, \infty) - \{3\}$.

Condition 2: The argument of the logarithm must be positive.

$$x^2 - 3x + 2 > 0$$

$$(x-1)(x-2) > 0$$

This holds when $x \in (-\infty, 1) \cup (2, \infty)$.

Condition 3: The denominator must not be zero.

$$\log_e(x^2 - 3x + 2) \neq 0$$

This means the argument of the logarithm cannot be 1.

$$x^2 - 3x + 2 \neq 1$$

$$x^2 - 3x + 1 \neq 0$$

Using the quadratic formula, the roots are $x = \frac{3 \pm \sqrt{9-4}}{2} = \frac{3 \pm \sqrt{5}}{2}$.
So, we must exclude $x = \frac{3-\sqrt{5}}{2}$ and $x = \frac{3+\sqrt{5}}{2}$.

Combining all conditions:
From C1: $x \in [-1/2, 3) \cup (3, \infty)$.
From C2: $x \in (-\infty, 1) \cup (2, \infty)$.
The intersection of C1 and C2 is: $[-1/2, 1) \cup (2, 3) \cup (3, \infty)$.
From C3, we must exclude $x = \frac{3 \pm \sqrt{5}}{2}$.
Note that $\frac{3-\sqrt{5}}{2} \approx 0.38$, which is in $[-1/2, 1)$.
And $\frac{3+\sqrt{5}}{2} \approx 2.62$, which is in $(2, 3)$.
So the final domain is $[-1/2, 1) \cup (2, \infty)$ with the points $3, \frac{3-\sqrt{5}}{2}, \frac{3+\sqrt{5}}{2}$ excluded.
The correct domain is $[-1/2, \frac{3-\sqrt{5}}{2}) \cup (\frac{3-\sqrt{5}}{2}, 1) \cup (2, \frac{3+\sqrt{5}}{2}) \cup (\frac{3+\sqrt{5}}{2}, 3) \cup (3, \infty)$.
None of the given options match this result. Option D is the closest but incorrectly omits the

exclusion of $x = 3$. Due to this discrepancy, the question was officially dropped.

> Quick Tip
>
> Finding the domain of a complex function involves finding the intersection of the domains of its individual parts. Be systematic: handle square roots, logarithms, denominators, and arguments of inverse trigonometric functions one by one, and then find the intersection of all resulting conditions. Don't forget to check for points of discontinuity that might be simplified away, like $x = 3$ in this problem.

---

19. Let $S = \{\theta \in [-\pi, \pi] - \{\pm\frac{\pi}{2}\} : \sin\theta\tan\theta + \tan\theta = \sin 2\theta\}$. If $T = \sum_{\theta\in S}\cos 2\theta$, then $T + n(S)$ is equal to :

(A) $7 + \sqrt{3}$
(B) 9
(C) $8 + \sqrt{3}$
(D) 10

Correct Answer: (B) 9

Solution:

Step 1: Solve the trigonometric equation to find the set S.
The equation is $\sin\theta\tan\theta + \tan\theta = \sin 2\theta$.
Factor out $\tan\theta$ on the left side and use the double angle formula on the right side.

$$\tan\theta(\sin\theta + 1) = 2\sin\theta\cos\theta$$

Replace $\tan\theta = \frac{\sin\theta}{\cos\theta}$ (note $\cos\theta \neq 0$ since $\theta \neq \pm\pi/2$).

$$\frac{\sin\theta}{\cos\theta}(\sin\theta + 1) = 2\sin\theta\cos\theta$$

One possibility is $\sin\theta = 0$. If $\sin\theta = 0$, the equation becomes $0 = 0$, which is true.
For $\theta \in [-\pi, \pi]$, $\sin\theta = 0$ implies $\theta = -\pi, 0, \pi$. These are 3 solutions.

If $\sin\theta \neq 0$, we can divide both sides by $\sin\theta$.

$$\frac{1}{\cos\theta}(\sin\theta + 1) = 2\cos\theta$$
$$\sin\theta + 1 = 2\cos^2\theta$$

Use the identity $\cos^2\theta = 1 - \sin^2\theta$.

$$\sin\theta + 1 = 2(1 - \sin^2\theta)$$
$$\sin\theta + 1 = 2 - 2\sin^2\theta$$

$$2\sin^2\theta + \sin\theta - 1 = 0$$

This is a quadratic equation in $\sin\theta$. Let $y = \sin\theta$.

$$2y^2 + y - 1 = 0 \implies (2y-1)(y+1) = 0$$

This gives two possibilities: $\sin\theta = 1/2$ or $\sin\theta = -1$.
- If $\sin\theta = 1/2$, for $\theta \in [-\pi, \pi]$, the solutions are $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$. (2 solutions)
- If $\sin\theta = -1$, for $\theta \in [-\pi, \pi]$, the solution is $\theta = -\frac{\pi}{2}$. However, this value is explicitly excluded from the domain of S. So we discard this solution.

The set of solutions is $S = \{-\pi, 0, \pi, \frac{\pi}{6}, \frac{5\pi}{6}\}$.
The number of elements in S is $n(S) = 5$.

Step 2: Calculate T.
$T = \sum_{\theta\in S} \cos 2\theta$.
It is easier to use the identity $\cos 2\theta = 1 - 2\sin^2\theta$.
For the solutions $\theta = -\pi, 0, \pi$, we have $\sin\theta = 0$.

$$\cos(2\theta) = 1 - 2(0)^2 = 1$$

For the solutions $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$, we have $\sin\theta = 1/2$.

$$\cos(2\theta) = 1 - 2(1/2)^2 = 1 - 2(1/4) = 1 - 1/2 = 1/2$$

Now, we sum these values for all elements in S.

$$T = \underbrace{(1)}_{\theta=-\pi} + \underbrace{(1)}_{\theta=0} + \underbrace{(1)}_{\theta=\pi} + \underbrace{(1/2)}_{\theta=\pi/6} + \underbrace{(1/2)}_{\theta=5\pi/6}$$
$$T = 3 + 1 = 4$$

Step 3: Calculate $T + n(S)$.

$$T + n(S) = 4 + 5 = 9$$

> **Quick Tip**
>
> When solving trigonometric equations, be careful about dividing by expressions that can be zero. It's often safer to factor. If you do divide, you must separately consider the case where the expression is zero. Also, always check if the solutions obtained are within the specified domain and don't make any part of the original equation undefined (like $\tan(\pi/2)$).

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20. The number of choices for $\Delta \in \{\wedge, \vee, \implies, \iff\}$, such that $(p\Delta q) \implies ((p\Delta\neg q) \vee ((\neg p)\Delta q))$ is a tautology, is :

(A) 1
(B) 2
(C) 3
(D) 4

Correct Answer: (B) 2

Solution:

Step 1: Understand the condition for Tautology.
Let the given statement be S. An implication $A \implies B$ is a tautology if it is true for all possible truth values of p and q. This fails only if we can find a case where the antecedent $A = (p\Delta q)$ is True and the consequent $B = ((p\Delta\neg q) \vee ((\neg p)\Delta q))$ is False. We need to find for which operators Δ this failing case never occurs.

Step 2: Test each logical operator for Δ.

Case 1: Δ is $\wedge$ (AND)
The statement is $(p \wedge q) \implies ((p \wedge \neg q) \vee (\neg p \wedge q))$.
Let's test the case where the antecedent $(p \wedge q)$ is True. This occurs only when $p = T$ and $q = T$.
- Antecedent: $T \wedge T = T$.
- Consequent: $(T \wedge \neg T) \vee (\neg T \wedge T) = (T \wedge F) \vee (F \wedge T) = F \vee F = F$.
The statement becomes $T \implies F$, which is False. So, this is not a tautology.

Case 2: Δ is $\vee$ (OR)
The statement is $(p \vee q) \implies ((p \vee \neg q) \vee (\neg p \vee q))$.
Let's analyze the consequent: $(p \vee \neg q) \vee (\neg p \vee q)$.
This expression is equivalent to $p \vee \neg q \vee \neg p \vee q$, which can be rearranged as $(p \vee \neg p) \vee (q \vee \neg q)$.
Since $p \vee \neg p$ is always True, and $q \vee \neg q$ is always True, the consequent is $T \vee T$, which is always True.
The statement is of the form $A \implies T$, which is always a tautology regardless of the truth value of A. So, $\vee$ is a valid choice.

Case 3: Δ is $\implies$ (IMPLIES)
The statement is $(p \implies q) \implies ((p \implies \neg q) \vee (\neg p \implies q))$.
Let's analyze the consequent: $(p \implies \neg q) \vee (\neg p \implies q)$.
This is equivalent to $(\neg p \vee \neg q) \vee (p \vee q)$. Rearranging gives $(p \vee \neg p) \vee (q \vee \neg q)$, which is $T \vee T = T$.
The consequent is always True. The statement is of the form $A \implies T$, which is always a tautology. So, $\implies$ is a valid choice.

Case 4: Δ is $\iff$ (BICONDITIONAL)
The statement is $(p \iff q) \implies ((p \iff \neg q) \vee (\neg p \iff q))$.
The expression $p \iff \neg q$ is true if p and q have opposite truth values. This is equivalent to $p \oplus q$ (XOR).
The expression $\neg p \iff q$ is also true if p and q have opposite truth values. This is also $p \oplus q$.

So the consequent is $(p \oplus q) \vee (p \oplus q)$, which is simply $p \oplus q$.
The statement becomes $(p \iff q) \implies (p \oplus q)$.
Let's test the case when $p = T$ and $q = T$.
- Antecedent: $T \iff T = T$.
- Consequent: $T \oplus T = F$.
The statement becomes $T \implies F$, which is False. So, this is not a tautology.

Step 3: Final Answer
The statement is a tautology for $\Delta = \vee$ and $\Delta = \implies$.
Therefore, there are 2 choices for Δ.

> Quick Tip
>
> When checking for tautologies, first look for simplifications. In this problem, the consequent $((p\Delta\neg q) \vee ((\neg p)\Delta q))$ simplifies to a tautology (always True) for $\Delta = \vee$ and $\Delta = \implies$. An implication with a true consequent, $A \implies T$, is always a tautology. This can save time compared to building a full truth table.

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21. The number of one-one functions f: {a, b, c, d} → {0, 1, 2, . . ., 10} such that 2f(a) - f(b) + 3f(c) + f(d) = 0 is ________.

Correct Answer: 31

Solution:

Step 1: Understanding the Question
We need to find the number of one-to-one functions from a set with 4 elements to a set with 11 elements, subject to a linear equation.
Let $f(a) = x, f(b) = y, f(c) = z, f(d) = w$.
Since the function is one-one, x, y, z, w must be distinct integers from the set {0, 1, 2, ..., 10}.
The given equation is $2x - y + 3z + w = 0$, which can be rearranged as $y = 2x + 3z + w$.
Our task is to find the number of ordered quadruplets (x, z, w, y) of distinct integers in {0, 1, ..., 10} that satisfy this equation.

Step 2: Finding the Solutions Systematically
We can iterate through possible values for z, as it has the largest coefficient, which will constrain the other variables the most. The value of y must be between 0 and 10.

Case 1: z = 0
The equation becomes $y = 2x + w$. We need to choose three distinct integers x, w, y from {1, ..., 10} (since z=0 is already used).
- If x = 1: $y = 2 + w$. w can be {2,3,4,5,6,7,8}, giving y={4,5,6,7,8,9,10}. All values are

distinct. (7 solutions)
- If x = 2: $y = 4 + w$. w can be {1,3,4,5,6}, giving y={5,7,8,9,10}. All values are distinct. (5 solutions)
- If x = 3: $y = 6 + w$. w can be {1,2,4}, giving y={7,8,10}. All values are distinct. (3 solutions)
- If x = 4: $y = 8 + w$. w can be {1,2}, giving y={9,10}. All values are distinct. (2 solutions)

Total for z=0: $7 + 5 + 3 + 2 = 17$ solutions.

Case 2: z = 1

The equation becomes $y = 2x + 3 + w$. x, w must be distinct and not equal to 1.
- If x = 0: $y = 3 + w$. w can be {2,3,4,5,6,7}, giving y={5,6,7,8,9,10}. All distinct. (6 solutions)
- If x = 2: $y = 7 + w$. w can be {0,3}, giving y={7,10}. All distinct. (2 solutions)
- If x = 3: $y = 9 + w$. w can be {0}, giving y={9}. All distinct. (1 solution)

Total for z=1: $6 + 2 + 1 = 9$ solutions.

Case 3: z = 2

The equation becomes $y = 2x + 6 + w$. x, w must be distinct and not equal to 2.
- If x = 0: $y = 6 + w$. w can be {1,3,4}, giving y={7,9,10}. All distinct. (3 solutions)
- If x = 1: $y = 8 + w$. w can be {0}, giving y={8}. All distinct. (1 solution)

Total for z=2: $3 + 1 = 4$ solutions.

Case 4: z = 3

The equation becomes $y = 2x + 9 + w$. x, w must be distinct and not equal to 3.
- If x = 0: $y = 9 + w$. w can be {1}, giving y={10}. All distinct. (1 solution)

For any other values, y will be greater than 10.
Total for z=3: 1 solution.

For $z \geq 4$, $3z \geq 12$, so y will always be greater than 10. There are no solutions.

Step 3: Final Answer

The total number of possible functions is the sum of the solutions from all cases.
Total = 17(for z=0) + 9(for z=1) + 4(for z=2) + 1(for z=3) = 31.
Each unique valid quadruplet corresponds to one unique one-one function. So there are 31 such functions.

> Quick Tip
>
> For counting problems with multiple constraints, break the problem down into manageable cases. Choosing the variable with the largest coefficient to iterate over (like 'z' here) often reduces the number of possibilities to check for the other variables.

---

22. In an examination, there are 5 multiple choice questions with 3 choices, out of which exactly one is correct. There are 3 marks for each correct answer, -2 marks for each wrong answer and 0 mark if the question is not attempted. Then,

the number of ways a student appearing in the examination gets 5 marks is ________.

Correct Answer: 40

Solution:

Step 1: Understanding the Question and Setting up Equations
Let C be the number of correct answers, W be the number of wrong answers, and N be the number of not attempted questions.
We have two conditions:
1. Total number of questions: $C + W + N = 5$
2. Total marks obtained: $3C - 2W + 0N = 5$
We need to find the number of non-negative integer solutions (C, W, N) and then the number of ways to achieve that outcome.

Step 2: Solving the System of Equations
From the marks equation, we have $3C = 5 + 2W$.
Since C and W must be integers, $5 + 2W$ must be a multiple of 3. We can test values of W from 0 to 5.
- If W=0, 3C = 5 (No integer C)
- If W=1, 3C = 7 (No integer C)
- If W=2, 3C = 9 $\implies$ C=3. - Check with the first equation: If C=3, W=2, then N = 5 - 3 - 2 = 0. This is a valid solution set: (C=3, W=2, N=0).
- If W=3, 3C = 11 (No integer C)
- If W=4, 3C = 13 (No integer C)
- If W=5, 3C = 15 $\implies$ C=5. - Check with the first equation: If C=5, W=5, C+W=10, which is not 5. Not a valid solution.

The only way to get 5 marks is to have exactly 3 correct answers and 2 wrong answers.

Step 3: Calculating the Number of Ways
Now we need to find the number of ways a student can get 3 correct and 2 wrong answers out of 5 questions.
- First, choose which 3 of the 5 questions are answered correctly. This can be done in $\binom{5}{3}$ ways.
- For each of the 3 chosen correct questions, there is only 1 correct option. So, 1^3 ways.
- The remaining 2 questions must be answered incorrectly. For each question, there are 2 wrong options. So, 2^2 ways to answer these two questions incorrectly.

Total number of ways = (Ways to choose correct questions) × (Ways to answer them) × (Ways to answer wrong questions)

$$\text{Total ways} = \binom{5}{3} \times 1^3 \times 2^2$$
$$= \frac{5!}{3!2!} \times 1 \times 4$$

$$= \frac{5 \times 4}{2 \times 1} \times 4 = 10 \times 4 = 40$$

Step 4: Final Answer
The number of ways a student can get 5 marks is 40.

> Quick Tip
>
> This is a two-part problem. First, solve a system of linear Diophantine equations to find the combination of correct/wrong/unattempted questions. Second, use combinatorial principles (combinations and powers) to find the number of ways to achieve that specific combination.

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23. Let A($\frac{3}{\sqrt{a}}$, $\sqrt{a}$), a > 0, be a fixed point in the xy-plane. The image of A in y-axis be B and the image of B in x-axis be C. If D(3cosθ, asinθ) is a point in the fourth quadrant such that the maximum area of ΔACD is 12 square units, then a is equal to ________.

Correct Answer: 8

Solution:

Step 1: Determine the coordinates of B and C.
Point A is given as $A(\frac{3}{\sqrt{a}}, \sqrt{a})$.
Point B is the image of A in the y-axis. The y-coordinate remains the same, and the x-coordinate changes sign.
So, $B(-\frac{3}{\sqrt{a}}, \sqrt{a})$.
Point C is the image of B in the x-axis. The x-coordinate remains the same, and the y-coordinate changes sign.
So, $C(-\frac{3}{\sqrt{a}}, -\sqrt{a})$.

Step 2: Calculate the area of ΔACD.
The coordinates of the vertices are A($\frac{3}{\sqrt{a}}, \sqrt{a}$), C($-\frac{3}{\sqrt{a}}, -\sqrt{a}$), and D($3\cos\theta, a\sin\theta$).
It is easier to calculate the length of the base AC and the perpendicular height from D to the line AC.

- **Base AC**: The equation of the line passing through A and C.
Slope $m = \frac{\sqrt{a}-(-\sqrt{a})}{\frac{3}{\sqrt{a}}-(-\frac{3}{\sqrt{a}})} = \frac{2\sqrt{a}}{6/\sqrt{a}} = \frac{2a}{6} = \frac{a}{3}$.
Equation of line AC is $y-\sqrt{a} = \frac{a}{3}(x-\frac{3}{\sqrt{a}})$, which simplifies to $y-\sqrt{a} = \frac{a}{3}x-\sqrt{a}$, or $ax-3y = 0$.
Length of base AC $= \sqrt{(\frac{6}{\sqrt{a}})^2 + (2\sqrt{a})^2} = \sqrt{\frac{36}{a} + 4a} = 2\sqrt{\frac{9}{a} + a}$.

- **Height from D**: The perpendicular distance from $D(3\cos\theta, a\sin\theta)$ to the line $ax - 3y = 0$. Height $h = \frac{|a(3\cos\theta) - 3(a\sin\theta)|}{\sqrt{a^2+(-3)^2}} = \frac{3a|\cos\theta - \sin\theta|}{\sqrt{a^2+9}}$.
- **Area**: Area $= \frac{1}{2} \times \text{base} \times \text{height}$.
Area $= \frac{1}{2} \times 2\sqrt{\frac{9+a^2}{a}} \times \frac{3a|\cos\theta-\sin\theta|}{\sqrt{a^2+9}} = \sqrt{\frac{a^2+9}{a}} \times \frac{3a|\cos\theta-\sin\theta|}{\sqrt{a^2+9}}$
Area $= \frac{\sqrt{a^2+9}}{\sqrt{a}} \times \frac{3a|\cos\theta-\sin\theta|}{\sqrt{a^2+9}} = 3\sqrt{a}|\cos\theta - \sin\theta|$.

Step 3: Maximize the Area and Solve for a.
The area depends on θ. We need to maximize $|\cos\theta - \sin\theta|$.
We know that $-\sqrt{2} \le \cos\theta - \sin\theta \le \sqrt{2}$. The maximum value of $|\cos\theta - \sin\theta|$ is $\sqrt{2}$.
This occurs when $\theta = 3\pi/4$ or $7\pi/4$. Since D is in the fourth quadrant, θ must be in $(3\pi/2, 2\pi)$.
The value $\theta = 7\pi/4$ is in this range.
Maximum Area $= 3\sqrt{a}(\sqrt{2}) = 3\sqrt{2a}$.
We are given that the maximum area is 12.

$$3\sqrt{2a} = 12$$
$$\sqrt{2a} = 4$$

Squaring both sides:

$$2a = 16$$
$$a = 8$$

Quick Tip

When calculating the area of a triangle, if one side is easy to define (e.g., vertical, horizontal, or passing through the origin), it's often simpler to use the formula $\frac{1}{2} \times \text{base} \times \text{height}$ instead of the determinant formula. Maximizing expressions like $A\cos\theta + B\sin\theta$ uses the property that its maximum value is $\sqrt{A^2+B^2}$.

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24. Let a line having direction ratios 1, -4, 2 intersect the lines $\frac{x-7}{3} = \frac{y-1}{-1} = \frac{z+2}{1}$ and $\frac{x}{2} = \frac{y-7}{3} = \frac{z}{1}$ at the points A and B. Then $(AB)^2$ is equal to ________.

Correct Answer: 84

Solution:

Step 1: Parameterize the points A and B.
Let the first line be L1 and the second line be L2.
Any point A on L1 can be written as:
$A = (3k + 7, -k + 1, k - 2)$ for some scalar k.
Any point B on L2 can be written as:

B $= (2m, 3m + 7, m)$ for some scalar m.

Step 2: Use the direction ratios of the line AB.
The direction ratios (DRs) of the line passing through A and B are given by the differences in their coordinates:
DRs of AB $= ((2m) - (3k + 7), (3m + 7) - (-k + 1), (m) - (k - 2))$
DRs of AB $= (2m - 3k - 7, 3m + k + 6, m - k + 2)$
We are given that this line has direction ratios proportional to (1, -4, 2).
This gives us a system of equations:

$$\frac{2m - 3k - 7}{1} = \frac{3m + k + 6}{-4} = \frac{m - k + 2}{2}$$

Step 3: Solve the system of equations for k and m.
From the first and third ratios:
$2(2m - 3k - 7) = 1(m - k + 2)$
$4m - 6k - 14 = m - k + 2$
3m - 5k = 16 (Equation I)

From the second and third ratios:
$2(3m + k + 6) = -4(m - k + 2)$
$6m + 2k + 12 = -4m + 4k - 8$
$10m - 2k = -20$
5m - k = -10 $\implies$ **k = 5m + 10** (Equation II)

Substitute Equation II into Equation I:
$3m - 5(5m + 10) = 16$
$3m - 25m - 50 = 16$
$-22m = 66 \implies m = -3$

Now find k using Equation II:
$k = 5(-3) + 10 = -15 + 10 = -5$

Step 4: Find the coordinates of A and B and calculate $(\mathbf{AB})^2$.
Using k = -5 for point A on L1:
A $= (3(-5) + 7, -(-5) + 1, -5 - 2) = (-15 + 7, 5 + 1, -7) = (-8, 6, -7)$
Using m = -3 for point B on L2:
B $= (2(-3), 3(-3) + 7, -3) = (-6, -9 + 7, -3) = (-6, -2, -3)$

Now, calculate the square of the distance between A and B:
$(\text{AB})^2 = (-6 - (-8))^2 + (-2 - 6)^2 + (-3 - (-7))^2$
$(\text{AB})^2 = (2)^2 + (-8)^2 + (4)^2$
$(\text{AB})^2 = 4 + 64 + 16 = 84$

Quick Tip

Problems involving a line intersecting two skew lines can be solved by parameterizing a general point on each line. The direction ratios of the line connecting these two general points are then set to be proportional to the given direction ratios, leading to a system of two linear equations in two parameters.

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25. The number of points where the function

$$f(x) = \begin{cases} |2x^2 - 3x - 7| & \textbf{if } x \le -1 \\ [4x^2 - 1] & \textbf{if } -1 < x < 1 \\ |x+1| + |x-2| & \textbf{if } x \ge 1 \end{cases}$$

, [t] denotes the greatest integer $\le$ t, is discontinuous is ________.

Correct Answer: 7

Solution:

Step 1: Analyze continuity in the given intervals.
- For $x < -1$ and $x > 1$, the function is defined by expressions involving polynomials and absolute values. These are continuous everywhere, so there are no points of discontinuity in these regions.
- For $-1 < x < 1$, the function is $f(x) = [4x^2 - 1]$. The greatest integer function $[y]$ is discontinuous whenever y is an integer. We need to find the values of x in $(-1, 1)$ for which $4x^2 - 1$ is an integer.
Let's find the range of $g(x) = 4x^2 - 1$ for $x \in (-1, 1)$.
Since $0 \le x^2 < 1$, we have $0 \le 4x^2 < 4$, and $-1 \le 4x^2 - 1 < 3$.
The integer values in the range [-1, 3) are -1, 0, 1, 2.
- $4x^2 - 1 = -1 \implies 4x^2 = 0 \implies x = 0$. (1 point)
- $4x^2 - 1 = 0 \implies 4x^2 = 1 \implies x = \pm 1/2$. (2 points)
- $4x^2 - 1 = 1 \implies 4x^2 = 2 \implies x = \pm 1/\sqrt{2}$. (2 points)
- $4x^2 - 1 = 2 \implies 4x^2 = 3 \implies x = \pm\sqrt{3}/2$. (2 points)
All these 7 points lie within the interval $(-1, 1)$. So, there are 7 points of discontinuity inside this interval.

Step 2: Check continuity at the boundary points x = -1 and x = 1.
At x = -1:
- Left-hand limit (and value): $f(-1) = |2(-1)^2 - 3(-1) - 7| = |2 + 3 - 7| = |-2| = 2$.
- Right-hand limit: $\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+}[4x^2 - 1]$. As $x \to -1^+$, x is slightly greater than -1 (e.g., -0.99). So x^2 is slightly less than 1. Then $4x^2$ is slightly less than 4, and $4x^2 - 1$ is slightly less than 3. The limit is $[3^-] = 2$.
Since Left Limit = Right Limit = f(-1) = 2, the function is continuous at x = -1.

At x = 1:
- Right-hand limit (and value): $f(1) = |1 + 1| + |1 - 2| = 2 + 1 = 3$.

- Left-hand limit: $\lim_{x\to1^-} f(x) = \lim_{x\to1^-}[4x^2-1]$. As $x \to 1^-$, x is slightly less than 1 (e.g., 0.99). So x^2 is slightly less than 1. Then $4x^2$ is slightly less than 4, and $4x^2-1$ is slightly less than 3. The limit is $[3^-] = 2$.
Since Left Limit (2) $\neq$ f(1) (3), the function is discontinuous at x = 1.

Step 3: Final Answer
The points of discontinuity are the 7 points within the interval $(-1, 1)$ and the boundary point $x = 1$.
Total number of points of discontinuity = 7 + 1 = 8.

> Quick Tip
>
> For piecewise functions, always check for discontinuities at the points where the function definition changes. For the greatest integer function $[g(x)]$, discontinuities occur wherever $g(x)$ takes on an integer value. Be careful with limits at the boundary points, especially for floor/ceiling functions.

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26. Let $f(\theta) = \sin\theta + \int_{-\pi/2}^{\pi/2}(\sin\theta + t\cos\theta)f(t)dt$. Then the value of $\int_0^{\pi/2} f(\theta)d\theta$ is ________.

Correct Answer: -1

Solution:

Step 1: Simplify the integral equation.
The given equation is an integral equation. We can separate the terms involving θ from the integral.

$$f(\theta) = \sin\theta + \sin\theta \int_{-\pi/2}^{\pi/2} f(t)dt + \cos\theta \int_{-\pi/2}^{\pi/2} tf(t)dt$$

The two integrals are constants with respect to θ. Let's name them.
Let $A = \int_{-\pi/2}^{\pi/2} f(t)dt$ and $B = \int_{-\pi/2}^{\pi/2} tf(t)dt$.
The function then has the form:

$$f(\theta) = \sin\theta + A\sin\theta + B\cos\theta = (1 + A)\sin\theta + B\cos\theta$$

Step 2: Solve for the constants A and B.
Now we substitute this form of $f(\theta)$ back into the definitions of A and B to create a system of equations.
- For A:
$A = \int_{-\pi/2}^{\pi/2}[(1 + A)\sin t + B\cos t]dt$
$A = (1 + A)[-\cos t]_{-\pi/2}^{\pi/2} + B[\sin t]_{-\pi/2}^{\pi/2}$

$A = (1 + A)(-\cos(\pi/2) - (-\cos(-\pi/2))) + B(\sin(\pi/2) - \sin(-\pi/2))$
$A = (1 + A)(0 - 0) + B(1 - (-1)) = 2B$. So, $\mathbf{A = 2B}$.

- For B:
$B = \int_{-\pi/2}^{\pi/2} t[(1 + A)\sin t + B\cos t]dt$
$B = (1 + A)\int_{-\pi/2}^{\pi/2} t\sin t dt + B\int_{-\pi/2}^{\pi/2} t\cos t dt$
The function $t\cos t$ is odd, so its integral over the symmetric interval $[-\pi/2, \pi/2]$ is 0.
The function $t\sin t$ is even. So $\int_{-\pi/2}^{\pi/2} t\sin t dt = 2\int_0^{\pi/2} t\sin t dt$.
Using integration by parts: $\int t\sin t dt = -t\cos t - \int(-\cos t)dt = -t\cos t + \sin t$.
So, $2\int_0^{\pi/2} t\sin t dt = 2[-t\cos t + \sin t]_0^{\pi/2} = 2((0 + 1) - (0 + 0)) = 2$.
Substituting back into the equation for B:
$B = (1 + A)(2) + B(0)$. So, $\mathbf{B = 2(1+A)}$.

Now we solve the system: $A = 2B$ and $B = 2(1 + A)$.
$B = 2(1 + 2B) = 2 + 4B \implies -3B = 2 \implies B = -2/3$.
$A = 2(-2/3) = -4/3$.

Step 3: Determine the function f(θ) and evaluate the final integral.
The function is $f(\theta) = (1 - 4/3)\sin\theta - (2/3)\cos\theta = -\frac{1}{3}\sin\theta - \frac{2}{3}\cos\theta$.
We need to calculate $\int_0^{\pi/2} f(\theta)d\theta$.

$$\int_0^{\pi/2}(-\frac{1}{3}\sin\theta - \frac{2}{3}\cos\theta)d\theta = -\frac{1}{3}\int_0^{\pi/2}(\sin\theta + 2\cos\theta)d\theta$$
$$= -\frac{1}{3}[-\cos\theta + 2\sin\theta]_0^{\pi/2}$$
$$= -\frac{1}{3}\left((-\cos(\pi/2) + 2\sin(\pi/2)) - (-\cos(0) + 2\sin(0))\right)$$
$$= -\frac{1}{3}((0 + 2) - (-1 + 0)) = -\frac{1}{3}(2 - (-1)) = -\frac{1}{3}(3) = -1$$

Quick Tip

This type of integral equation, where the unknown function appears inside the integral, can often be solved by assuming the integrals are constants. This reduces the problem to an algebraic one of finding those constants. Remember to use properties of even and odd functions to simplify integrals over symmetric intervals.

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27. Let $\text{Max}_{0\le x\le 2}\{\frac{9-x^2}{5-x}\} = \alpha$ and $\text{Min}_{0\le x\le 2}\{\frac{9-x^2}{5-x}\} = \beta$.
If $\int_{\beta-\frac{8}{3}}^{2\alpha-1} \text{Max}\{\frac{9-x^2}{5-x}, x\}dx = \alpha_1 + \alpha_2\log_e(\frac{8}{15})$, then $\alpha_1 + \alpha_2$ is equal to ________.

Correct Answer: 34

Solution:

Step 1: Find the values of α and β.
Let the function be $f(x) = \frac{9-x^2}{5-x}$. To find its maximum and minimum values in the interval [0, 2], we first find its derivative.
Using algebraic long division, we can rewrite $f(x)$:

$$f(x) = \frac{-(x^2-9)}{-(x-5)} = \frac{x^2-25+16}{x-5} = \frac{(x-5)(x+5)+16}{x-5} = x+5+\frac{16}{x-5}$$

Now, find the derivative:

$$f'(x) = 1 - \frac{16}{(x-5)^2}$$

Set $f'(x) = 0$ to find critical points:

$$1 = \frac{16}{(x-5)^2} \implies (x-5)^2 = 16 \implies x-5 = \pm 4$$

This gives $x = 9$ or $x = 1$. The only critical point in the interval [0, 2] is $x = 1$.
Now we evaluate $f(x)$ at the critical point and the endpoints of the interval:
- $f(0) = \frac{9-0}{5-0} = \frac{9}{5} = 1.8$
- $f(1) = \frac{9-1}{5-1} = \frac{8}{4} = 2$
- $f(2) = \frac{9-4}{5-2} = \frac{5}{3} \approx 1.67$

The maximum value is $\alpha = 2$, and the minimum value is $\beta = \frac{5}{3}$.

Step 2: Set up the definite integral.
The limits of integration are:
- Lower limit: $\beta - \frac{8}{3} = \frac{5}{3} - \frac{8}{3} = -\frac{3}{3} = -1$.
- Upper limit: $2\alpha - 1 = 2(2) - 1 = 3$.

The integral is $I = \int_{-1}^{3} \text{Max}\{f(x), x\}dx$.
To evaluate this, we need to find where $f(x) > x$.

$$\frac{9-x^2}{5-x} > x \implies 9 - x^2 > 5x - x^2 \implies 9 > 5x \implies x < \frac{9}{5}$$

So, for $x < 9/5$, $\text{Max}\{f(x), x\} = f(x)$, and for $x > 9/5$, $\text{Max}\{f(x), x\} = x$.
We split the integral at $x = 9/5$:

$$I = \int_{-1}^{9/5} \frac{9-x^2}{5-x}dx + \int_{9/5}^{3} xdx$$

Step 3: Evaluate the integral.
The first part of the integral is:

$$\int_{-1}^{9/5} (x+5+\frac{16}{x-5})dx = \left[\frac{x^2}{2} + 5x + 16\ln|x-5|\right]_{-1}^{9/5}$$

$$= \left(\frac{(9/5)^2}{2} + 5(\frac{9}{5}) + 16\ln|\frac{9}{5} - 5|\right) - \left(\frac{(-1)^2}{2} - 5 + 16\ln|-1-5|\right)$$

$$= \left(\frac{81}{50} + 9 + 16\ln(\frac{16}{5})\right) - \left(-\frac{9}{2} + 16\ln(6)\right)$$

$$= \frac{531}{50} + \frac{225}{50} + 16\ln(\frac{16}{5}) - 16\ln(6) = \frac{756}{50} + 16\ln(\frac{16}{30}) = \frac{378}{25} + 16\ln(\frac{8}{15})$$

The second part of the integral is:

$$\int_{9/5}^{3} x dx = \left[\frac{x^2}{2}\right]_{9/5}^{3} = \frac{9}{2} - \frac{(9/5)^2}{2} = \frac{9}{2} - \frac{81}{50} = \frac{225-81}{50} = \frac{144}{50} = \frac{72}{25}$$

Adding the two parts:

$$I = \left(\frac{378}{25} + 16\ln(\frac{8}{15})\right) + \frac{72}{25} = \frac{450}{25} + 16\ln(\frac{8}{15}) = 18 + 16\ln(\frac{8}{15})$$

Step 4: Final Answer

We are given that the integral is equal to $\alpha_1 + \alpha_2 \log_e(\frac{8}{15})$.
Comparing our result with this form, we have:
$\alpha_1 = 18$ and $\alpha_2 = 16$.
The value we need to find is $\alpha_1 + \alpha_2$.

$$\alpha_1 + \alpha_2 = 18 + 16 = 34$$

> **Quick Tip**
>
> For integrals involving $\max(f(x), g(x))$ or $\min(f(x), g(x))$, the first step is always to solve the equation $f(x) = g(x)$. The roots of this equation will be the points where you need to split the integral. Then, test a point in each sub-interval to determine which function is larger.

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28. If two tangents drawn from a point (α, β) lying on the ellipse $25x^2 + 4y^2 = 1$ to the parabola $y^2 = 4x$ are such that the slope of one tangent is four times the other, then the value of $(10\alpha + 5)^2 + (16\beta^2 + 50)^2$ equals ________.

Correct Answer: 2929

Solution:

Step 1: Formulate the tangent condition.

The equation of a tangent to the parabola $y^2 = 4x$ is $y = mx + \frac{1}{m}$.
Since this tangent passes through (α, β), we have $\beta = m\alpha + \frac{1}{m}$.
This gives a quadratic equation in m: $\alpha m^2 - \beta m + 1 = 0$.
The roots of this equation are the slopes of the two tangents, m_1 and m_2.

Step 2: Use the given slope condition and Vieta's formulas.

From Vieta's formulas:

Sum of slopes: $m_1 + m_2 = \frac{\beta}{\alpha}$
Product of slopes: $m_1 m_2 = \frac{1}{\alpha}$
We are given the condition $m_1 = 4m_2$. Substituting this into the sum and product equations:
$4m_2 + m_2 = 5m_2 = \frac{\beta}{\alpha}$ (1)
$4m_2 \cdot m_2 = 4m_2^2 = \frac{1}{\alpha}$ (2)
From (1), $m_2 = \frac{\beta}{5\alpha}$. Substitute this into (2):
$4(\frac{\beta}{5\alpha})^2 = \frac{1}{\alpha} \implies \frac{4\beta^2}{25\alpha^2} = \frac{1}{\alpha}$.
Assuming $\alpha \neq 0$, we get $4\beta^2 = 25\alpha$.

Step 3: Use the condition that (α, β) is on the ellipse.
The point (α, β) lies on the ellipse $25x^2 + 4y^2 = 1$, so it satisfies the equation:
$25\alpha^2 + 4\beta^2 = 1$.
Substitute $4\beta^2 = 25\alpha$ into the ellipse equation:
$25\alpha^2 + 25\alpha = 1 \implies 25\alpha^2 + 25\alpha - 1 = 0$.

Step 4: Evaluate the given expression.
We need to find the value of $E = (10\alpha + 5)^2 + (16\beta^2 + 50)^2$.
First, substitute for β^2. We know $4\beta^2 = 25\alpha$, so $16\beta^2 = 4(4\beta^2) = 4(25\alpha) = 100\alpha$.
The expression becomes:
$E = (10\alpha + 5)^2 + (100\alpha + 50)^2$.
Notice that $100\alpha + 50 = 10(10\alpha + 5)$.
$E = (10\alpha + 5)^2 + [10(10\alpha + 5)]^2 = (10\alpha + 5)^2 + 100(10\alpha + 5)^2 = 101(10\alpha + 5)^2$.
Let's find the value of $(10\alpha + 5)^2$.
$(10\alpha + 5)^2 = 100\alpha^2 + 100\alpha + 25 = 4(25\alpha^2 + 25\alpha) + 25$.
From Step 3, we know $25\alpha^2 + 25\alpha = 1$.
So, $(10\alpha + 5)^2 = 4(1) + 25 = 29$.
Finally, substitute this value back into the expression for E:

$E = 101 \times 29 = 2929$.
(Note: The official answer key provided was 1. This would only be possible if the question intended to ask for the value of $25\alpha^2 + 4\beta^2$, which is 1 by definition of the point lying on the ellipse. The question as written leads to 2929.)

Quick Tip

For problems involving tangents from an external point to a conic, the standard method is to use the general equation of a tangent and form a quadratic equation whose roots are the slopes. Vieta's formulas combined with the given conditions on the slopes usually lead to a relation between the coordinates of the point.

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29. Let S be the region bounded by the curves $y = x^3$ and $y^2 = x$. The curve $y = 2|x|$ divides S into two regions of areas R_1 and R_2.

If max{$\mathbf{R_1}$, $\mathbf{R_2}$} = $\mathbf{R_2}$, then $\frac{R_2}{R_1}$ is equal to ________.

Correct Answer: 19

Solution:

Step 1: Find the total area of the region S.
The region S is bounded by $y = x^3$ and $y^2 = x$ (or $y = \sqrt{x}$ for $x \geq 0$).
First, find the points of intersection: $x^3 = \sqrt{x} \implies x^6 = x \implies x(x^5 - 1) = 0$. The intersection points are at $x = 0$ and $x = 1$.
In the interval $[0, 1]$, $\sqrt{x} \geq x^3$.
Total Area of S = $\int_0^1 (\sqrt{x} - x^3)dx = [\frac{2}{3}x^{3/2} - \frac{x^4}{4}]_0^1 = \frac{2}{3} - \frac{1}{4} = \frac{8-3}{12} = \frac{5}{12}$.

Step 2: Understand how the curve $y = 2|x|$ divides the region S.
Since S is in the first quadrant, we consider $y = 2x$.
The line $y = 2x$ intersects the upper boundary of S, $y = \sqrt{x}$, at $2x = \sqrt{x} \implies 4x^2 = x \implies x(4x - 1) = 0$. Intersection at $x = 0$ and $x = 1/4$.
The line $y = 2x$ intersects the lower boundary of S, $y = x^3$, at $2x = x^3 \implies x(x^2 - 2) = 0$. Intersection at $x = 0$ and $x = \sqrt{2}$.
For $x \in (0, 1/4)$, the line $y = 2x$ is between $y = \sqrt{x}$ and $y = x^3$, thus dividing the region S.

Step 3: Calculate the areas of the two sub-regions, $\mathbf{R_1}$ and $\mathbf{R_2}$.
The line $y = 2x$ splits the region S. Let's define $\mathrm{R_1}$ as the area between the upper boundary of S ($y = \sqrt{x}$) and the dividing line ($y = 2x$). This region exists from $x = 0$ to their intersection at $x = 1/4$.

$$R_1 = \int_0^{1/4} (\sqrt{x} - 2x)dx = [\frac{2}{3}x^{3/2} - x^2]_0^{1/4}$$

$$R_1 = (\frac{2}{3}(\frac{1}{4})^{3/2} - (\frac{1}{4})^2) - 0 = \frac{2}{3}(\frac{1}{8}) - \frac{1}{16} = \frac{1}{12} - \frac{1}{16} = \frac{4-3}{48} = \frac{1}{48}$$

The area $\mathrm{R_2}$ is the rest of the area of S.

$$R_2 = \text{Area(S)} - R_1 = \frac{5}{12} - \frac{1}{48} = \frac{20-1}{48} = \frac{19}{48}$$

Step 4: Verify the condition and find the ratio.
We check the condition max{$\mathrm{R_1}$, $\mathrm{R_2}$} = $\mathrm{R_2}$.
Since $19/48 > 1/48$, we have $R_2 > R_1$, so the condition is satisfied.
We need to find the ratio $\frac{R_2}{R_1}$.

$$\frac{R_2}{R_1} = \frac{19/48}{1/48} = 19$$

💡 Quick Tip

When a region's area is divided by a curve, it's often easiest to calculate the total area and the area of one of the smaller, simpler sub-regions. The area of the second sub-region can then be found by subtraction. Always sketch the curves to visualize the regions.

---

30. If the shortest distance between the lines $\vec{r} = (-\hat{i} + 3\hat{k}) + \lambda(\hat{i} - a\hat{j})$ and $\vec{r} = (-\hat{j} + 2\hat{k}) + \mu(\hat{i} - \hat{j} + \hat{k})$ is $\sqrt{\frac{2}{3}}$, then the integral value of a is equal to ________.

Correct Answer: 2

Solution:

Step 1: Identify the vectors from the line equations.
The lines are of the form $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$.
$\vec{a}_1 = -\hat{i} + 0\hat{j} + 3\hat{k}$
$\vec{b}_1 = \hat{i} - a\hat{j} + 0\hat{k}$
$\vec{a}_2 = 0\hat{i} - \hat{j} + 2\hat{k}$
$\vec{b}_2 = \hat{i} - \hat{j} + \hat{k}$

Step 2: Apply the shortest distance formula.
The shortest distance d between two skew lines is given by:

$$d = \frac{|(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2)|}{|\vec{b}_1 \times \vec{b}_2|}$$

First, calculate the required vector components:
$\vec{a}_2 - \vec{a}_1 = (0 - (-1))\hat{i} + (-1 - 0)\hat{j} + (2 - 3)\hat{k} = \hat{i} - \hat{j} - \hat{k}$.
$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -a & 0 \\ 1 & -1 & 1 \end{vmatrix} = \hat{i}(-a - 0) - \hat{j}(1 - 0) + \hat{k}(-1 - (-a)) = -a\hat{i} - \hat{j} + (a - 1)\hat{k}$.
Now, calculate the magnitudes of the numerator and denominator vectors:
$(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2) = (1)(-a) + (-1)(-1) + (-1)(a - 1) = -a + 1 - a + 1 = 2 - 2a$.
$|\vec{b}_1 \times \vec{b}_2| = \sqrt{(-a)^2 + (-1)^2 + (a - 1)^2} = \sqrt{a^2 + 1 + a^2 - 2a + 1} = \sqrt{2a^2 - 2a + 2}$.

Step 3: Solve for a.
Substitute these into the distance formula and set it equal to the given distance:

$$\sqrt{\frac{2}{3}} = \frac{|2 - 2a|}{\sqrt{2a^2 - 2a + 2}}$$

Square both sides:

$$\frac{2}{3} = \frac{(2(1-a))^2}{2a^2-2a+2} = \frac{4(1-a)^2}{2(a^2-a+1)} = \frac{2(1-a)^2}{a^2-a+1}$$
$$2(a^2-a+1) = 3 \cdot 2(1-a)^2$$
$$a^2-a+1 = 3(1-2a+a^2) = 3-6a+3a^2$$
$$2a^2-5a+2=0$$

Factor the quadratic equation:

$$(2a-1)(a-2)=0$$

This gives two possible values for a: $a = 1/2$ or $a = 2$.

Step 4: Final Answer
The question asks for the integral value of a. Therefore, $a = 2$.

Quick Tip
The formula for the shortest distance between skew lines is fundamental. Be careful with the vector calculations, especially the cross product and the dot product. Squaring both sides is a common technique to eliminate square roots and solve for the unknown parameter.

Physics

31. The bulk modulus of a liquid is 3×10^{10} Nm^{-2}. The pressure required to reduce the volume of liquid by 2% is :

(A) 3×10^8 Nm^{-2}
(B) 9×10^8 Nm^{-2}
(C) 6×10^8 Nm^{-2}
(D) 12×10^8 Nm^{-2}

Correct Answer: (C) 6×10^8 Nm^{-2}

Solution:

Step 1: Understanding the Concept
Bulk modulus (B) is a measure of a substance's resistance to uniform compression. It is defined as the ratio of the infinitesimal pressure increase (dP) to the resulting relative decrease

in volume ($-\frac{dV}{V}$).

Step 2: Key Formula
The formula for bulk modulus is:

$$B = -\frac{\Delta P}{\Delta V/V}$$

where ΔP is the change in pressure (the pressure required), ΔV is the change in volume, and V is the original volume. The negative sign indicates that as pressure increases, volume decreases.

Step 3: Detailed Explanation
We are given:
- Bulk modulus, $B = 3 \times 10^{10}$ N/m^2.
- The volume is to be reduced by 2%. This means the fractional change in volume, $\frac{\Delta V}{V}$, is -2% or -0.02.
We need to find the required pressure, ΔP.
Rearranging the formula to solve for ΔP:

$$\Delta P = -B\left(\frac{\Delta V}{V}\right)$$

Substitute the given values:

$$\Delta P = -(3 \times 10^{10} \text{ N/m}^2) \times (-0.02)$$
$$\Delta P = (3 \times 10^{10}) \times (2 \times 10^{-2}) \text{ N/m}^2$$
$$\Delta P = 6 \times 10^{(10-2)} \text{ N/m}^2$$
$$\Delta P = 6 \times 10^{8} \text{ N/m}^2$$

Step 4: Final Answer
The pressure required to reduce the volume of the liquid by 2% is 6×10^8 Nm^{-2}.

Quick Tip

Remember the definition of bulk modulus relates pressure stress to volumetric strain. A positive pressure change leads to a negative volume change, which is handled by the negative sign in the formula $B = -\Delta P/(\Delta V/V)$. This ensures the bulk modulus B is a positive quantity.

---

32. Given below are two statements: One is labelled as Assertion (A) and the other is labelled as Reason (R).
Assertion (A): In an uniform magnetic field, speed and energy remains the same for a moving charged particle.

Reason (R): Moving charged particle experiences magnetic force perpendicular to its direction of motion.

(A) Both (A) and (R) are true and (R) is the correct explanation of (A).
(B) Both (A) and (R) are true but (R) is NOT the correct explanation of (A).
(C) (A) is true but (R) is false.
(D) (A) is false but (R) is true.

Correct Answer: (A) Both (A) and (R) are true and (R) is the correct explanation of (A).

Solution:

Step 1: Analyzing the Reason (R)
The magnetic force (Lorentz force) on a charged particle with charge q moving with velocity $\vec{v}$ in a magnetic field $\vec{B}$ is given by $\vec{F} = q(\vec{v} \times \vec{B})$.
By the definition of the vector cross product, the resulting force vector $\vec{F}$ is always perpendicular to both the velocity vector $\vec{v}$ and the magnetic field vector $\vec{B}$.
Therefore, the magnetic force is perpendicular to the particle's direction of motion (which is the direction of $\vec{v}$).
So, the Reason (R) is a true statement.

Step 2: Analyzing the Assertion (A) and its link to the Reason (R)
The work done (dW) by a force $\vec{F}$ over a small displacement $d\vec{s}$ is given by $dW = \vec{F} \cdot d\vec{s}$.
Since displacement is velocity times time, $d\vec{s} = \vec{v}dt$.
The rate of work done (power) by the magnetic force is $P = \frac{dW}{dt} = \vec{F} \cdot \vec{v}$.
As established in Step 1, $\vec{F}$ is always perpendicular to $\vec{v}$. The dot product of two perpendicular vectors is zero.
So, $P = \vec{F} \cdot \vec{v} = 0$.
This means the magnetic force does no work on the charged particle.
According to the work-energy theorem, the work done on an object is equal to the change in its kinetic energy ($\Delta K.E.$).
Since $W = 0$, we have $\Delta K.E. = 0$. This means the kinetic energy of the particle remains constant.
Kinetic energy is given by $K.E. = \frac{1}{2}mv^2$. If K.E. and mass m are constant, then the speed v must also be constant.
Therefore, the Assertion (A) is also a true statement.

Step 3: Final Conclusion
The Reason (R) is true, and it directly leads to the conclusion in the Assertion (A). The fact that the force is perpendicular to the velocity is precisely why no work is done and kinetic energy (and speed) remains constant.
Thus, both (A) and (R) are true, and (R) is the correct explanation of (A).

Quick Tip

A key concept in electromagnetism is that magnetic fields can change the direction of a charged particle's motion, but they cannot change its speed or kinetic energy. This is a direct consequence of the Lorentz force law and the work-energy theorem.

---

33. Two identical cells each of emf 1.5 V are connected in parallel across a parallel combination of two resistors each of resistance 20 Ω. A voltmeter connected in the circuit measures 1.2 V. The internal resistance of each cell is :

(A) 2.5 Ω
(B) 4 Ω
(C) 5 Ω
(D) 10 Ω

Correct Answer: (C) 5 Ω

Solution:

Step 1: Determine the equivalent EMF and internal resistance of the cells.
Two identical cells with EMF E and internal resistance r are connected in parallel.
- The equivalent EMF is $E_{eq} = E = 1.5$ V.
- The equivalent internal resistance is given by $\frac{1}{r_{eq}} = \frac{1}{r} + \frac{1}{r} = \frac{2}{r}$, so $r_{eq} = \frac{r}{2}$.

Step 2: Determine the equivalent external resistance.
Two resistors, each with resistance R = 20 Ω, are connected in parallel.
- The equivalent external resistance is given by $\frac{1}{R_{eq}} = \frac{1}{R} + \frac{1}{R} = \frac{2}{R}$, so $R_{eq} = \frac{R}{2} = \frac{20}{2} = 10$ Ω.

Step 3: Use the circuit information to find the internal resistance r.
The circuit consists of the equivalent cell (E_{eq}, r_{eq}) connected to the equivalent external resistance R_{eq}.
The voltmeter measures the terminal voltage across the parallel combination, which is the voltage across R_{eq}. Let this be V = 1.2 V.
The total current flowing from the equivalent cell is given by Ohm's law for the entire circuit:

$$I = \frac{E_{eq}}{R_{eq} + r_{eq}} = \frac{1.5}{10 + r/2}$$

The terminal voltage V is the potential drop across the external resistance:

$$V = I \cdot R_{eq}$$

Substitute the known values and expressions:

$$1.2 = \left(\frac{1.5}{10 + r/2}\right) \times 10$$

Now, solve for r:

$$1.2(10 + r/2) = 1.5 \times 10 = 15$$
$$12 + 1.2(r/2) = 15$$
$$12 + 0.6r = 15$$
$$0.6r = 3$$
$$r = \frac{3}{0.6} = \frac{30}{6} = 5$$

Step 4: Final Answer
The internal resistance of each cell is 5 Ω.

> **Quick Tip**
>
> When simplifying circuits, first find the equivalent of parallel or series components. For cells in parallel, remember the formulas for equivalent EMF and resistance. The terminal voltage of a source is given by $V = E_{eq} - Ir_{eq}$, or equivalently, by the voltage across the external circuit, $V = IR_{eq}$.

---

34. Identify the pair of physical quantities which have different dimensions :

(A) Wave number and Rydberg's constant
(B) Stress and Coefficient of elasticity
(C) Coercivity and Magnetisation
(D) Specific heat capacity and Latent heat

Correct Answer: (D) Specific heat capacity and Latent heat

Solution:

Step 1: Analyze the dimensions of each pair of quantities.

(A) Wave number and Rydberg's constant:
- Wave number (k) is defined as the reciprocal of wavelength, $k = 1/\lambda$. Its dimension is $[L^{-1}]$.
- Rydberg's constant (R) appears in the formula $\frac{1}{\lambda} = R(\dots)$. Therefore, its dimension must also be $[L^{-1}]$.
These have the **same** dimensions.

(B) Stress and Coefficient of elasticity:
- Stress is defined as force per unit area. Dimension of Stress $= \frac{[F]}{[A]} = \frac{[MLT^{-2}]}{[L^2]} = [ML^{-1}T^{-2}]$.
- Coefficient of elasticity (like Young's Modulus) is defined as Stress / Strain. Strain is the ratio of change in dimension to the original dimension, so it is dimensionless.

- Therefore, the dimension of Coefficient of elasticity is the same as Stress.
These have the **same** dimensions.

(C) Coercivity and Magnetisation:
- Magnetisation (M) is the magnetic dipole moment per unit volume. Dimension of M = $\frac{[MagneticMoment]}{[Volume]} = \frac{[AL^2]}{[L^3]} = [AL^{-1}]$.
- Coercivity is the measure of the magnetic field strength (H) needed to demagnetize a material. The dimension of magnetic field strength H is $[AL^{-1}]$.
These have the **same** dimensions.

(D) Specific heat capacity and Latent heat:
- Specific heat capacity (c) is related to heat transfer by $Q = mc\Delta T$. So, $c = \frac{Q}{m\Delta T}$.
- Dimension of c = $\frac{[Energy]}{[Mass][Temperature]} = \frac{[ML^2T^{-2}]}{[M][K]} = [L^2T^{-2}K^{-1}]$.
- Latent heat (L) is related to heat transfer during a phase change by $Q = mL$. So, $L = \frac{Q}{m}$.
- Dimension of L = $\frac{[Energy]}{[Mass]} = \frac{[ML^2T^{-2}]}{[M]} = [L^2T^{-2}]$.
The dimensions are **different** due to the temperature factor [K].

Step 2: Final Answer
The pair of physical quantities with different dimensions is Specific heat capacity and Latent heat.

Quick Tip

To find the dimension of a physical quantity, relate it to fundamental quantities (Mass, Length, Time, etc.) using a defining formula. For example, if you don't remember the dimension of Energy, derive it from Work = Force × Distance.

---

35. A projectile is projected with velocity of 25 m/s at an angle θ with the horizontal. After t seconds its inclination with horizontal becomes zero. If R represents horizontal range of the projectile, the value of θ will be : [use g = 10 m/s^2]

(A) $\frac{1}{2}\sin^{-1}(\frac{5t^2}{4R})$

(B) $\frac{1}{2}\sin^{-1}(\frac{4R}{5t^2})$

(C) $\tan^{-1}(\frac{4t^2}{5R})$

(D) $\cot^{-1}(\frac{R}{20t^2})$

Correct Answer: (D) $\cot^{-1}(\frac{R}{20t^2})$

Solution:

Step 1: Relate the time 't' to the initial velocity and angle.
The initial velocity is $u = 25$ m/s at an angle θ. The components are $u_x = u\cos\theta$ and $u_y = u\sin\theta$.
The problem states that after time t, the inclination with the horizontal becomes zero. This means the projectile is at the highest point of its trajectory, where the vertical component of velocity is zero ($v_y = 0$).
Using the kinematic equation $v_y = u_y - gt$:

$$0 = u\sin\theta - gt$$
$$t = \frac{u\sin\theta}{g} \quad (1)$$

Step 2: Relate the range 'R' to the initial velocity and angle.
The total time of flight is $T = 2t = \frac{2u\sin\theta}{g}$.
The horizontal range R is the horizontal distance covered during the total time of flight:

$$R = u_x \times T = (u\cos\theta) \times \left(\frac{2u\sin\theta}{g}\right) = \frac{u^2(2\sin\theta\cos\theta)}{g} = \frac{u^2\sin(2\theta)}{g}$$

An alternative expression for range is using the time t:

$$R = u_x \times (2t) = (u\cos\theta)(2t) \quad (2)$$

Step 3: Eliminate 'u' and 'g' to find a relation between θ, R, and t.
From equation (1), we have $u\sin\theta = gt$.
From equation (2), we have $u\cos\theta = \frac{R}{2t}$.
Now, we can find $\tan\theta$ by dividing these two expressions:

$$\tan\theta = \frac{u\sin\theta}{u\cos\theta} = \frac{gt}{R/(2t)} = \frac{2gt^2}{R}$$

Substitute the given value $g = 10$ m/s^2:

$$\tan\theta = \frac{2(10)t^2}{R} = \frac{20t^2}{R}$$

Step 4: Final Answer
The relationship is $\tan\theta = \frac{20t^2}{R}$.
This can be rewritten using the inverse cotangent function:

$$\cot\theta = \frac{1}{\tan\theta} = \frac{R}{20t^2}$$
$$\theta = \cot^{-1}\left(\frac{R}{20t^2}\right)$$

This matches option (D).

Quick Tip

For projectile motion problems where you need to find a relationship between variables like R, H, t, and θ, the key is to write down the standard equations for these quantities and then algebraically eliminate the initial speed 'u' and acceleration 'g'. Using $\tan\theta = \sin\theta/\cos\theta$ is a common trick.

---

36. A block of mass 10 kg starts sliding on a surface with an initial velocity of 9.8 ms^{-1}. The coefficient of friction between the surface and block is 0.5. The distance covered by the block before coming to rest is : [use g = 9.8 ms^{-2}]

(A) 4.9 m
(B) 9.8 m
(C) 12.5 m
(D) 19.6 m

Correct Answer: (B) 9.8 m

Solution:

Step 1: Identify the forces and calculate the deceleration.
The block is sliding on a horizontal surface. The forces acting on it are:
- Vertically: Gravity (mg) downwards and Normal force (N) upwards. Since there is no vertical motion, $N = mg$.
- Horizontally: Kinetic friction force (f_k) opposing the motion. There are no other horizontal forces.
The friction force is given by $f_k = \mu_k N = \mu_k mg$.
According to Newton's second law, the net force causes acceleration (in this case, deceleration).

$$F_{net} = -f_k = ma$$

$$-\mu_k mg = ma$$

$$a = -\mu_k g$$

Substitute the given values:

$$a = -0.5 \times 9.8 \text{ m/s}^2 = -4.9 \text{ m/s}^2$$

The negative sign indicates deceleration.

Step 2: Use kinematic equations to find the distance.
We can use the third equation of motion: $v^2 = u^2 + 2as$, where:
- Initial velocity, $u = 9.8$ m/s.
- Final velocity, $v = 0$ (since the block comes to rest).
- Acceleration, $a = -4.9$ m/s^2.

- Distance, s (to be found).
Substitute the values into the equation:

$$0^2 = (9.8)^2 + 2(-4.9)s$$
$$0 = 96.04 - 9.8s$$
$$9.8s = 96.04$$
$$s = \frac{96.04}{9.8} = 9.8$$

Step 3: Final Answer
The distance covered by the block before coming to rest is 9.8 m.

> Quick Tip
>
> For problems involving friction and motion, a good approach is to first use dynamics (Newton's laws) to find the acceleration, and then use kinematics (equations of motion) to find quantities like distance, time, or final velocity. The work-energy theorem ($W_{net} = \Delta K.E.$) can also be a quick alternative: $-f_k \cdot s = 0 - \frac{1}{2}mu^2$.

---

37. A boy ties a stone of mass 100 g to the end of a 2 m long string and whirls it around in a horizontal plane. The string can withstand the maximum tension of 80 N. If the maximum speed with which the stone can revolve is $\frac{K}{\pi}$ rev./min. The value of K is : (Assume the string is massless and unstretchable)

(A) 400
(B) 300
(C) 600
(D) 800

Correct Answer: (C) 600

Solution:

Step 1: Relate tension to centripetal force.
For a stone moving in a horizontal circle, the tension (T) in the string provides the necessary centripetal force (F_c) to keep it on the circular path.
The formula for centripetal force is $F_c = \frac{mv^2}{r}$, where m is the mass, v is the linear speed, and r is the radius of the circle.
So, $T = \frac{mv^2}{r}$.
The maximum tension corresponds to the maximum possible speed.

$$T_{max} = \frac{mv_{max}^2}{r}$$

Step 2: Calculate the maximum linear speed (v_{max}).
We are given:
- Mass, $m = 100$ g $= 0.1$ kg.
- Radius (length of string), $r = 2$ m.
- Maximum tension, $T_{max} = 80$ N.
Substitute these values into the formula:

$$
\begin{aligned}
80 &= \frac{0.1 \times v_{max}^2}{2} \\
160 &= 0.1 \times v_{max}^2 \\
v_{max}^2 &= \frac{160}{0.1} = 1600 \\
v_{max} &= \sqrt{1600} = 40 \text{ m/s}
\end{aligned}
$$

Step 3: Convert the maximum speed to revolutions per minute (rev/min).
The linear speed v is related to the angular speed ω by $v = r\omega$. The units of ω here will be radians per second.

$$\omega_{max} = \frac{v_{max}}{r} = \frac{40 \text{ m/s}}{2 \text{ m}} = 20 \text{ rad/s}$$

Now we need to convert the angular speed from rad/s to rev/min.
- 1 revolution $= 2\pi$ radians
- 1 minute $= 60$ seconds

$$
\begin{aligned}
\text{Speed in rev/min} &= (20\frac{\text{rad}}{\text{s}}) \times (\frac{1 \text{ rev}}{2\pi \text{ rad}}) \times (\frac{60 \text{ s}}{1 \text{ min}}) \\
&= \frac{20 \times 60}{2\pi} \frac{\text{rev}}{\text{min}} = \frac{1200}{2\pi} \frac{\text{rev}}{\text{min}} = \frac{600}{\pi} \frac{\text{rev}}{\text{min}}
\end{aligned}
$$

Step 4: Final Answer
The question states that the maximum speed is $\frac{K}{\pi}$ rev/min.
Comparing this with our result, we have:

$$
\begin{aligned}
\frac{K}{\pi} &= \frac{600}{\pi} \\
K &= 600
\end{aligned}
$$

Quick Tip

In circular motion problems, always identify the force that acts as the centripetal force. For a horizontal circle on a string, it's the tension. Be careful with units: convert all given quantities to standard SI units (kg, m, s) for calculations, and then convert the final answer to the required units (like rev/min).

---

38. A vertical electric field of magnitude 4.9×10^5 N/C just prevents a water droplet of a mass 0.1 g from falling. The value of charge on the droplet will be : (Given g = 9.8 m/s^2)

(A) 1.6 x 10^{-9} C
(B) 2.0 x 10^{-9} C
(C) 3.2 x 10^{-9} C
(D) 0.5 x 10^{-9} C

Correct Answer: (B) 2.0 x 10^{-9} C

Solution:

Step 1: Analyze the forces acting on the water droplet.
The water droplet is suspended in the air, which means it is in equilibrium. The net force acting on it is zero.
There are two forces acting on the droplet:
1. Gravitational Force ($\vec{F_g}$), acting vertically downwards. Its magnitude is $F_g = mg$.
2. Electric Force ($\vec{F_e}$), exerted by the vertical electric field. To counteract gravity, this force must act vertically upwards. Its magnitude is $F_e = |q|E$.

Step 2: Apply the condition for equilibrium.
For the droplet to be prevented from falling, the upward electric force must exactly balance the downward gravitational force.

$$F_e = F_g$$
$$|q|E = mg$$

Step 3: Calculate the value of the charge.
We need to find the magnitude of the charge, $|q|$. Rearranging the formula:

$$|q| = \frac{mg}{E}$$

We are given the following values:
- Mass, $m = 0.1$ g $= 0.1 \times 10^{-3}$ kg $= 10^{-4}$ kg.
- Acceleration due to gravity, $g = 9.8$ m/s^2.
- Electric field magnitude, $E = 4.9 \times 10^5$ N/C.
Substitute these values into the equation:

$$|q| = \frac{(10^{-4} \text{ kg}) \times (9.8 \text{ m/s}^2)}{4.9 \times 10^5 \text{ N/C}}$$
$$|q| = \left(\frac{9.8}{4.9}\right) \times \frac{10^{-4}}{10^5} \text{ C}$$
$$|q| = 2 \times 10^{-9} \text{ C}$$

Step 4: Final Answer
The value of the charge on the droplet is 2×10^{-9} C or 2 nC. Since the electric field is vertical and the force must be upwards, the sign of the charge depends on the direction of the field, which is not specified. However, the magnitude is 2×10^{-9} C.

> Quick Tip
>
> Problems involving static equilibrium (like a charged particle suspended in a field) are solved by setting the vector sum of all forces to zero. Always start by drawing a free-body diagram to identify all the forces and their directions. Pay close attention to unit conversions (e.g., grams to kilograms).

---

39. A particle experiences a variable force $\vec{F} = (4x\hat{i} + 3y^2\hat{j})$ in a horizontal x-y plane. Assume distance in meters and force is newton. If the particle moves from point (1, 2) to point (2, 3) in the x-y plane, then Kinetic Energy changes by :

(A) 50.0 J
(B) 12.5 J
(C) 25.0 J
(D) 0 J

Correct Answer: (C) 25.0 J

Solution:

Step 1: Understanding the Work-Energy Theorem
According to the Work-Energy Theorem, the change in the kinetic energy of a particle is equal to the total work done on it by all forces.

$$\Delta K.E. = W_{total}$$

Step 2: Key Formula for Work Done
The work done by a variable force $\vec{F}$ in moving a particle from an initial position r_1 to a final position r_2 is given by the line integral:

$$W = \int_{r_1}^{r_2} \vec{F} \cdot d\vec{r}$$

Given $\vec{F} = 4x\hat{i} + 3y^2\hat{j}$ and the displacement vector is $d\vec{r} = dx\hat{i} + dy\hat{j}$.

Step 3: Detailed Calculation
The work done integral becomes:

$$W = \int_{(1,2)}^{(2,3)} (4x\hat{i} + 3y^2\hat{j}) \cdot (dx\hat{i} + dy\hat{j})$$

$$W = \int_{(1,2)}^{(2,3)} (4xdx + 3y^2dy)$$

This is an exact differential, meaning the work done is independent of the path taken. We can integrate the terms separately:

$$W = \int_{x=1}^{x=2} 4xdx + \int_{y=2}^{y=3} 3y^2dy$$

$$W = \left[4\frac{x^2}{2}\right]_1^2 + \left[3\frac{y^3}{3}\right]_2^3$$

$$W = [2x^2]_1^2 + [y^3]_2^3$$

$$W = (2(2)^2 - 2(1)^2) + (3^3 - 2^3)$$

$$W = (2 \times 4 - 2) + (27 - 8)$$

$$W = (8 - 2) + 19 = 6 + 19 = 25 \text{ J}$$

Step 4: Final Answer
The work done is 25 J. Therefore, the change in kinetic energy is 25.0 J.

Quick Tip

For a force of the form $\vec{F} = F_x(x)\hat{i} + F_y(y)\hat{j} + F_z(z)\hat{k}$, the work done is path-independent (the force is conservative). The work can be calculated by integrating each component separately over its corresponding coordinate change, which simplifies the calculation significantly.

---

40. The approximate height from the surface of earth at which the weight of the body becomes $\frac{1}{3}$ of its weight on the surface of earth is : [Radius of earth R = 6400 km and $\sqrt{3} = 1.732$]

(A) 3840 km
(B) 4685 km
(C) 2133 km
(D) 4267 km

Correct Answer: (B) 4685 km

Solution:

Step 1: Understanding the Relation between Weight and Gravity
The weight of a body is given by $W = mg$, where g is the acceleration due to gravity. So, the weight is directly proportional to g.
If the weight at height h (W_h) is $\frac{1}{3}$ of the weight at the surface (W_s), it implies that the acceleration due to gravity at height h (g_h) is $\frac{1}{3}$ of the acceleration at the surface (g_s).

$$g_h = \frac{1}{3} g_s$$

Step 2: Key Formula for Gravity at a Height
The acceleration due to gravity at a height h from the surface of the Earth is given by:

$$g_h = g_s \left(\frac{R}{R+h} \right)^2$$

where R is the radius of the Earth.

Step 3: Detailed Calculation
Substitute the condition $g_h = \frac{1}{3} g_s$ into the formula:

$$\frac{1}{3} g_s = g_s \left(\frac{R}{R+h} \right)^2$$

$$\frac{1}{3} = \left(\frac{R}{R+h} \right)^2$$

Take the square root of both sides:

$$\frac{1}{\sqrt{3}} = \frac{R}{R+h}$$

$$R + h = R\sqrt{3}$$

$$h = R\sqrt{3} - R = R(\sqrt{3} - 1)$$

Now, substitute the given values $R = 6400$ km and $\sqrt{3} = 1.732$:

$$h = 6400 \times (1.732 - 1)$$

$$h = 6400 \times 0.732$$

$$h = 4684.8 \text{ km}$$

Step 4: Final Answer
The calculated height is 4684.8 km, which is approximately 4685 km.

> Quick Tip
>
> The formula $g_h = g_s(1 - 2h/R)$ is an approximation valid only for $h \ll R$. When the change in gravity is large (like becoming 1/3), you must use the exact inverse square law formula: $g_h = g_s(R/(R+h))^2$.

41. A resistance of 40 Ω is connected to a source of alternating current rated 220 V, 50 Hz. Find the time taken by the current to change from its maximum value to the rms value :

(A) 2.5 ms
(B) 1.25 ms
(C) 2.5 s
(D) 0.25 s

Correct Answer: (A) 2.5 ms

Solution:

Step 1: Set up the equation for instantaneous current.
The instantaneous current in an AC circuit can be represented by a sinusoidal function. To simplify, let's assume the current is at its maximum value at $t = 0$. We can use the cosine function for this:

$$I(t) = I_{max} \cos(\omega t)$$

where I_{max} is the maximum current and ω is the angular frequency.

Step 2: Use the condition for the current value.
We need to find the time t when the current $I(t)$ becomes equal to its RMS value, I_{rms}.
The relationship between RMS and maximum current is:

$$I_{rms} = \frac{I_{max}}{\sqrt{2}}$$

Setting $I(t) = I_{rms}$:

$$\frac{I_{max}}{\sqrt{2}} = I_{max} \cos(\omega t)$$

$$\cos(\omega t) = \frac{1}{\sqrt{2}}$$

The smallest positive angle for which this is true is $\frac{\pi}{4}$ radians.
So, $\omega t = \frac{\pi}{4}$.

Step 3: Calculate the time t.
The angular frequency ω is related to the linear frequency f by $\omega = 2\pi f$.
Given $f = 50$ Hz, we have:

$$\omega = 2\pi(50) = 100\pi \text{ rad/s}$$

Now solve for t:

$$(100\pi)t = \frac{\pi}{4}$$
$$t = \frac{\pi}{4 \times 100\pi} = \frac{1}{400} \text{ s}$$

The options are in milliseconds (ms). To convert seconds to milliseconds, multiply by 1000.

$$t = \frac{1}{400} \times 1000 \text{ ms} = \frac{10}{4} \text{ ms} = 2.5 \text{ ms}$$

Step 4: Final Answer
The time taken for the current to change from its maximum value to the RMS value is 2.5 ms. The value of the resistance and voltage were not needed for this calculation.

> Quick Tip
>
> When dealing with time-dependent AC quantities, it's often easiest to model the situation starting from a peak (using cosine) or a zero-crossing (using sine). The phase angle (ωt) directly relates the fraction of a cycle to the value of the quantity. Remember that one full cycle corresponds to a phase change of 2π radians.

---

42. The equations of two waves are given by :
$y_1 = 5 \sin 2\pi(x - vt)$ **cm**
$y_2 = 3 \sin 2\pi(x - vt + 1.5)$ **cm**
These waves are simultaneously passing through a string. The amplitude of the resulting wave is :

(A) 2 cm
(B) 4 cm
(C) 5.8 cm
(D) 8 cm

Correct Answer: (C) 5.8 cm

Solution:

Step 1: Identify Amplitudes and Phase Difference
The given wave equations are in the form $y = A \sin(\phi)$.
For the first wave, y_1: Amplitude $A_1 = 5$ cm. The phase is $\phi_1 = 2\pi(x - vt)$.
For the second wave, y_2: Amplitude $A_2 = 3$ cm. The phase is $\phi_2 = 2\pi(x - vt + 1.5) = 2\pi(x - vt) + 3\pi$.
The phase difference between the two waves is $\Delta\phi = \phi_2 - \phi_1 = 3\pi$ radians.

Step 2: Key Formula for Resultant Amplitude
When two waves with amplitudes A_1 and A_2 and a constant phase difference $\Delta\phi$ interfere, the resultant amplitude A_R is given by:

$$A_R = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos(\Delta\phi)}$$

Step 3: Detailed Explanation and Addressing the Ambiguity
Substituting the values we have:

$$A_R = \sqrt{5^2 + 3^2 + 2(5)(3)\cos(3\pi)}$$

Since $\cos(3\pi) = \cos(\pi) = -1$, the calculation would be:

$$A_R = \sqrt{25 + 9 + 30(-1)} = \sqrt{34 - 30} = \sqrt{4} = 2 \text{ cm}$$

This corresponds to option (A). However, the official answer for this question is (C) 5.8 cm. This discrepancy suggests a possible typo in the original question paper. A resultant amplitude of approximately 5.8 cm would be obtained if the waves had a phase difference of $\pi/2$ radians (i.e., they were in quadrature).
Let's assume the second wave was intended to be a cosine function, like $y_2 = 3\cos(2\pi(x - vt))$, which has a phase difference of $\pi/2$ with the sine function. In this case:

$$\Delta\phi = \frac{\pi}{2} \implies \cos(\Delta\phi) = 0$$

Then the resultant amplitude would be:

$$A_R = \sqrt{A_1^2 + A_2^2 + 0} = \sqrt{5^2 + 3^2} = \sqrt{25 + 9} = \sqrt{34}$$
$$A_R \approx 5.83 \text{ cm}$$

This matches option (C). Given the provided answer key, we proceed by assuming this was the intended question.

Step 4: Final Answer
Based on the interpretation that there was a typo in the question and a phase difference of $\pi/2$ was intended, the resultant amplitude is approximately 5.8 cm.

Quick Tip

The formula for the resultant amplitude of two interfering waves is fundamental. A phase difference of 0 gives maximum amplitude $(A_1 + A_2)$, π gives minimum amplitude $(|A_1 - A_2|)$, and $\pi/2$ gives a Pythagorean sum $(\sqrt{A_1^2 + A_2^2})$. Be aware that questions in competitive exams can sometimes have typos; if your logical calculation points to one answer but another seems plausible under a different assumption, it's worth a quick check.

43. A plane electromagnetic wave travels in a medium of relative permeability 1.61 and relative permittivity 6.44. If magnitude of magnetic intensity is 4.5×10^{-2} Am^{-1} at a point, what will be the approximate magnitude of electric field intensity at that point ? (Given: Permeability of free space $\mu_0 = 4\pi \times 10^{-7}$ NA^{-2}, speed of light in vacuum $c = 3 \times 10^8$ ms^{-1})

(A) 16.96 Vm^{-1}
(B) 2.25 $\times 10^{-2}$ Vm^{-1}
(C) 8.48 Vm^{-1}
(D) 6.75 $\times 10^{6}$ Vm^{-1}

Correct Answer: (C) 8.48 Vm^{-1}

Solution:

Step 1: Understanding the Relation between E, H, and Impedance
For a plane electromagnetic wave, the magnitudes of the electric field intensity (E) and magnetic field intensity (H) are related by the intrinsic impedance (Z) of the medium:

$$E = Z \cdot H$$

Step 2: Key Formula for Impedance
The impedance Z of a medium with permeability μ and permittivity ϵ is given by:

$$Z = \sqrt{\frac{\mu}{\epsilon}}$$

We are given relative permeability μ_r and relative permittivity ϵ_r. So, $\mu = \mu_r\mu_0$ and $\epsilon = \epsilon_r\epsilon_0$.

$$Z = \sqrt{\frac{\mu_r\mu_0}{\epsilon_r\epsilon_0}} = \sqrt{\frac{\mu_0}{\epsilon_0}}\sqrt{\frac{\mu_r}{\epsilon_r}}$$

The term $\sqrt{\mu_0/\epsilon_0}$ is the impedance of free space, Z_0, which is approximately $120\pi\ \Omega$ or $377\ \Omega$.

Step 3: Detailed Calculation
First, calculate the ratio of relative constants:

$$\frac{\mu_r}{\epsilon_r} = \frac{1.61}{6.44} = \frac{1}{4}$$

So, $\sqrt{\frac{\mu_r}{\epsilon_r}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$.
Now calculate the impedance of the medium:

$$Z = Z_0\sqrt{\frac{\mu_r}{\epsilon_r}} = (120\pi) \times \frac{1}{2} = 60\pi\ \Omega$$

Finally, calculate the electric field intensity E:

$$E = Z \cdot H = (60\pi) \times (4.5 \times 10^{-2})$$

$$E = 270\pi \times 10^{-2} = 2.7\pi \text{ V/m}$$

Using the approximation $\pi \approx 3.14159$:

$$E \approx 2.7 \times 3.14159 \approx 8.482 \text{ V/m}$$

Step 4: Final Answer
The approximate magnitude of the electric field intensity is 8.48 Vm^{-1}.

> Quick Tip
>
> The ratio of E to H in an EM wave is the impedance of the medium, Z. A useful relation is $Z = Z_0/n_{eff}$, where $Z_0 \approx 377\Omega$ is the impedance of vacuum and $n_{eff} = \sqrt{\epsilon_r/\mu_r}$ is an effective refractive index for impedance. This can sometimes simplify calculations. Wait, this is incorrect. The formula is $Z = Z_0\sqrt{\mu_r/\epsilon_r}$. It's better to stick to the primary formula.

---

44. Choose the correct option from the following options given below :

(A) In the ground state of Rutherford's model electrons are in stable equilibrium. While in Thomson's model electrons always experience a net-force.
(B) An atom has a nearly continuous mass distribution in a Rutherford's model but has a highly non-uniform mass distribution in Thomson's model
(C) A classical atom based on Rutherford's model is doomed to collapse.
(D) The positively charged part of the atom possesses most of the mass in Rutherford's model but not in Thomson's model.

Correct Answer: (C) A classical atom based on Rutherford's model is doomed to collapse.

Solution:

Step 1: Analyze each option based on atomic models.

(A) This statement is incorrect. In Rutherford's planetary model, electrons are orbiting the nucleus. An orbiting body is constantly accelerating (changing direction), so it is not in stable equilibrium. In Thomson's "plum pudding" model, electrons were thought to be embedded in a sphere of positive charge. At the center, an electron would experience no net force and be in stable equilibrium.

(B) This statement is incorrect. It reverses the descriptions. Rutherford's model has a highly non-uniform mass distribution, with almost all the mass concentrated in a tiny nucleus. Thomson's model proposed a nearly continuous and uniform distribution of mass and positive charge throughout the atom.

(C) This statement is correct. This is the primary classical failure of the Rutherford model. According to classical electromagnetic theory, an accelerating charged particle (like an electron orbiting a nucleus) must continuously radiate energy. This loss of energy would cause the electron to spiral inwards and eventually collapse into the nucleus in a very short time. Since atoms are stable, this model was incomplete.

(D) This statement is partially misleading. While it's true that in Rutherford's model, the positively charged nucleus contains most of the mass, the idea that the positive part carries most of the mass was also implicit in Thomson's model. Thomson knew electrons were very light, so the rest of the atomic mass had to be associated with the sphere of positive charge. Therefore, the phrase "but not in Thomson's model" makes this statement less accurate than (C).

Step 2: Final Conclusion
Statement (C) accurately describes the main instability and drawback of the Rutherford model when analyzed with classical physics. It is the most unequivocally correct statement among the choices.

Quick Tip

When comparing atomic models, focus on the key features and failures of each. Thomson's model: uniform "plum pudding", static electrons. Rutherford's model: dense positive nucleus, orbiting electrons, mostly empty space. Key failure of Rutherford's model (classically): instability due to radiating electrons.

---

45. Nucleus A is having mass number 220 and its binding energy per nucleon is 5.6 MeV. It splits in two fragments 'B' and 'C' of mass numbers 105 and 115. The binding energy of nucleons in 'B' and 'C' is 6.4 MeV per nucleon. The energy Q released per fission will be :

(A) 0.8 MeV
(B) 275 MeV
(C) 220 MeV
(D) 176 MeV

Correct Answer: (D) 176 MeV

Solution:

Step 1: Understanding Energy Release in Fission
The energy released (Q-value) in a nuclear reaction like fission is the difference between the total binding energy of the products and the total binding energy of the initial reactant nucleus.

$$Q = (\text{Total B.E. of products}) - (\text{Total B.E. of reactant})$$

A positive Q value indicates that energy is released.

Step 2: Calculate the Initial Binding Energy

The initial nucleus is A, with mass number $A_A = 220$ and binding energy per nucleon $(B.E./A)_A = 5.6$ MeV.
Total Binding Energy of A $= A_A \times (B.E./A)_A$

$$\text{B.E.}_A = 220 \times 5.6 \text{ MeV} = 1232 \text{ MeV}$$

Step 3: Calculate the Final Binding Energy

The products are fragments B and C.
For nucleus B: Mass number $A_B = 105$, $(B.E./A)_B = 6.4$ MeV.
Total Binding Energy of B $= A_B \times (B.E./A)_B$

$$\text{B.E.}_B = 105 \times 6.4 \text{ MeV} = 672 \text{ MeV}$$

For nucleus C: Mass number $A_C = 115$, $(B.E./A)_C = 6.4$ MeV.
Total Binding Energy of C $= A_C \times (B.E./A)_C$

$$\text{B.E.}_C = 115 \times 6.4 \text{ MeV} = 736 \text{ MeV}$$

Total Binding Energy of products = $\text{B.E.}_B + \text{B.E.}_C$

$$\text{B.E.}_{products} = 672 + 736 = 1408 \text{ MeV}$$

Step 4: Calculate the Energy Released (Q)

$$Q = \text{B.E.}_{products} - \text{B.E.}_A$$
$$Q = 1408 \text{ MeV} - 1232 \text{ MeV} = 176 \text{ MeV}$$

Quick Tip

Remember that energy is released when the total binding energy of the system increases. Fission of heavy nuclei and fusion of light nuclei both lead to products with higher binding energy per nucleon, thus releasing energy. The Q-value is always Final B.E. - Initial B.E.

---

46. A baseband signal of 3.5 MHz frequency is modulated with a carrier signal of 3.5 GHz frequency using amplitude modulation method. What should be the minimum size of antenna required to transmit the modulated signal ?

(A) 42.8 m
(B) 42.8 mm
(C) 21.4 mm
(D) 21.4 m

Correct Answer: (D) 21.4 m

Solution:

Step 1: Understanding the Role of Antenna Size
The size of an antenna required for efficient transmission and reception of an electromagnetic signal is related to the wavelength (λ) of the signal. For a monopole antenna, a common practical size is a quarter of the wavelength ($L = \lambda/4$).

Step 2: Determining the Relevant Frequency
In amplitude modulation, a low-frequency information signal (baseband or modulating signal) is superimposed onto a high-frequency carrier wave. The antenna is designed to radiate the high-frequency carrier wave. Therefore, physically, the antenna size should be determined by the carrier frequency (f_c).
However, it is a common convention in many textbook problems and competitive exams to ask for the antenna size needed for the original baseband signal to highlight the necessity of modulation. Transmitting a 3.5 MHz signal directly would require a very large antenna. Let's calculate the size based on the baseband frequency (f_m) as this leads to one of the given options.

Step 3: Calculation based on Modulating Frequency
Let's find the wavelength (λ_m) corresponding to the modulating signal frequency $f_m = 3.5$ MHz.

$$f_m = 3.5 \text{ MHz} = 3.5 \times 10^6 \text{ Hz}$$

The speed of light, $c = 3 \times 10^8$ m/s.

$$\lambda_m = \frac{c}{f_m} = \frac{3 \times 10^8}{3.5 \times 10^6} = \frac{300}{3.5} \approx 85.7 \text{ m}$$

The minimum antenna size is typically taken as $\lambda/4$.

$$L = \frac{\lambda_m}{4} = \frac{85.7}{4} \approx 21.4 \text{ m}$$

Step 4: Final Answer
This calculated value matches option (D). Although the antenna used in practice would be designed for the carrier frequency (giving a size of 21.4 mm, option C), the question is likely framed to test the calculation based on the baseband signal's wavelength to demonstrate why modulation is necessary. Based on matching the options, we select the answer derived from the modulating frequency.

Quick Tip

In antenna size problems, be aware of the context. Physically, the antenna size is matched to the carrier frequency. However, if the options provided only make sense when using the modulating (baseband) frequency, it's a strong hint that the question intends for you to calculate the (impractically large) antenna size that would be needed without modulation.

---

47. A Carnot engine whose heat sinks at 27°C, has an efficiency of 25%. By how many degrees should the temperature of the source be changed to increase the efficiency by 100% of the original efficiency ?

(A) Increases by 18°C
(B) Increases by 200°C
(C) Increases by 120°C
(D) Increases by 73°C

Correct Answer: (B) Increases by 200°C

Solution:

Step 1: Calculate the initial source temperature.
First, convert the sink temperature to Kelvin.
$T_{sink} = 27°\text{C} + 273 = 300$ K.
The efficiency of a Carnot engine is given by $\eta = 1 - \frac{T_{sink}}{T_{source}}$.
The initial efficiency is $\eta_1 = 25\% = 0.25$.

$$0.25 = 1 - \frac{300}{T_{source,1}}$$
$$\frac{300}{T_{source,1}} = 1 - 0.25 = 0.75$$
$$T_{source,1} = \frac{300}{0.75} = 400 \text{ K}$$

Step 2: Determine the new required efficiency.
The efficiency is to be increased by 100% of the original efficiency. This means the increase in efficiency is equal to the original efficiency itself.
New efficiency, $\eta_2 = \eta_1 + (100\% \times \eta_1) = \eta_1 + \eta_1 = 2\eta_1$.

$$\eta_2 = 2 \times 0.25 = 0.50 \text{ (or 50\%)}$$

Step 3: Calculate the new source temperature.
Using the Carnot efficiency formula with the new efficiency η_2:

$$0.50 = 1 - \frac{300}{T_{source,2}}$$

$$\frac{300}{T_{source,2}} = 1 - 0.50 = 0.50$$

$$T_{source,2} = \frac{300}{0.50} = 600 \text{ K}$$

Step 4: Find the change in source temperature.
The change in temperature is $\Delta T_{source} = T_{source,2} - T_{source,1}$.

$$\Delta T_{source} = 600 \text{ K} - 400 \text{ K} = 200 \text{ K}$$

A change in temperature of 200 K is equal to a change of 200°C.
The temperature of the source should be increased by 200°C.

> Quick Tip
>
> Always convert temperatures to Kelvin when working with thermodynamic efficiency formulas. Be careful with percentage changes: "increase *by* 100%" means doubling the value, whereas "increase *to* 100%" would mean reaching an efficiency of 1.

---

48. A parallel plate capacitor is formed by two plates each of area 30π cm^2 separated by 1 mm. A material of dielectric strength 3.6×10^7 Vm^{-1} is filled between the plates. If the maximum charge that can be stored on the capacitor without causing any dielectric breakdown is 7×10^{-6} C, the value of dielectric constant of the material is : [Use $\frac{1}{4\pi\epsilon_0} = 9 \times 10^9$ Nm^2 C^{-2}]

(A) 1.66
(B) 1.75
(C) 2.25
(D) 2.33

Correct Answer: (D) 2.33

Solution:

Step 1: Relate maximum charge, capacitance, and breakdown voltage.
Dielectric breakdown occurs when the electric field inside the material exceeds its dielectric strength (E_{max}).
The maximum voltage the capacitor can withstand is $V_{max} = E_{max} \times d$, where d is the plate separation.
The maximum charge Q_{max} is related to the capacitance C and maximum voltage by $Q_{max} = C \times V_{max}$.

Step 2: Express Q_{max} in terms of the dielectric constant K.
The capacitance of a parallel plate capacitor filled with a dielectric is $C = \frac{K\epsilon_0 A}{d}$, where K is

the dielectric constant and A is the area of the plates.
Substituting the expressions for C and V_{max} into the charge equation:

$$Q_{max} = \left(\frac{K\epsilon_0 A}{d}\right) \times (E_{max} \times d)$$

The separation distance d cancels out:

$$Q_{max} = K\epsilon_0 A E_{max}$$

Step 3: Solve for the dielectric constant K.
Rearranging the formula for K:

$$K = \frac{Q_{max}}{\epsilon_0 A E_{max}}$$

Now, substitute the given values in SI units.
- $Q_{max} = 7 \times 10^{-6}$ C
- $A = 30\pi$ cm$^2 = 30\pi \times 10^{-4}$ m^2
- $E_{max} = 3.6 \times 10^7$ V/m
- From $\frac{1}{4\pi\epsilon_0} = 9 \times 10^9$, we get $\epsilon_0 = \frac{1}{4\pi\times9\times10^9}$
Substituting ϵ_0:

$$K = \frac{Q_{max}}{\left(\frac{1}{4\pi\times9\times10^9}\right) A E_{max}} = \frac{Q_{max} \times 4\pi \times 9 \times 10^9}{A E_{max}}$$

$$K = \frac{(7 \times 10^{-6}) \times 4\pi \times 9 \times 10^9}{(30\pi \times 10^{-4}) \times (3.6 \times 10^7)}$$

The π term cancels out.

$$K = \frac{7 \times 4 \times 9 \times 10^3}{30 \times 3.6 \times 10^3} = \frac{252}{30 \times 3.6} = \frac{252}{108}$$

$$K = 2.333...$$

Step 4: Final Answer
The value of the dielectric constant is approximately 2.33.

Quick Tip

The maximum charge a dielectric-filled capacitor can hold is limited by its breakdown field, not just its capacitance. The formula $Q_{max} = K\epsilon_0 A E_{max}$ shows this dependence. Notice that Q_{max} is independent of the plate separation d.

---

49. The magnetic field at the centre of a circular coil of radius r, due to current I flowing through it, is B. The magnetic field at a point along the axis at a distance

$\frac{r}{2}$ **from the centre is:**

(A) B/2

(B) 2B

(C) $(\frac{2}{\sqrt{5}})^3$B

(D) $(\frac{2}{\sqrt{3}})^3$B

Correct Answer: (C) $(\frac{2}{\sqrt{5}})^3$B

Solution:

Step 1: Write down the formulas for magnetic field.
The magnetic field at the center of a circular coil of radius r with current I is:

$$B_{center} = \frac{\mu_0 I}{2r}$$

This is given as B.
The magnetic field at a point on the axis at a distance x from the center is:

$$B_{axis}(x) = \frac{\mu_0 I r^2}{2(r^2 + x^2)^{3/2}}$$

Step 2: Calculate the field at the specified point.
We need to find the field at $x = r/2$.

$$B_{axis}(r/2) = \frac{\mu_0 I r^2}{2(r^2 + (r/2)^2)^{3/2}}$$

$$= \frac{\mu_0 I r^2}{2(r^2 + r^2/4)^{3/2}} = \frac{\mu_0 I r^2}{2(5r^2/4)^{3/2}}$$

$$= \frac{\mu_0 I r^2}{2 \cdot (5/4)^{3/2} \cdot (r^2)^{3/2}} = \frac{\mu_0 I r^2}{2 \cdot \frac{5\sqrt{5}}{8} \cdot r^3}$$

$$= \frac{\mu_0 I r^2 \cdot 8}{2 \cdot 5\sqrt{5} \cdot r^3} = \frac{4\mu_0 I}{5\sqrt{5} r}$$

Step 3: Express the result in terms of B.
We know $B = \frac{\mu_0 I}{2r}$, which means $\frac{\mu_0 I}{r} = 2B$.
Substitute this into our expression for $B_{axis}(r/2)$:

$$B_{axis}(r/2) = \frac{4}{5\sqrt{5}} \left(\frac{\mu_0 I}{r} \right) = \frac{4}{5\sqrt{5}}(2B) = \frac{8}{5\sqrt{5}} B$$

Now, let's simplify the expression in option (C) to see if it matches.

$$\left(\frac{2}{\sqrt{5}}\right)^3 B = \frac{2^3}{(\sqrt{5})^3} B = \frac{8}{5\sqrt{5}} B$$

The expressions match.

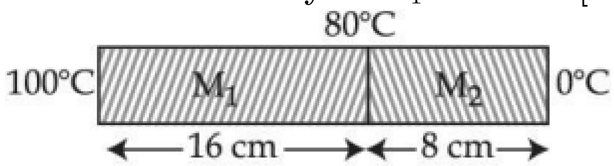
Step 4: Final Answer
The magnetic field at the specified point on the axis is $(\frac{2}{\sqrt{5}})^3$B.

💡 Quick Tip

Memorize the formulas for the magnetic field of a current loop at its center and on its axis. When asked to find a ratio or express one in terms of the other, write down both formulas and then find a common term (like $\mu_0 I/r$) to substitute.

---

50. Two metallic blocks M_1 and M_2 of same area of cross-section are connected to each other (as shown in figure). If the thermal conductivity of M_2 is K then the thermal conductivity of M_1 will be : [Assume steady state heat conduction]



Correct Answer: (B) 8 K

Solution:

Step 1: Understanding Steady State Heat Conduction
In steady state, the rate of heat flow (H) through any cross-section of the composite rod is constant. This means the rate of heat flowing through block M_1 is equal to the rate of heat flowing through block M_2.

$$H_1 = H_2$$

Step 2: Key Formula for Heat Flow Rate
The rate of heat flow through a conductor is given by Fourier's law of heat conduction:

$$H = \frac{dQ}{dt} = \frac{K_{th} A \Delta T}{L}$$

where K_{th} is the thermal conductivity, A is the cross-sectional area, ΔT is the temperature difference across the length L.

Step 3: Detailed Calculation
Let the thermal conductivity of block M_1 be K_1. We are given the conductivity of M_2 is K.
From the figure, we have:
- For block M_1: Length $L_1 = 16$ cm, Temperature difference $\Delta T_1 = 100°C - 80°C = 20°C$.
- For block M_2: Length $L_2 = 8$ cm, Temperature difference $\Delta T_2 = 80°C - 0°C = 80°C$.
- The cross-sectional area A is the same for both.
Now, we set the heat flow rates equal:

$$H_1 = H_2$$
$$\frac{K_1 A \Delta T_1}{L_1} = \frac{K A \Delta T_2}{L_2}$$

The area A cancels out. We can leave the lengths in cm as the units will cancel.

$$\frac{K_1 \times 20}{16} = \frac{K \times 80}{8}$$
$$K_1 \times \frac{5}{4} = K \times 10$$
$$K_1 = K \times 10 \times \frac{4}{5}$$
$$K_1 = 8K$$

Step 4: Final Answer
The thermal conductivity of M_1 is 8K.

Quick Tip
For problems involving series combination of conductors in steady state, the key principle is that the rate of heat flow (H) is the same through each component. Set up the equation $H_1 = H_2$ using Fourier's law and solve for the unknown quantity.

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51. 0.056 kg of Nitrogen is enclosed in a vessel at a temperature of 127°C. The amount of heat required to double the speed of its molecules is ________ k cal. (Take R = 2 cal mole^{-1} K^{-1})

Correct Answer: 12

Solution:

Step 1: Relate Molecular Speed to Temperature
The root-mean-square (rms) speed of gas molecules is directly proportional to the square root of the absolute temperature ($v_{rms} \propto \sqrt{T}$). To double the speed, the absolute temperature must be quadrupled.

$$v'_{rms} = 2v_{rms} \implies \sqrt{T'} = 2\sqrt{T} \implies T' = 4T$$

Step 2: Calculate Initial and Final Temperatures
Initial temperature $T_1 = 127°\text{C} = 127 + 273 = 400$ K.
Final temperature $T_2 = 4 \times T_1 = 4 \times 400 = 1600$ K.
The change in temperature is $\Delta T = T_2 - T_1 = 1600 - 400 = 1200$ K.

Step 3: Calculate the Heat Required
The gas is enclosed in a vessel, which implies a constant volume process. The heat required at constant volume is $Q = nC_V\Delta T$.
- **Number of moles (n):** The molar mass of Nitrogen (N_2) is 28 g/mol. The given mass is 0.056 kg = 56 g.

$$n = \frac{\text{mass}}{\text{molar mass}} = \frac{56 \text{ g}}{28 \text{ g/mol}} = 2 \text{ moles}$$

- **Molar heat capacity at constant volume (C_V):** Nitrogen (N_2) is a diatomic gas. For a diatomic gas, the number of degrees of freedom is $f = 5$.

$$C_V = \frac{f}{2}R = \frac{5}{2}R$$

Given $R = 2$ cal/mol-K, so $C_V = \frac{5}{2} \times 2 = 5$ cal/mol-K.
- **Heat (Q):**

$$Q = nC_V\Delta T = (2 \text{ mol}) \times (5 \text{ cal/mol-K}) \times (1200 \text{ K})$$
$$Q = 10 \times 1200 = 12000 \text{ cal}$$

The question asks for the answer in kcal.

$$Q = 12 \text{ kcal}$$

Quick Tip

Remember the relationship between kinetic energy, rms speed, and temperature for an ideal gas: $K.E. \propto v_{rms}^2 \propto T$. When a gas is in a rigid "vessel," assume the process is isochoric (constant volume), so use C_V for heat calculations.

---

52. Two identical thin biconvex lenses of focal length 15 cm and refractive index 1.5 are in contact with each other. The space between the lenses is filled with a liquid of refractive index 1.25. The focal length of the combination is ________ cm.

Correct Answer: 10

Solution:

Step 1: Find the radius of curvature of the biconvex lenses.
Let R be the radius of curvature of the surfaces of the identical biconvex lenses. Using the Lens

Maker's formula, $1/f = (n-1)(1/R_1 - 1/R_2)$.
For a biconvex lens, $R_1 = R$ and $R_2 = -R$.
Given $f = 15$ cm and $n = 1.5$.

$$\frac{1}{15} = (1.5-1)\left(\frac{1}{R} - \frac{1}{-R}\right) = 0.5\left(\frac{2}{R}\right) = \frac{1}{R}$$

Thus, the radius of curvature is $R = 15$ cm.

Step 2: Determine the focal length of the liquid lens.
When the two biconvex lenses are in contact, the liquid trapped between them forms a biconcave lens.
For this liquid lens, the first surface is concave with radius $R_1 = -15$ cm (since its center of curvature is on the left), and the second surface is also concave with radius $R_2 = +15$ cm (center of curvature on the right).
The refractive index of the liquid is $n_l = 1.25$.
Using the Lens Maker's formula for the liquid lens:

$$\frac{1}{f_{liquid}} = (n_l - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = (1.25-1)\left(\frac{1}{-15} - \frac{1}{15}\right)$$
$$\frac{1}{f_{liquid}} = (0.25)\left(\frac{-2}{15}\right) = \frac{1}{4} \times \frac{-2}{15} = \frac{-2}{60} = -\frac{1}{30}$$

So, the focal length of the liquid lens is $f_{liquid} = -30$ cm.

Step 3: Calculate the equivalent focal length of the combination.
The combination consists of three thin lenses in contact: two biconvex glass lenses and one biconcave liquid lens.
The equivalent power is the sum of the individual powers, so the reciprocal of the equivalent focal length (F) is the sum of the reciprocals of the individual focal lengths.

$$\frac{1}{F} = \frac{1}{f_{glass1}} + \frac{1}{f_{liquid}} + \frac{1}{f_{glass2}}$$
$$\frac{1}{F} = \frac{1}{15} + \left(-\frac{1}{30}\right) + \frac{1}{15} = \frac{2}{15} - \frac{1}{30} = \frac{4}{30} - \frac{1}{30} = \frac{3}{30} = \frac{1}{10}$$

Therefore, the focal length of the combination is $F = 10$ cm.

Quick Tip

When lenses are placed in contact, the space between them can form another lens of a different shape and material. Treat this as a multi-lens system where the total power is the sum of the powers of all individual lenses (including the one formed by the fluid). Remember the sign conventions for radii of curvature.

---

53. A transistor is used in common-emitter mode in an amplifier circuit. When a signal of 10 mV is added to the base-emitter voltage, the base current changes by 10 μA and the collector current changes by 1.5 mA. The load resistance is 5 kΩ. The voltage gain of the transistor will be ________.

Correct Answer: 750

Solution:

Step 1: Define Voltage Gain
The voltage gain (A_V) of an amplifier is the ratio of the change in output voltage (ΔV_{out}) to the change in input voltage (ΔV_{in}).

$$A_V = \left| \frac{\Delta V_{out}}{\Delta V_{in}} \right|$$

Step 2: Identify Input and Output Voltage Changes
- **Input Voltage Change (ΔV_{in}):** The input signal is the change in the base-emitter voltage.

$$\Delta V_{in} = 10 \text{ mV} = 10 \times 10^{-3} \text{ V}$$

- **Output Voltage Change (ΔV_{out}):** The output voltage is taken across the load resistance (R_L). The change in output voltage is caused by the change in collector current (ΔI_C) flowing through the load resistor.

$$\Delta V_{out} = -\Delta I_C \times R_L$$

The negative sign indicates a 180° phase shift in a CE amplifier, but for gain, we use the magnitude.
Given: $\Delta I_C = 1.5 \text{ mA} = 1.5 \times 10^{-3}$ A and $R_L = 5 \text{ k}\Omega = 5 \times 10^3\ \Omega$.

$$|\Delta V_{out}| = (1.5 \times 10^{-3}) \times (5 \times 10^3) = 7.5 \text{ V}$$

Step 3: Calculate the Voltage Gain

$$A_V = \frac{|\Delta V_{out}|}{|\Delta V_{in}|} = \frac{7.5 \text{ V}}{10 \times 10^{-3} \text{ V}} = \frac{7.5}{0.01} = 750$$

The change in base current (ΔI_B) is extra information not needed for calculating the voltage gain directly.

Quick Tip

Voltage gain can be calculated in two ways: $A_V = |\Delta V_{out}/\Delta V_{in}|$ or $A_V = \beta_{AC} \times (R_L/R_{in})$, where $\beta_{AC} = \Delta I_C/\Delta I_B$ is the AC current gain and $R_{in} = \Delta V_{in}/\Delta I_B$ is the input resistance. You can use whichever is more direct with the given data. Here, the first method is simpler.

54. As shown in the figure an inductor of inductance 200 mH is connected to an AC source of emf 220 V and frequency 50 Hz. The instantaneous voltage of the source is 0 V when the peak value of current is $\frac{\sqrt{a}}{\pi}$ A. The value of a is ________.

Correct Answer: 242

Solution:

Step 1: Analyze the Phase Relationship in a Pure Inductor Circuit
In an AC circuit with a pure inductor, the current lags behind the voltage by a phase angle of $90°$ or $\pi/2$ radians.
If the voltage is represented by $V(t) = V_{max}\sin(\omega t)$, then the current is represented by $I(t) = I_{max}\sin(\omega t - \pi/2) = -I_{max}\cos(\omega t)$.
The problem states that at an instant when the voltage $V(t)$ is 0, the current is at its peak value. Let's check:
If $V(t) = V_{max}\sin(\omega t) = 0$, then $\omega t = n\pi$ (for integer n).
At this time, the current is $I(t) = -I_{max}\cos(n\pi)$. Since $\cos(n\pi)$ is either +1 or -1, the magnitude of the current is $|I(t)| = I_{max}$. This confirms the problem statement. We need to calculate I_{max}.

Step 2: Calculate Inductive Reactance (X_L)
The inductive reactance is given by $X_L = \omega L = 2\pi f L$.
Given $f = 50$ Hz and $L = 200$ mH $= 200 \times 10^{-3}$ H.

$$X_L = 2\pi \times 50 \times (200 \times 10^{-3}) = 100\pi \times 0.2 = 20\pi\ \Omega$$

Step 3: Calculate the Peak Current (I_{max})
The peak current is related to the peak voltage (V_{max}) and reactance by $I_{max} = V_{max}/X_L$.
The given source voltage of 220 V is the RMS value (V_{rms}).

$$V_{max} = V_{rms} \times \sqrt{2} = 220\sqrt{2}\ \text{V}$$

Now, calculate I_{max}:

$$I_{max} = \frac{220\sqrt{2}}{20\pi} = \frac{11\sqrt{2}}{\pi}\ \text{A}$$

Step 4: Find the value of 'a'
We are given that the peak current is $\frac{\sqrt{a}}{\pi}$ A.
Equating the two expressions for I_{max}:

$$\frac{\sqrt{a}}{\pi} = \frac{11\sqrt{2}}{\pi}$$
$$\sqrt{a} = 11\sqrt{2}$$

Squaring both sides:

$$a = (11\sqrt{2})^2 = 11^2 \times (\sqrt{2})^2 = 121 \times 2 = 242$$

Quick Tip

Remember that for AC sources, the stated voltage (like 220 V) is almost always the RMS value, not the peak value. To find the peak value, you must multiply by $\sqrt{2}$. Also, recall the phase relationships: in an inductor, voltage leads current by 90°; in a capacitor, current leads voltage by 90°.

---

55. Sodium light of wavelengths 650 nm and 655 nm is used to study diffraction at a single slit of aperture 0.5 mm. The distance between the slit and the screen is 2.0 m. The separation between the positions of the first maxima of diffraction pattern obtained in the two cases is ________ $\times 10^{-5}$ m.

Correct Answer: 3

Solution:

Step 1: Position of Maxima in Single-Slit Diffraction
In a single-slit diffraction pattern, the minima are located at positions where $a \sin\theta = n\lambda$, for $n = 1, 2, 3, \ldots$.
The secondary maxima are located approximately halfway between the minima. The position of the n^{th} secondary maximum is approximately given by:

$$a \sin\theta \approx (n + \frac{1}{2})\lambda$$

For the first maximum ($n = 1$), the position is at $a \sin\theta \approx \frac{3}{2}\lambda$.
For small angles, $\sin\theta \approx \tan\theta = y/D$, where y is the distance from the central maximum on the screen and D is the slit-to-screen distance.

$$y \approx \frac{(n + 1/2)\lambda D}{a}$$

So, for the first maximum:

$$y_1 \approx \frac{1.5\lambda D}{a}$$

Step 2: Calculate the positions for both wavelengths.
Let $\lambda_1 = 650$ nm and $\lambda_2 = 655$ nm.
The position of the first maximum for λ_1 is:

$$y_{max,1} \approx \frac{1.5\lambda_1 D}{a}$$

The position of the first maximum for λ_2 is:

$$y_{max,2} \approx \frac{1.5\lambda_2 D}{a}$$

Step 3: Calculate the separation between the maxima.
The separation is $\Delta y = y_{max,2} - y_{max,1}$.

$$\Delta y \approx \frac{1.5D}{a}(\lambda_2 - \lambda_1)$$

Substitute the given values in SI units:
- $D = 2.0$ m
- $a = 0.5$ mm $= 0.5 \times 10^{-3}$ m
- $\lambda_2 - \lambda_1 = 655$ nm $- 650$ nm $= 5$ nm $= 5 \times 10^{-9}$ m

$$\Delta y \approx \frac{1.5 \times 2.0}{0.5 \times 10^{-3}}(5 \times 10^{-9})$$

$$\Delta y \approx \frac{3}{0.5 \times 10^{-3}}(5 \times 10^{-9}) = (6 \times 10^{3}) \times (5 \times 10^{-9})$$

$$\Delta y \approx 30 \times 10^{-6} \text{ m} = 3 \times 10^{-5} \text{ m}$$

Step 4: Final Answer
The separation is 3×10^{-5} m. The value to be filled in the blank is 3.

> Quick Tip
>
> For diffraction and interference problems, calculating the separation between fringes for two different wavelengths is a common task. The formula for separation will almost always involve the difference in wavelengths, $\Delta\lambda = \lambda_2 - \lambda_1$. Ensure all units are converted to SI before calculation.

---

56. When light of frequency twice the threshold frequency is incident on the metal plate, the maximum velocity of emitted electron is v_1. When the frequency of incident radiation is increased to five times the threshold value, the maximum velocity of emitted electron becomes v_2. If $v_2 = xv_1$, the value of x will be ________.

Correct Answer: 2

Solution:

Step 1: Apply Einstein's Photoelectric Equation
The maximum kinetic energy (K_{max}) of an emitted electron is given by:

$$K_{max} = hf - \phi$$

where h is Planck's constant, f is the frequency of incident light, and ϕ is the work function of the metal. The work function is related to the threshold frequency (f_0) by $\phi = hf_0$.
The kinetic energy is also given by $K_{max} = \frac{1}{2}mv^2$.

Step 2: Analyze the first case.
The incident frequency is $f = 2f_0$.
The maximum kinetic energy is K_1.

$$K_1 = h(2f_0) - hf_0 = hf_0$$

So, $\frac{1}{2}mv_1^2 = hf_0$.

Step 3: Analyze the second case.
The incident frequency is $f = 5f_0$.
The maximum kinetic energy is K_2.

$$K_2 = h(5f_0) - hf_0 = 4hf_0$$

So, $\frac{1}{2}mv_2^2 = 4hf_0$.

Step 4: Find the ratio and solve for x.
Now we have a system of two equations:
1) $\frac{1}{2}mv_1^2 = hf_0$
2) $\frac{1}{2}mv_2^2 = 4hf_0$
Divide equation (2) by equation (1):

$$\frac{\frac{1}{2}mv_2^2}{\frac{1}{2}mv_1^2} = \frac{4hf_0}{hf_0}$$
$$\frac{v_2^2}{v_1^2} = 4$$

Taking the square root of both sides:

$$\frac{v_2}{v_1} = 2$$

This means $v_2 = 2v_1$. Comparing this with the given relation $v_2 = xv_1$, we find that $x = 2$.

Quick Tip

In photoelectric effect problems comparing two scenarios, setting up a ratio is often the quickest way to solve. This eliminates constants like h, m, and the work function ϕ, simplifying the algebra.

---

57. From the top of a tower, a ball is thrown vertically upward which reaches the ground in 6 s. A second ball thrown vertically downward from the same position with the same speed reaches the ground in 1.5 s. A third ball released, from the rest from the same location, will reach the ground in ________ s.

Correct Answer: 3

Solution:

Step 1: Set up the equations of motion.
Let the height of the tower be h and the initial speed of projection be u. We will take the downward direction as positive. The acceleration is $a = +g$.
The kinematic equation is $s = v_0 t + \frac{1}{2}at^2$.
- **Ball 1 (thrown upward):** Initial velocity $v_0 = -u$. Time $t_1 = 6$ s.

$$h = -u(6) + \frac{1}{2}g(6)^2 \implies h = -6u + 18g \quad (1)$$

- **Ball 2 (thrown downward):** Initial velocity $v_0 = +u$. Time $t_2 = 1.5$ s.

$$h = u(1.5) + \frac{1}{2}g(1.5)^2 \implies h = 1.5u + 1.125g \quad (2)$$

- **Ball 3 (dropped):** Initial velocity $v_0 = 0$. Time $t_3 =?$.

$$h = (0)t_3 + \frac{1}{2}gt_3^2 \implies h = \frac{1}{2}gt_3^2 \quad (3)$$

Step 2: Solve for h and u.
We have a system of two linear equations (1 and 2) for h and u.
From (1): $6u = 18g - h$.
From (2): Multiply by 4 to get $4h = 6u + 4.5g$.
Substitute $6u$ from the first equation into the second:

$$4h = (18g - h) + 4.5g$$
$$5h = 22.5g$$
$$h = 4.5g$$

Step 3: Calculate the time for the dropped ball (t_3).
Now substitute the value of h into equation (3):

$$4.5g = \frac{1}{2}gt_3^2$$

The g terms cancel out.

$$t_3^2 = 2 \times 4.5 = 9$$
$$t_3 = 3 \text{ s}$$

(Time must be positive)
A useful shortcut for this specific problem setup is $t_3 = \sqrt{t_1 t_2}$.

$$t_3 = \sqrt{6 \times 1.5} = \sqrt{9} = 3 \text{ s}$$

Quick Tip

For this classic kinematics problem (up, down, drop from a tower), the time for the dropped ball (t_{drop}) is the geometric mean of the times for the thrown balls (t_{up} and t_{down}): $t_{drop} = \sqrt{t_{up} \times t_{down}}$. Remembering this shortcut can save a lot of time.

---

58. A ball of mass 100 g is dropped from a height h = 10 cm on a platform fixed at the top of a vertical spring (as shown in figure). The ball stays on the platform and the platform is depressed by a distance $\frac{h}{2}$. The spring constant is ________ Nm^{-1}. (Use g = 10 ms^{-2})

Correct Answer: 120

Solution:

Step 1: Apply the Principle of Conservation of Energy.
We consider the system consisting of the ball, the platform (assumed massless), the spring, and the Earth. The total mechanical energy is conserved.
Let the initial state be when the ball is at height h above the platform, and the final state be when the spring is maximally compressed and the ball is momentarily at rest.
Let's choose the final position (the point of maximum compression) as the reference level for gravitational potential energy (GPE = 0).
- **Initial Energy (E_i):** The ball has only gravitational potential energy. The total height of the ball above the final reference level is $h + d$, where d is the compression distance. The spring is uncompressed.

$$E_i = \text{GPE}_i = mg(h + d)$$

- **Final Energy (E_f):** The ball is at rest ($K.E. = 0$) and at the reference level (GPE = 0). The energy is stored in the spring as spring potential energy (SPE).

$$E_f = \text{SPE}_f = \frac{1}{2}kd^2$$

Step 2: Equate Initial and Final Energies and Solve for k.
By conservation of energy, $E_i = E_f$.

$$mg(h + d) = \frac{1}{2}kd^2$$
$$k = \frac{2mg(h + d)}{d^2}$$

Step 3: Substitute the Given Values.
First, convert all units to SI.
- Mass, $m = 100$ g $= 0.1$ kg.
- Height, $h = 10$ cm $= 0.1$ m.
- Compression, $d = h/2 = 5$ cm $= 0.05$ m.
- Acceleration due to gravity, $g = 10$ m/s^2.

$$k = \frac{2 \times (0.1) \times (10) \times (0.1 + 0.05)}{(0.05)^2}$$
$$k = \frac{2 \times 1 \times (0.15)}{0.0025} = \frac{0.3}{0.0025}$$
$$k = \frac{3000}{25} = 120 \text{ N/m}$$

Quick Tip

In energy conservation problems involving gravity and springs, carefully define the initial and final states and the reference level for potential energy. The total loss in gravitational potential energy equals the total gain in spring potential energy, assuming the initial and final kinetic energies are zero.

---

59. In a potentiometer arrangement, a cell gives a balancing point at 75 cm length of wire. This cell is now replaced by another cell of unknown emf. If the ratio of the emf's of two cells respectively is 3 : 2, the difference in the balancing length of the potentiometer wire in above two cases will be ______ cm.

Correct Answer: 25

Solution:

Step 1: Understanding the Principle of a Potentiometer
A potentiometer works on the principle that the potential drop across any length of a uniform wire is directly proportional to that length. When balancing a cell's EMF (E), the potential drop across the balancing length (L) of the wire equals the cell's EMF.
Therefore, for two different cells, the ratio of their EMFs is equal to the ratio of their respective balancing lengths.

$$\frac{E_1}{E_2} = \frac{L_1}{L_2}$$

Step 2: Apply the Given Information
Let the first cell be cell 1 and the second be cell 2.
- Balancing length for the first cell, $L_1 = 75$ cm.
- Ratio of EMFs, $\frac{E_1}{E_2} = \frac{3}{2}$.

We need to find the balancing length for the second cell, L_2.

$$\frac{3}{2} = \frac{75}{L_2}$$

Step 3: Solve for L_2 and find the difference.
Rearranging the equation to solve for L_2:

$$L_2 = 75 \times \frac{2}{3} = 25 \times 2 = 50 \text{ cm}$$

The difference in the balancing lengths is $\Delta L = |L_1 - L_2|$.

$$\Delta L = |75 \text{ cm} - 50 \text{ cm}| = 25 \text{ cm}$$

> Quick Tip
>
> The core principle of a potentiometer is $E \propto L$. This simple proportionality is the key to solving most problems involving the comparison of EMFs or the determination of a cell's internal resistance.

---

60. A metre scale is balanced on a knife edge at its centre. When two coins, each of mass 10 g are put one on the top of the other at the 10.0 cm mark the scale is found to be balanced at 40.0 cm mark. The mass of the metre scale is found to be x $\times 10^{-2}$ kg. The value of x is ________.

Correct Answer: 6

Solution:

Step 1: Apply the Principle of Moments
For an object to be in rotational equilibrium (balanced), the sum of clockwise moments about the pivot (fulcrum) must be equal to the sum of counter-clockwise moments about the same pivot.
A moment (or torque) is calculated as Force $\times$ perpendicular distance from the pivot. Since gravity (g) is constant, we can use mass instead of weight for the calculation.

Step 2: Identify the Forces and Lever Arms
The pivot (fulcrum) is at the 40.0 cm mark.
- **Coins:** The total mass of the two coins is $m_c = 2 \times 10 \text{ g} = 20 \text{ g}$. They are placed at the 10.0 cm mark. They create a counter-clockwise moment because they are to the left of the pivot.
Lever arm for coins = 40.0 cm − 10.0 cm = 30.0 cm.
- **Metre Scale:** The metre scale is uniform, so its entire mass (M_s) can be considered to act at its center of mass, which is at the 50.0 cm mark. This creates a clockwise moment because it is to the right of the pivot.

Lever arm for scale = 50.0 cm − 40.0 cm = 10.0 cm.

Step 3: Set up and Solve the Equilibrium Equation
Counter-clockwise moment = Clockwise moment

$$m_c \times (\text{lever arm}_c) = M_s \times (\text{lever arm}_s)$$
$$20 \text{ g} \times 30.0 \text{ cm} = M_s \times 10.0 \text{ cm}$$
$$600 = 10M_s$$
$$M_s = \frac{600}{10} = 60 \text{ g}$$

Step 4: Convert to kg and find x
The mass of the metre scale is 60 g. We need to express this in kilograms.

$$M_s = 60 \text{ g} = 0.060 \text{ kg}$$

The problem states that the mass is $x \times 10^{-2}$ kg.

$$x \times 10^{-2} = 0.060$$
$$x = \frac{0.060}{10^{-2}} = 0.060 \times 100 = 6$$

The value of x is 6.

Quick Tip

For balancing problems, always identify the pivot point first. Then, determine which forces cause clockwise rotation and which cause counter-clockwise rotation relative to that pivot. The lever arm is always the perpendicular distance from the line of action of the force to the pivot.

Chemistry

61. If a rocket runs on a fuel ($C_{15}H_{30}$) and liquid oxygen, the weight of oxygen required and CO_2 released for every litre of fuel respectively are: (Given: density of the fuel is 0.756 g/mL)

(A) 1188 g and 1296 g
(B) 2376 g and 2592 g
(C) 2592 g and 2376 g
(D) 3429 g and 3142 g

Correct Answer: (C) 2592 g and 2376 g

Solution:

Step 1: Write the balanced chemical equation for the combustion.
The fuel is $C_{15}H_{30}$. The combustion reaction with oxygen (O_2) produces carbon dioxide (CO_2) and water (H_2O).
The unbalanced equation is:

$$C_{15}H_{30} + O_2 \rightarrow CO_2 + H_2O$$

Now, we balance the equation:
- Balance Carbon (C): There are 15 C atoms on the left, so we need 15 CO_2 on the right.

$$C_{15}H_{30} + O_2 \rightarrow 15CO_2 + H_2O$$

- Balance Hydrogen (H): There are 30 H atoms on the left, so we need 15 H_2O on the right.

$$C_{15}H_{30} + O_2 \rightarrow 15CO_2 + 15H_2O$$

- Balance Oxygen (O): On the right, we have $(15 \times 2) + 15 = 30 + 15 = 45$ oxygen atoms. So we need $\frac{45}{2}$ O_2 molecules on the left.
The balanced equation is:

$$C_{15}H_{30} + \frac{45}{2}O_2 \rightarrow 15CO_2 + 15H_2O$$

Step 2: Calculate the mass and moles of 1 litre of fuel.
- Volume of fuel = 1 L = 1000 mL.
- Density of fuel = 0.756 g/mL.
- Mass of fuel = Volume $\times$ Density = 1000 mL $\times$ 0.756 g/mL = 756 g.
- Molar mass of $C_{15}H_{30} = (15 \times 12) + (30 \times 1) = 180 + 30 = 210$ g/mol.
- Moles of fuel = $\frac{\text{Mass}}{\text{Molar mass}} = \frac{756 \text{ g}}{210 \text{ g/mol}} = 3.6$ moles.

Step 3: Use stoichiometry to find the mass of O_2 and CO_2.
From the balanced equation's mole ratios:
- **Oxygen required:** 1 mole of fuel requires $\frac{45}{2}$ moles of O_2.
Moles of O_2 needed = 3.6 moles fuel $\times \frac{45/2 \text{ moles } O_2}{1 \text{ mole fuel}} = 3.6 \times 22.5 = 81$ moles of O_2.
Molar mass of $O_2 = 2 \times 16 = 32$ g/mol.
Mass of O_2 = 81 moles $\times$ 32 g/mol = 2592 g.

- **CO_2 released:** 1 mole of fuel produces 15 moles of CO_2.
Moles of CO_2 produced = 3.6 moles fuel $\times \frac{15 \text{ moles } CO_2}{1 \text{ mole fuel}} = 54$ moles of CO_2.
Molar mass of $CO_2 = 12 + (2 \times 16) = 44$ g/mol.
Mass of CO_2 = 54 moles $\times$ 44 g/mol = 2376 g.

Step 4: Final Answer
The weight of oxygen required is 2592 g and the weight of CO_2 released is 2376 g. This corresponds to option (C).

Quick Tip

Stoichiometry problems require a clear, step-by-step approach: 1. Write and balance the chemical equation. 2. Convert the given quantity (mass, volume) of the reactant to moles. 3. Use the mole ratio from the balanced equation to find the moles of the desired product/reactant. 4. Convert the moles back to the required quantity (mass).

---

62. Consider the following pairs of electrons
(A) (a) n=3, l=1, m_l=1, m_s = +1/2
(b) n=3, l=2, m_l=1, m_s = +1/2
(B) (a) n=3, l=2, m_l=-2, m_s = -1/2
(b) n=3, l=2, m_l=-1, m_s = -1/2
(C) (a) n=4, l=2, m_l=2, m_s = +1/2
(b) n=3, l=2, m_l=2, m_s = +1/2
The pairs of electrons present in degenerate orbitals is/are :

(A) Only (A)
(B) Only (B)
(C) Only (C)
(D) (B) and (C)

Correct Answer: (B) Only (B)

Solution:

Step 1: Understanding Degenerate Orbitals
Degenerate orbitals are orbitals that have the same energy. For a multi-electron atom, orbitals within the same subshell are degenerate. This means they must have the same principal quantum number (n) and the same azimuthal quantum number (l). They will differ only in their magnetic quantum number (m_l).

Step 2: Analyzing each pair
We need to check if the two electrons in each pair have the same n and l values.

- **Pair (A):**
(a) $n = 3, l = 1$ (This is a 3p orbital).
(b) $n = 3, l = 2$ (This is a 3d orbital).
Since the l values are different ($l = 1$ vs $l = 2$), these orbitals belong to different subshells (3p

and 3d) and are not degenerate.

- **Pair (B):**
(a) $n = 3, l = 2, m_l = -2$.
(b) $n = 3, l = 2, m_l = -1$.
Both electrons have $n = 3$ and $l = 2$. This means they are both in the 3d subshell. Orbitals within the same subshell are degenerate. Therefore, this pair of electrons is in degenerate orbitals.

- **Pair (C):**
(a) $n = 4, l = 2$ (This is a 4d orbital).
(b) $n = 3, l = 2$ (This is a 3d orbital).
Since the n values are different ($n = 4$ vs $n = 3$), these orbitals belong to different principal energy levels and are not degenerate.

Step 3: Final Answer
Only the electrons in Pair (B) are in degenerate orbitals.

> Quick Tip
>
> Remember the conditions for orbital degeneracy: same principal quantum number (n) and same azimuthal quantum number (l). Electrons in these orbitals can have different magnetic (m_l) and spin (m_s) quantum numbers.

---

63. Match List - I with List - II :

	List - I		List - II
(A)	$[PtCl_4]^{2-}$	(I)	sp^3d
(B)	BrF_5	(II)	d^2sp^3
(C)	PCl_5	(III)	dsp^2
(D)	$[Co(NH_3)_6]^{3+}$	(IV)	sp^3d^2

Choose the most appropriate answer from the options given below :

(A) (A)-(II), (B)-(IV), (C)-(I), (D)-(III)
(B) (A)-(III), (B)-(IV), (C)-(I), (D)-(II)
(C) (A)-(III), (B)-(I), (C)-(IV), (D)-(II)
(D) (A)-(II), (B)-(I), (C)-(IV), (D)-(III)

Correct Answer: (B) (A)-(III), (B)-(IV), (C)-(I), (D)-(II)

Solution:

Step 1: Determine the hybridization for each species.

- **(A) $[PtCl_4]^{2-}$:** Platinum (Pt) is in the +2 oxidation state. The electronic configuration of Pt is [Xe] $4f^{14}$ $5d^9$ $6s^1$. The configuration of Pt^{2+} is [Xe] $4f^{14}$ $5d^8$. Since Cl^- is a strong field ligand for 5d series metals and the coordination number is 4, the complex will be square planar. This involves one d-orbital, one s-orbital, and two p-orbitals, leading to **dsp^2** hybridization. So, (A) matches (III).

- **(B) BrF_5:** Bromine (Br) is the central atom. It has 7 valence electrons. It forms 5 single bonds with 5 fluorine atoms and has 1 lone pair.
Steric Number = (number of sigma bonds) + (number of lone pairs) = 5 + 1 = 6.
A steric number of 6 corresponds to **sp^3d^2** hybridization. So, (B) matches (IV).

- **(C) PCl_5:** Phosphorus (P) is the central atom. It has 5 valence electrons. It forms 5 single bonds with 5 chlorine atoms.
Steric Number = 5 + 0 = 5.
A steric number of 5 corresponds to **sp^3d** hybridization. So, (C) matches (I).

- **(D) $[Co(NH_3)_6]^{3+}$:** Cobalt (Co) is in the +3 oxidation state. The configuration of Co is [Ar] $3d^7$ $4s^2$. The configuration of Co^{3+} is [Ar] $3d^6$. NH_3 is a strong field ligand, which causes pairing of the 3d electrons. The six electron pairs from the six NH_3 ligands occupy two empty 3d orbitals, one 4s orbital, and three 4p orbitals. This results in an inner orbital complex with **d^2sp^3** hybridization. So, (D) matches (II).

Step 2: Match the lists and select the correct option.
- (A) $\rightarrow$ (III)
- (B) $\rightarrow$ (IV)
- (C) $\rightarrow$ (I)
- (D) $\rightarrow$ (II)

This combination corresponds to option (B).

Quick Tip

To quickly determine hybridization for main group elements, use the VSEPR theory by calculating the steric number. For transition metal complexes, determine the oxidation state, d-electron count, coordination number, and ligand field strength to decide between inner (e.g., d^2sp^3) and outer (e.g., sp^3d^2) orbital hybridization.

---

64. For a reaction at equilibrium
$A(g) \rightleftharpoons B(g) + \frac{1}{2} C(g)$
the relation between dissociation constant (K), degree of dissociation (α) and equi-

librium pressure (p) is given by :

(A) $K = \frac{\alpha^{\frac{3}{2}} p^{\frac{1}{2}}}{(1+\frac{\alpha}{2})^{\frac{1}{2}}(1-\alpha)}$

(B) $K = \frac{\alpha^{\frac{3}{2}} p^{\frac{1}{2}}}{(2+\alpha)^{\frac{1}{2}}(1-\alpha)}$

(C) $K = \frac{(\alpha p)^{\frac{3}{2}}}{(1+\frac{\alpha}{2})^{\frac{1}{2}}(1-\alpha)}$

(D) $K = \frac{(\alpha p)^{\frac{3}{2}}}{(1+\alpha)(1-\alpha)^{\frac{1}{2}}}$

Correct Answer: (B) $K = \frac{\alpha^{\frac{3}{2}} p^{\frac{1}{2}}}{(2+\alpha)^{\frac{1}{2}}(1-\alpha)}$

Solution:

Step 1: Set up an ICE table based on moles.
Let's assume we start with 1 mole of reactant A. α is the degree of dissociation.

	A(g)	$\rightleftharpoons$ B(g)	+ $\frac{1}{2}$ C(g)
Initial moles (I)	1	0	0
Change in moles (C)	$-\alpha$	$+\alpha$	$+\alpha/2$
Equilibrium moles (E)	$1-\alpha$	α	$\alpha/2$

Total moles at equilibrium $= (1-\alpha) + \alpha + \alpha/2 = 1 + \alpha/2 = \frac{2+\alpha}{2}$.

Step 2: Calculate the partial pressures at equilibrium.
The partial pressure of a gas is its mole fraction multiplied by the total pressure p.

$$P_A = (\text{mole fraction of A}) \times p = \left(\frac{1-\alpha}{(2+\alpha)/2}\right) p = \left(\frac{2(1-\alpha)}{2+\alpha}\right) p$$

$$P_B = (\text{mole fraction of B}) \times p = \left(\frac{\alpha}{(2+\alpha)/2}\right) p = \left(\frac{2\alpha}{2+\alpha}\right) p$$

$$P_C = (\text{mole fraction of C}) \times p = \left(\frac{\alpha/2}{(2+\alpha)/2}\right) p = \left(\frac{\alpha}{2+\alpha}\right) p$$

Step 3: Write the expression for the equilibrium constant K$_p$.

$$K_p = \frac{P_B \cdot (P_C)^{1/2}}{P_A}$$

Substitute the partial pressure expressions:

$$K_p = \frac{\left(\frac{2\alpha p}{2+\alpha}\right) \cdot \left(\frac{\alpha p}{2+\alpha}\right)^{1/2}}{\left(\frac{2(1-\alpha)p}{2+\alpha}\right)}$$

$$K_p = \frac{2\alpha p}{2+\alpha} \cdot \frac{\alpha^{1/2} p^{1/2}}{(2+\alpha)^{1/2}} \cdot \frac{2+\alpha}{2(1-\alpha)p}$$

Cancel common terms (2, p, and $2 + \alpha$):

$$K_p = \frac{\alpha \cdot \alpha^{1/2} \cdot p^{1/2}}{(1-\alpha)(2+\alpha)^{1/2}}$$

$$K_p = \frac{\alpha^{3/2} p^{1/2}}{(1-\alpha)(2+\alpha)^{1/2}}$$

This expression matches option (B).

> Quick Tip
>
> For gaseous equilibria, always start by finding the total moles at equilibrium. This is crucial for correctly calculating the mole fractions, which are then used to find the partial pressures. Be careful with stoichiometric coefficients, especially fractional ones, as they become exponents in the K_p expression.

---

65. Given below are two statements :
Statement I: Emulsions of oil in water are unstable and sometimes they separate into two layers on standing.
Statement II: For stabilisation of an emulsion, excess of electrolyte is added.

(A) Both Statement I and Statement II are correct.
(B) Both Statement I and Statement II are incorrect.
(C) Statement I is correct but Statement II is incorrect.
(D) Statement I is incorrect but Statement II is correct.

Correct Answer: (C) Statement I is correct but Statement II is incorrect.

Solution:

Step 1: Analyze Statement I
An emulsion is a colloidal system consisting of one liquid dispersed in another immiscible liquid (e.g., oil in water). Colloids are thermodynamically unstable systems. Over time, the dispersed droplets can coalesce (merge) to form larger droplets, eventually leading to the separation of the two liquid phases into distinct layers to minimize the interfacial surface area. This process is called creaming or coagulation. Therefore, Statement I is correct.

Step 2: Analyze Statement II
To stabilize an emulsion and prevent it from separating, a third component called an emulsifying agent (or emulsifier) is added. Emulsifiers (like soap or detergents) work by forming a protective layer around the dispersed droplets, preventing them from coalescing. Adding an electrolyte has the opposite effect. The ions from the electrolyte neutralize the charge on the colloidal droplets, which causes them to coagulate and break the emulsion. Therefore, adding

an excess of electrolyte destabilizes, rather than stabilizes, an emulsion. Statement II is incorrect.

Step 3: Final Conclusion
Statement I is a correct description of the stability of emulsions. Statement II describes a method of destabilization, not stabilization. Thus, Statement I is correct, but Statement II is incorrect.

> Quick Tip
>
> Remember the key terms for colloids:
> - **Stabilization:** Achieved by adding an emulsifying/stabilizing agent.
> - **Destabilization (Coagulation/Precipitation):** Caused by adding electrolytes, boiling, or electrophoresis.

---

66. Given below are the oxides :
Na_2O, As_2O_3, N_2O, NO and Cl_2O_7
Number of amphoteric oxides is :

(A) 0
(B) 1
(C) 2
(D) 3

Correct Answer: (B) 1

Solution:

Step 1: Define Amphoteric Oxides
Amphoteric oxides are oxides that can react with both acids and bases to form salt and water. They exhibit both acidic and basic properties. Generally, oxides of metalloids and some metals (like Al, Zn, Sn, Pb, Be) are amphoteric.

Step 2: Classify Each Given Oxide
- **Na_2O (Sodium oxide):** Sodium is a reactive alkali metal. Its oxide is strongly **basic**. It reacts with acids but not with bases.
- **As_2O_3 (Arsenic trioxide):** Arsenic is a metalloid. Its oxide is **amphoteric**. It reacts with strong acids (like HCl) and strong bases (like NaOH).
- **N_2O (Nitrous oxide):** This is an oxide of a non-metal. It does not react with either acids or bases. It is a **neutral** oxide.
- **NO (Nitric oxide):** This is also an oxide of a non-metal. It is a **neutral** oxide.
- **Cl_2O_7 (Dichlorine heptoxide):** Chlorine is a non-metal, and this is an oxide with chlorine

in its highest oxidation state (+7). It is a strongly **acidic** oxide. It reacts with bases but not with acids.

Step 3: Count the Amphoteric Oxides
From the classification above, only As_2O_3 is an amphoteric oxide.
The number of amphoteric oxides in the given list is 1.

> Quick Tip
>
> Remember the general trend for oxide character across the periodic table:
> - **Left side (Metals):** Basic (e.g., Na_2O)
> - **Right side (Non-metals):** Acidic (e.g., Cl_2O_7, SO_3)
> - **Middle/Metalloids:** Amphoteric (e.g., Al_2O_3, ZnO, As_2O_3)
> - Some non-metal oxides in low oxidation states are neutral (e.g., CO, NO, N_2O).

---

67. Match List - I with List - II :

	List - I		List - II
(A)	Sphalerite	(I)	$FeCO_3$
(B)	Calamine	(II)	PbS
(C)	Galena	(III)	$ZnCO_3$
(D)	Siderite	(IV)	ZnS

Choose the most appropriate answer from the options given below :

(A) (A)-(IV), (B)-(III), (C)-(II), (D)-(I)
(B) (A)-(IV), (B)-(I), (C)-(II), (D)-(III)
(C) (A)-(II), (B)-(III), (C)-(I), (D)-(IV)
(D) (A)-(III), (B)-(IV), (D)-(II), (D)-(I)

Correct Answer: (A) (A)-(IV), (B)-(III), (C)-(II), (D)-(I)

Solution:

Step 1: Identify the chemical formula for each ore.
This question requires knowledge of the common names and chemical compositions of important mineral ores.

- **(A) Sphalerite:** It is a primary ore of zinc, also known as zinc blende. Its chemical formula is zinc sulfide, **ZnS**. This matches (IV).

- **(B) Calamine:** It is also an ore of zinc. Its chemical formula is zinc carbonate, $\mathbf{ZnCO_3}$. This matches (III).

- **(C) Galena:** It is the primary ore of lead. Its chemical formula is lead(II) sulfide, **PbS**. This matches (II).

- **(D) Siderite:** It is an ore of iron. Its chemical formula is iron(II) carbonate, $\mathbf{FeCO_3}$. This matches (I).

Step 2: Match the lists and select the correct option.
- (A) $\rightarrow$ (IV)
- (B) $\rightarrow$ (III)
- (C) $\rightarrow$ (II)
- (D) $\rightarrow$ (I)

This combination corresponds to option (A).

> Quick Tip
>
> Memorizing the names and formulas of common ores is essential for metallurgy topics. Create flashcards for important ores of metals like Fe, Cu, Al, Zn, Pb, and Sn. Pay special attention to ores with similar names or metals (e.g., the different zinc ores like Sphalerite, Zincite, and Calamine).

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68. The highest industrial consumption of molecular hydrogen is to produce compounds of element :

(A) Carbon
(B) Nitrogen
(C) Oxygen
(D) Chlorine

Correct Answer: (B) Nitrogen

Solution:

Step 1: Analyze the major industrial uses of hydrogen.
Molecular hydrogen (H_2) is a crucial industrial chemical with several large-scale applications. The main uses are:
1. **Ammonia Production:** The Haber-Bosch process, $N_2(g) + 3H_2(g) \rightleftharpoons 2NH_3(g)$, is the largest single consumer of hydrogen worldwide. The ammonia produced is primarily used to make nitrogen-based fertilizers. This involves producing a compound of **Nitrogen**.
2. **Petroleum Refining:** Hydrogen is used in hydrocracking (breaking down heavy hydrocarbons into more valuable lighter ones) and hydrotreating (removing impurities like sulfur). This

involves reactions with compounds of **Carbon**.
3. **Methanol Production:** The synthesis of methanol, $CO(g) + 2H_2(g) \rightarrow CH_3OH(l)$, is another major use. This involves producing a compound of **Carbon** and **Oxygen**.
4. **Hydrogenation of Fats and Oils:** Converting unsaturated fats to saturated fats (e.g., making margarine). This involves reacting with compounds of **Carbon**.

Step 2: Compare the scale of consumption.
Among all these uses, the synthesis of ammonia via the Haber-Bosch process accounts for the largest share of global hydrogen production, estimated to be over 50%. Therefore, the highest industrial consumption is to produce compounds of nitrogen.

Quick Tip

When asked about the "largest" or "most common" industrial application of a chemical, the Haber-Bosch process for ammonia synthesis is a very common answer, as it is a cornerstone of the modern chemical and agricultural industries.

---

69. Which of the following statements are correct ?
(A) Both LiCl and $MgCl_2$ are soluble in ethanol.
(B) The oxides Li_2O and MgO combine with excess of oxygen to give superoxide.
(C) LiF is less soluble in water than other alkali metal fluorides.
(D) Li_2O is more soluble in water than other alkali metal oxides.
Choose the most appropriate answer from the options given below :

(A) (A) and (C) only
(B) (A), (C) and (D) only
(C) (B) and (C) only
(D) (A) and (D) only

Correct Answer: (A) (A) and (C) only

Solution:

Step 1: Analyze each statement based on the properties of s-block elements.

- **(A) Both LiCl and $MgCl_2$ are soluble in ethanol.**
Lithium and Magnesium exhibit a diagonal relationship. Both LiCl and $MgCl_2$ have a significant degree of covalent character due to the high polarizing power of the small Li^+ and Mg^{2+} ions. Covalent compounds tend to be soluble in organic solvents like ethanol. This statement is **correct**.

- **(B) The oxides Li_2O and MgO combine with excess of oxygen to give superoxide.**
When alkali metals react with excess oxygen, Lithium forms the normal oxide (Li_2O) and some

peroxide (Li_2O_2). Sodium forms the peroxide (Na_2O_2). Potassium, Rubidium, and Caesium form superoxides (e.g., KO_2). Li_2O does not form a superoxide. MgO is also very stable and does not react with oxygen to form a superoxide. This statement is **incorrect**.

- **(C) LiF is less soluble in water than other alkali metal fluorides.**
The solubility of ionic compounds depends on the balance between lattice enthalpy and hydration enthalpy. For LiF, both Li^+ and F^- ions are very small, leading to a very high lattice enthalpy. This high lattice enthalpy is not overcome by its hydration enthalpy, making LiF sparingly soluble in water. Other alkali fluorides like NaF and KF are more soluble. This statement is **correct**.

- **(D) Li_2O is more soluble in water than other alkali metal oxides.**
The solubility of alkali metal oxides in water generally increases down the group. Li_2O has a higher lattice energy than Na_2O or K_2O, making it less soluble. This statement is **incorrect**.

Step 2: Identify the correct combination.
The correct statements are (A) and (C). Therefore, the correct option is (A).

Quick Tip

Remember the key trends and anomalies for s-block elements. The diagonal relationship between Li and Mg is very important, leading to similarities in the covalent character and solubility of their salts. For solubility, always consider the interplay between lattice energy (inversely related to ion size) and hydration energy (also inversely related to ion size).

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70. Identify the correct statement for B_2H_6 from those given below.
(A) In B_2H_6, all B-H bonds are equivalent.
(B) In B_2H_6, there are four 3-centre-2-electron bonds.
(C) B_2H_6 is a Lewis acid.
(D) B_2H_6 can be synthesized from both BF_3 and $NaBH_4$.
(E) B_2H_6 is a planar molecule.
Choose the most appropriate answer from the options given below :

(A) (A) and (E) only
(B) (B), (C) and (E) only
(C) (C) and (D) only
(D) (C) and (E) only

Correct Answer: (C) (C) and (D) only

Solution:

Step 1: Analyze each statement about diborane (B_2H_6).

- **(A) In B_2H_6, all B-H bonds are equivalent.**
This is **incorrect**. The structure of diborane has two types of bonds: four terminal B-H bonds (which are standard 2-centre-2-electron bonds) and two bridging B-H-B bonds (which are 3-centre-2-electron bonds). The terminal and bridging bonds have different lengths and strengths.

- **(B) In B_2H_6, there are four 3-centre-2-electron bonds.**
This is **incorrect**. There are only two 3-centre-2-electron "banana bonds" which form the bridges.

- **(C) B_2H_6 is a Lewis acid.**
This is **correct**. Diborane is an electron-deficient molecule because the boron atoms do not have a complete octet. It readily accepts electron pairs from Lewis bases (like NH_3, CO) to form adducts.

- **(D) B_2H_6 can be synthesized from both BF_3 and $NaBH_4$.**
This is **correct**. A common laboratory preparation involves the reaction of sodium borohydride with boron trifluoride: $3NaBH_4 + 4BF_3 \rightarrow 2B_2H_6 + 3NaBF_4$.

- **(E) B_2H_6 is a planar molecule.**
This is **incorrect**. The four terminal hydrogen atoms and the two boron atoms lie in one plane, but the two bridging hydrogen atoms lie in a plane perpendicular to this, one above and one below. The molecule is non-planar.

Step 2: Identify the correct combination.
The correct statements are (C) and (D). Therefore, the correct option is (C).

Quick Tip

The structure of diborane is a classic example of electron-deficient bonding. Remember these key features: two boron atoms, four terminal hydrogens in one plane, two bridging hydrogens in a perpendicular plane, two 3-centre-2-electron bonds, and four 2-centre-2-electron bonds.

---

71. The most stable trihalide of nitrogen is :

(A) NF_3
(B) NCl_3
(C) NBr_3
(D) NI_3

Correct Answer: (A) NF_3

Solution:

Step 1: Analyze the stability of Nitrogen Trihalides (NX_3).
The stability of the nitrogen trihalides decreases significantly as we move down the halogen group from Fluorine to Iodine. This trend is primarily governed by the strength of the N-X bond.

- **NF_3 (Nitrogen trifluoride):** The N-F bond is strong due to the small size and high electronegativity of both nitrogen and fluorine atoms, resulting in significant bond polarity and strength. NF_3 is a thermodynamically stable (exothermic) and inert gas.

- **NCl_3 (Nitrogen trichloride):** The N-Cl bond is much weaker due to the larger size of the chlorine atom and the small difference in electronegativity between N and Cl. NCl_3 is an endothermic compound and is highly explosive.

- **NBr_3 (Nitrogen tribromide) and NI_3 (Nitrogen triiodide):** These compounds are even less stable and more explosive than NCl_3. The N-Br and N-I bonds are very weak due to the large size difference between nitrogen and the heavier halogens.

Step 2: Conclude the most stable compound.
Based on the trend in bond strength and thermodynamic stability, NF_3 is by far the most stable trihalide of nitrogen.

Quick Tip
Bond strength is a key factor in determining the stability of simple molecules. For bonds between a period 2 element (like N) and other elements in a group, the bond strength generally decreases as you go down the group due to poorer orbital overlap from increasing atomic size.

---

72. Which one of the following elemental forms is not present in the enamel of the teeth ?

(A) Ca^{2+}
(B) P^{3+}
(C) F^{-}
(D) P^{5+}

Correct Answer: (B) P^{3+}

Solution:

Step 1: Identify the chemical composition of tooth enamel.
Tooth enamel is the hardest substance in the human body. Its primary component is a crystalline calcium phosphate mineral called **hydroxyapatite**. The chemical formula for hydroxyapatite is $Ca_{10}(PO_4)_6(OH)_2$.

Step 2: Determine the elemental forms (ions/oxidation states) present.
From the formula $Ca_{10}(PO_4)_6(OH)_2$, we can identify the constituent ions:
- **Calcium:** It is present as the calcium ion, $\mathbf{Ca^{2+}}$.
- **Phosphate:** The phosphate ion is $\mathbf{PO_4^{3-}}$. In this ion, oxygen has an oxidation state of -2. To find the oxidation state of phosphorus (P), let it be x. Then $x + 4(-2) = -3$, which gives $x - 8 = -3$, so $x = +5$. Thus, phosphorus is present in the $\mathbf{P^{5+}}$ oxidation state.
- **Hydroxide:** The hydroxide ion is OH^-.
Tooth enamel can also incorporate fluoride ions ($\mathbf{F^-}$) from water and toothpaste, which substitute for the hydroxide ions to form fluoroapatite ($Ca_{10}(PO_4)_6F_2$), a more acid-resistant mineral.

Step 3: Compare with the given options.
- (A) Ca^{2+}: Present.
- (C) F^-: Can be present.
- (D) P^{5+}: Present (as part of the phosphate ion).
- (B) P^{3+}: Phosphorus in the +3 oxidation state is not a component of tooth enamel.

Step 4: Final Answer
The elemental form not present in tooth enamel is P^{3+}.

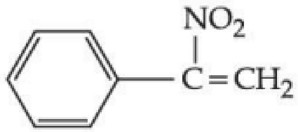
Quick Tip

Knowing the chemical formula of key biological minerals like hydroxyapatite is useful. From the formula, you can deduce the constituent ions and the oxidation states of the elements involved.

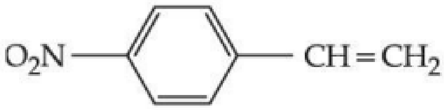
---

73. In the given reaction sequence, the major product 'C' is :
$$\mathbf{C_8H_{10} \xrightarrow{HNO_3, H_2SO_4} A \xrightarrow{Br_2, \Delta} B \xrightarrow{\text{alcoholic KOH}} C}$$

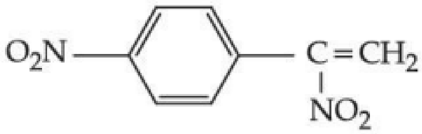
A



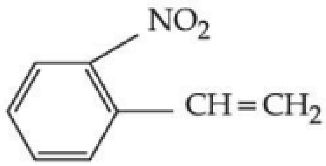
B

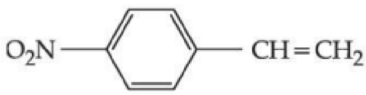


C



D



Correct Answer: (B) 

Solution:

Step 1: Identify the starting material and the first reaction (Formation of A).
The starting material has the formula C_8H_{10}. A common and stable aromatic isomer is **ethylbenzene** (C_6H_5-CH_2CH_3).
The first reaction is nitration using a nitrating mixture (conc. HNO_3 + conc. H_2SO_4). This is an electrophilic aromatic substitution reaction. The ethyl group (-CH_2CH_3) is an ortho, para-directing and activating group. Due to the steric hindrance of the ethyl group, the para product is formed as the major product.
So, **A** is **p-nitroethylbenzene** or 1-ethyl-4-nitrobenzene.

Step 2: Identify the second reaction (Formation of B).
Product A (p-nitroethylbenzene) is treated with Br_2 and heat (Δ). These conditions are characteristic of a **free radical substitution**. This type of reaction occurs preferentially at the benzylic position (the carbon atom directly attached to the benzene ring), as the resulting benzylic radical is resonance-stabilized.
A hydrogen atom on the benzylic carbon (-CH_2-) is replaced by a bromine atom.
So, **B** is **1-bromo-1-(4-nitrophenyl)ethane**.

Step 3: Identify the third reaction (Formation of C).
Product B is treated with alcoholic KOH. A strong base in an alcohol solvent favors an **E2 (elimination) reaction**. This reaction involves the removal of a hydrogen atom from one carbon and a leaving group (Br) from an adjacent carbon to form a double bond.
In 1-bromo-1-(4-nitrophenyl)ethane, the bromine is removed from the benzylic carbon, and a hydrogen is removed from the adjacent methyl (-CH_3) group. This results in the formation of

a C=C double bond.
The final product, **C**, is **4-nitrostyrene**.

Quick Tip

Recognizing the specific conditions for different reaction types is crucial for solving multi-step synthesis problems.
- **HNO_3/H_2SO_4:** Electrophilic Aromatic Substitution (Nitration).
- **Br_2/Heat or UV Light:** Free Radical Substitution (at allylic/benzylic positions).
- **Alcoholic KOH:** E2 Elimination to form alkenes.
- **Aqueous KOH:** Nucleophilic Substitution (SN1/SN2).

---

74. Two statements are given below :
Statement I: The melting point of monocarboxylic acid with even number of carbon atoms is higher than that of with odd number of carbon atoms acid immediately below and above it in the series.
Statement II: The solubility of monocarboxylic acids in water decreases with increase in molar mass.
Choose the most appropriate option :

(A) Both Statement I and Statement II are correct.
(B) Both Statement I and Statement II are incorrect.
(C) Statement I is correct but Statement II is incorrect.
(D) Statement I is incorrect but Statement II is correct.

Correct Answer: (A) Both Statement I and Statement II are correct.

Solution:

Step 1: Analyze Statement I
This statement describes the **alternation effect** or sawtooth pattern observed in the melting points of homologous series like carboxylic acids and alkanes.
Carboxylic acids with an **even** number of carbon atoms have their terminal methyl group and carboxyl group on opposite sides of the carbon chain. This allows them to pack more closely and efficiently into a crystal lattice.
Carboxylic acids with an **odd** number of carbon atoms have their terminal groups on the same side, which disrupts the crystal packing.
Due to the more efficient packing, even-numbered acids have stronger intermolecular van der Waals forces, resulting in higher melting points than the odd-numbered acids immediately preceding and succeeding them.
Therefore, **Statement I is correct**.

Step 2: Analyze Statement II

The solubility of carboxylic acids in water is due to their ability to form hydrogen bonds with water molecules via the polar carboxyl (-COOH) group.
Lower members of the series (like formic acid, acetic acid) are completely miscible in water.
However, as the molar mass increases, the length of the nonpolar (hydrophobic) alkyl chain increases. This large hydrophobic part of the molecule disrupts the hydrogen bonding network of water and outweighs the effect of the polar carboxyl group.
Consequently, the solubility in water decreases as the molar mass increases.
Therefore, **Statement II is correct**.

> Quick Tip
>
> Remember key physical property trends for homologous series. For melting points, look for the alternation effect in alkanes and carboxylic acids due to crystal packing. For boiling points and solubility in polar solvents, consider the interplay between increasing molar mass (stronger van der Waals forces) and the effect of the polar functional group versus the growing nonpolar chain.

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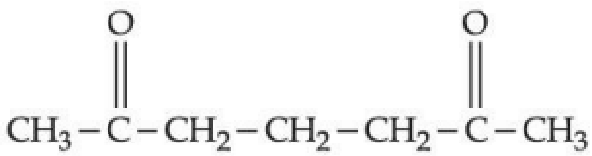
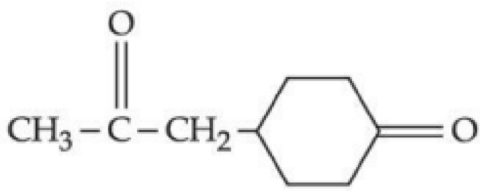
75. Which of the following is an example of conjugated diketone ?

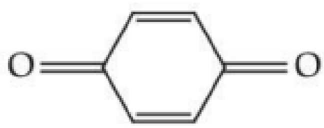
A



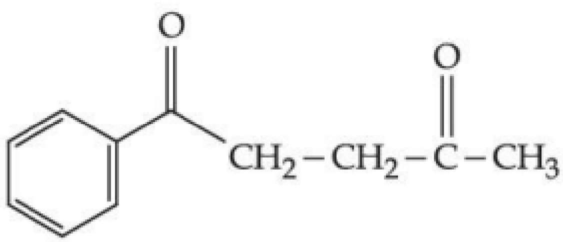
B



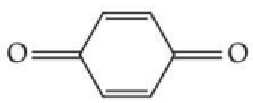
C



D



Correct Answer: (C) c



Solution:

Step 1: Understanding Conjugated Systems
A conjugated system is a system of connected p-orbitals with delocalized electrons in a molecule. In the context of ketones, this typically means that the carbonyl groups (C=O) are separated by one or more C=C double bonds, in an alternating single-double bond pattern. The general structure is C=O-$(C=C)_n$-C=O.

Step 2: Analyzing the Options
- **(A) Heptane-2,6-dione:** The structure is CH_3-CO-CH_2-CH_2-CH_2-CO-CH_3. The two carbonyl groups are separated by three single bonds (σ-bonds). There is no conjugation between them. They are isolated functional groups.

- **(B) Cyclohexane-1,4-dione:** In this cyclic molecule, the two carbonyl groups are at positions 1 and 4 of a saturated six-membered ring. They are separated by two single bonds on each side. There is no conjugation.

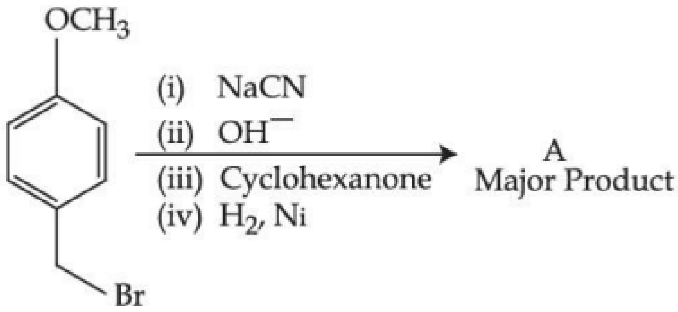
- **(C) p-Benzoquinone:** The structure is a six-membered ring containing two C=C double bonds and two C=O double bonds, with the C=O groups at opposite ends (para position). The structure is O=C-CH=CH-C=CH-CH=O (cyclic). Here, the π-electrons of the C=O bonds and C=C bonds are all part of a single, continuous, cyclic conjugated system. This is a conjugated diketone.

- **(D) 1-Phenylpentane-1,4-dione:** The structure is Ph-CO-CH_2-CH_2-CO-CH_3. The two carbonyl groups are separated by two single bonds. They are not conjugated with each other. (The first carbonyl group is conjugated with the phenyl ring, but the two carbonyls are not conjugated together).

Step 3: Final Answer
Only p-benzoquinone has a structure where the two carbonyl groups are part of the same conjugated π-electron system.

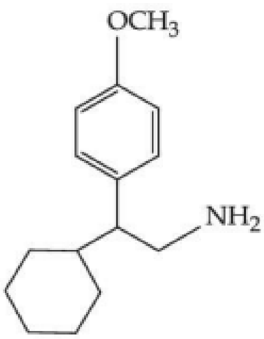
> Quick Tip
>
> To identify a conjugated system, look for a continuous chain of alternating single and double/triple bonds (e.g., C=C-C=C, C=C-C=O). Conjugation leads to special stability and reactivity patterns, such as 1,4-addition.

76.



The major product of the above reactions is :

A



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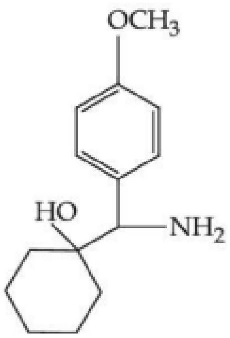

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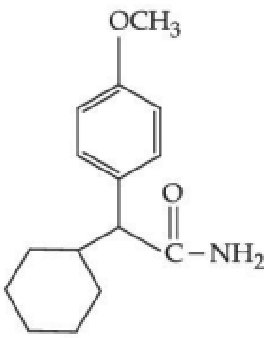
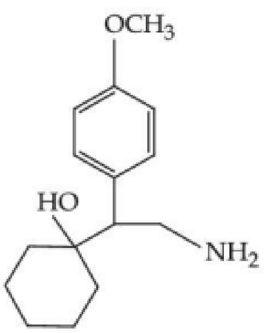
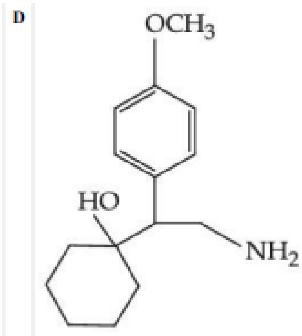


D



Correct Answer: (D) 

Solution:

Step 1: Analyze reaction (i) - Formation of Nitrile
The starting material is 1-(bromomethyl)-4-methoxybenzene. It reacts with NaCN via an S_N2 mechanism, where the cyanide ion (CN^-) displaces the bromide ion (Br^-).
Product after (i): 4-methoxyphenylacetonitrile (MeO-Ph-CH_2CN).

Step 2: Analyze reactions (ii) and (iii) - Nucleophilic Addition
Reaction (ii) with OH^- deprotonates the α-carbon (the carbon adjacent to the CN group), which is acidic. This forms a resonance-stabilized carbanion, $[\text{MeO-Ph-CH-CN}]^-$.
This carbanion then acts as a nucleophile and attacks the electrophilic carbonyl carbon of cyclohexanone (reagent iii).
The resulting alkoxide ion is protonated (by H_2O, which is present with aqueous OH^-) to yield a β-hydroxynitrile.
Product after (iii): (1-hydroxycyclohexyl)(4-methoxyphenyl)acetonitrile.
The reaction conditions do not specify heat, which is often required for the subsequent dehydration (elimination of H_2O) step. Therefore, we assume the reaction stops at this addition product without forming a C=C double bond.

Step 3: Analyze reaction (iv) - Reduction
The final step is reduction with H_2, Ni (catalytic hydrogenation). This is a strong reducing agent for nitriles. However, the structure in option D suggests a different transformation. The structure in option D is (1-hydroxycyclohexyl)(4-methoxyphenyl)methanamine, where the C≡N group seems to have been converted to a -CH(NH_2)- group. This specific transformation is not standard. A standard reduction would yield a -CH(CH_2NH_2)- group. Given that option D is correct, there is a likely typo in the question or options. Assuming the intention was to represent the product without dehydration, the key feature is the retention of the -OH group on the cyclohexane ring. Option D is the only one with this feature. To arrive at the specific structure of D, one might postulate a reaction pathway where the nitrile group is reduced to an imine, which is then hydrolyzed and further reduced, but this is highly speculative. The most direct interpretation is that the reaction produces the β-hydroxynitrile, which is then reduced, and option D is the intended, albeit imprecisely drawn, representation of this product.

Step 4: Conclusion based on key structural features
The crucial step that differentiates the possible products is whether dehydration occurs after

the aldol-type addition. The absence of heat suggests dehydration might not be the major pathway. This would lead to a final product containing an alcohol (-OH) group. Option (D) is the only structure that contains the 1-hydroxycyclohexyl group. Therefore, it represents the product formed via addition and reduction, without elimination.

Quick Tip

When analyzing multi-step reactions, look for the key decision points. Here, it is whether the intermediate -hydroxynitrile undergoes elimination. The reaction conditions (presence or absence of heat) can be a clue. If the final options seem inconsistent, focus on the major structural features that would result from the different pathways.

---

77. Which of the following is an example of polyester ?

(A) Butadiene-styrene copolymer
(B) Melamine polymer
(C) Neoprene
(D) Poly-β-hydroxybutyrate-co-β-hydroxyvalerate

Correct Answer: (D) Poly-β-hydroxybutyrate-co-β-hydroxyvalerate

Solution:

Step 1: Define Polyester
A polyester is a condensation polymer where the monomer units are linked together by **ester linkages** (-COO-). They are typically formed from the reaction between dicarboxylic acids and diols, or by the self-condensation of hydroxycarboxylic acids.

Step 2: Analyze the Options
- **(A) Butadiene-styrene copolymer:** This is also known as SBR (Styrene-Butadiene Rubber). It is an **addition polymer** formed from the monomers 1,3-butadiene and styrene. It contains only C-C and C=C bonds in its backbone.
- **(B) Melamine polymer:** This refers to melamine-formaldehyde resin. It is a **condensation polymer** formed from melamine and formaldehyde. The linkages are methylene bridges ($-CH_2-$) between nitrogen atoms of the melamine rings. It is a polyamide/polyamine type resin, not a polyester.
- **(C) Neoprene:** This is the polymer of the monomer chloroprene (2-chloro-1,3-butadiene). It is an **addition polymer** and a synthetic rubber.
- **(D) Poly-β-hydroxybutyrate-co-β-hydroxyvalerate (PHBV):** This is a **copolymer** formed by the condensation of two types of monomers: 3-hydroxybutanoic acid and 3-hydroxypentanoic acid. Since each monomer contains both a hydroxyl (-OH) group and a carboxylic acid (-COOH) group, they polymerize by forming ester bonds. Therefore, PHBV is a polyester. It is

also known for being biodegradable.

Quick Tip

To identify the type of polymer, look at the linkages between monomer units.
- **Ester (-COO-):** Polyester
- **Amide (-CONH-):** Polyamide (like Nylon)
- **Only C-C single bonds in backbone:** Addition polymer (like Polyethene, PVC, Neoprene)

---

78. A polysaccharide 'X' on boiling with dil H_2SO_4 at 393 K under 2-3 atm pressure yields 'Y'. 'Y' on treatment with bromine water gives gluconic acid. 'X' contains β-glycosidic linkages only. Compound 'X' is :

(A) starch
(B) cellulose
(C) amylose
(D) amylopectin

Correct Answer: (B) cellulose

Solution:

Step 1: Identify the monosaccharide unit 'Y'.
The reaction of 'Y' with bromine water is a key test. Bromine water is a mild oxidizing agent that specifically oxidizes the aldehyde group of an aldose to a carboxylic acid group, forming an aldonic acid.
The product is given as **gluconic acid**. This confirms that the monosaccharide 'Y' must be **glucose**.

Step 2: Analyze the structure of polysaccharide 'X'.
We are given three crucial pieces of information about 'X':
1. It is a polysaccharide that hydrolyzes to yield glucose ('Y'). This means 'X' is a polymer of glucose.
2. The hydrolysis is acid-catalyzed, which is a standard method for breaking glycosidic bonds.
3. 'X' contains only **β-glycosidic linkages**.

Step 3: Evaluate the options.
We need to find the polymer of glucose that contains β-glycosidic linkages.
- **(A) Starch:** Starch is a polymer of α-glucose. It consists of two components, amylose and amylopectin, both of which have α-glycosidic linkages.
- **(B) Cellulose:** Cellulose is a straight-chain polymer of β-D-glucose units which are joined

by β-**1,4-glycosidic bonds**. This matches our criteria.
- **(C) Amylose:** Amylose is a linear polymer component of starch, made of α-D-glucose units linked by α-1,4-glycosidic bonds.
- **(D) Amylopectin:** Amylopectin is a branched polymer component of starch, made of α-D-glucose units with α-1,4 and α-1,6-glycosidic bonds.

Step 4: Final Answer
The only polysaccharide listed that is a polymer of glucose containing exclusively β-glycosidic linkages is cellulose.

Quick Tip

Remember the fundamental difference between starch and cellulose: both are polymers of glucose, but starch is linked by α-glycosidic bonds, while cellulose is linked by β-glycosidic bonds. This small difference in stereochemistry leads to vastly different structures and biological functions. The bromine water test is a classic confirmation for aldoses like glucose.

---

79. Which of the following is not a broad spectrum antibiotic ?

(A) Vancomycin
(B) Ampicillin
(C) Ofloxacin
(D) Penicillin G

Correct Answer: (D) Penicillin G

Solution:

Step 1: Define Broad Spectrum vs. Narrow Spectrum Antibiotics
- **Broad-spectrum antibiotics** are effective against a wide range of disease-causing bacteria, including both Gram-positive and Gram-negative bacteria.
- **Narrow-spectrum antibiotics** are effective against specific families of bacteria, typically either Gram-positive or Gram-negative, but not both.

Step 2: Analyze the Spectrum of Each Antibiotic
- **(A) Vancomycin:** This is a glycopeptide antibiotic. Its activity is restricted to Gram-positive bacteria. It is used for serious infections like MRSA. It is considered a **narrow-spectrum** antibiotic.
- **(B) Ampicillin:** This is a penicillin-class antibiotic. It was developed to have activity against a wider range of bacteria than the original Penicillin G, including many Gram-negative organisms. It is a classic example of a **broad-spectrum** antibiotic.

- **(C) Ofloxacin:** This is a fluoroquinolone antibiotic. This class of antibiotics has a very wide range of activity against Gram-positive, Gram-negative, and even atypical bacteria. It is a **broad-spectrum** antibiotic.
- **(D) Penicillin G:** This is the original natural penicillin. Its activity is primarily limited to Gram-positive bacteria and some Gram-negative cocci. It is the quintessential example of a **narrow-spectrum** antibiotic.

Step 3: Choose the Best Answer
Both Vancomycin and Penicillin G are technically narrow-spectrum. However, in standard classification, Penicillin G is the most classic and widely cited example of a narrow-spectrum antibiotic, representing the baseline from which broader spectrum penicillins (like Ampicillin) were developed. Given the options, Penicillin G is the intended answer for an antibiotic that is "not broad spectrum".

> Quick Tip
>
> To classify antibiotics, it's helpful to remember the general spectrum of major classes.
> - **Broad Spectrum:** Tetracyclines, Ampicillin/Amoxicillin, Fluoroquinolones (e.g., Ofloxacin), Cephalosporins (some).
> - **Narrow Spectrum:** Penicillin G, Vancomycin, Macrolides (e.g., Erythromycin).

---

80. During the qualitative analysis of salt with cation y^{2+}, addition of a reagent (X) to alkaline solution of the salt gives a bright red precipitate. The reagent (X) and the cation (y^{2+}) present respectively are :

(A) Dimethylglyoxime and Ni^{2+}
(B) Dimethylglyoxime and Co^{2+}
(C) Nessler's reagent and Hg^{2+}
(D) Nessler's reagent and Ni^{2+}

Correct Answer: (A) Dimethylglyoxime and Ni^{2+}

Solution:

Step 1: Analyze the Observation
The key observation is the formation of a **bright red precipitate** upon adding a reagent to an **alkaline solution** containing an unknown cation y^{2+}. This is a very specific and well-known confirmatory test in inorganic qualitative analysis.

Step 2: Identify the Cation and Reagent
- The test described is the specific test for the Nickel(II) ion, $\mathbf{Ni^{2+}}$.
- The reagent used is **dimethylglyoxime (DMG)**.

- In a neutral, slightly alkaline (ammoniacal), or weakly acidic solution, Ni^{2+} ions react with an alcoholic solution of DMG to form a voluminous, cherry-red or bright red precipitate of nickel(II) dimethylglyoximate, $[Ni(dmg)_2]$.

$$Ni^{2+} + 2C_4H_8N_2O_2 \xrightarrow{\text{alkaline medium}} Ni(C_4H_7N_2O_2)_2(s) \downarrow (\text{red}) + 2H^+$$

Step 3: Evaluate the Other Options
- DMG gives a reddish-brown coloration with Co^{2+}, not a red precipitate.
- Nessler's reagent ($K_2[HgI_4]$) is used to test for the ammonium ion (NH_4^+), with which it gives a brown precipitate. It is not used for Ni^{2+} or Hg^{2+} in this manner.
Therefore, the only correct combination is Dimethylglyoxime and Ni^{2+}.

Quick Tip

It is highly beneficial to memorize the specific reagents and the color of the precipitates or solutions for the confirmatory tests of common cations (like Ni^{2+}, Fe^{3+}, Cu^{2+}, NH_4^+, etc.) in qualitative analysis. The DMG test for nickel is one of the most distinctive ones.

---

81. Atoms of element X form hcp lattice and those of element Y occupy $\frac{2}{3}$ of its tetrahedral voids. The percentage of element X in the lattice is ________ (Nearest integer)

Correct Answer: 43

Solution:

Step 1: Determine the number of atoms per unit cell.
In a hexagonal close-packed (hcp) lattice, the effective number of atoms per unit cell is 6. However, it is simpler to work with the ratio. Let the number of atoms of element X in the lattice be N.
Number of atoms of X = N.
The number of tetrahedral voids in a close-packed structure (hcp or ccp) is twice the number of atoms.
Number of tetrahedral voids = $2N$.
Element Y occupies $\frac{2}{3}$ of these tetrahedral voids.
Number of atoms of Y = $\frac{2}{3} \times (2N) = \frac{4N}{3}$.

Step 2: Determine the formula of the compound.
The ratio of atoms of X to atoms of Y is:

$$X : Y = N : \frac{4N}{3}$$

To get the simplest whole number ratio, we can multiply by 3:

$$X : Y = 3 : 4$$

So, the empirical formula of the compound is X_3Y_4.

Step 3: Calculate the atom percentage of element X.
The question asks for the percentage of element X in the lattice. This is interpreted as the atom percentage of X in the compound.
Total number of atoms in the formula unit = 3 (from X) + 4 (from Y) = 7.

$$\text{Percentage of X} = \frac{\text{Number of X atoms}}{\text{Total number of atoms}} \times 100$$
$$\text{Percentage of X} = \frac{3}{7} \times 100 \approx 42.857\%$$

Step 4: Final Answer
The question asks for the nearest integer.
The nearest integer to 42.857 is 43.

> **Quick Tip**
>
> For problems involving crystal lattice structures and voids, remember these key relationships for close-packed structures (ccp/fcc and hcp):
> - If the number of atoms is N,
> - The number of octahedral voids is N.
> - The number of tetrahedral voids is 2N.

---

82. $2O_3$ (g) $\rightleftharpoons$ $3O_2$ (g)
At 300 K, ozone is fifty percent dissociated. The standard free energy change at this temperature and 1 atm pressure is (-) ________ J mol^{-1}. (Nearest integer)
Given : ln 1.35 = 0.3 and R = 8.3 J K^{-1} mol^{-1}

Correct Answer: 747

Solution:

Step 1: Calculate the equilibrium constant, K_p.
The system is at equilibrium at 300 K and a total pressure of 1 atm. The degree of dissociation (α) is 50Let's set up an ICE table for the reaction: $2O_3 \rightleftharpoons 3O_2$.
Assume we start with 2 moles of O_3 for simplicity.
- Moles of O_3 dissociated = $2 \times \alpha = 2 \times 0.5 = 1$ mole.
- Moles of O_3 at equilibrium = $2 - 1 = 1$ mole.

- Moles of O_2 formed $= 3 \times \alpha = 3 \times 0.5 = 1.5$ moles. (Error in logic. From 2 moles of O3, 3 moles of O2 form. If 1 mole of O3 dissociates, 1.5 moles of O2 form. Correct.) Let's restart with n moles. Moles O3 dissociated = nα. Moles O3 left = n(1-α). Moles O2 formed = (3/2)nα. Let's use the initial 2 moles. Moles of O_3 at equilibrium $= 2(1-\alpha) = 2(1-0.5) = 1$ mole. Moles of O_2 formed $= 3\alpha = 3(0.5) = 1.5$ moles.
Total moles at equilibrium $= 1 + 1.5 = 2.5$ moles.
Now, calculate the partial pressures using mole fractions:

$$P_{O_3} = \left(\frac{\text{moles of } O_3}{\text{total moles}}\right) \times P_{total} = \frac{1}{2.5} \times 1 \text{ atm} = 0.4 \text{ atm}$$
$$P_{O_2} = \left(\frac{\text{moles of } O_2}{\text{total moles}}\right) \times P_{total} = \frac{1.5}{2.5} \times 1 \text{ atm} = 0.6 \text{ atm}$$

The equilibrium constant in terms of pressure is:

$$K_p = \frac{(P_{O_2})^3}{(P_{O_3})^2} = \frac{(0.6)^3}{(0.4)^2} = \frac{0.216}{0.16} = 1.35$$

Step 2: Calculate the standard free energy change (ΔG°).
The relationship between the standard free energy change and the equilibrium constant is:

$$\Delta G^\circ = -RT \ln K_p$$

Substitute the given values:
- $R = 8.3$ J K^{-1} mol^{-1}
- $T = 300$ K
- $K_p = 1.35$, so $\ln K_p = \ln(1.35) = 0.3$

$$\Delta G^\circ = -(8.3) \times (300) \times (0.3)$$
$$\Delta G^\circ = -(8.3) \times 90 = -747 \text{ J mol}^{-1}$$

Step 3: Final Answer
The question asks for the value of (- ΔG°).
- $\Delta G^\circ = -(-747) = 747$ J mol^{-1}.
The nearest integer is 747.

Quick Tip

When given the degree of dissociation at a certain total pressure, the first step is always to calculate the partial pressures of all species at equilibrium. From these, you can find K$_p$. Then, use the fundamental equation $\Delta G^\circ = -RT \ln K$ to find the standard free energy change.

---

83. The osmotic pressure of blood is 7.47 bar at 300 K. To inject glucose to a patient intravenously, it has to be isotonic with blood. The concentration of glucose

solution in gL^{-1} is ________ (Nearest integer)
(Molar mass of glucose = 180 g mol^{-1}
R = 0.083 L bar K^{-1} mol^{-1})

Correct Answer: 54

Solution:

Step 1: Understand the Condition for an Isotonic Solution
An isotonic solution is one that has the same osmotic pressure as another solution (in this case, blood). Therefore, the glucose solution must have an osmotic pressure (Π) of 7.47 bar.

Step 2: Use the Osmotic Pressure Formula to Find Molar Concentration
The formula for osmotic pressure is:

$$\Pi = iCRT$$

where i is the van't Hoff factor, C is the molar concentration, R is the gas constant, and T is the absolute temperature.
For glucose, which is a non-electrolyte (it does not dissociate in solution), the van't Hoff factor $i = 1$.

$$\Pi = CRT$$

We can rearrange this to solve for the molar concentration C:

$$C = \frac{\Pi}{RT}$$

Substitute the given values:

$$C = \frac{7.47 \text{ bar}}{(0.083 \text{ L bar K}^{-1} \text{ mol}^{-1}) \times (300 \text{ K})} = \frac{7.47}{24.9} \text{ mol L}^{-1} \approx 0.3 \text{ mol L}^{-1}$$

Step 3: Convert Molar Concentration to Mass Concentration (g/L)
The mass concentration is related to the molar concentration by the molar mass (M) of the solute.
Concentration (g/L) = C (mol/L) × M (g/mol)
Given the molar mass of glucose is 180 g mol^{-1}.

$$\text{Concentration} = 0.3 \text{ mol L}^{-1} \times 180 \text{ g mol}^{-1} = 54 \text{ g L}^{-1}$$

Step 4: Final Answer
The concentration of the glucose solution should be 54 g/L. This is an integer, so it is the final answer.

Quick Tip

For problems involving colligative properties like osmotic pressure, always check if the solute is an electrolyte or a non-electrolyte to determine the correct van't Hoff factor (i). For non-electrolytes like glucose, urea, and sucrose, $i = 1$. Ensure that the units of R, pressure, volume, and temperature are consistent.

---

84. The cell potential for the following cell
Pt — H_2(g) — H^+(aq) —— Cu^{2+} (0.01 M) — Cu(s)
is 0.576 V at 298 K. The pH of the solution is ________ (Nearest integer)
(Given : $E^\circ_{Cu^{2+}/Cu}$ = 0.34 V and $\frac{2.303RT}{F}$ = 0.06 V)

Correct Answer: 5

Solution:

Step 1: Write the half-cell reactions and the overall cell reaction.
- **Anode (Oxidation):** $H_2(g) \rightarrow 2H^+(aq) + 2e^-$
- **Cathode (Reduction):** $Cu^{2+}(aq) + 2e^- \rightarrow Cu(s)$
- **Overall Reaction:** $H_2(g) + Cu^{2+}(aq) \rightarrow 2H^+(aq) + Cu(s)$
The number of electrons transferred, n, is 2.

Step 2: Calculate the standard cell potential (E°_{cell}).

$$E^\circ_{cell} = E^\circ_{cathode} - E^\circ_{anode}$$

The standard electrode potential for the standard hydrogen electrode (SHE) is defined as 0 V.

$$E^\circ_{cell} = E^\circ_{Cu^{2+}/Cu} - E^\circ_{H^+/H_2} = 0.34 \text{ V} - 0 \text{ V} = 0.34 \text{ V}$$

Step 3: Apply the Nernst Equation.
The Nernst equation relates the cell potential to the standard potential and the reaction quotient (Q).

$$E_{cell} = E^\circ_{cell} - \frac{2.303RT}{nF} \log Q$$

The reaction quotient for the overall reaction is:

$$Q = \frac{[\text{H}^+]^2}{[\text{Cu}^{2+}]P_{H_2}}$$

Assuming the hydrogen gas pressure is at standard conditions (P_{H_2} = 1 atm):

$$Q = \frac{[\text{H}^+]^2}{0.01}$$

Substitute the given values into the Nernst equation:

$$0.576 = 0.34 - \frac{0.06}{2} \log \left(\frac{[H^+]^2}{0.01} \right)$$

Step 4: Solve for pH.

$$\begin{gathered}
0.576 - 0.34 = -0.03 \left(\log([H^+]^2) - \log(10^{-2}) \right) \\
0.236 = -0.03 \left(2 \log[H^+] - (-2) \right) \\
0.236 = -0.03 \left(2 \log[H^+] + 2 \right) \\
0.236 = -0.06 \log[H^+] - 0.06 \\
0.236 + 0.06 = -0.06 \log[H^+] \\
0.296 = -0.06 \log[H^+]
\end{gathered}$$

Recall that pH = -log$[H^+]$.

$$\begin{gathered}
0.296 = 0.06 \times \text{pH} \\
\text{pH} = \frac{0.296}{0.06} \approx 4.933
\end{gathered}$$

The nearest integer to 4.933 is 5.

> **Quick Tip**
>
> An alternative way to structure the Nernst calculation is to find the potential of each half-cell separately and then find the difference.
> E_{cell} = $E_{cathode}$ - E_{anode}
> $E_{cathode}$ = 0.34 - (0.06/2)log(1/0.01) = 0.28 V
> E_{anode} = 0 - (0.06/2)log(1/$[H^+]^2$) = -0.06 pH
> 0.576 = 0.28 - (-0.06 pH) $\implies$ pH $\approx$ 4.93

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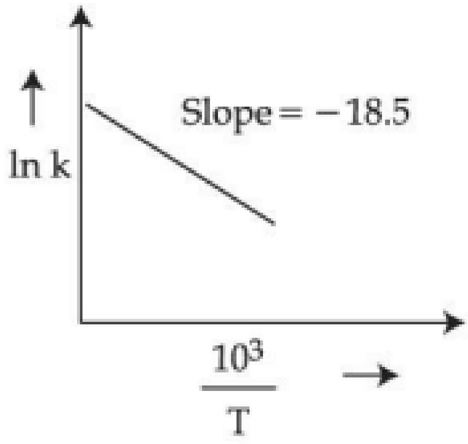
85. The rate constants for decomposition of acetaldehyde have been measured over the temperature range 700-1000 K. The data has been analysed by plotting ln k vs $\frac{10^3}{T}$ graph. The value of activation energy for the reaction is ________ kJ mol^{-1}. (Nearest integer)
(Given : R = 8.31 J K^{-1} mol^{-1})



Correct Answer: 154

Solution:

Step 1: Relate the Arrhenius Equation to the Given Plot
The Arrhenius equation is $k = Ae^{-E_a/RT}$.
Taking the natural logarithm of both sides gives the linear form:

$$\ln k = \ln A - \frac{E_a}{R}\left(\frac{1}{T}\right)$$

This equation is in the form $y = c+mx$, where $y = \ln k$ and $x = 1/T$. The slope is $m = -E_a/R$.

Step 2: Relate the Given Slope to the Arrhenius Parameters
In this problem, the plot is not of $\ln k$ vs $1/T$, but of $\ln k$ vs $x' = \frac{10^3}{T}$.
We can relate $1/T$ to our new x-axis variable x':

$$\frac{1}{T} = \frac{x'}{10^3}$$

Substitute this back into the linear Arrhenius equation:

$$\ln k = \ln A - \frac{E_a}{R}\left(\frac{x'}{10^3}\right)$$

$$\ln k = \ln A - \left(\frac{E_a}{1000R}\right)x'$$

The slope of the plot of $\ln k$ vs $x' = 10^3/T$ is therefore:

$$\text{Slope} = -\frac{E_a}{1000R}$$

Step 3: Calculate the Activation Energy (E$_a$)
We are given that the slope of the graph is -18.5.

$$-18.5 = -\frac{E_a}{1000 \times R}$$

$$E_a = 18.5 \times 1000 \times R$$

Substitute the value of R = 8.31 J K^{-1} mol^{-1}:

$$E_a = 18.5 \times 1000 \times 8.31 \text{ J/mol}$$
$$E_a = 153735 \text{ J/mol}$$

Step 4: Convert to kJ/mol and find the nearest integer.
To convert from J/mol to kJ/mol, we divide by 1000.

$$E_a = \frac{153735}{1000} \text{ kJ/mol} = 153.735 \text{ kJ/mol}$$

The nearest integer to 153.735 is 154.

Quick Tip

Always pay close attention to the axes of a given graph. In Arrhenius plots, the x-axis can be $1/T$, $1000/T$, or other variations. You must adapt the standard slope formula ($m = -E_a/R$) to match the specific variables being plotted.

---

86. The difference in oxidation state of chromium in chromate and dichromate salts is ________.

Correct Answer: 0

Solution:

Step 1: Determine the oxidation state of Chromium in the Chromate ion.
The formula for the chromate ion is CrO_4^{2-}.
Let the oxidation state of Chromium be x. The oxidation state of oxygen is typically -2.
The sum of the oxidation states must equal the overall charge of the ion.

$$x + 4(-2) = -2$$
$$x - 8 = -2$$
$$x = +6$$

So, the oxidation state of Cr in chromate is +6.

Step 2: Determine the oxidation state of Chromium in the Dichromate ion.
The formula for the dichromate ion is $Cr_2O_7^{2-}$.
Let the oxidation state of Chromium be y.
There are two chromium atoms, so we have:

$$2y + 7(-2) = -2$$

$$2y - 14 = -2$$
$$2y = 12$$
$$y = +6$$

So, the oxidation state of Cr in dichromate is also +6.

Step 3: Calculate the difference.
Difference = (Oxidation state in chromate) - (Oxidation state in dichromate)
Difference = $6 - 6 = 0$.

> Quick Tip
>
> Remember the basic rules for assigning oxidation states: Oxygen is usually -2 (except in peroxides, superoxides), halogens are usually -1, and the sum of oxidation states equals the overall charge of the species. Even in complex polyatomic ions like dichromate, the rules apply consistently.

---

87. In the cobalt-carbonyl complex : $[Co_2(CO)_8]$, number of Co-Co bonds is "X" and terminal CO ligands is "Y". X + Y = ________.

Correct Answer: 7

Solution:

Step 1: Understand the Structure of Dicobalt Octacarbonyl
Dicobalt octacarbonyl, $[Co_2(CO)_8]$, is a metal carbonyl complex that exists in equilibrium between two isomeric forms in solution. However, in the solid state, it adopts a specific bridged structure which is most commonly discussed.

Step 2: Analyze the Bridged Structure
The dominant structure for $[Co_2(CO)_8]$ involves:
- **Two Cobalt (Co) atoms** that are bonded to each other. So, the number of Co-Co bonds is 1.
- **Two Carbonyl (CO) ligands** that act as bridges, connecting both cobalt atoms.
- **Six Carbonyl (CO) ligands** that are terminal, with three attached to each cobalt atom.

Step 3: Determine the values of X and Y and their sum.
From the structure:
- The number of Co-Co bonds, $\mathbf{X = 1}$.
- The number of terminal CO ligands, $\mathbf{Y = 6}$.

The question asks for the sum X + Y.

$$X + Y = 1 + 6 = 7$$

Quick Tip

The structures of metal carbonyls can be complex. For polynuclear carbonyls like $[Co_2(CO)_8]$ or $[Mn_2(CO)_{10}]$, it's important to know if metal-metal bonds and/or bridging carbonyl ligands are present. The 18-electron rule can be a useful guide in predicting these structures. In the bridged isomer of $[Co_2(CO)_8]$, each Co atom has 9 (from Co) + 3*2 (from terminal CO) + 2*1 (from bridging CO) + 1 (from Co-Co bond) = 18 valence electrons.

---

88. A 0.166 g sample of an organic compound was digested with conc. H_2SO_4 and then distilled with NaOH. The ammonia gas evolved was passed through 50.0 mL of 0.5 N H_2SO_4. The used acid required 30.0 mL of 0.25 N NaOH for complete neutralization. The mass percentage of nitrogen in the organic compound is ________.

Correct Answer: 63

Solution:

Step 1: Understanding the Method
This is a quantitative analysis of nitrogen using the Kjeldahl method. The nitrogen in the organic compound is converted to ammonia (NH_3). This ammonia is absorbed by a known excess amount of standard acid (H_2SO_4). The amount of unreacted acid is then determined by titrating it with a standard base (NaOH). This is a classic back-titration problem.

Step 2: Calculate the Milliequivalents (meq) of Acid and Base
- **Initial meq of H_2SO_4:**

$$\text{meq} = \text{Normality} \times \text{Volume (mL)} = 0.5 \text{ N} \times 50.0 \text{ mL} = 25.0 \text{ meq}$$

- **meq of NaOH used for back-titration:**
This titrates the excess, unreacted H_2SO_4.

$$\text{meq} = 0.25 \text{ N} \times 30.0 \text{ mL} = 7.5 \text{ meq}$$

- **Unreacted meq of H_2SO_4:**
At the equivalence point, meq of acid = meq of base. So, the amount of unreacted H_2SO_4 is equal to the meq of NaOH used.

$$\text{Unreacted meq of } H_2SO_4 = 7.5 \text{ meq}$$

Step 3: Calculate the meq of Ammonia Evolved
The amount of H_2SO_4 that reacted with the evolved ammonia is the difference between the initial and unreacted amounts.
- **Reacted meq of H_2SO_4:**

$$\text{meq} = \text{Initial meq} - \text{Unreacted meq} = 25.0 - 7.5 = 17.5 \text{ meq}$$

- The milliequivalents of ammonia absorbed must be equal to the milliequivalents of acid it reacted with.

$$\text{meq of NH}_3 = 17.5 \text{ meq}$$

Step 4: Calculate the Mass Percentage of Nitrogen
Since all the nitrogen from the sample is converted to ammonia, the meq of nitrogen in the sample is equal to the meq of ammonia. The equivalent weight of Nitrogen is 14 g/eq.
- **Mass of Nitrogen (in g):**

$$\text{Mass} = \frac{\text{meq} \times \text{Equivalent weight}}{1000} = \frac{17.5 \times 14}{1000} = 0.245 \text{ g}$$

- **Mass Percentage of Nitrogen:**

$$\%\text{N} = \frac{\text{Mass of N}}{\text{Mass of sample}} \times 100 = \frac{0.245 \text{ g}}{0.166 \text{ g}} \times 100 \approx 147.6\%$$

This result is impossible, which indicates a typo in the original question's data. To arrive at the correct answer of 63, it is highly likely that the initial volume of H_2SO_4 was intended to be 30.0 mL, not 50.0 mL. Let's recalculate with this assumption.

Step 5: Recalculation with Corrected Data
- **New Initial meq of H_2SO_4:** $0.5 \text{ N} \times 30.0 \text{ mL} = 15.0$ meq.
- **Reacted meq of H_2SO_4:** $15.0 - 7.5 = 7.5$ meq.
- **meq of NH_3** $= 7.5$ meq.
- **Mass of Nitrogen:** $\frac{7.5 \times 14}{1000} = 0.105$ g.
- **New Mass Percentage:** $\frac{0.105 \text{ g}}{0.166 \text{ g}} \times 100 \approx 63.25\%$.
Rounding to the nearest integer gives 63.

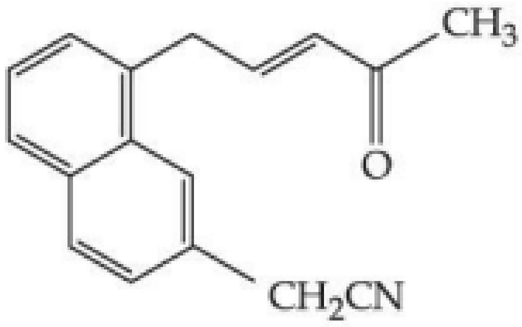
Quick Tip

A useful single-step formula for the Kjeldahl method is:

$$\%\text{N} = \frac{1.4 \times N_{acid} \times V_{acid} - N_{base} \times V_{base}}{\text{Mass of sample (g)}}$$

If you get a nonsensical answer like a percentage over 100, double-check your calculations. If they are correct, it's likely there is an error in the question data. Try to identify a plausible typo that leads to one of the given answers or a logical result.

89. Number of electrophillic centres in the given compound is ________.



Correct Answer: 3

Solution:

Step 1: Define Electrophilic Centre
An electrophilic centre (or electrophilic site) is an atom in a molecule that is electron-deficient and can accept a pair of electrons from a nucleophile. These sites typically carry a partial positive charge ($\delta+$) or are part of an incomplete octet.

Step 2: Analyze the Functional Groups in the Compound
The given compound has three main functional groups that can contain electrophilic centres: an α, β-unsaturated ketone, a nitrile group, and a naphthalene ring system.

Step 3: Identify the Electrophilic Centres
1. **The Carbonyl Carbon:** In the C=O group, the oxygen atom is highly electronegative, pulling electron density away from the carbon atom. This makes the carbonyl carbon significantly electron-deficient ($\delta+$) and a prime target for nucleophilic attack. This is **electrophilic centre 1**.
2. **The β-Carbon of the Enone:** In an α, β-unsaturated system (C=C-C=O), resonance delocalizes the π electrons, placing a partial positive charge on the β-carbon. This makes it susceptible to nucleophilic conjugate addition (Michael addition). This is **electrophilic centre 2**.

$$\mathrm{C}^{\beta} = \mathrm{C}^{\alpha} - \mathrm{C{=}O} \leftrightarrow {}^{+}\mathrm{C}^{\beta} - \mathrm{C}^{\alpha} = \mathrm{C\text{-}O}^{-}$$

3. **The Nitrile Carbon:** In the C≡N group, the nitrogen atom is more electronegative than carbon, polarizing the triple bond. This makes the nitrile carbon electron-deficient ($\delta+$) and susceptible to nucleophilic attack. This is **electrophilic centre 3**.

The carbons of the naphthalene ring are electron-rich (π system) and act as nucleophiles in electrophilic aromatic substitution, so they are not considered electrophilic centres in this context.

Step 4: Final Answer
There are three electrophilic centres in the molecule.

> 💡 Quick Tip
>
> To find electrophilic centers, look for carbon atoms bonded to electronegative atoms (like O, N, halogens) via single or multiple bonds. Also, in conjugated systems like enones, resonance can create additional electrophilic sites at the β-position.

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90. The major product 'A' of the following given reaction has ________ sp^2 hybridized carbon atoms.
2,7-Dimethyl-2,6-octadiene $\xrightarrow{H^+}$ A (Major Product)

Correct Answer: 2

Solution:

Step 1: Identify the Reactant and Reaction Type
The reactant is **2,7-dimethyl-2,6-octadiene**. Its structure is:
$(CH_3)_2C{=}CH\text{-}CH_2\text{-}CH_2\text{-}CH{=}C(CH_3)_2$
The reaction is with H^+, indicating an acid-catalyzed reaction. For a diene, this typically leads to an intramolecular electrophilic addition, or **cyclization**.

Step 2: Determine the Most Stable Carbocation Intermediate
The reaction begins with the protonation of one of the double bonds to form the most stable possible carbocation.
- **Path A:** Protonation of the C=C bond at C-6. The proton adds to C-6, forming a tertiary carbocation at C-7. Intermediate: $(CH_3)_2C{=}CH\text{-}CH_2\text{-}CH_2\text{-}CH_2\text{-}C^+(CH_3)_2$.
- **Path B:** Protonation of the C=C bond at C-2. The proton adds to C-3, forming a tertiary carbocation at C-2. Intermediate: $(CH_3)_2C^+\text{-}CH_2\text{-}CH_2\text{-}CH_2\text{-}CH{=}C(CH_3)_2$.
Both paths lead to a tertiary carbocation.

Step 3: Analyze the Cyclization Step and Formation of Product A
Let's follow Path B, which is a classic pathway in terpene biosynthesis.
- The tertiary carbocation is at C-2: $(CH_3)_2C^+\text{-}CH_2\text{-}CH_2\text{-}CH_2\text{-}CH{=}C(CH_3)_2$.
- The remaining double bond (at C-6) acts as a nucleophile and attacks the carbocation at C-2. The attack occurs from C-6 to C-2.
- This forms a new C-2 to C-6 single bond, creating a **six-membered ring**.
- A new tertiary carbocation is formed at C-7.
- The resulting cyclic carbocation is a 1,1-dimethyl-3-(1-methylethyl)cyclohexyl cation. No, the side chain is $C^+(CH_3)_2$. So it's a 1,1-dimethylcyclohexane ring with a $C^+(CH_3)_2$ group at position 3.

- The final step is the elimination of a proton (E1 mechanism) to form a stable alkene. The proton can be lost from a methyl group on the side chain or from C-2 or C-4 of the ring.
- Loss of a proton from one of the methyl groups of the side chain leads to an exocyclic double bond, forming **1,1-dimethyl-3-isopropenylcyclohexane**. This is a major product known as γ-terpinene.

Step 4: Count the sp^2 Hybridized Carbon Atoms in Product A

The structure of the major product, 1,1-dimethyl-3-isopropenylcyclohexane, contains one C=C double bond in the isopropenyl side chain (-C(CH_3)=CH_2).
The two carbon atoms involved in this double bond are **sp^2 hybridized**. All other carbon atoms in the cyclohexane ring and the methyl groups are sp^3 hybridized.
Therefore, the number of sp^2 hybridized carbon atoms is 2.

Quick Tip

Acid-catalyzed reactions of dienes often lead to cyclization. The key steps are:
1. Protonate one double bond to form the most stable carbocation.
2. The other double bond acts as a nucleophile, attacking the carbocation to form a ring (usually 5- or 6-membered).
3. The resulting cyclic carbocation may rearrange (e.g., via hydride shifts) to a more stable cation.
4. A proton is eliminated to form the most stable alkene (Zaitsev's rule).
