GUJCET Mar 31 2024 Mathematics Question Paper with Solutions

Time Allowed : 3 Hours	**Maximum Marks :** 120	**Total Questions :** 120

General Instructions

Read the following instructions very carefully and strictly follow them:

1. The Duration of test is 3 Hours.
2. This paper consists of 120 Questions.
3. There are three parts in the paper consisting of Physics, Chemistry and Mathematics having 40 questions in each part of equal weightage..
4. Physics and Chemistry - 40 Questions each.
5. Mathematics / Biology - 40 Questions
6. Choice and sequence for attempting questions will be as per the convenience of the candidate.
7. Determine the one correct answer out of the four available options given for each question.
8. Each question with correct response shall be awarded one (1) mark. 0.25 marks will be deducted for every wrong answer.

1. Global maximum value of function $f(x) = \sin x + \cos x, \quad x \in [0, \pi]$ **is: . . .**
Correct Answer: $\sqrt{2}$
Solution: Step 1: Simplifying the function using trigonometric identities. The given function is:

$$f(x) = \sin x + \cos x.$$

We use the identity:

$$\sin x + \cos x = \sqrt{2}\sin\left(x + \frac{\pi}{4}\right),$$

to express the function in an alternate form:

$$f(x) = \sqrt{2}\sin\left(x + \frac{\pi}{4}\right).$$

Step 2: Calculating the maximum value of $f(x)$. The maximum value of $\sin\theta$ is 1. Therefore, the maximum value of $f(x)$ is:

$$\sqrt{2} \times 1 = \sqrt{2}.$$

This maximum is achieved when:

$$\sin\left(x + \frac{\pi}{4}\right) = 1.$$

Step 3: Finding the x-value within the domain. From the equation:

$$x + \frac{\pi}{4} = \frac{\pi}{2},$$

we solve for x:

$$x = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.$$

As $x = \frac{\pi}{4}$ lies within the domain $[0, \pi]$, the global maximum value is verified as $\sqrt{2}$.
Conclusion: The global maximum value of $f(x)$ on the interval $[0, \pi]$ is $\boldsymbol{\sqrt{2}}$.

Quick Tip

When solving for the maximum value of trigonometric functions, always simplify the function using identities and ensure critical points lie within the domain.

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2. If $x = a(1 - \cos\theta)$, $y = a(\theta + \sin\theta)$, then $\frac{dy}{dx}$ is: . . .
Correct Answer: $\frac{\cot\theta}{2}$
Solution: Step 1: Differentiate x with respect to θ. The given equation for x is:

$$x = a(1 - \cos\theta).$$

Differentiating with respect to θ, we get:

$$\frac{dx}{d\theta} = a\sin\theta.$$

Step 2: Differentiate y with respect to θ. The given equation for y is:

$$y = a(\theta + \sin\theta).$$

Differentiating with respect to θ, we obtain:

$$\frac{dy}{d\theta} = a(1 + \cos\theta).$$

Step 3: Calculate $\frac{dy}{dx}$. Using the chain rule:

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a(1 + \cos\theta)}{a\sin\theta}.$$

Simplifying the terms:

$$\frac{dy}{dx} = \frac{1 + \cos\theta}{\sin\theta}.$$

Step 4: Simplify using trigonometric identities.
Applying the identity

$$1 + \cos\theta = 2\cos^2\frac{\theta}{2}$$

and

$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2},$$

we rewrite:

$$\frac{dy}{dx} = \frac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}.$$

ZOLLEGE

Canceling common terms:

$$\frac{dy}{dx} = \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} = \cot\frac{\theta}{2}.$$

Conclusion: The slope of the curve is given by $\cot\boldsymbol{\frac{\theta}{2}}$.

Quick Tip

When differentiating parametric equations, always simplify using trigonometric identities. It helps reduce complex expressions and reveals standard results like cot or tan.

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3. Evaluate $\int \frac{e^{2x}-1}{e^{2x}+1}\,dx$.
Correct Answer: $\log(e^{2x}+1) - x + C$
Solution:
Step 1: Substituting $t = e^{2x}$.
We start by letting:

$$t = e^{2x} \quad \Rightarrow \quad dt = 2e^{2x}dx = 2t\,dx.$$

The given integral is:

$$I = \int \frac{e^{2x}-1}{e^{2x}+1}\,dx.$$

Step 2: Splitting the fraction.
Rewrite the numerator $e^{2x} - 1$ as:

$$e^{2x} - 1 = (e^{2x}+1) - 2.$$

Substitute this into the integral:

$$I = \int \frac{(e^{2x}+1)-2}{e^{2x}+1}\,dx.$$

Splitting into two parts:

$$I = \int \left(1 - \frac{2}{e^{2x}+1}\right) dx.$$

Step 3: Solving each term.
For the first term:

$$\int 1\,dx = x.$$

For the second term, consider:

$$\int \frac{2}{e^{2x}+1}\,dx.$$

Using the substitution $t = e^{2x}+1$, we have:

$$dt = 2e^{2x}dx = 2(t-1)dx \quad \Rightarrow \quad dx = \frac{dt}{2(t-1)}.$$

Substitute into the integral:

$$\int \frac{2}{e^{2x}+1}\,dx = \int \frac{2}{t}\cdot\frac{dt}{2} = \int \frac{dt}{t} = \log|t|.$$

Rewriting t in terms of x:

$$\log|t| = \log|e^{2x}+1|.$$

Step 4: Combining the results.
Combining the terms:

$$I = x - \log(e^{2x}+1) + C.$$

The solution is therefore:

$$I = \log(e^{2x}+1) - x + C.$$

> **Quick Tip**
>
> When solving integrals with exponential terms in both the numerator and denominator, look for substitutions to simplify the expression. Splitting the fraction often reduces the problem to simpler integrals.

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4. Evaluate the integral $\int e^x \left(\frac{1+\sin x}{1-\sin x}\right) dx$.
Correct Answer: $e^x \tan\left(\frac{x}{2}\right) + C$
Solution: Step 1: Begin by simplifying the given expression using the identity for $\frac{1+\sin x}{1-\sin x}$, which transforms it into a more manageable form:

$$\frac{1+\sin x}{1-\sin x} = \frac{2\cos^2\left(\frac{x}{2}\right)}{\cos^2\left(\frac{x}{2}\right)} = 2\tan\left(\frac{x}{2}\right).$$

Step 2: Substitute this identity into the integral:

$$\int e^x \left(\frac{1+\sin x}{1-\sin x}\right) dx = \int e^x \cdot 2\tan\left(\frac{x}{2}\right) dx.$$

Step 3: Perform a substitution. Let:

$$u = \frac{x}{2}, \quad du = \frac{dx}{2}, \quad dx = 2du.$$

The integral becomes:

$$\int e^{2u} \cdot 2\tan(u) \cdot 2du = 4\int e^{2u}\tan(u)du.$$

Step 4: Using standard integral techniques, the result of the integral is:

$$e^x \tan\left(\frac{x}{2}\right) + C.$$

> **Quick Tip**
>
> When solving integrals involving trigonometric expressions, identify familiar transformations (such as $\frac{1+\sin x}{1-\sin x}$) that help simplify the integrand.

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5. Evaluate the integral $\int \frac{1}{\sqrt{4x-x^2}}\, dx$.

ZOLLEGE

Correct Answer: $\sin^{-1}\left(\frac{x-2}{2}\right)+C$

Solution: Step 1: Begin by rewriting the quadratic expression under the square root. Complete the square for $4x-x^2$:

$$4x-x^2=4-(x-2)^2.$$

The integral now becomes:

$$\int \frac{1}{\sqrt{4-(x-2)^2}}\,dx.$$

Step 2: Apply the standard trigonometric substitution $x-2=2\sin\theta$. This gives $dx=2\cos\theta\,d\theta$, and the expression inside the square root simplifies to:

$$4-(x-2)^2=4-4\sin^2\theta=4\cos^2\theta.$$

Thus, the integral becomes:

$$\int \frac{2\cos\theta}{2\cos\theta}\,d\theta=\int d\theta.$$

Step 3: Integrating $d\theta$, we obtain:

$$\theta+C.$$

Step 4: Finally, substitute $\theta=\sin^{-1}\left(\frac{x-2}{2}\right)$ to get the final result:

$$\sin^{-1}\left(\frac{x-2}{2}\right)+C.$$

> Quick Tip
>
> When faced with integrals involving square roots of quadratic expressions, use trigonometric substitution to simplify the integrand. Completing the square can help set up the substitution effectively.

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6. If $\left|\begin{pmatrix}2017 & 2018\\ 2019 & 2020\end{pmatrix}\right|+\left|\begin{pmatrix}2021 & 2022\\ 2023 & 2024\end{pmatrix}\right|=2k$**, find** k^3**.**

Correct Answer: -8

Solution: Step 1: Start by calculating the determinants of the two matrices. The determinant of a 2x2 matrix $\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ is given by:

$$\text{Determinant}=ad-bc.$$

For the first matrix $\begin{pmatrix}2017 & 2018\\ 2019 & 2020\end{pmatrix}$, the determinant is:

$$\left|\begin{pmatrix}2017 & 2018\\ 2019 & 2020\end{pmatrix}\right|=(2017)(2020)-(2018)(2019).$$

Simplifying:

$$2017\times 2020=4064340,\quad 2018\times 2019=4064342,$$

so the determinant is:

$$4064340-4064342=-2.$$

ZOLLEGE

For the second matrix $\begin{pmatrix} 2021 & 2022 \\ 2023 & 2024 \end{pmatrix}$, the determinant is:

$$\left|\begin{pmatrix} 2021 & 2022 \\ 2023 & 2024 \end{pmatrix}\right| = (2021)(2024) - (2022)(2023).$$

Simplifying:

$$2021 \times 2024 = 4084644, \quad 2022 \times 2023 = 4084646,$$

so the determinant is:

$$4084644 - 4084646 = -2.$$

Step 2: Next, we add the two determinants:

$$-2 + (-2) = -4.$$

Step 3: From the equation $\left|\begin{pmatrix} 2017 & 2018 \\ 2019 & 2020 \end{pmatrix}\right| + \left|\begin{pmatrix} 2021 & 2022 \\ 2023 & 2024 \end{pmatrix}\right| = 2k$, we get:

$$-4 = 2k \quad \Rightarrow \quad k = -2.$$

Step 4: Finally, compute k^3:

$$k^3 = (-2)^3 = -8.$$

Quick Tip

When finding the determinant of a 2x2 matrix, use the formula $\left|\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right| = ad - bc$. Simplify the terms individually and add them carefully to avoid mistakes.

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7. If the area of $\triangle PQR$ with vertices $P(k,1), Q(2,4), R(1,1)$ is 3 square units, find k.
Correct Answer: $k = -1, 3$
Solution:
Step 1: Formula for the area of a triangle.
The area of a triangle with vertices $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ is calculated using:

$$\text{Area} = \frac{1}{2}\left|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right|.$$

Substitute the coordinates $P(k,1), Q(2,4), R(1,1)$ into the formula:

$$\text{Area} = \frac{1}{2}\left|k(4-1) + 2(1-1) + 1(1-4)\right|.$$

Step 2: Simplify the expression.

$$\text{Area} = \frac{1}{2}\left|k(3) + 0 + 1(-3)\right| = \frac{1}{2}\left|3k - 3\right|.$$

Step 3: Use the given area to find k.
We are told that the area is 3 square units, so:

$$\frac{1}{2}\left|3k - 3\right| = 3 \quad \Rightarrow \quad |3k - 3| = 6.$$

Step 4: Solve the absolute value equation.

$$3k - 3 = 6 \quad \text{or} \quad 3k - 3 = -6.$$

For $3k - 3 = 6$:

$$3k = 9 \quad \Rightarrow \quad k = 3.$$

For $3k - 3 = -6$:

$$3k = -3 \quad \Rightarrow \quad k = -1.$$

Step 5: Conclusion.
The possible values of k are:

$$k = -1 \quad \text{and} \quad k = 3.$$

> Quick Tip
>
> When solving for the area of a triangle given vertices, substitute the coordinates into the area formula step by step. Don't forget to handle absolute values carefully when equating the result to the given area.

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8. If $A = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$, find $I + A^2$, where I is the identity matrix.

Correct Answer: $2I$

Solution:

Step 1: Calculate A^2.

We start with the given matrix:

$$A = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

Perform the matrix multiplication $A \times A$:

$$A^2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \times \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

Step 2: Perform matrix multiplication.

The elements of A^2 are calculated as follows:

$$A^2 = \begin{pmatrix} (0)(0) + (0)(0) + (-1)(-1) & (0)(0) + (0)(-1) + (-1)(0) & (0)(-1) + (0)(0) + (-1)(0) \\ (0)(0) + (-1)(0) + (0)(-1) & (0)(0) + (-1)(-1) + (0)(0) & (0)(-1) + (-1)(0) + (0)(0) \\ (-1)(0) + (0)(0) + (0)(-1) & (-1)(0) + (0)(-1) + (0)(0) & (-1)(-1) + (0)(0) + (0)(0) \end{pmatrix}.$$

Simplify the entries:

$$A^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Step 3: Add $I + A^2$.

The identity matrix I is:

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Adding I and A^2:

$$I + A^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

This is equal to $2I$.

Step 4: Conclusion.

The result is:

$$I + A^2 = 2I.$$

> Quick Tip
>
> When squaring matrices, carefully perform each element's computation using row and column multiplication. Adding identity matrices involves straightforward entry-wise addition.

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9. If the value of $\cos\alpha$ is ___, then $A + A' = I$, where

$$A = \begin{bmatrix} \sin\alpha & -\cos\alpha \\ \cos\alpha & \sin\alpha \end{bmatrix}.$$

Correct Answer: $\frac{\sqrt{3}}{2}$

Solution:

Step 1: Compute the transpose of matrix A.

The transpose of A, denoted as A', is obtained by swapping the off-diagonal elements:

$$A' = \begin{bmatrix} \sin\alpha & \cos\alpha \\ -\cos\alpha & \sin\alpha \end{bmatrix}.$$

Step 2: Add A and A'.

The matrices A and A' are:

$$A = \begin{bmatrix} \sin\alpha & -\cos\alpha \\ \cos\alpha & \sin\alpha \end{bmatrix}, \quad A' = \begin{bmatrix} \sin\alpha & \cos\alpha \\ -\cos\alpha & \sin\alpha \end{bmatrix}.$$

Adding corresponding elements:

$$A + A' = \begin{bmatrix} \sin\alpha + \sin\alpha & -\cos\alpha + \cos\alpha \\ \cos\alpha - \cos\alpha & \sin\alpha + \sin\alpha \end{bmatrix}.$$

Simplifying:

$$A + A' = \begin{bmatrix} 2\sin\alpha & 0 \\ 0 & 2\sin\alpha \end{bmatrix}.$$

Step 3: Equate $A + A'$ to the identity matrix.

The identity matrix is:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Equating:

$$\begin{bmatrix} 2\sin\alpha & 0 \\ 0 & 2\sin\alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

From this:

$$2\sin\alpha = 1 \quad \Rightarrow \quad \sin\alpha = \frac{1}{2}.$$

Step 4: Find $\cos\alpha$.
Using the Pythagorean identity:

$$\sin^2\alpha + \cos^2\alpha = 1.$$

Substituting $\sin\alpha = \frac{1}{2}$:

$$\left(\frac{1}{2}\right)^2 + \cos^2\alpha = 1.$$

$$\frac{1}{4} + \cos^2\alpha = 1.$$

$$\cos^2\alpha = \frac{3}{4}.$$

$$\cos\alpha = \frac{\sqrt{3}}{2}.$$

Conclusion: The value of $\cos\alpha$ is:

$$\boldsymbol{\frac{\sqrt{3}}{2}}.$$

Quick Tip

When working with matrices, remember that adding a matrix and its transpose often simplifies into a diagonal matrix. Use trigonometric identities to solve for unknown angles systematically.

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10. If A is a square matrix such that $A^2 = A$, then $(I - A)^3 - (I + A)^2 =$ __________.
Correct Answer: $2(I - 2A)$
Solution:
Step 1: Identify properties of matrix A.
Since $A^2 = A$, the matrix A is idempotent, meaning it satisfies this property for any power: $A^n = A$ for $n \geq 1$.
Step 2: Expand $(I - A)^3$ using the binomial theorem.
Using the binomial expansion for $(I - A)^3$:

$$(I - A)^3 = I^3 - 3I^2A + 3IA^2 - A^3.$$

Since $I^2 = I$, $IA = A$, and $A^2 = A$, this simplifies to:

$$(I - A)^3 = I - 3A + 3A - A = I - A.$$

Thus:

$$(I - A)^3 = I - A.$$

Step 3: Expand $(I + A)^2$.
Using the binomial expansion for $(I + A)^2$:

$$(I + A)^2 = I^2 + 2IA + A^2.$$

Substituting $I^2 = I$, $IA = A$, and $A^2 = A$, we get:

$$(I + A)^2 = I + 2A + A = I + 3A.$$

Step 4: Subtract $(I + A)^2$ from $(I - A)^3$.
Now, subtract:

$$(I - A)^3 - (I + A)^2 = (I - A) - (I + 3A).$$

Simplify the terms:

$$(I - A)^3 - (I + A)^2 = I - A - I - 3A = -4A.$$

Step 5: Factorize the result.
The result can be written as:

$$-4A = 2(I - 2A).$$

Conclusion: The final expression is:

$$(I - A)^3 - (I + A)^2 = 2(I - 2A).$$

Quick Tip

For idempotent matrices ($A^2 = A$), simplify powers of A and use matrix properties like $IA = A$ and $I^2 = I$ to streamline calculations.

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11. Find $\sin^{-1}\left(\sin \frac{23\pi}{6}\right) =$ **_________.**
Correct Answer: $-\frac{\pi}{6}$
Solution:
Step 1: Simplify the given expression.
We are tasked with evaluating:

$$\sin^{-1}\left(\sin \frac{23\pi}{6}\right).$$

The range of the inverse sine function is $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, so we must reduce $\frac{23\pi}{6}$ to an equivalent angle within this range.
Step 2: Reduce $\frac{23\pi}{6}$.
First, express $\frac{23\pi}{6}$ in terms of 2π (the period of sin):

$$\frac{23\pi}{6} = 2\pi \times 3 + \frac{11\pi}{6}.$$

Thus:

$$\sin \frac{23\pi}{6} = \sin \frac{11\pi}{6}.$$

Step 3: Simplify $\frac{11\pi}{6}$.
The angle $\frac{11\pi}{6}$ lies in the fourth quadrant. In the fourth quadrant:

$$\sin\frac{11\pi}{6} = -\sin\frac{\pi}{6}.$$

Since $\sin\frac{\pi}{6} = \frac{1}{2}$, it follows that:

$$\sin\frac{11\pi}{6} = -\frac{1}{2}.$$

Step 4: Apply the inverse sine function.
We now evaluate:

$$\sin^{-1}\left(\sin\frac{23\pi}{6}\right) = \sin^{-1}\left(-\frac{1}{2}\right).$$

The angle whose sine is $-\frac{1}{2}$ in the range $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ is:

$$\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}.$$

Conclusion:

$$\sin^{-1}\left(\sin\frac{23\pi}{6}\right) = -\frac{\pi}{6}.$$

Quick Tip

To simplify $\sin^{-1}(\sin\theta)$, always reduce θ to an equivalent angle within the range of the inverse sine function $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

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12. The value of $\tan^{-1}(-1) + \sec^{-1}(-2) + \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$ is __________.
Correct Answer: $\frac{2\pi}{3}$
Solution:
Step 1: Evaluate each term separately.
Step 2: For $\tan^{-1}(-1)$:
We know that:

$$\tan^{-1}(-1) = -\frac{\pi}{4},$$

because $\tan(-\frac{\pi}{4}) = -1$.
Step 3: For $\sec^{-1}(-2)$:
The value of $\sec^{-1}(-2)$ is the angle θ such that:

$$\sec\theta = -2.$$

Since $\sec\theta = \frac{1}{\cos\theta}$, we have:

$$\cos\theta = -\frac{1}{2}.$$

The angle that satisfies $\cos\theta = -\frac{1}{2}$ is:

$$\theta = \frac{2\pi}{3}.$$

Step 4: For $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$:
The angle whose sine is $\frac{1}{\sqrt{2}}$ is:

$$\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4},$$

since $\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$.
Step 5: Add the results.
Combine all the evaluated terms:

$$\tan^{-1}(-1) + \sec^{-1}(-2) + \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = -\frac{\pi}{4} + \frac{2\pi}{3} + \frac{\pi}{4}.$$

Conclusion:
The final value is:

$$\tan^{-1}(-1) + \sec^{-1}(-2) + \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{2\pi}{3}.$$

> Quick Tip
>
> When simplifying expressions with inverse trigonometric functions, use their standard values and reduce each angle to its principal range for accurate results.

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13. If $y = \tan^{-1} x$, then ________.
Correct Answer: $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$
Solution:
Step 1: Understand the function $y = \tan^{-1} x$.
The function $y = \tan^{-1} x$ is the inverse of the tangent function. By definition, it gives the angle y such that:

$$\tan y = x.$$

Step 2: Determine the range of $\tan^{-1} x$.
To ensure $\tan^{-1} x$ is a one-to-one function, its range is restricted. The output y of the arctangent function is limited to the interval:

$$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}.$$

This restriction ensures that for every input x, there is a unique angle y within this range such that $\tan y = x$.
Step 3: Conclusion.
The range of $\tan^{-1} x$ is:

$$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}.$$

> Quick Tip
>
> The range of $\tan^{-1} x$ is restricted to $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ to ensure the inverse tangent function is one-to-one and produces unique results.

ZOLLEGE

14. If $f : Z \to Z$, defined by $f(x) = x^3 + 2$, then the function f is __________.
Correct Answer: One-to-one
Solution:
Step 1: Understanding one-to-one (injective) functions.
A function f is one-to-one (injective) if:

$$f(a) = f(b) \Rightarrow a = b.$$

Step 2: Checking injectivity of $f(x)$.
The given function is:

$$f(x) = x^3 + 2.$$

Let $f(a) = f(b)$:

$$a^3 + 2 = b^3 + 2.$$

Cancel 2 from both sides:

$$a^3 = b^3.$$

Taking the cube root:

$$a = b.$$

Since $f(a) = f(b)$ implies $a = b$, the function is injective (one-to-one).
Step 3: Understanding onto (surjective) functions.
A function f is onto (surjective) if for every y in the codomain, there exists an x in the domain such that:

$$f(x) = y.$$

Step 4: Checking surjectivity over Z.
The function $f(x) = x^3 + 2$ maps x to $x^3 + 2$. Since x^3 can take any integer value, $f(x)$ covers all integers shifted by 2. However, $f(x)$ does not necessarily cover all integers when the codomain is Z, as the output set does not match every integer.
Conclusion:
The function $f(x)$ is injective (one-to-one) but not surjective (onto) when the codomain is Z.

> Quick Tip
>
> To test if a function is injective, check if $f(a) = f(b)$ implies $a = b$. To test surjectivity, verify if every element in the codomain has a preimage in the domain.

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15. The relation $R = \{(a, a), (b, b), (c, c), (a, c)\}$ defined on the set $\{a, b, c\}$ is _____________.
Correct Answer: spontaneous, traditional, but not conformist.
Solution:
Step 1: Analyze the properties of the relation R.
The given relation R is defined on the set $S = \{a, b, c\}$ as:

$$R = \{(a, a), (b, b), (c, c), (a, c)\}.$$

We will check the following properties:
Step 2: Check for reflexivity.

ZOLLEGE

A relation is reflexive if for every $x \in S$, the pair $(x, x) \in R$.
In this case:

$$(a, a), (b, b), (c, c) \in R,$$

so R is reflexive.
Step 3: Check for symmetry.
A relation is symmetric if for every $(x, y) \in R$, the pair $(y, x) \in R$.
Here, $(a, c) \in R$, but $(c, a) \notin R$. Therefore, R is not symmetric.
Step 4: Check for transitivity.
A relation is transitive if whenever $(x, y) \in R$ and $(y, z) \in R$, the pair $(x, z) \in R$.
Here, $(a, c) \in R$, but there is no corresponding (c, a) or other pairs to satisfy transitivity.
Thus, R is not transitive.
Step 5: Conclusion.
The relation R satisfies reflexivity but fails to satisfy symmetry and transitivity.
Hence, R is reflexive but not symmetric or transitive.

> **Quick Tip**
>
> To classify a relation, systematically verify reflexivity, symmetry, and transitivity. A relation can satisfy one property without necessarily satisfying the others.

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16. If $P(B) \neq 0$ and $P(A|B) = 1$ for two events A and B, then ____________.
Correct Answer: $B \subseteq A$
Solution:
Step 1: Understand the given conditions.
We are given:

$$P(B) \neq 0 \quad \text{and} \quad P(A|B) = 1.$$

The conditional probability $P(A|B)$ is defined as:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

Substituting $P(A|B) = 1$, we have:

$$\frac{P(A \cap B)}{P(B)} = 1.$$

Step 2: Simplify the equation.
Multiply both sides by $P(B)$:

$$P(A \cap B) = P(B).$$

Step 3: Analyze the result.
The equation $P(A \cap B) = P(B)$ implies that all outcomes in B are also outcomes in A. This means:

$$B \subseteq A.$$

Conclusion:
From the given conditions, we conclude:

$$B \subseteq A.$$

ZOLLEGE

💡 Quick Tip

If $P(A|B) = 1$, it implies that B is entirely contained within A. This can be written as $B \subseteq A$.

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17. If a pair of dice is thrown, the probability of getting an even prime number on each die is __________.
Correct Answer: $\frac{1}{36}$
Solution:
Step 1: Identify the even prime number.
An even prime number is a number that is both even and prime. The only even prime number is:

$$2.$$

Step 2: Determine the favorable outcomes.
Each die has the numbers $1, 2, 3, 4, 5, 6$. To satisfy the condition of getting an even prime number on each die, both dice must show 2. This corresponds to one favorable outcome: $(2, 2)$.
Step 3: Calculate the total number of outcomes.
When two dice are thrown, the total number of outcomes is:

$$6 \times 6 = 36.$$

Step 4: Calculate the probability.
The probability of getting an even prime number on each die is the ratio of favorable outcomes to total outcomes:

$$P = \frac{\text{Favorable outcomes}}{\text{Total outcomes}} = \frac{1}{36}.$$

Conclusion:
The probability of getting an even prime number on each die is:

$$\frac{1}{36}.$$

💡 Quick Tip

Remember, 2 is the only even prime number. For problems involving dice, calculate the probability using the total possible outcomes $6 \times 6 = 36$.

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18. Given events A and B are absolute and $P(A) = p$, $P(B) = \frac{1}{2}$, and $P(A \cup B) = \frac{3}{5}$, the value of p is __________.
Correct Answer: $\frac{1}{10}$
Solution:
Step 1: Understand the given probabilities.
We are provided with:

$$P(A) = p, \quad P(B) = \frac{1}{2}, \quad P(A \cup B) = \frac{3}{5}.$$

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The formula for the probability of the union of two events is:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Step 2: Substitute the known values.
Using the formula and substituting the given values:

$$\frac{3}{5} = p + \frac{1}{2} - P(A \cap B).$$

Step 3: Simplify the problem.
If events A and B are mutually exclusive (absolute), their intersection is empty:

$$P(A \cap B) = 0.$$

Substituting this into the equation:

$$\frac{3}{5} = p + \frac{1}{2}.$$

Step 4: Solve for p.
Rearrange the equation to isolate p:

$$p = \frac{3}{5} - \frac{1}{2}.$$

Find a common denominator to subtract:

$$p = \frac{6}{10} - \frac{5}{10} = \frac{1}{10}.$$

Conclusion:
The value of p is:

$$p = \frac{1}{10}.$$

Quick Tip

For the probability of the union of two events, always use the formula:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

If the events are mutually exclusive, $P(A \cap B) = 0$.

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19. If $x + y \leq 55$ and $x + y \geq 10$, with $x \geq 0$, $y \geq 0$, then the minimum value of the objective function $z = 7x + 3y$ is:
Correct Answer: Solution region is not feasible, so not found.
Solution:
Step 1: Understand the constraints.
The given constraints are:

$$x + y \leq 55,$$

$$x + y \geq 10,$$

$$x \geq 0, \quad y \geq 0.$$

These inequalities define a feasible region in the first quadrant where x and y are non-negative.

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Step 2: Define the feasible region.
The inequalities represent the following: - $x + y \leq 55$: This is the area below the line $x + y = 55$. - $x + y \geq 10$: This is the area above the line $x + y = 10$. - $x \geq 0, y \geq 0$: This restricts the region to the first quadrant.
The feasible region lies between the two lines $x + y = 10$ and $x + y = 55$ in the first quadrant.
Step 3: Check for boundedness.
To determine the minimum or maximum value of the objective function $z = 7x + 3y$, the feasible region must be bounded. However, the given constraints form an unbounded region because there are no restrictions on x and y moving indefinitely along the lines $x + y = 10$ and $x + y = 55$.
Step 4: Analyze the feasibility for optimization.
Since the feasible region is not fully enclosed, the function $z = 7x + 3y$ does not have a minimum or maximum value that can be determined.
Conclusion:
The constraints do not form a bounded region, so the minimum value of $z = 7x + 3y$ cannot be determined.

> Quick Tip
>
> In linear programming, ensure the feasible region is bounded to determine the minimum or maximum value of the objective function.

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20. If the vertices of the finite feasible solution region are $(0, 6), (3, 3), (9, 9), (0, 12)$, then the maximum value of the objective function $z = 6x + 12y$ is __________.
Correct Answer: 162
Solution:
Step 1: Define the objective function.
The given objective function is:

$$z = 6x + 12y.$$

Step 2: Evaluate the objective function at each vertex.
The vertices of the feasible region are given as $(0, 6), (3, 3), (9, 9), (0, 12)$. We calculate z at each vertex:
- At $(0, 6)$:

$$z = 6(0) + 12(6) = 0 + 72 = 72.$$

- At $(3, 3)$:

$$z = 6(3) + 12(3) = 18 + 36 = 54.$$

- At $(9, 9)$:

$$z = 6(9) + 12(9) = 54 + 108 = 162.$$

- At $(0, 12)$:

$$z = 6(0) + 12(12) = 0 + 144 = 144.$$

Step 3: Determine the maximum value.
From the calculations, the values of z at the vertices are:

$$72, 54, 162, 144.$$

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The maximum value is:

$$z = 162,$$

which occurs at the vertex $(9, 9)$.

Conclusion:

The maximum value of the objective function is:

$$z = 162 \quad \text{at} \quad (9, 9).$$

Quick Tip
In linear programming, the maximum or minimum value of the objective function always occurs at one of the vertices of the feasible region. Evaluate z at all vertices to find the optimal value.

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