CBSE Class 12 Mathematics Set 3 (65/1/3) Question Paper with Solutions

Time Allowed : 3 Hour	**Maximum Marks :** 80	**Total Questions :** 38

General Instructions

Read the following instructions very carefully and strictly follow them:

1. This question paper contains 38 questions. All questions are compulsory.
2. This question paper is divided into five Sections – A, B, C, D and E.
3. In Section A, Questions no. 1 to 18 are multiple choice questions (MCQs) andquestions number 19 and 20 are Assertion-Reason based questions of 1 markeach.
4. In Section B, Questions no. 21 to 25 are very short answer (VSA) type questions, carrying 2 marks each.
5. In Section C, Questions no. 26 to 31 are short answer (SA) type questions,carrying 3 marks each.
6. In Section D, Questions no. 32 to 35 are long answer (LA) type questions carrying 5 marks each.
7. In Section E, Questions no. 36 to 38 are case study based questions carrying 4 marks each.
8. There is no overall choice. However, an internal choice has been provided in 2 questions in Section B, 3 questions in Section C, 2 questions in Section D and 2 questions in Section E.
9. Use of calculators is not allowed.

SECTION A

This section comprises multiple choice questions (MCQs) of 1 mark each.

1. If $x = at, y = \frac{a}{t}$, then $\frac{dy}{dx}$ is:

Options:

1. t^2
2. $-t^2$
3. $\frac{1}{t^2}$
4. $-\frac{1}{t^2}$

Correct Answer: (D) $-\frac{1}{t^2}$.

Solution:

We are given:

$$x = at, \quad y = \frac{a}{t}.$$

Differentiate $x = at$ with respect to t:

$$\frac{dx}{dt} = a.$$

Differentiate $y = \frac{a}{t}$ with respect to t:

$$\frac{dy}{dt} = -\frac{a}{t^2}.$$

Using the chain rule, we find $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}.$$

Substitute the values of $\frac{dy}{dt}$ and $\frac{dx}{dt}$:

$$\frac{dy}{dx} = \frac{-\frac{a}{t^2}}{a}.$$

Simplify:

$$\frac{dy}{dx} = -\frac{1}{t^2}.$$

Hence, the correct answer is (D) $-\frac{1}{t^2}$.

> **Quick Tip**
>
> When using the chain rule, always ensure that the derivatives with respect to the independent variable t are correctly calculated before simplifying.

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2. The solution of the differential equation $\frac{dy}{dx} = \frac{1}{\log y}$ is:

(A) $\log y = x + c$

(B) $y \log y - y = x + c$

(C) $\log y - y = x + c$

(D) $y \log y + y = x + c$

Correct Answer: (B) $y \log y - y = x + c$.

Solution: The given differential equation is:

$$\frac{dy}{dx} = \frac{1}{\log y}.$$

Rewriting, we have:

$$\log y \, dy = dx.$$

Integrate both sides:

$$\int \log y \, dy = \int dx.$$

Using integration by parts for $\int \log y \, dy$, let $u = \log y$ and $dv = dy$:

$$\int \log y \, dy = y \log y - \int y \, dy = y \log y - y + C_1,$$

where C_1 is the constant of integration. Substituting back, we get:

$$y \log y - y = x + C_1.$$

Let $C = C_1$, resulting in:

$$y \log y - y = x + C.$$

Hence, the correct answer is (B) $y \log y - y = x + c$.

Quick Tip

For differential equations involving $\frac{dy}{dx}$, rewrite to separate variables, integrate, and simplify.

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3. The vector with terminal point A (2, -3, 5) and initial point B (3, -4, 7) is:
(A) $\hat{i} - \hat{j} + 2\hat{k}$
(B) $\hat{i} + \hat{j} + 2\hat{k}$
(C) $-\hat{i} - \hat{j} - 2\hat{k}$
(D) $-\hat{i} + \hat{j} - 2\hat{k}$

Correct Answer: (C) $-\hat{i} - \hat{j} - 2\hat{k}$.

Solution: The formula for the vector from the initial point $B(x_1, y_1, z_1)$ to the terminal point $A(x_2, y_2, z_2)$ is:

$$\vec{AB} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}.$$

Substitute $A(2, -3, 5)$ and $B(3, -4, 7)$:

$$\vec{AB} = (2 - 3)\hat{i} + (-3 + 4)\hat{j} + (5 - 7)\hat{k}.$$

Simplify each component:

$$\vec{AB} = -\hat{i} - \hat{j} - 2\hat{k}.$$

Hence, the correct answer is (C) $-\hat{i} - \hat{j} - 2\hat{k}$.

Quick Tip

To find a vector between two points, subtract the coordinates of the initial point from the terminal point.

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4. The distance of point $P(a, b, c)$ from the y-axis is:

Options:

1. b
2. b^2
3. $\sqrt{a^2 + c^2}$

4. $a^2 + c^2$

Correct Answer: (C) $\sqrt{a^2 + c^2}$.
Solution:
The distance of a point $P(a, b, c)$ from the y-axis is determined by the perpendicular distance from the point to the axis. Since the y-axis lies along the direction where b varies while a and c remain constant, the distance is given by:

$$\text{Distance} = \sqrt{a^2 + c^2}.$$

Hence, the correct answer is (C) $\sqrt{a^2 + c^2}$.

> Quick Tip
>
> For finding the distance from an axis, exclude the coordinate along the axis of interest, and compute the Euclidean distance using the remaining coordinates.

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5. The number of corner points of the feasible region determined by constraints $x \geq 0, y \geq 0, x + y \geq 4$ is:
(A) 0
(B) 1
(C) 2
(D) 3

Correct Answer: (C) 2.

Solution: The constraints given are:

$$x \geq 0, \quad y \geq 0, \quad x + y \geq 4.$$

These represent: 1. $x \geq 0$: The region to the right of the y-axis. 2. $y \geq 0$: The region above the x-axis. 3. $x + y \geq 4$: A line passing through points $(4, 0)$ and $(0, 4)$ with the feasible region above this line.
The feasible region lies in the first quadrant and satisfies all three constraints. The intersection points (corner points) of the feasible region are: 1. $(4, 0)$ 2. $(0, 4)$
Hence, the number of corner points is 2.

> Quick Tip
>
> The number of corner points of a feasible region is determined by solving the intersection of the constraints.

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6. If matrices A and B are of order 1×3 and 3×1 respectively, then the order of $A'B'$ is:
(A) 1×1
(B) 3×1
(C) 1×3

(D) 3×3

Correct Answer: (D) 3×3.

Solution: The transpose of a matrix changes its dimensions: - For matrix A of order 1×3, A' has order 3×1. - For matrix B of order 3×1, B' has order 1×3.
The product $A'B'$ has the order determined by the dimensions of A' and B':

$$A' \text{ is } (3 \times 1), \quad B' \text{ is } (1 \times 3).$$

The resulting matrix $A'B'$ will have order 3×3.
Hence, the correct answer is (D) 3×3.

Quick Tip

The order of the product of matrices depends on the number of rows of the first matrix and the number of columns of the second matrix.

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7. A relation R defined on a set of human beings as $R = \{(x, y) : x$ is 5 cm shorter than $y\}$ is:

Options:

1. Reflexive only
2. Reflexive and transitive
3. Symmetric and transitive
4. Neither transitive, nor symmetric, nor reflexive

Correct Answer: (D) Neither transitive, nor symmetric, nor reflexive.

Solution:
- **Reflexive:** For R to be reflexive, (x, x) must belong to R. However, x cannot be 5 cm shorter than itself. Hence, R is not reflexive. - **Symmetric:** For R to be symmetric, if $(x, y) \in R$, then $(y, x) \in R$. If x is 5 cm shorter than y, y cannot be 5 cm shorter than x. Hence, R is not symmetric. - **Transitive:** For R to be transitive, if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$. However, if x is 5 cm shorter than y and y is 5 cm shorter than z, x is 10 cm shorter than z. Thus, R is not transitive.
Hence, the correct answer is (D) Neither transitive, nor symmetric, nor reflexive.

Quick Tip

Check the definitions of reflexive, symmetric, and transitive properties carefully for any given relation.

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8. If a matrix has 36 elements, the number of possible orders it can have, is:
(A) 13

(B) 3
(C) 5
(D) 9

Correct Answer: (D) 9.

Solution: If a matrix has 36 elements, it means the product of the number of rows (m) and the number of columns (n) must equal 36:

$$m \times n = 36.$$

To determine the possible orders (m, n), we find all pairs of positive integers whose product is 36:

$$(1, 36), (2, 18), (3, 12), (4, 9), (6, 6), (9, 4), (12, 3), (18, 2), (36, 1).$$

Thus, there are 9 possible orders: $1 \times 36, 2 \times 18, 3 \times 12, 4 \times 9, 6 \times 6, 9 \times 4, 12 \times 3, 18 \times 2, 36 \times 1$. Hence, the correct answer is (D) 9.

Quick Tip

To find the number of possible orders for a matrix, factorize the total number of elements into all possible integer pairs.

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9. Which of the following statements is true for the function:

$$f(x) = \begin{cases} x^2 + 3, & \text{if } x \neq 0, \\ 1, & \text{if } x = 0? \end{cases}$$

Options:

1. $f(x)$ is continuous and differentiable $\forall x \in R$.
2. $f(x)$ is continuous $\forall x \in R$.
3. $f(x)$ is continuous and differentiable $\forall x \in R - \{0\}$.
4. $f(x)$ is discontinuous at infinitely many points.

Correct Answer: (C) $f(x)$ is continuous and differentiable $\forall x \in R - \{0\}$.
Solution:
To determine continuity and differentiability:
1. **Continuity at $x = 0$:**

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} (x^2 + 3) = 3.$$

However, $f(0) = 1$. Since $\lim_{x \to 0} f(x) \neq f(0)$, $f(x)$ is not continuous at $x = 0$.
2. **Differentiability at $x = 0$:** Since $f(x)$ is not continuous at $x = 0$, it cannot be differentiable at $x = 0$.
3. **Continuity and Differentiability for $x \neq 0$:** For $x \neq 0$, $f(x) = x^2 + 3$, which is a polynomial function. Polynomials are both continuous and differentiable for all x.
Hence, the function $f(x)$ is continuous and differentiable $\forall x \in R - \{0\}$.

Thus, the correct answer is (C) $f(x)$ is continuous and differentiable $\forall x \in R - \{0\}$.

> **Quick Tip**
>
> Check for continuity by comparing the left-hand limit, right-hand limit, and function value at the point of interest. Differentiability implies continuity.

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10. Let $f(x)$ be a continuous function on $[a, b]$ and differentiable on (a, b). Then, this function $f(x)$ is strictly increasing in (a, b) if:

Options:

1. $f'(x) < 0$, $\forall x \in (a, b)$
2. $f'(x) > 0$, $\forall x \in (a, b)$
3. $f'(x) = 0$, $\forall x \in (a, b)$
4. $f(x) > 0$, $\forall x \in (a, b)$

Correct Answer: (B) $f'(x) > 0$, $\forall x \in (a, b)$.
Solution:
A function $f(x)$ is strictly increasing on an interval (a, b) if for any $x_1, x_2 \in (a, b)$, where $x_1 < x_2$, we have $f(x_1) < f(x_2)$.
The derivative $f'(x)$ gives the slope of the tangent to the curve. If $f'(x) > 0$ for all $x \in (a, b)$, the slope is positive, meaning $f(x)$ is strictly increasing.
Hence, the correct answer is (B) $f'(x) > 0$, $\forall x \in (a, b)$.

> **Quick Tip**
>
> A strictly increasing function has a positive derivative throughout the interval. A derivative equal to zero implies a constant function.

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11. If

$$\begin{bmatrix} x+y & 2 \\ 5 & xy \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 5 & 8 \end{bmatrix},$$

then the value of

$$\left(\frac{24}{x} + \frac{24}{y}\right)$$

is:
(A) 7
(B) 6
(C) 8
(D) 18

Correct Answer: (D) 18.

Solution: From the given matrix equation:

$$\begin{bmatrix} x+y & 2 \\ 5 & xy \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 5 & 8 \end{bmatrix},$$

we equate the corresponding elements: 1. $x + y = 6$, 2. $xy = 8$.
We are tasked to find:

$$\frac{24}{x} + \frac{24}{y}.$$

Using the relationship $x + y = 6$ and $xy = 8$, we know:

$$\frac{24}{x} + \frac{24}{y} = 24\left(\frac{1}{x} + \frac{1}{y}\right).$$

From the property of reciprocals:

$$\frac{1}{x} + \frac{1}{y} = \frac{x+y}{xy}.$$

Substitute $x + y = 6$ and $xy = 8$:

$$\frac{1}{x} + \frac{1}{y} = \frac{6}{8} = \frac{3}{4}.$$

Thus:

$$\frac{24}{x} + \frac{24}{y} = 24 \times \frac{3}{4} = 18.$$

Hence, the correct answer is (D) 18.

> Quick Tip
>
> To solve such problems, use the properties of matrices to equate elements, and apply algebraic manipulation for the final expression.

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12. If $f(x)$ is an odd function, then:

$$\int_{-\pi/2}^{\pi/2} f(x)\cos^3 x\, dx \text{ equals:}$$

Options:

1. $2\int_0^{\pi/2} f(x)\cos^3 x\, dx$
2. 0
3. $2\int_0^{\pi/2} f(x)\, dx$
4. $2\int_0^{\pi/2} \cos^3 x\, dx$

Correct Answer: (B) 0.
Solution:
If $f(x)$ is an odd function, then $f(-x) = -f(x)$. The limits of integration are symmetric about $x = 0$, i.e., $[-a, a]$. For any odd function $f(x)$, the integral over symmetric limits is:

$$\int_{-a}^{a} f(x)\, dx = 0.$$

Since $f(x)\cos^3 x$ is the product of an odd function $f(x)$ and an even function $\cos^3 x$, the result is an odd function. Therefore:

$$\int_{-\pi/2}^{\pi/2} f(x)\cos^3 x\,dx = 0.$$

Hence, the correct answer is (B) 0.

Quick Tip

The integral of an odd function over symmetric limits is always zero.

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13. Let θ be the angle between two unit vectors $\hat{a}$ and $\hat{b}$ such that $\sin\theta = \frac{3}{5}$. Then, $\hat{a}\cdot\hat{b}$ is equal to:

Options:

1. $\pm\frac{3}{5}$
2. $\pm\frac{3}{4}$
3. $\pm\frac{4}{5}$
4. $\pm\frac{4}{3}$

Correct Answer: (C) $\pm\frac{4}{5}$.
Solution:
The relationship between $\sin\theta$ and $\cos\theta$ is given by:

$$\sin^2\theta + \cos^2\theta = 1.$$

Substitute $\sin\theta = \frac{3}{5}$:

$$\left(\frac{3}{5}\right)^2 + \cos^2\theta = 1.$$

Simplify:

$$\frac{9}{25} + \cos^2\theta = 1.$$

$$\cos^2\theta = 1 - \frac{9}{25} = \frac{16}{25}.$$

$$\cos\theta = \pm\frac{4}{5}.$$

For unit vectors $\hat{a}$ and $\hat{b}$, the dot product is:

$$\hat{a}\cdot\hat{b} = \cos\theta.$$

Thus:

$$\hat{a}\cdot\hat{b} = \pm\frac{4}{5}.$$

Hence, the correct answer is (C) $\pm\frac{4}{5}$.

Quick Tip

Use the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$ to find $\cos\theta$ when $\sin\theta$ is given.

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14. The integrating factor of the differential equation $(1-x^2)\frac{dy}{dx}+xy = ax$, $-1 < x < 1$, **is:**

(A) $\frac{1}{x^2-1}$
(B) $\frac{1}{\sqrt{x^2-1}}$
(C) $\frac{1}{1-x^2}$
(D) $\frac{1}{\sqrt{1-x^2}}$

Correct Answer: (D) $\frac{1}{\sqrt{1-x^2}}$.

Solution: The given differential equation is:

$$(1-x^2)\frac{dy}{dx} + xy = ax.$$

Divide through by $1-x^2$ to simplify:

$$\frac{dy}{dx} + \frac{xy}{1-x^2} = \frac{ax}{1-x^2}.$$

This is a linear differential equation of the form:

$$\frac{dy}{dx} + P(x)y = Q(x),$$

where $P(x) = \frac{x}{1-x^2}$.
The integrating factor (IF) is given by:

$$\text{IF} = e^{\int P(x)\,dx}.$$

Substitute $P(x) = \frac{x}{1-x^2}$:

$$\int P(x)\,dx = \int \frac{x}{1-x^2}\,dx.$$

Let $u = 1-x^2$, so $du = -2x\,dx$:

$$\int \frac{x}{1-x^2}\,dx = -\frac{1}{2}\int \frac{1}{u}\,du = -\frac{1}{2}\ln|u| = -\frac{1}{2}\ln|1-x^2|.$$

The integrating factor becomes:

$$\text{IF} = e^{-\frac{1}{2}\ln|1-x^2|} = (1-x^2)^{-\frac{1}{2}} = \frac{1}{\sqrt{1-x^2}}.$$

Hence, the correct answer is (D) $\frac{1}{\sqrt{1-x^2}}$.

Quick Tip

To find the integrating factor, use $e^{\int P(x)\,dx}$, where $P(x)$ is the coefficient of y in a linear differential equation.

15. If the direction cosines of a line are $\sqrt{3}k, \sqrt{3}k, \sqrt{3}k$, then the value of k is:

Options:

1. ± 1
2. $\pm\sqrt{3}$
3. ± 3
4. $\pm\frac{1}{3}$

Correct Answer: (D) $\pm\frac{1}{3}$.
Solution:
For a line in 3D space, the direction cosines l, m, n satisfy:

$$l^2 + m^2 + n^2 = 1.$$

Substitute $l = \sqrt{3}k$, $m = \sqrt{3}k$, $n = \sqrt{3}k$:

$$(\sqrt{3}k)^2 + (\sqrt{3}k)^2 + (\sqrt{3}k)^2 = 1.$$

Simplify:

$$3k^2 + 3k^2 + 3k^2 = 1.$$

$$9k^2 = 1.$$

$$k^2 = \frac{1}{9}.$$

$$k = \pm\frac{1}{3}.$$

Hence, the correct answer is (D) $\pm\frac{1}{3}$.

Quick Tip

The direction cosines l, m, n must always satisfy $l^2 + m^2 + n^2 = 1$.

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16. A linear programming problem deals with the optimization of a/an:

Options:

1. Logarithmic function
2. Linear function
3. Quadratic function
4. Exponential function

Correct Answer: (B) Linear function.
Solution:
Linear programming involves optimizing (maximizing or minimizing) a linear objective function, subject to a set of linear constraints. The objective function typically takes the form:

$$Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n,$$

where $c_1, c_2, \ldots, c_n$ are constants and $x_1, x_2, \ldots, x_n$ are decision variables.
Hence, the correct answer is (B) Linear function.

> Quick Tip
>
> Linear programming problems always involve a linear objective function and linear constraints.

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17. If $P(A|B) = P(A'|B)$, then which of the following statements is true?

Options:

1. $P(A) = P(A')$
2. $P(A) = 2P(B)$
3. $P(A \cap B) = \frac{1}{2}P(B)$
4. $P(A \cap B) = 2P(B)$

Correct Answer: (C) $P(A \cap B) = \frac{1}{2}P(B)$.
Solution:
Given $P(A|B) = P(A'|B)$, we know:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \quad P(A'|B) = \frac{P(A' \cap B)}{P(B)}.$$

Since A and A' are complementary events:

$$P(A \cap B) + P(A' \cap B) = P(B).$$

Substitute $P(A|B) = P(A'|B)$:

$$\frac{P(A \cap B)}{P(B)} = \frac{P(A' \cap B)}{P(B)}.$$

This implies:

$$P(A \cap B) = P(A' \cap B).$$

Using $P(A \cap B) + P(A' \cap B) = P(B)$:

$$P(A \cap B) + P(A \cap B) = P(B).$$

$$2P(A \cap B) = P(B).$$

$$P(A \cap B) = \frac{1}{2}P(B).$$

Hence, the correct answer is (C) $P(A \cap B) = \frac{1}{2}P(B)$.

> Quick Tip
>
> When $P(A|B) = P(A'|B)$, use the complement rule and the fact that probabilities sum to 1 to simplify.

18.

$$\begin{vmatrix} x+1 & x-1 \\ x^2+x+1 & x^2-x+1 \end{vmatrix}$$

is equal to:
(A) $2x^3$
(B) 2
(C) 0
(D) $2x^3 - 2$

Correct Answer: (B) 2.

Solution: The determinant of a 2×2 matrix is given by:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

For the given matrix:

$$\begin{vmatrix} x+1 & x-1 \\ x^2+x+1 & x^2-x+1 \end{vmatrix},$$

we have:

$$a = x+1, \quad b = x-1, \quad c = x^2+x+1, \quad d = x^2-x+1.$$

The determinant is:

$$\text{Determinant} = (x+1)(x^2-x+1) - (x-1)(x^2+x+1).$$

Expand both terms:

$$(x+1)(x^2-x+1) = x^3 - x^2 + x + x^2 - x + 1 = x^3 + x + 1,$$

$$(x-1)(x^2+x+1) = x^3 + x^2 + x - x^2 - x - 1 = x^3 - 1.$$

Subtract the second term from the first:

$$\text{Determinant} = (x^3+x+1) - (x^3-1) = x^3 + x + 1 - x^3 + 1 = 2.$$

Hence, the correct answer is (B) 2.

> Quick Tip
>
> The determinant of a 2×2 matrix is calculated as the product of the main diagonal elements minus the product of the off-diagonal elements.

Questions number 19 and 20 are Assertion and Reason based questions. Two statements are given, one labelled Assertion (A) and the other labelled Reason (R). Select the correct answer from the codes (A), (B), (C) and (D) as given below.

(A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).

(B) Both Assertion (A) and Reason (R) are true, but Reason (R) is *not* the correct explanation of the Assertion (A).

(C) Assertion (A) is true, but Reason (R) is false.

(D) Assertion (A) is false, but Reason (R) is true.

19. Assertion (A): For the matrix

$$A = \begin{bmatrix} 1 & \cos\theta & 1 \\ -\cos\theta & 1 & \cos\theta \\ -1 & -\cos\theta & 1 \end{bmatrix}, \quad \text{where } \theta \in [0, 2\pi],$$

$|A| \in [2, 4]$.

Reason (R): $\cos\theta \in [-1, 1],\ \forall\ \theta \in [0, 2\pi]$.

Solution:
To verify the given assertion and reason, we calculate the determinant of matrix A:

$$|A| = \begin{vmatrix} 1 & \cos\theta & 1 \\ -\cos\theta & 1 & \cos\theta \\ -1 & -\cos\theta & 1 \end{vmatrix}.$$

Using cofactor expansion along the first row:

$$|A| = 1 \cdot \begin{vmatrix} 1 & \cos\theta \\ -\cos\theta & 1 \end{vmatrix} - \cos\theta \cdot \begin{vmatrix} -\cos\theta & \cos\theta \\ -1 & 1 \end{vmatrix} + 1 \cdot \begin{vmatrix} -\cos\theta & 1 \\ -1 & -\cos\theta \end{vmatrix}.$$

1. Compute the first minor:

$$\begin{vmatrix} 1 & \cos\theta \\ -\cos\theta & 1 \end{vmatrix} = (1)(1) - (-\cos\theta)(\cos\theta) = 1 + \cos^2\theta.$$

2. Compute the second minor:

$$\begin{vmatrix} -\cos\theta & \cos\theta \\ -1 & 1 \end{vmatrix} = (-\cos\theta)(1) - (\cos\theta)(-1) = -\cos\theta + \cos\theta = 0.$$

3. Compute the third minor:

$$\begin{vmatrix} -\cos\theta & 1 \\ -1 & -\cos\theta \end{vmatrix} = (-\cos\theta)(-\cos\theta) - (1)(-1) = \cos^2\theta + 1.$$

Substitute back into the determinant:

$$|A| = 1 \cdot (1 + \cos^2\theta) - \cos\theta \cdot 0 + 1 \cdot (1 + \cos^2\theta).$$

Simplify:

$$|A| = (1 + \cos^2\theta) + (1 + \cos^2\theta) = 2 + 2\cos^2\theta.$$

Since $\cos\theta \in [-1, 1]$, we have:

$$\cos^2\theta \in [0, 1].$$

Thus, the determinant $|A|$ varies as:

$$|A| = 2 + 2\cos^2\theta \in [2, 4].$$

Verification of Assertion (A): The determinant $|A|$ lies in the interval $[2, 4]$, so the assertion is **true**.

Verification of Reason (R): The cosine function satisfies $\cos\theta \in [-1, 1]$ for all $\theta \in [0, 2\pi]$, so the reason is also **true**.

Conclusion: Both Assertion (A) and Reason (R) are true, and the Reason (R) correctly explains the Assertion (A).

> Quick Tip
>
> When calculating determinants, simplify using cofactor expansion and evaluate the minors carefully. Use the properties of trigonometric functions like $\cos\theta \in [-1, 1]$ to determine the range of values for expressions involving $\cos^2\theta$.

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20. Assertion (A): A line in space cannot be drawn perpendicular to x, y, and z axes simultaneously.
Reason (R): For any line making angles α, β, γ with the positive directions of x, y, and z axes respectively,

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

Solution:
A line in three-dimensional space cannot be perpendicular to all three axes simultaneously. If a line is perpendicular to all three axes, the direction cosines $\cos\alpha, \cos\beta, \cos\gamma$ would all be zero, which would violate the fundamental relation of direction cosines:

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

The given equation $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$ ensures that at least one of the direction cosines is non-zero, indicating that the line cannot be simultaneously perpendicular to x, y, and z axes.
Conclusion: Both Assertion (A) and Reason (R) are true, and Reason (R) is the correct explanation of Assertion (A).
Answer: (A)

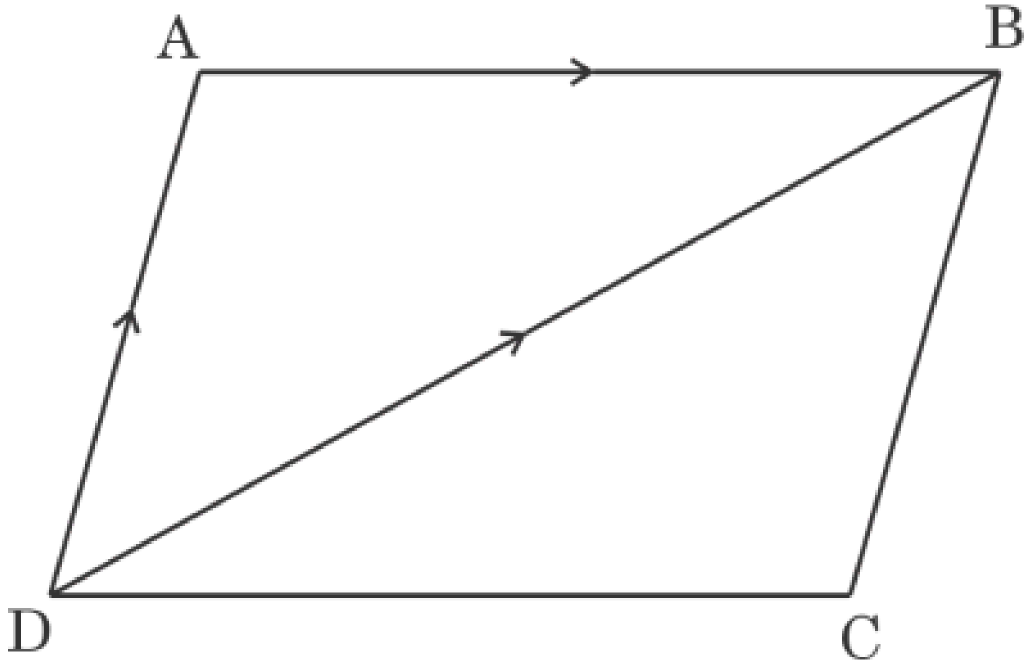
Quick Tip

In 3D geometry, the direction cosines of a line $(\cos\alpha, \cos\beta, \cos\gamma)$ satisfy the fundamental relation $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$. This property ensures that no line can be perpendicular to all three axes simultaneously.

SECTION B

This section comprises very short answer (VSA) type questions of 2 marks each.

21. In the given figure, ABCD is a parallelogram. If $\vec{AB} = 2\hat{i} - 4\hat{j} + 5\hat{k}$ and $\vec{DB} = 3\hat{i} - 6\hat{j} + 2\hat{k}$, then find $\vec{AD}$ and hence find the area of parallelogram ABCD.



Solution: To find $\vec{AD}$, we use the relationship:

$$\vec{AD} = \vec{AB} + \vec{DB}.$$

Given that $\vec{AB} = 2\hat{i} - 4\hat{j} + 5\hat{k}$ and $\vec{DB} = 3\hat{i} - 6\hat{j} + 2\hat{k}$, we calculate $\vec{AD}$:

$$\vec{AD} = (2\hat{i} - 4\hat{j} + 5\hat{k}) + (3\hat{i} - 6\hat{j} + 2\hat{k}).$$

Simplifying:

$$\vec{AD} = (2+3)\hat{i} + (-4-6)\hat{j} + (5+2)\hat{k} = 5\hat{i} - 10\hat{j} + 7\hat{k}.$$

The area of parallelogram ABCD is given by the magnitude of the cross product of vectors $\vec{AB}$ and $\vec{AD}$:

$$\text{Area} = |\vec{AB} \times \vec{AD}|.$$

The cross product of $\vec{AB} = 2\hat{i} - 4\hat{j} + 5\hat{k}$ and $\vec{AD} = 5\hat{i} - 10\hat{j} + 7\hat{k}$ is computed as follows:

$$\vec{AB} \times \vec{AD} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -4 & 5 \\ 5 & -10 & 7 \end{vmatrix}.$$

Expanding the determinant:

$$\vec{AB} \times \vec{AD} = \hat{i} \begin{vmatrix} -4 & 5 \\ -10 & 7 \end{vmatrix} - \hat{j} \begin{vmatrix} 2 & 5 \\ 5 & 7 \end{vmatrix} + \hat{k} \begin{vmatrix} 2 & -4 \\ 5 & -10 \end{vmatrix}.$$

Calculating each 2x2 determinant:

$$\hat{i} = (-4)(7) - (5)(-10) = -28 + 50 = 22,$$

$$\hat{j} = (2)(7) - (5)(5) = 14 - 25 = -11,$$

$$\hat{k} = (2)(-10) - (-4)(5) = -20 + 20 = 0.$$

Thus,

$$\vec{AB} \times \vec{AD} = 22\hat{i} + 11\hat{j} + 0\hat{k}.$$

The magnitude of the cross product is:

$$|\vec{AB} \times \vec{AD}| = \sqrt{22^2 + 11^2} = \sqrt{484 + 121} = \sqrt{605}.$$

Answer: The magnitude of the cross product is $\sqrt{605}$, so the area of parallelogram ABCD is $\sqrt{605}$.

Quick Tip

To find the area of a parallelogram using vectors, compute the cross product of two adjacent sides, and the magnitude of this vector gives the area. Use $|\vec{AB} \times \vec{AD}|$ for the solution.

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22. (a) Check the differentiability of the function $f(x) = \lfloor x \rfloor$ at $x = -3$, where $\lfloor \cdot \rfloor$ denotes the greatest integer function.
Solution:
The greatest integer function $f(x) = \lfloor x \rfloor$ gives the greatest integer less than or equal to x. For $x = -3$, we have:

$$f(x) = \lfloor x \rfloor = \begin{cases} -4, & -4 \leq x < -3, \\ -3, & x = -3. \end{cases}$$

To check differentiability, we need to verify the left-hand derivative (LHD) and right-hand derivative (RHD) at $x = -3$.
1. **Left-hand derivative (LHD):**

$$\text{LHD} = \lim_{h \to 0^-} \frac{f(-3+h) - f(-3)}{h}.$$

For $h \to 0^-$, $-3 + h \in (-4, -3)$, so $f(-3+h) = -4$. Substituting:

$$\text{LHD} = \lim_{h \to 0^-} \frac{-4 - (-3)}{h} = \lim_{h \to 0^-} \frac{-1}{h}.$$

Since $h \to 0^-$, the denominator is negative, and LHD $\to \infty$.
2. **Right-hand derivative (RHD):**

$$\text{RHD} = \lim_{h \to 0^+} \frac{f(-3+h) - f(-3)}{h}.$$

For $h \to 0^+$, $-3 + h \in (-3, -2)$, so $f(-3+h) = -3$. Substituting:

$$\text{RHD} = \lim_{h \to 0^+} \frac{-3 - (-3)}{h} = \lim_{h \to 0^+} \frac{0}{h} = 0.$$

Since LHD $\neq$ RHD, the function $f(x) = \lfloor x \rfloor$ is not differentiable at $x = -3$.

> **Quick Tip**
>
> The greatest integer function is not differentiable at integer values of x because of the discontinuity in the derivative at those points.

(b) If $x^{1/3} + y^{1/3} = 1$, find $\frac{dy}{dx}$ at the point $\left(\frac{1}{8}, \frac{1}{8}\right)$.
Solution:
Differentiate the given equation implicitly with respect to x:

$$\frac{d}{dx}\left(x^{1/3} + y^{1/3}\right) = \frac{d}{dx}(1).$$

Using the chain rule:

$$\frac{1}{3}x^{-2/3} + \frac{1}{3}y^{-2/3}\frac{dy}{dx} = 0.$$

Rearrange to solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = -\frac{x^{-2/3}}{y^{-2/3}}.$$

Substitute $x = \frac{1}{8}$ and $y = \frac{1}{8}$:

$$x^{-2/3} = \left(\frac{1}{8}\right)^{-2/3} = (8)^{2/3} = 4, \quad y^{-2/3} = \left(\frac{1}{8}\right)^{-2/3} = (8)^{2/3} = 4.$$

$$\frac{dy}{dx} = -\frac{4}{4} = -1.$$

Hence, $\frac{dy}{dx} = -1$ at the point $\left(\frac{1}{8}, \frac{1}{8}\right)$.

> **Quick Tip**
>
> For implicit differentiation, always isolate $\frac{dy}{dx}$ after applying the chain rule to each term.

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23. Find the local maximum value and local minimum value (whichever exists) for the function $f(x) = 4x^2 + \frac{1}{x}$ $(x \neq 0)$.
Solution:

To find the critical points, differentiate $f(x)$ with respect to x:

$$f'(x) = \frac{d}{dx}\left(4x^2 + \frac{1}{x}\right) = 8x - \frac{1}{x^2}.$$

Set $f'(x) = 0$ to find the critical points:

$$8x - \frac{1}{x^2} = 0 \quad \Rightarrow \quad 8x^3 = 1 \quad \Rightarrow \quad x = \frac{1}{2}.$$

To classify the critical point, compute the second derivative:

$$f''(x) = \frac{d}{dx}\left(8x - \frac{1}{x^2}\right) = 8 + \frac{2}{x^3}.$$

Substitute $x = \frac{1}{2}$:

$$f''\left(\frac{1}{2}\right) = 8 + \frac{2}{\left(\frac{1}{2}\right)^3} = 8 + 16 = 24 > 0.$$

Since $f''(x) > 0$, the function has a local minimum at $x = \frac{1}{2}$.
Find the value of $f(x)$ at $x = \frac{1}{2}$:

$$f\left(\frac{1}{2}\right) = 4\left(\frac{1}{2}\right)^2 + \frac{1}{\frac{1}{2}} = 4\left(\frac{1}{4}\right) + 2 = 1 + 2 = 3.$$

Conclusion: The local minimum value of $f(x)$ is 3.

Quick Tip

Use the first derivative test to find critical points and the second derivative test to classify them as local maxima or minima.

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24. (a) Find:

$$\int x\sqrt{1+2x}\,dx$$

Solution: Let $I = \int x\sqrt{1+2x}\,dx$. Using substitution, let $u = 1 + 2x$. Then,

$$du = 2\,dx \quad \text{and} \quad x = \frac{u-1}{2}.$$

Substitute into the integral:

$$I = \int \frac{u-1}{2}\sqrt{u} \cdot \frac{1}{2}\,du = \frac{1}{4}\int (u-1)u^{\frac{1}{2}}\,du.$$

Simplify:

$$I = \frac{1}{4}\int \left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right)\,du.$$

Split the integral:

$$I = \frac{1}{4}\left(\int u^{\frac{3}{2}}\,du - \int u^{\frac{1}{2}}\,du\right).$$

Integrate each term:

$$\int u^{\frac{3}{2}}\,du = \frac{2}{5}u^{\frac{5}{2}}, \quad \int u^{\frac{1}{2}}\,du = \frac{2}{3}u^{\frac{3}{2}}.$$

Substitute back:

$$I = \frac{1}{4}\left(\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right).$$

Simplify and substitute $u = 1 + 2x$:

$$I = \frac{1}{10}(1+2x)^{\frac{5}{2}} - \frac{1}{6}(1+2x)^{\frac{3}{2}} + C.$$

Answer:

$$\int x\sqrt{1+2x}\,dx = \frac{1}{10}(1+2x)^{\frac{5}{2}} - \frac{1}{6}(1+2x)^{\frac{3}{2}} + C.$$

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24. (b) Evaluate:

$$\int_0^{\frac{\pi^2}{4}} \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$$

Solution: Let $I = \int_0^{\frac{\pi^2}{4}} \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$. Using substitution, let $t = \sqrt{x}$. Then,

$$x = t^2, \quad dx = 2t\,dt, \quad \sqrt{x} = t.$$

Substitute into the integral:

$$I = \int_0^{\frac{\pi}{2}} \frac{\sin t}{t} \cdot 2t\,dt = 2\int_0^{\frac{\pi}{2}} \sin t\,dt.$$

Simplify:

$$I = 2\left[-\cos t\right]_0^{\frac{\pi}{2}}.$$

Evaluate:

$$I = 2\left[-\cos\left(\frac{\pi}{2}\right) + \cos(0)\right] = 2\left[0+1\right] = 2.$$

Answer:

$$\int_0^{\frac{\pi^2}{4}} \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx = 2.$$

Quick Tip

For definite integrals, simplify the integrand using substitution or trigonometric identities before integrating. Ensure the limits of integration are updated when substitution is used.

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25. If $\vec{a}$ and $\vec{b}$ are two non-zero vectors such that $(\vec{a}+\vec{b}) \perp \vec{a}$ and $(2\vec{a}+\vec{b}) \perp \vec{b}$, then prove that $|\vec{b}| = \sqrt{2}|\vec{a}|$.

Correct answer: $|\vec{b}| = \sqrt{2}|\vec{a}|$.

Solution:
1. **Condition 1: $(\vec{a}+\vec{b}) \perp \vec{a}$:**

$$(\vec{a}+\vec{b}) \cdot \vec{a} = 0 \quad \Rightarrow \quad \vec{a}\cdot\vec{a} + \vec{a}\cdot\vec{b} = 0.$$

$$|\vec{a}|^2 + \vec{a}\cdot\vec{b} = 0 \quad \Rightarrow \quad \vec{a}\cdot\vec{b} = -|\vec{a}|^2.$$

2. **Condition 2: $(2\vec{a}+\vec{b}) \perp \vec{b}$:**

$$(2\vec{a}+\vec{b}) \cdot \vec{b} = 0 \quad \Rightarrow \quad 2(\vec{a}\cdot\vec{b}) + |\vec{b}|^2 = 0.$$

Substitute $\vec{a}\cdot\vec{b} = -|\vec{a}|^2$:

$$2(-|\vec{a}|^2) + |\vec{b}|^2 = 0 \quad \Rightarrow \quad -2|\vec{a}|^2 + |\vec{b}|^2 = 0.$$

Rearrange:

$$|\vec{b}|^2 = 2|\vec{a}|^2 \quad \Rightarrow \quad |\vec{b}| = \sqrt{2}|\vec{a}|.$$

Quick Tip

Use dot product properties and given perpendicularity conditions to derive relationships between the magnitudes of vectors.

SECTION C

This section comprises short answer (SA) type questions of 3 marks each.

26. Solve the following linear programming problem graphically:
Minimise $z = 5x - 2y$
Subject to the constraints:

$$x + 2y \leq 120, \quad x + y \geq 60, \quad x - 2y \geq 0, \quad x \geq 0, \quad y \geq 0.$$

Solution: To solve this linear programming problem graphically, follow these steps:
1. **Plot the constraints:** - $x + 2y \leq 120$: Rewrite as $y \leq \frac{120-x}{2}$. This is a line passing through $(0, 60)$ and $(120, 0)$. - $x+y \geq 60$: Rewrite as $y \geq 60-x$. This is a line passing through $(0, 60)$ and $(60, 0)$. - $x - 2y \geq 0$: Rewrite as $y \leq \frac{x}{2}$. This is a line passing through $(0, 0)$ and $(120, 60)$. - $x \geq 0, y \geq 0$: These are the x- and y-axes, restricting the feasible region to the first quadrant.
2. **Identify the feasible region:** The feasible region is the area that satisfies all the constraints. Plot the lines and shade the overlapping region that satisfies the inequalities.
3. **Determine the corner points of the feasible region:** The corner points are the intersections of the constraint lines: - Intersection of $x + 2y = 120$ and $x + y = 60$: Solve:

$$x + 2y = 120, \quad x + y = 60.$$

Subtract the second equation from the first:

$$y = 60, \quad x = 0.$$

Point: $(0, 60)$. - Intersection of $x + y = 60$ and $x - 2y = 0$: Solve:

$$x + y = 60, \quad x - 2y = 0.$$

Solve for $x = 2y$ and substitute:

$$2y + y = 60 \implies y = 20, \quad x = 40.$$

Point: $(40, 20)$. - Intersection of $x - 2y = 0$ and $x + 2y = 120$: Solve:

$$x - 2y = 0, \quad x + 2y = 120.$$

Solve for $x = 2y$ and substitute:

$$2y + 2y = 120 \implies y = 30, \quad x = 60.$$

Point: $(60, 30)$.
4. **Evaluate the objective function at each corner point:** The objective function is $z = 5x - 2y$: - At $(0, 60)$: $z = 5(0) - 2(60) = -120$. - At $(40, 20)$: $z = 5(40) - 2(20) = 200 - 40 = 160$. - At $(60, 30)$: $z = 5(60) - 2(30) = 300 - 60 = 240$.
5. **Conclusion:** The minimum value of $z = 5x - 2y$ occurs at $(0, 60)$ with $z = -120$.

Quick Tip

In graphical linear programming, identify the feasible region, locate the corner points, and evaluate the objective function at each corner point to find the optimal solution.

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27. E and F are two independent events such that $P(\overline{E}) = 0.6$ and $P(E \cup F) = 0.6$. Find $P(F)$ and $P(\overline{E} \cup \overline{F})$.
Solution:
1. **Step 1: Find $P(E)$:**

$$P(E) = 1 - P(\overline{E}) = 1 - 0.6 = 0.4.$$

2. **Step 2: Use the formula for $P(E \cup F)$:**

$$P(E \cup F) = P(E) + P(F) - P(E \cap F).$$

For independent events:

$$P(E \cap F) = P(E) \cdot P(F).$$

Substitute:

$$0.6 = 0.4 + P(F) - (0.4 \cdot P(F)).$$

Simplify:

$$0.6 = 0.4 + P(F) - 0.4P(F).$$

$$0.6 - 0.4 = P(F)(1 - 0.4).$$

$$0.2 = 0.6P(F).$$

$$P(F) = \frac{0.2}{0.6} = \frac{1}{3}.$$

3. **Step 3: Find $P(\overline{E} \cup \overline{F})$:**

$$P(\overline{E} \cup \overline{F}) = P(\overline{E}) + P(\overline{F}) - P(\overline{E} \cap \overline{F}).$$

For independent events:

$$P(\overline{E} \cap \overline{F}) = P(\overline{E}) \cdot P(\overline{F}).$$

Substitute $P(\overline{F}) = 1 - P(F) = 1 - \frac{1}{3} = \frac{2}{3}$:

$$P(\overline{E} \cup \overline{F}) = 0.6 + \frac{2}{3} - (0.6 \cdot \frac{2}{3}).$$

Simplify:

$$P(\overline{E} \cup \overline{F}) = 0.6 + 0.6667 - 0.4 = 0.8667.$$

Conclusion:

$$P(F) = \frac{1}{3}, \quad P(\overline{E} \cup \overline{F}) \approx 0.867.$$

Quick Tip

For independent events, use $P(E \cap F) = P(E) \cdot P(F)$.

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28. **(a)** A relation R on set $A = \{1, 2, 3, 4, 5\}$ is defined as

$$R = \{(x, y) : |x^2 - y^2| < 8\}.$$

Check whether the relation R is reflexive, symmetric, and transitive.
Solution:
- **Reflexive:** A relation is reflexive if for every element $x \in A$, (x, x) is in R. This means we need to check if $|x^2 - x^2| < 8$ for all $x \in A$. Since $|x^2 - x^2| = 0$, which is less than 8, we conclude that $(x, x) \in R$ for all $x \in A$. Hence, the relation is reflexive.
- **Symmetric:** A relation is symmetric if for every pair $(x, y) \in R$, the pair (y, x) is also in R. In our case, $|x^2 - y^2| < 8$, which is the same as $|y^2 - x^2| < 8$. Therefore, the relation is symmetric because $|x^2 - y^2| = |y^2 - x^2|$.
- **Transitive:** A relation is transitive if whenever $(x, y) \in R$ and $(y, z) \in R$, we also have $(x, z) \in R$. We need to check if $|x^2 - z^2| < 8$ whenever $|x^2 - y^2| < 8$ and $|y^2 - z^2| < 8$. For example, consider the elements $x = 1, y = 2, z = 3$. We have:

$$|1^2 - 2^2| = |1 - 4| = 3 < 8, \quad |2^2 - 3^2| = |4 - 9| = 5 < 8.$$

But:

$$|1^2 - 3^2| = |1 - 9| = 8 \not< 8.$$

Therefore, the relation is **not transitive**.
Answer: The relation R is reflexive and symmetric, but not transitive.

Quick Tip

Part (a): Check reflexivity, symmetry, and transitivity by verifying conditions for all elements in the relation.

28. (b) A function f is defined from $R \to R$ as $f(x) = ax + b$, such that $f(1) = 1$ and $f(2) = 3$. Find the function $f(x)$. Hence, check whether the function $f(x)$ is one-one and onto.

Solution:

From the given conditions, we have two equations:

$$f(1) = a(1) + b = 1 \quad \Rightarrow \quad a + b = 1, \tag{1}$$

$$f(2) = a(2) + b = 3 \quad \Rightarrow \quad 2a + b = 3. \tag{2}$$

Solving equations (1) and (2) simultaneously: - From equation (1): $b = 1 - a$. - Substitute this into equation (2):

$$2a + (1 - a) = 3 \quad \Rightarrow \quad 2a + 1 - a = 3 \quad \Rightarrow \quad a = 2.$$

- Substituting $a = 2$ into equation (1):

$$2 + b = 1 \quad \Rightarrow \quad b = -1.$$

Thus, the function is:

$$f(x) = 2x - 1.$$

- **One-one (Injective):** A function is one-one if distinct inputs lead to distinct outputs. Since $f(x) = 2x - 1$ is a linear function with a non-zero slope, it is one-one.
- **Onto (Surjective):** A function is onto if for every element $y \in R$, there exists $x \in R$ such that $f(x) = y$. For $f(x) = 2x - 1$, for any $y \in R$, we can solve $y = 2x - 1$ for x, which gives $x = \frac{y+1}{2}$. Hence, the function is onto.

Answer: The function $f(x) = 2x - 1$ is both one-one and onto.

Quick Tip

Part (b): For a function $f(x) = ax + b$, use the given points to form and solve simultaneous equations for a and b. Check one-to-one by verifying $f'(x) \neq 0$ and onto by solving $f(x) = y$ for all $y \in R$.

29(a). If $\sqrt{1 - x^2} + \sqrt{1 - y^2} = a(x - y)$, prove that $\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$.

Solution:

Differentiate the given equation implicitly with respect to x:

$$\frac{d}{dx}\left(\sqrt{1 - x^2} + \sqrt{1 - y^2}\right) = \frac{d}{dx}\left(a(x - y)\right).$$

Use the chain rule:

$$\frac{-x}{\sqrt{1 - x^2}} + \frac{-y}{\sqrt{1 - y^2}} \cdot \frac{dy}{dx} = a\left(1 - \frac{dy}{dx}\right).$$

Rearrange terms:

$$\frac{-x}{\sqrt{1 - x^2}} - a = \frac{dy}{dx}\left(a - \frac{y}{\sqrt{1 - y^2}}\right).$$

Solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{\frac{-x}{\sqrt{1-x^2}} - a}{a - \frac{y}{\sqrt{1-y^2}}}.$$

For $a = 0$, simplify further to:

$$\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}.$$

Conclusion: The result is proved.

Quick Tip

Apply implicit differentiation systematically and simplify carefully.

29(b). If $y = (\tan x)^x$, then find $\frac{dy}{dx}$.

Solution:

Let $y = (\tan x)^x$. Take the natural logarithm on both sides:

$$\ln y = \ln\left((\tan x)^x\right).$$

Using the logarithmic property $\ln(a^b) = b \ln a$:

$$\ln y = x \ln(\tan x).$$

Differentiate both sides with respect to x using the chain rule:

$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}\left(x \ln(\tan x)\right).$$

Differentiate the right-hand side using the product rule:

$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}(x) \cdot \ln(\tan x) + x \cdot \frac{d}{dx}(\ln(\tan x)).$$

Simplify:

$$\frac{1}{y}\frac{dy}{dx} = \ln(\tan x) + x \cdot \frac{1}{\tan x} \cdot \sec^2 x.$$

Multiply through by y:

$$\frac{dy}{dx} = y\left(\ln(\tan x) + x \cdot \frac{\sec^2 x}{\tan x}\right).$$

Substitute $y = (\tan x)^x$:

$$\frac{dy}{dx} = (\tan x)^x \left(\ln(\tan x) + x \cdot \frac{\sec^2 x}{\tan x}\right).$$

Final Answer:

$$\frac{dy}{dx} = (\tan x)^x \left(\ln(\tan x) + x \cdot \frac{\sec^2 x}{\tan x}\right).$$

Quick Tip

When differentiating functions of the form $y = f(x)^{g(x)}$, take the natural logarithm on both sides and use the product rule.

30(a). Find:

$$\int \frac{x^2}{(x^2+4)(x^2+9)}\,dx.$$

Solution:
1. **Step 1: Decompose the fraction into partial fractions:**

$$\frac{x^2}{(x^2+4)(x^2+9)} = \frac{A}{x^2+4} + \frac{B}{x^2+9}.$$

Multiply through by $(x^2+4)(x^2+9)$:

$$x^2 = A(x^2+9) + B(x^2+4).$$

Expand and simplify:

$$x^2 = Ax^2 + 9A + Bx^2 + 4B.$$

Combine terms:

$$x^2 = (A+B)x^2 + (9A+4B).$$

Equating coefficients of x^2 and the constant term:

$$A + B = 1, \quad 9A + 4B = 0. \tag{1}$$

2. **Step 2: Solve for A and B:**
From $A + B = 1$, we get $B = 1 - A$. Substitute $B = 1 - A$ into $9A + 4B = 0$:

$$9A + 4(1-A) = 0.$$

$$9A + 4 - 4A = 0 \quad \Rightarrow \quad 5A = -4 \quad \Rightarrow \quad A = -\frac{4}{5}.$$

Substitute $A = -\frac{4}{5}$ into $B = 1 - A$:

$$B = 1 - \left(-\frac{4}{5}\right) = 1 + \frac{4}{5} = \frac{9}{5}.$$

Thus, the partial fraction decomposition is:

$$\frac{x^2}{(x^2+4)(x^2+9)} = \frac{-\frac{4}{5}}{x^2+4} + \frac{\frac{9}{5}}{x^2+9}.$$

3. **Step 3: Integrate each term:**

$$\int \frac{x^2}{(x^2+4)(x^2+9)}\,dx = -\frac{4}{5}\int \frac{1}{x^2+4}\,dx + \frac{9}{5}\int \frac{1}{x^2+9}\,dx.$$

The standard integral formulas are:

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right).$$

Substitute $a = 2$ for the first term and $a = 3$ for the second term:

$$\int \frac{x^2}{(x^2+4)(x^2+9)}\,dx = -\frac{4}{5}\cdot\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + \frac{9}{5}\cdot\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right).$$

Simplify:

$$\int \frac{x^2}{(x^2+4)(x^2+9)}\,dx = -\frac{2}{5}\tan^{-1}\left(\frac{x}{2}\right) + \frac{3}{5}\tan^{-1}\left(\frac{x}{3}\right) + C.$$

Final Answer:

$$\int \frac{x^2}{(x^2+4)(x^2+9)}\,dx = -\frac{2}{5}\tan^{-1}\left(\frac{x}{2}\right) + \frac{3}{5}\tan^{-1}\left(\frac{x}{3}\right) + C.$$

30(b). Evaluate:

$$\int_1^3 (|x-1| + |x-2| + |x-3|)\,dx.$$

Solution:
The given integral involves absolute values, so split the integration range based on the points where the arguments of the absolute values change sign: $x = 1$, $x = 2$, and $x = 3$.
1. **Break the integral into intervals:**

$$\int_1^3 (|x-1|+|x-2|+|x-3|)\,dx = \int_1^2 (x-1+2-x+3-x)\,dx + \int_2^3 (x-1+x-2+3-x)\,dx.$$

Simplify each term in the intervals: - For $x \in [1,2]$:

$$|x-1| = x-1, \quad |x-2| = 2-x, \quad |x-3| = 3-x.$$

$$|x-1| + |x-2| + |x-3| = x-1+2-x+3-x = 4-x.$$

- For $x \in [2,3]$:

$$|x-1| = x-1, \quad |x-2| = x-2, \quad |x-3| = 3-x.$$

$$|x-1| + |x-2| + |x-3| = x-1+x-2+3-x = x.$$

2. **Evaluate the integrals:**

$$\int_1^3 (|x-1| + |x-2| + |x-3|)\,dx = \int_1^2 (4-x)\,dx + \int_2^3 x\,dx.$$

- First integral:

$$\int_1^2 (4-x)\,dx = \left[4x - \frac{x^2}{2}\right]_1^2 = \left(4(2) - \frac{2^2}{2}\right) - \left(4(1) - \frac{1^2}{2}\right).$$

$$= (8-2) - (4-0.5) = 6 - 3.5 = 2.5.$$

- Second integral:

$$\int_2^3 x\,dx = \left[\frac{x^2}{2}\right]_2^3 = \frac{3^2}{2} - \frac{2^2}{2} = \frac{9}{2} - \frac{4}{2} = \frac{5}{2}.$$

3. **Add the results:**

$$\int_1^3 (|x-1| + |x-2| + |x-3|)\,dx = 2.5 + 2.5 = 5.$$

Final Answer:

$$\int_1^3 (|x-1| + |x-2| + |x-3|)\, dx = 5.$$

---

31. Solve the following differential equation:

$$(\tan^{-1} y - x)\, dy = (1 + y^2)\, dx.$$

Solution: Rearrange the given equation to separate the variables x and y:

$$\frac{\tan^{-1} y - x}{1 + y^2}\, dy = dx.$$

Rewrite as:

$$\frac{\tan^{-1} y}{1 + y^2}\, dy - x\, dx = 0.$$

Step 1: Split the equation The first term involves y, and the second term involves x. Separate these:

$$\frac{\tan^{-1} y}{1 + y^2}\, dy = x\, dx.$$

Step 2: Integrate both sides - For the left-hand side:

$$\int \frac{\tan^{-1} y}{1 + y^2}\, dy.$$

Let $u = \tan^{-1} y$, so $du = \frac{1}{1+y^2}\, dy$:

$$\int \frac{\tan^{-1} y}{1 + y^2}\, dy = \int u\, du = \frac{u^2}{2} + C_1 = \frac{(\tan^{-1} y)^2}{2} + C_1.$$

- For the right-hand side:

$$\int x\, dx = \frac{x^2}{2} + C_2.$$

Step 3: Combine the results The equation becomes:

$$\frac{(\tan^{-1} y)^2}{2} = \frac{x^2}{2} + C,$$

where $C = C_2 - C_1$.
Multiply through by 2 to simplify:

$$(\tan^{-1} y)^2 = x^2 + 2C.$$

Let $2C = C_1$, so:

$$(\tan^{-1} y)^2 - x^2 = C_1.$$

Final Solution:

$$(\tan^{-1} y)^2 - x^2 = C_1,$$

where C_1 is the constant of integration.

💡 Quick Tip

To solve separable differential equations, rearrange the terms to isolate variables, integrate each side, and combine the results.

SECTION D

This section comprises long answer type questions (LA) of 5 marks each.

32. Find the equation of a line l_2 which is the mirror image of the line l_1 with respect to line $l : \frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$, given that line l_1 passes through the point $P(1, 6, 3)$ and is parallel to line l.

Solution:

1. **Step 1: Parametric form of the given line l:**

The given line l is:

$$\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}.$$

Its parametric equations are:

$$x = t, \quad y = 1 + 2t, \quad z = 2 + 3t,$$

where t is the parameter.

2. **Step 2: Direction ratios of line l:**

The direction ratios (DRs) of line l are $(1, 2, 3)$.

3. **Step 3: Equation of line l_1:**

Line l_1 passes through the point $P(1, 6, 3)$ and is parallel to l, so its DRs are also $(1, 2, 3)$. The equation of l_1 is:

$$\frac{x-1}{1} = \frac{y-6}{2} = \frac{z-3}{3}.$$

4. **Step 4: Find the foot of the perpendicular from $P(1, 6, 3)$ to l:**

Let the foot of the perpendicular be $Q(t, 1 + 2t, 2 + 3t)$ on line l. The line joining $P(1, 6, 3)$ to $Q(t, 1+2t, 2+3t)$ is perpendicular to line l. The DRs of PQ are $(t-1, (1+2t)-6, (2+3t)-3) = (t-1, 2t-5, 3t-1)$.

Using the condition for perpendicularity of two lines, $DR_{PQ} \cdot DR_l = 0$:

$$(t-1)(1) + (2t-5)(2) + (3t-1)(3) = 0.$$

Simplify:

$$t - 1 + 4t - 10 + 9t - 3 = 0.$$

$$14t - 14 = 0 \quad \Rightarrow \quad t = 1.$$

Substitute $t = 1$ into $Q(t, 1 + 2t, 2 + 3t)$:

$$Q(1, 1 + 2(1), 2 + 3(1)) = Q(1, 3, 5).$$

5. **Step 5: Reflection of $P(1, 6, 3)$ about $Q(1, 3, 5)$:**

The reflection point $P'(x', y', z')$ of $P(x_1, y_1, z_1)$ across $Q(x_2, y_2, z_2)$ is:

$$x' = 2x_2 - x_1, \quad y' = 2y_2 - y_1, \quad z' = 2z_2 - z_1.$$

Substitute $P(1,6,3)$ and $Q(1,3,5)$:

$$x' = 2(1) - 1 = 1, \quad y' = 2(3) - 6 = 0, \quad z' = 2(5) - 3 = 7.$$

Thus, the reflected point is $P'(1,0,7)$.
6. **Step 6: Equation of line l_2:**
Line l_2 passes through $P'(1,0,7)$ and is parallel to l, so its DRs are $(1,2,3)$. The equation of l_2 is:

$$\frac{x-1}{1} = \frac{y-0}{2} = \frac{z-7}{3}.$$

Final Answer:

$$\frac{x-1}{1} = \frac{y}{2} = \frac{z-7}{3}.$$

Quick Tip

To find the mirror image of a line with respect to another, calculate the foot of the perpendicular and use reflection formulas for the points.

---

33. (a) If $A = \begin{bmatrix} 1 & -2 & 0 \\ 2 & -1 & -1 \\ 0 & -2 & 1 \end{bmatrix}$, find A^{-1} and use it to solve the following system of equations:

$$x - 2y = 10, \quad 2x - y - z = 8, \quad -2y + z = 7.$$

Solution:
Step 1: Represent the system in matrix form. The given system of equations can be written as:

$$A \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 8 \\ 7 \end{bmatrix},$$

where

$$A = \begin{bmatrix} 1 & -2 & 0 \\ 2 & -1 & -1 \\ 0 & -2 & 1 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 10 \\ 8 \\ 7 \end{bmatrix} \text{ is the constant matrix.}$$

Step 2: Find A^{-1}. The inverse of a 3×3 matrix A is given by:

$$A^{-1} = \frac{1}{\det(A)} \cdot \text{adj}(A),$$

where $\det(A)$ is the determinant of A and $\text{adj}(A)$ is the adjugate of A.
(a) Compute $\det(A)$:

$$\det(A) = \begin{vmatrix} 1 & -2 & 0 \\ 2 & -1 & -1 \\ 0 & -2 & 1 \end{vmatrix}.$$

Expanding along the first row:

$$\det(A) = 1 \cdot \begin{vmatrix} -1 & -1 \\ -2 & 1 \end{vmatrix} - (-2) \cdot \begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} + 0 \cdot \begin{vmatrix} 2 & -1 \\ 0 & -2 \end{vmatrix}.$$

Compute the minors:

$$\begin{vmatrix} -1 & -1 \\ -2 & 1 \end{vmatrix} = (-1)(1) - (-1)(-2) = -1 - 2 = -3,$$

$$\begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} = (2)(1) - (-1)(0) = 2 - 0 = 2.$$

Substitute back:

$$\det(A) = 1(-3) - (-2)(2) + 0 = -3 + 4 = 1.$$

(b) Compute $\operatorname{adj}(A)$: The adjugate of A is the transpose of the cofactor matrix. Compute the cofactors for each element of A:

$$\text{Cofactor matrix of } A = \begin{bmatrix} -3 & 2 & 4 \\ 1 & 1 & -2 \\ 4 & 2 & 5 \end{bmatrix}.$$

Thus:

$$\operatorname{adj}(A) = \begin{bmatrix} -3 & 1 & 4 \\ 2 & 1 & 2 \\ 4 & -2 & 5 \end{bmatrix}.$$

(c) Compute A^{-1}:

$$A^{-1} = \frac{1}{\det(A)} \cdot \operatorname{adj}(A) = \operatorname{adj}(A),$$

as $\det(A) = 1$. Thus:

$$A^{-1} = \begin{bmatrix} -3 & 1 & 4 \\ 2 & 1 & 2 \\ 4 & -2 & 5 \end{bmatrix}.$$

Step 3: Solve for $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Using the formula:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1} \cdot \begin{bmatrix} 10 \\ 8 \\ 7 \end{bmatrix},$$

compute the product:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 & 1 & 4 \\ 2 & 1 & 2 \\ 4 & -2 & 5 \end{bmatrix} \cdot \begin{bmatrix} 10 \\ 8 \\ 7 \end{bmatrix}.$$

Perform the multiplication:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3(10) + 1(8) + 4(7) \\ 2(10) + 1(8) + 2(7) \\ 4(10) - 2(8) + 5(7) \end{bmatrix}.$$

Simplify each term:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -30 + 8 + 28 \\ 20 + 8 + 14 \\ 40 - 16 + 35 \end{bmatrix} = \begin{bmatrix} 6 \\ 42 \\ 59 \end{bmatrix}.$$

Final Answer:

$$x = 6, \quad y = 42, \quad z = 59.$$

💡 Quick Tip

To solve a system of linear equations using the inverse matrix method, express the system in the form $A \cdot \vec{x} = \vec{b}$, compute A^{-1}, and use $\vec{x} = A^{-1} \cdot \vec{b}$.

---

33. (b) If $A = \begin{bmatrix} -1 & a & 2 \\ 1 & 2 & x \\ 3 & 1 & 1 \end{bmatrix}$ **and** $A^{-1} = \begin{bmatrix} 1 & -1 & 1 \\ -8 & 7 & -5 \\ b & y & 3 \end{bmatrix}$**, find the value of** $(a+x)-(b+y)$**.**

Solution:
Step 1: Use the property of matrix inverses. The product of a matrix A and its inverse A^{-1} is the identity matrix:

$$A \cdot A^{-1} = I_3,$$

where I_3 is the 3×3 identity matrix.
Step 2: Multiply A and A^{-1}. Compute the product $A \cdot A^{-1}$:

$$\begin{bmatrix} -1 & a & 2 \\ 1 & 2 & x \\ 3 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & 1 \\ -8 & 7 & -5 \\ b & y & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Step 3: Analyze each element of the product. From the first row of the product:

$$[-1(1) + a(-8) + 2(b)] = 1, \quad [-1(-1) + a(7) + 2(y)] = 0, \quad [-1(1) + a(-5) + 2(3)] = 0.$$

Simplify each equation: 1. $-1 - 8a + 2b = 1 \implies -8a + 2b = 2 \implies 4a - b = -1$. 2. $1 + 7a + 2y = 0 \implies 7a + 2y = -1$. 3. $-1 - 5a + 6 = 0 \implies -5a = -5 \implies a = 1$.
From the second row of the product:

$$[1(1) + 2(-8) + x(b)] = 0, \quad [1(-1) + 2(7) + x(y)] = 1, \quad [1(1) + 2(-5) + x(3)] = 0.$$

Simplify each equation: 1. $1 - 16 + xb = 0 \implies xb = 15$. 2. $-1 + 14 + xy = 1 \implies xy = -12$. 3. $1 - 10 + 3x = 0 \implies 3x = 9 \implies x = 3$.
From the third row of the product:

$$[3(1) + 1(-8) + 1(b)] = 0, \quad [3(-1) + 1(7) + 1(y)] = 0, \quad [3(1) + 1(-5) + 1(3)] = 1.$$

Simplify each equation: 1. $3 - 8 + b = 0 \implies b = 5$. 2. $-3 + 7 + y = 0 \implies y = -4$.
Step 4: Compute $(a + x) - (b + y)$. Substitute the values $a = 1$, $x = 3$, $b = 5$, and $y = -4$:

$$(a + x) - (b + y) = (1 + 3) - (5 + (-4)).$$

Simplify:

$$(a + x) - (b + y) = 4 - (5 - 4) = 4 - 1 = 3.$$

Final Answer:

$$(a + x) - (b + y) = 3.$$

Quick Tip

To solve problems involving matrix inverses, use the property $A \cdot A^{-1} = I$, and analyze the resulting equations row by row. Simplify systematically to find unknown elements.

34(a). Find:

$$\int \frac{(3\cos x - 2)\sin x}{5 - \sin^2 x - 4\cos x}\, dx.$$

Solution:

1. **Step 1: Simplify the integrand:**

Let the denominator $D = 5 - \sin^2 x - 4\cos x$. Since $\sin^2 x = 1 - \cos^2 x$, substitute this into D:

$$D = 5 - (1 - \cos^2 x) - 4\cos x = 5 - 1 + \cos^2 x - 4\cos x.$$

$$D = \cos^2 x - 4\cos x + 4.$$

Thus, the integral becomes:

$$\int \frac{(3\cos x - 2)\sin x}{\cos^2 x - 4\cos x + 4}\, dx.$$

2. **Step 2: Substitution:**

Let $u = \cos x$, so that $du = -\sin x\, dx$. Rewrite the integral:

$$\int \frac{(3u-2)(-\sin x)}{u^2 - 4u + 4}\, dx = -\int \frac{3u - 2}{u^2 - 4u + 4}\, du.$$

The denominator can be factored as:

$$u^2 - 4u + 4 = (u-2)^2.$$

The integral becomes:

$$-\int \frac{3u-2}{(u-2)^2}\, du.$$

3. **Step 3: Simplify the fraction:**

Split the numerator $3u - 2$ as:

$$3u - 2 = 3(u-2) + 4.$$

Thus, the integral becomes:

$$-\int \frac{3(u-2)}{(u-2)^2}\, du - \int \frac{4}{(u-2)^2}\, du.$$

Simplify each term: - For the first term:

$$-\int \frac{3(u-2)}{(u-2)^2}\, du = -\int \frac{3}{u-2}\, du = -3\ln|u-2|.$$

- For the second term:

$$-\int \frac{4}{(u-2)^2}\, du = -\int 4(u-2)^{-2}\, du.$$

Using the formula $\int x^n\,dx = \frac{x^{n+1}}{n+1}$, for $n = -2$:

$$-\int 4(u-2)^{-2}\,du = \frac{4}{u-2}.$$

4. **Step 4: Combine the results:**
Substitute back $u = \cos x$:

$$\int \frac{(3\cos x - 2)\sin x}{5 - \sin^2 x - 4\cos x}\,dx = -3\ln|\cos x - 2| + \frac{4}{\cos x - 2} + C,$$

where C is the constant of integration.
Final Answer:

$$\int \frac{(3\cos x - 2)\sin x}{5 - \sin^2 x - 4\cos x}\,dx = -3\ln|\cos x - 2| + \frac{4}{\cos x - 2} + C.$$

—

34(b). Evaluate:

$$\int_{-2}^{2} \frac{x^3 + |x| + 1}{x^2 + 4|x| + 4}\,dx.$$

Solution:
1. **Step 1: Analyze the denominator and absolute value:**
The denominator $x^2+4|x|+4$ depends on the value of x: - For $x \geq 0$, $|x| = x$, so $x^2+4|x|+4 = x^2 + 4x + 4 = (x+2)^2$. - For $x < 0$, $|x| = -x$, so $x^2 + 4|x| + 4 = x^2 - 4x + 4 = (x-2)^2$.
2. **Step 2: Split the integral into two regions:**
Split the integral as:

$$\int_{-2}^{2} \frac{x^3 + |x| + 1}{x^2 + 4|x| + 4}\,dx = \int_{-2}^{0} \frac{x^3 - x + 1}{(x-2)^2}\,dx + \int_{0}^{2} \frac{x^3 + x + 1}{(x+2)^2}\,dx.$$

3. **Step 3: Simplify and evaluate each integral:**
For simplicity, observe the symmetry of the function. Notice that the numerator $x^3 + |x| + 1$ is neither odd nor even, so we need to calculate both parts explicitly. Evaluate each integral using substitution and standard methods.

---

35. Using integration, find the area of the ellipse:

$$\frac{x^2}{16} + \frac{y^2}{4} = 1,$$

included between the lines $x = -2$ and $x = 2$.

Solution:
The equation of the ellipse is:

$$\frac{x^2}{16} + \frac{y^2}{4} = 1.$$

Rearrange to solve for y^2:

$$\frac{y^2}{4} = 1 - \frac{x^2}{16}.$$

$$y^2 = 4\left(1 - \frac{x^2}{16}\right) = 4 - \frac{x^2}{4}.$$

$$y = \pm\sqrt{4 - \frac{x^2}{4}}.$$

Step 1: Use symmetry to simplify the calculation. The ellipse is symmetric about the x-axis. The area between $x = -2$ and $x = 2$ can be calculated as twice the area above the x-axis:

$$\text{Area} = 2\int_{-2}^{2} \sqrt{4 - \frac{x^2}{4}}\, dx.$$

Step 2: Change the limits and integrate. Since the integrand is even (symmetric about the y-axis), we can further simplify:

$$\text{Area} = 4\int_{0}^{2} \sqrt{4 - \frac{x^2}{4}}\, dx.$$

Step 3: Substitution for simplification. Let:

$$u = 4 - \frac{x^2}{4}, \quad \text{so} \quad du = -\frac{x}{2}\, dx \quad \text{and} \quad x\, dx = -2\, du.$$

When $x = 0$, $u = 4$, and when $x = 2$, $u = 4 - \frac{2^2}{4} = 3$.
The integral becomes:

$$\int_{0}^{2} \sqrt{4 - \frac{x^2}{4}}\, dx = \int_{4}^{3} \sqrt{u} \cdot (-2)\, du.$$

Simplify:

$$\int_{0}^{2} \sqrt{4 - \frac{x^2}{4}}\, dx = 2\int_{3}^{4} \sqrt{u}\, du.$$

Step 4: Evaluate the integral. The integral of $\sqrt{u}$ is:

$$\int \sqrt{u}\, du = \frac{2}{3}u^{3/2}.$$

Evaluate from $u = 3$ to $u = 4$:

$$\int_{3}^{4} \sqrt{u}\, du = \frac{2}{3}\left[4^{3/2} - 3^{3/2}\right].$$

Simplify:

$$4^{3/2} = (2^2)^{3/2} = 2^3 = 8, \quad 3^{3/2} = \sqrt{3^3} = \sqrt{27}.$$

Thus:

$$\int_{3}^{4} \sqrt{u}\, du = \frac{2}{3}\left[8 - \sqrt{27}\right].$$

Step 5: Final area. Substitute back into the expression for the area:

$$\text{Area} = 4 \cdot 2 \cdot \frac{2}{3}\left[8 - \sqrt{27}\right] = \frac{16}{3}\left[8 - \sqrt{27}\right].$$

Final Answer:

$$\text{Area} = \frac{16}{3}\left[8 - \sqrt{27}\right].$$

Quick Tip

To find the area of a region using integration, utilize symmetry to simplify calculations and make substitutions to handle square root expressions effectively.

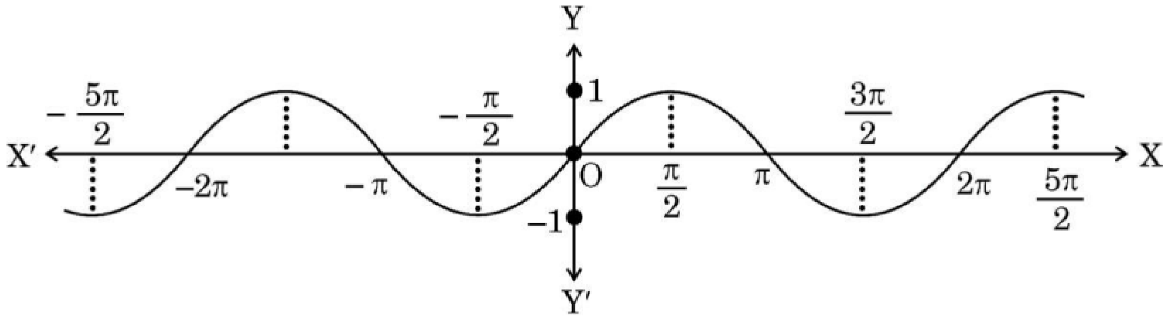
SECTION E

This section comprises 3 case study based questions of 4 marks each.

Case Study - 1

36. If a function $f : X \to Y$ defined as $f(x) = y$ is one-one and onto, then we can define a unique function $g : Y \to X$ such that $g(y) = x$, where $x \in X$ and $y = f(x)$, $y \in Y$. Function g is called the inverse of function f.

The domain of sine function is R and function sine : $R \to R$ is neither one-one nor onto. The following graph shows the sine function.



y = sin x

Let sine function be defined from set A to $[-1, 1]$ such that inverse of sine function exists, i.e., $\sin^{-1} x$ is defined from $[-1, 1]$ to A.

On the basis of the above information, answer the following questions:

If A is the interval other than principal value branch, give an example of one such interval.

If $\sin^{-1}(x)$ is defined from $[-1, 1]$ to its principal value branch, find the value of $\sin^{-1}\left(-\frac{1}{2}\right) - \sin^{-1}(1)$.

Draw the graph of $\sin^{-1} x$ from $[-1, 1]$ to its principal value branch.

OR Find the domain and range of $f(x) = 2\sin^{-1}(1 - x)$.

Solution:

Given Information: The sine function $y = \sin x$ is defined from R to $[-1, 1]$, but it is neither one-one nor onto over R. By restricting the domain to an interval, we can define the inverse $\sin^{-1} x$, which is a one-one and onto function.

—

(i) If A is an interval other than the principal value branch, give an example of one such interval. The principal value branch of the sine function is the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Another interval where the sine function is one-one and onto $[-1, 1]$ is:

$$A = [\frac{\pi}{2}, \frac{3\pi}{2}].$$

—

(ii) If $\sin^{-1}(x)$ is defined from $[-1, 1]$ to its principal value branch, find the value of:

$$\sin^{-1}\left(-\frac{1}{2}\right) - \sin^{-1}(1).$$

Step 1: Evaluate $\sin^{-1}\left(-\frac{1}{2}\right)$. From the definition of $\sin^{-1}$, $\sin^{-1}\left(-\frac{1}{2}\right)$ is the angle θ in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ such that:

$$\sin\theta = -\frac{1}{2}.$$

Thus:

$$\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}.$$

Step 2: Evaluate $\sin^{-1}(1)$. From the definition of $\sin^{-1}$, $\sin^{-1}(1)$ is the angle θ in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ such that:

$$\sin\theta = 1.$$

Thus:

$$\sin^{-1}(1) = \frac{\pi}{2}.$$

Step 3: Compute the difference.

$$\sin^{-1}\left(-\frac{1}{2}\right) - \sin^{-1}(1) = -\frac{\pi}{6} - \frac{\pi}{2}.$$

Simplify:

$$\sin^{-1}\left(-\frac{1}{2}\right) - \sin^{-1}(1) = -\frac{\pi}{6} - \frac{3\pi}{6} = -\frac{4\pi}{6} = -\frac{2\pi}{3}.$$

—

(iii) (a) Draw the graph of $\sin^{-1} x$ from $[-1, 1]$ to its principal value branch.
Step 1: Description of the graph. The graph of $\sin^{-1} x$ is obtained by reflecting the graph of $y = \sin x$ (restricted to $[-\frac{\pi}{2}, \frac{\pi}{2}]$) across the line $y = x$.

—

(iii) (b) Find the domain and range of $f(x) = 2\sin^{-1}(1 - x)$.
Step 1: Domain of $f(x)$. For $f(x) = 2\sin^{-1}(1 - x)$, the argument of $\sin^{-1}$ must lie within $[-1, 1]$, i.e.:

$$-1 \leq 1 - x \leq 1.$$

Simplify:

$$-1 - 1 \leq -x \leq 1 - 1 \implies -2 \leq -x \leq 0.$$

Multiplying through by -1 (and reversing inequalities):

$$0 \leq x \leq 2.$$

Thus, the domain is:

$$x \in [0, 2].$$

Step 2: Range of $f(x)$. The range of $\sin^{-1}(x)$ is $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Therefore:

$$f(x) = 2\sin^{-1}(1 - x) \implies f(x) \in \left[2 \cdot -\frac{\pi}{2}, 2 \cdot \frac{\pi}{2}\right].$$

Simplify:

$$f(x) \in [-\pi, \pi].$$

—

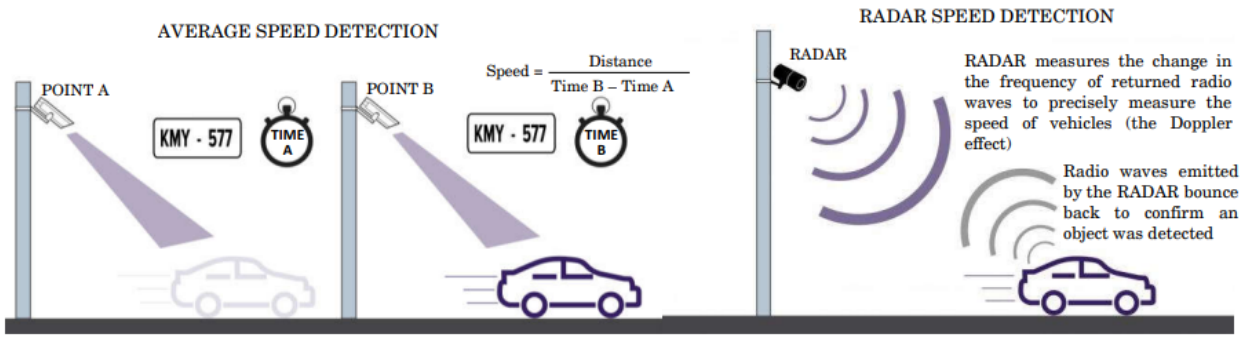
Final Answers: 1. Example of an interval other than the principal value branch: $[\frac{\pi}{2}, \frac{3\pi}{2}]$. 2. $\sin^{-1}\left(-\frac{1}{2}\right) - \sin^{-1}(1) = -\frac{2\pi}{3}$. 3. (a) The graph of $y = \sin^{-1} x$ is shown above. (b) The domain of $f(x) = 2\sin^{-1}(1-x)$ is $[0, 2]$, and the range is $[-\pi, \pi]$.

Quick Tip

To find the domain of functions involving inverse trigonometric functions, ensure the argument satisfies the range of the trigonometric function. For the range, scale the interval accordingly.

Case Study - 2

37. The traffic police has installed Over Speed Violation Detection (OSVD) system at various locations in a city. These cameras can capture a speeding vehicle from a distance of 300 m and even function in the dark.



A camera is installed on a pole at the height of 5 m. It detects a car travelling away from the pole at the speed of 20 m/s. At any point, x m away from the base of the pole, the angle of elevation of the speed camera from the car C is θ.

On the basis of the above information, answer the following questions:

(i)Express θ in terms of the height of the camera installed on the pole and x.

(ii) Find $\frac{d\theta}{dx}$.

(iii) (a) Find the rate of change of angle of elevation with respect to time at an instant when the car is 50 m away from the pole.

(iii) (b) If the rate of change of angle of elevation with respect to time of another car at a distance of 50 m from the base of the pole is $\frac{3}{101}$ rad/s, then find the speed of the car.

Solution:

A camera is installed on a pole at the height of 5 m. It detects a car traveling away from the pole at the speed of 20 m/s. At any point, x m away from the base of the pole, the angle of elevation of the camera to the car C is θ.

(i) Express θ in terms of the height of the camera and x. From the given setup, we can use the right triangle formed by the height of the pole and the distance of the car from the base. The tangent of the angle θ is given by:

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{5}{x}.$$

Thus:

$$\theta = \tan^{-1}\left(\frac{5}{x}\right).$$

(ii) Find $\frac{d\theta}{dx}$. Differentiating $\theta = \tan^{-1}\left(\frac{5}{x}\right)$ with respect to x, we use the chain rule:

$$\frac{d\theta}{dx} = \frac{1}{1+\left(\frac{5}{x}\right)^2} \cdot \frac{d}{dx}\left(\frac{5}{x}\right).$$

Simplify $\frac{5}{x}$:

$$\frac{d}{dx}\left(\frac{5}{x}\right) = -\frac{5}{x^2}.$$

Substitute:

$$\frac{d\theta}{dx} = \frac{1}{1+\frac{25}{x^2}} \cdot \left(-\frac{5}{x^2}\right).$$

Simplify further:

$$\frac{d\theta}{dx} = \frac{-\frac{5}{x^2}}{1+\frac{25}{x^2}} = \frac{-5}{x^2+25}.$$

(iii) (a) Find the rate of change of angle of elevation with respect to time when the car is 50 m away from the pole. Let $x = 50\,\text{m}$ and the car's speed be $\frac{dx}{dt} = 20\,\text{m/s}$. The rate of change of the angle of elevation with respect to time is:

$$\frac{d\theta}{dt} = \frac{d\theta}{dx} \cdot \frac{dx}{dt}.$$

From part (ii), $\frac{d\theta}{dx} = \frac{-5}{x^2+25}$. Substitute $x = 50$:

$$\frac{d\theta}{dx} = \frac{-5}{50^2+25} = \frac{-5}{2500+25} = \frac{-5}{2525} = \frac{-1}{505}.$$

Now:

$$\frac{d\theta}{dt} = \frac{-1}{505} \cdot 20 = \frac{-20}{505} = \frac{-4}{101}\,\text{rad/s}.$$

(iii) (b) If the rate of change of angle of elevation with respect to time for another car at 50 m from the base is $\frac{3}{101}$ rad/s, find the speed of the car. Let the speed of the car be $\frac{dx}{dt} = v$. From part (ii):

$$\frac{d\theta}{dt} = \frac{d\theta}{dx} \cdot \frac{dx}{dt}.$$

Substitute $\frac{d\theta}{dt} = \frac{3}{101}$ and $\frac{d\theta}{dx} = \frac{-1}{505}$:

$$\frac{3}{101} = \frac{-1}{505} \cdot v.$$

Solve for v:

$$v = \frac{3}{101} \cdot 505 = \frac{1515}{101} = 15\,\text{m/s}.$$

Final Answers: 1. $\theta = \tan^{-1}\left(\frac{5}{x}\right)$, 2. $\frac{d\theta}{dx} = \frac{-5}{x^2+25}$, 3. (a) $\frac{d\theta}{dt} = \frac{-4}{101}$ rad/s, (b) Speed of the car is 15 m/s.

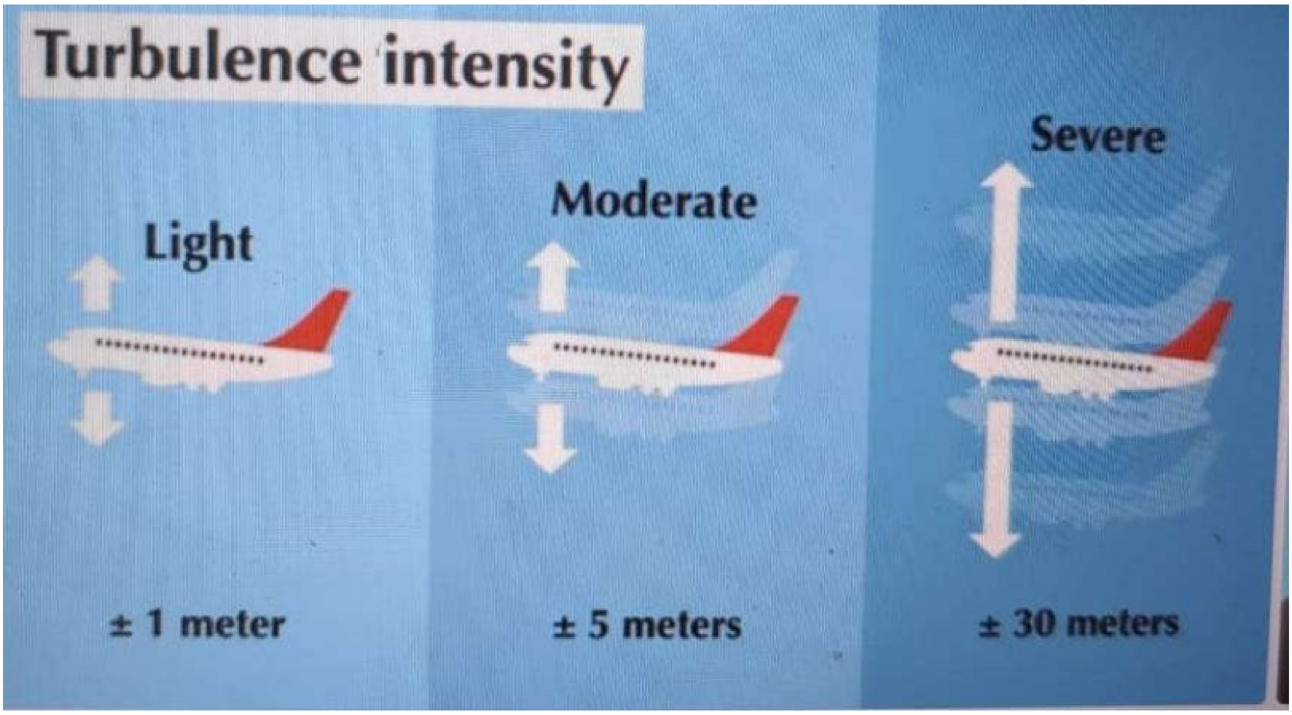
Quick Tip

For related rates problems involving trigonometric functions, differentiate using the chain rule, substitute given values, and simplify systematically.

Case Study - 3

38.According to recent research, air turbulence has increased in various regions around the world due to climate change. Turbulence makes flights bumpy and often delays the flights.

Assume that an airplane observes severe turbulence, moderate turbulence or light turbulence with equal probabilities. Further, the chance of an airplane reaching late to the destination are 55%, 37% and 17% due to severe, moderate and light turbulence respectively.



On the basis of the above information, answer the following questions:

(i) Find the probability that an airplane reached its destination

(ii)If the airplane reached its destination late, find the probability that it was due to moderate turbulence.

Solution:

Given Information: 1. Turbulence can be severe, moderate, or light, each occurring with equal probabilities:

$$P(\text{Severe}) = P(\text{Moderate}) = P(\text{Light}) = \frac{1}{3}.$$

2. The probability of an airplane reaching late due to: - Severe turbulence: $P(\text{Late}|\text{Severe}) = 0.55$, - Moderate turbulence: $P(\text{Late}|\text{Moderate}) = 0.37$, - Light turbulence: $P(\text{Late}|\text{Light}) = 0.17$.

(i) Find the probability that an airplane reached its destination late.
Using the law of total probability:

$$P(\text{Late}) = P(\text{Late}|\text{Severe})P(\text{Severe}) + P(\text{Late}|\text{Moderate})P(\text{Moderate}) + P(\text{Late}|\text{Light})P(\text{Light}).$$

Substitute the values:

$$P(\text{Late}) = (0.55 \cdot \frac{1}{3}) + (0.37 \cdot \frac{1}{3}) + (0.17 \cdot \frac{1}{3}).$$

Simplify:

$$P(\text{Late}) = \frac{0.55 + 0.37 + 0.17}{3} = \frac{1.09}{3}.$$

Thus:

$$P(\text{Late}) = 0.3633 \,(\text{approximately}).$$

(ii) If the airplane reached its destination late, find the probability that it was due to moderate turbulence.
Using Bayes' theorem:

$$P(\text{Moderate}|\text{Late}) = \frac{P(\text{Late}|\text{Moderate})P(\text{Moderate})}{P(\text{Late})}.$$

Substitute the values:

$$P(\text{Moderate}|\text{Late}) = \frac{(0.37 \cdot \frac{1}{3})}{0.3633}.$$

Simplify:

$$P(\text{Moderate}|\text{Late}) = \frac{0.37}{3 \cdot 0.3633} = \frac{0.37}{1.09}.$$

Thus:

$$P(\text{Moderate}|\text{Late}) = 0.3394 \,(\text{approximately}).$$

Final Answers: 1. The probability that an airplane reached its destination late is:

$$P(\text{Late}) = 0.3633.$$

2. The probability that the airplane was late due to moderate turbulence is:

$$P(\text{Moderate}|\text{Late}) = 0.3394.$$

Quick Tip

Use the law of total probability to calculate overall probabilities and Bayes' theorem for conditional probabilities when given dependent conditions.