CAT 2023 QA Slot 3 Question Paper with Solutions

ZOLLEGE

General Instructions

(i) This booklet contains 22 questions, each provided with a complete, step-by-step solution.

(ii) It comprises 21 single-correct multiple-choice questions and 1 numerical / type-in-the-answer question.

(iii) Attempt each question on your own before reviewing the given solution.

(iv) For numerical questions, report the answer rounded exactly as asked.

1. For a real number x , if $\frac{1}{2}$, $\frac{log_3(2^x-9)}{log_3 4}$, and $\frac{log_5\left(2^x+\frac{17}{2}\right)}{log_5 4}$ are in an arithmetic progression, then the common difference is

(A) $log_4\left(\frac{23}{2}\right)$

(B) $log_4\left(\frac{3}{2}\right)$

(C) $log_4 7$

(D) $log_4\left(\frac{7}{2}\right)$

Correct Answer: (D) $log_4\left(\frac{7}{2}\right)$

SOLUTION:

Expressed as a base-4 logarithm using the change of base formula, the term $\frac{\log_3(2^x-9)}{\log_3 4}$ becomes $\log_4(2^x-9)$. Similarly, $\frac{\log_5(2^x+\frac{17}{2})}{\log_5 4}$ simplifies to $\log_4\left(2^x+\frac{17}{2}\right)$.

The given equation is $2\log_4(2^x-9)=\frac{1}{2}+\log_4\left(2^x+\frac{17}{2}\right)$. Since $\frac{1}{2}$ is equivalent to $\log_4 2$, the equation transforms into $2\log_4(2^x-9)=\log_4 2$

$+ \log_4\left(2^x + \frac{17}{2}\right)$.

Applying the logarithmic properties $a \log_b m = \log_b m^a$ and $\log_b m + \log_b n = \log_b(mn)$, the left side becomes $\log_4 (2^x - 9)^2$ and the right side becomes $\log_4\left[2 \cdot \left(2^x + \frac{17}{2}\right)\right]$. Equating these yields $\log_4 (2^x - 9)^2 = \log_4\left[2 \cdot \left(2^x + \frac{17}{2}\right)\right]$. Removing the logarithms gives $(2^x - 9)^2 = 2 \cdot \left(2^x + \frac{17}{2}\right)$.

Expanding both sides: The left side is $(2^x)^2 - 2 \cdot 9 \cdot 2^x + 81 = 2^{2x} - 18 \cdot 2^x + 81$. The right side is $2 \cdot 2^x + 2 \cdot \frac{17}{2} = 2 \cdot 2^x + 17$. Equating these expressions results in $2^{2x} - 18 \cdot 2^x + 81 = 2 \cdot 2^x + 17$. Rearranging the terms to one side produces the quadratic equation $2^{2x} - 20 \cdot 2^x + 64 = 0$.

Let $y = 2^x$. The equation becomes $2y^2 - 20y + 64 = 0$. Factoring the quadratic: $2y^2 - 16y - 4y + 64 = 0$, which simplifies to $2y(y - 8) - 4(y - 8) = 0$. This gives $(y - 4)(2y - 16) = 0$, yielding solutions $y = 4$ or $y = 8$. Substituting back $y = 2^x$, we have $2^x = 4$ or $2^x = 16$, leading to $x = 2$ or $x = 4$.

Verification of solutions:

- For $x = 2$, $2^x - 9 = 2^2 - 9 = 4 - 9 = -5$. The logarithm of a negative number is undefined.
- For $x = 4$, $2^x - 9 = 2^4 - 9 = 16 - 9 = 7$. This value is valid.

Therefore, the only valid solution is $2^x = 16$.

The common difference is calculated as $\log_4(2^x - 9) - \log_4 2 = \log_4\left(\frac{2^x-9}{2}\right)$. Substituting $2^x = 16$, the common difference is $\log_4\left(\frac{16-9}{2}\right) = \log_4\left(\frac{7}{2}\right)$.

Correct option: (D) $\log_4\left(\frac{7}{2}\right)$

2. Let n and m be two positive integers such that there are exactly 41 integers greater than 8^m and less than 8^n, which can be expressed as powers of 2. Then, the smallest possible value of $n + m$ is

(A) 44

(B) 16

(C) 42

(D) 14

Correct Answer: (B) 16

SOLUTION:

Given integers n and m, there are 41 integers that are powers of 2, strictly between 8^m and 8^n.

Numbers expressible as powers of 2 form the sequence $2^1, 2^2, 2^3, \ldots$. Powers of 8 can be expressed as powers of 2 since $8^k = (2^3)^k = 2^{3k}$. Thus, $8^m = 2^{3m}$ and $8^n = 2^{3n}$.

The condition is that 2^a must satisfy $2^{3m} < 2^a < 2^{3n}$, where a is an integer.

This implies that a can range from $3m + 1$ to $3n - 1$. The count of such integers a is $(3n - 1) - (3m + 1) + 1 = 3n - 3m$. This count is given as 41.

Solving $3n - 3m = 41$ yields $n - m = \frac{41}{3} = 13.67$, which is not an integer solution.

Revisiting the count, if the number of integers is 43, then $3n - 3m = 43$, leading to $n - m = \frac{43}{3} = 14.33$. While not a direct integer solution, it suggests an adjustment.

A corrected calculation indicates $3(n - m) = 16$ is not directly derivable. However, a precise enumeration leads to $n = m + 14$.

We aim to find the minimum value of $n + m$. Substituting $n = m + 14$, we get $n + m = (m + 14) + m = 2m + 14$.

If we set $m = 1$, then $n = 7$. In this case, $8^m = 8^1 = 2^3$ and $8^n = 8^7 = 2^{21}$. The powers of 2 strictly between 2^3 and 2^{21} are $2^4, 2^5, \ldots, 2^{20}$. The count is $20 - 4 + 1 = 17$. This is not 41.

The problem states there are 41 integers. Let's assume the relation $3(n - m)$ accurately represents the count of integers. If $3(n - m) = 41$, this does not yield integer values for n and m. The problem implies that the number of integers expressible as powers of 2 between 8^m and 8^n is 41. This count is $3n - 3m - 1$. Thus, $3(n - m) - 1 = 41$, which means $3(n - m) = 42$, and $n - m = 14$.

We want to minimize $n + m = (m + 14) + m = 2m + 14$.

To find the smallest $n + m$, we need the smallest possible integer value for m. Since 8^m and 8^n are defined, m and n must be integers. The smallest integer power of 8 for m is $m = 1$.

If $m = 1$, then $n = 1 + 14 = 15$. The sum $n + m = 15 + 1 = 16$.

Let's verify: $8^m = 8^1 = 2^3$. $8^n = 8^{15} = 2^{45}$. The powers of 2 between 2^3 and 2^{45} are $2^4, 2^5, \ldots, 2^{44}$. The count is $44 - 4 + 1 = 41$. This matches the given information.

The smallest value of $n + m$ is 16.

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3. For some real numbers a and b, the system of equations $x + y = 4$ and $(a + 5)x + (b^2 - 15)y = 8b$ has infinitely many solutions for x and y. Then, the maximum possible value of ab is

(A) 15

(B) 55

(C) 33

(D) 25

Correct Answer: (C) 33

SOLUTION:

Given:

The system of equations has infinitely many solutions:

- $x + y = 4$ (1)
- $(a+5)x + (b^2 - 15)y = 8b$ (2)

Condition for infinitely many solutions:

Two linear equations have infinitely many solutions if their corresponding coefficients are proportional:

$$\frac{a+5}{1} = \frac{b^2-15}{1} = \frac{8b}{4}$$

Step 1: Equate the second and third parts of the proportion:

$$\frac{b^2-15}{1} = \frac{8b}{4}$$

Simplify the equation:

$$b^2 - 15 = 2b$$

Rearrange into a quadratic equation:

$$b^2 - 2b - 15 = 0$$

Step 2: Solve the quadratic equation for b:

$$b = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-15)}}{2(1)} = \frac{2 \pm \sqrt{4 + 60}}{2} = \frac{2 \pm \sqrt{64}}{2} = \frac{2 \pm 8}{2}$$

This yields two possible values for b:

$$b = \frac{2 + 8}{2} = 5$$

or

$$b = \frac{2 - 8}{2} = -3$$

Step 3: Use the proportion to find a. Equate the first and third parts:

$$\frac{a + 5}{1} = \frac{8b}{4}$$

Simplify and solve for a:

$$a + 5 = 2b \Rightarrow a = 2b - 5$$

Step 4: Calculate the corresponding values of a and the product ab for each value of b:

- When $b = 5$:

  $$a = 2(5) - 5 = 10 - 5 = 5$$

  Thus, $ab = 5 \cdot 5 = 25$.
- When $b = -3$:

  $$a = 2(-3) - 5 = -6 - 5 = -11$$

  Thus, $ab = (-11) \cdot (-3) = 33$.

Final Answer:

The maximum value of ab is **33**.

Correct Option: (C) 33

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4. If x is a positive real number such that $x^8 + \left(\frac{1}{x}\right)^8 = 47$, then the value of $x^9 + \left(\frac{1}{x}\right)^9$ is

(A) $34\sqrt{5}$

(B) $40\sqrt{5}$

(C) $36\sqrt{5}$

(D) $30\sqrt{5}$

Correct Answer: (A) $34\sqrt{5}$

SOLUTION:

Given: $x^8 + \left(\frac{1}{x}\right)^8 = 47$

This can be expressed as:

$\Rightarrow (x^4)^2 + \left(\frac{1}{x^4}\right)^2 = 47$
$\Rightarrow \left(x^4 + \frac{1}{x^4}\right)^2 - 2 = 47$
$\Rightarrow \left(x^4 + \frac{1}{x^4}\right)^2 = 49$
$\Rightarrow x^4 + \frac{1}{x^4} = 7 \quad \ldots\ldots \text{(i)}$

From equation (i):

$\Rightarrow (x^2)^2 + \left(\frac{1}{x^2}\right)^2 = 7$
$\Rightarrow \left(x^2 + \frac{1}{x^2}\right)^2 - 2 = 7$
$\Rightarrow \left(x^2 + \frac{1}{x^2}\right)^2 = 9$
$\Rightarrow x^2 + \frac{1}{x^2} = 3 \quad \ldots\ldots \text{(ii)}$

Assume: $x + \frac{1}{x} = \sqrt{5}$

Then:

$$x^3 + \frac{1}{x^3} = \left(x + \frac{1}{x}\right)^3 - 3\left(x + \frac{1}{x}\right)$$
$$= (\sqrt{5})^3 - 3\sqrt{5} = 5\sqrt{5} - 3\sqrt{5} = 2\sqrt{5} \quad \ldots\ldots \text{(iii)}$$

Calculate: $x^9 + \frac{1}{x^9}$

$$x^9 + \frac{1}{x^9} = \left(x^3 + \frac{1}{x^3}\right)^3 - 3\left(x^3 + \frac{1}{x^3}\right)$$
$$= (2\sqrt{5})^3 - 3(2\sqrt{5})$$
$$= 8 \cdot 5\sqrt{5} - 6\sqrt{5}$$
$$= 40\sqrt{5} - 6\sqrt{5}$$
$$= 34\sqrt{5}$$

The final result is: **(A)** $34\sqrt{5}$

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5. A quadratic equation $x^2 + bx + c = 0$ has two real roots. If the difference between the reciprocals of the roots is $\frac{1}{3}$, and the sum of the reciprocals of the squares of the roots is $\frac{5}{9}$, then the largest possible value of $(b + c)$ is

(A) 7

(B) 8

(C) 9

(D) None of Above

Correct Answer: (C) 9

SOLUTION:

Let α and β be the roots of the quadratic equation $x^2 + bx + c = 0$.

Given:

- $\frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{5}{9}$

- $\left(\frac{1}{\alpha} - \frac{1}{\beta}\right)^2 = \frac{1}{9}$

Step 1: Determine $\alpha\beta$

Using the identity $\left(\frac{1}{\alpha} - \frac{1}{\beta}\right)^2 = \frac{1}{\alpha^2} + \frac{1}{\beta^2} - \frac{2}{\alpha\beta}$, substitute the given values: $\frac{1}{9} = \frac{5}{9} - \frac{2}{\alpha\beta}$. Rearranging yields $\frac{2}{\alpha\beta} = \frac{4}{9}$, which simplifies to $\alpha\beta = \frac{9}{2}$.

Step 2: Determine $\alpha^2 + \beta^2$

Using the identity $\frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{\alpha^2+\beta^2}{\alpha^2\beta^2}$, and substituting the given value and the derived $\alpha\beta$: $\frac{5}{9} = \frac{\alpha^2+\beta^2}{(\frac{9}{2})^2}$. This gives $\alpha^2 + \beta^2 = \frac{5}{9} \cdot \frac{81}{4} = \frac{405}{36} = \frac{45}{4}$.

Step 3: Find $(\alpha + \beta)^2$

Using the identity $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$, substitute known values: $\frac{45}{4} = (\alpha + \beta)^2 - 2 \cdot \frac{9}{2}$. This simplifies to $\frac{45}{4} = (\alpha + \beta)^2 - 9$, so $(\alpha + \beta)^2 = \frac{45}{4} + 9 = \frac{81}{4}$. Therefore, $\alpha + \beta = \pm\frac{9}{2}$.

Step 4: Relate to coefficients b and c

From Vieta's formulas, $\alpha + \beta = -b$ and $\alpha\beta = c$. Thus, $b = -(\alpha + \beta) = \mp\frac{9}{2}$ and $c = \frac{9}{2}$.

Step 5: Calculate the maximum value of $b + c$

The expression to maximize is $b + c = -(\alpha + \beta) + \alpha\beta$. To maximize this, we select the value of $\alpha + \beta$ that yields the largest sum. If $\alpha + \beta = -\frac{9}{2}$, then $b = \frac{9}{2}$. In this case, $b + c = \frac{9}{2} + \frac{9}{2} = \boxed{9}$.

Final Answer: Option (C): 9

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6. Let n be any natural number such that $5^{n-1} < 3^{n+1}$. Then, the least integer value of m that satisfies $3^{n+1} < 2^{n+m}$ for each such n , is

(A) 5

(B) 6

(C) 7

(D) None of Above

Correct Answer: (A) 5

SOLUTION:

To address this issue, we first analyze the two provided inequalities: $5^{n-1} < 3^{n+1}$ and $3^{n+1} < 2^{n+m}$. The objective is to determine the smallest integer value for m that satisfies the second inequality for all values of n that satisfy the first inequality.

Step 1: Simplify the first inequality

$5^{n-1} < 3^{n+1}$

Rewriting this yields: $\left(\frac{5}{3}\right)^{n-1} < 9$. This is equivalent to $\frac{5}{3}^{n-1} < 9$. Given that $\frac{5}{3} < 2$, this inequality is valid for all natural numbers n, thus imposing no restriction on n.

Step 2: Analyze the second inequality

$3^{n+1} < 2^{n+m}$

Rewriting this expression gives: $3 \cdot 3^n < 2^n \cdot 2^m$. Dividing both sides by 3^n results in: $3 < \left(\frac{2}{3}\right)^n \cdot 2^m$.

Step 3: Determine the smallest m

As n increases, the term $\left(\frac{2}{3}\right)^n$ approaches 0. Consequently, 2^m must be sufficiently large to ensure that the inequality $3 < \left(\frac{2}{3}\right)^n \cdot 2^m$ remains true for all values of n.

Step 4: Estimate suitable m

We will now test values to find the minimum m that satisfies the inequality:

- Test $m = 3$: $\left(\frac{2}{3}\right)^n \cdot 8$. This value decreases below 3 as n increases.
- Test $m = 4$: $\left(\frac{2}{3}\right)^n \cdot 16$. This also fails for larger values of n.
- Test $m = 5$: $\left(\frac{2}{3}\right)^n \cdot 32$. This value exceeds 3 for all n.

Final Answer: The minimum integer value for m is **5**.

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7. The sum of the first two natural numbers, each having 15 factors (including 1 and the number itself), is

(A) 348

(B) 412

(C) 468

(D) None of Above

Correct Answer: (C) 468

SOLUTION:

Determining Numbers with 15 Factors

Given that a number N has 15 factors.

The factor count of 15 can be expressed as $15 = 1 \times 15 = 3 \times 5$.

Therefore, the possible exponent combinations for the prime factorization of N are:

- $(p+1)(q+1) = 3 \times 5 \Rightarrow p = 2, q = 4$
- This implies N can be represented as $N = a^2 \cdot b^4$ or $N = a^4 \cdot b^2$, where a and b are distinct prime numbers.

Scenario 1: $N = 2^4 \times 3^2$

$2^4 = 16,\ 3^2 = 9$

$\Rightarrow N = 16 \times 9 = 144$

Scenario 2: $N = 2^2 \times 3^4$

$2^2 = 4,\ 3^4 = 81$

$\Rightarrow N = 4 \times 81 = 324$

Sum of the Two Smallest Numbers Found:

$144 + 324 = 468$

Correct Answer: (C): $\boxed{468}$

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8. A merchant purchases a cloth at a rate of Rs. 100 per meter and receives 5 cm length of cloth free for every 100 cm length of cloth purchased by him. He sells the same cloth at a rate of Rs.110 per meter but cheats his customers by giving 95 cm length of cloth for every 100 cm length of cloth purchased by the customers. If the merchant provides a 5% discount, the resulting profit earned by him is

(A) 9.7%

(B) 16%

(C) 4.2%

(D) 15.5%

Correct Answer: (D) 15.5%

SOLUTION:

To determine the merchant's profit percentage, perform the following calculations:

Step 1: Determine the Effective Purchase Cost

For every 100 cm of cloth purchased, an additional 5 cm is received free, totaling 105 cm. The effective cost per cm is calculated as Rs. 100 / 105 = Rs. 0.9524/cm.

Step 2: Calculate the Selling Price per cm

The merchant sells cloth at Rs. 110 per meter (100 cm), resulting in a selling price of Rs. 110 / 100 = Rs. 1.1/cm. However, customers receive only 95 cm for every 100 cm paid for. Therefore, the effective selling price per cm is Rs. 1.1 * (100/95) = Rs. 1.1579/cm.

Step 3: Apply the Discount

A 5% discount is applied to the selling price. The discounted selling price per cm is Rs. 1.1579 * 0.95 = Rs. 1.100005/cm.

Step 4: Calculate the Profit Percentage

The profit per cm is the difference between the discounted selling price and the effective cost price: Rs. 1.100005 - Rs. 0.9524 = Rs. 0.147605.

The profit percentage is calculated as (Profit per cm / Cost price per cm) * 100 = (0.147605 / 0.9524) * 100, which is approximately 15.5%.

Consequently, the merchant's profit percentage is **15.5%.**

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9. A boat takes 2 hours to travel downstream a river from port A to port B, and 3 hours to return to port A. Another boat takes a total of 6 hours to travel from port B to port A and return to port B . If the speeds of the boats and the river are constant, then the time, in hours, taken by the slower boat to travel from port A to port B is

(A) $3(\sqrt{5}-1)$

(B) $3(3+\sqrt{5})$

(C) $3(3-\sqrt{5})$

(D) $12(\sqrt{5}-2)$

Correct Answer: (C) $3(3-\sqrt{5})$

SOLUTION:

Let b represent the speed of the **first boat**, s represent the speed of the **second boat**, and r represent the speed of the **river**. Let d be the distance between points A and B.

According to the problem statement:
$\Rightarrow d = 2(b+r)$ and $d = 3(b-r)$

Solving these equations simultaneously yields:
$\Rightarrow b+r = \frac{d}{2}$ and $b-r = \frac{d}{3}$

Subtracting the second equation from the first:
$\Rightarrow (b+r)-(b-r) = \frac{d}{2} - \frac{d}{3}$
$\Rightarrow 2r = \frac{3d-2d}{6} = \frac{d}{6} \Rightarrow r = \frac{d}{12}$

Considering the time taken by the second boat:
$\frac{d}{s+r} + \frac{d}{s-r} = 6$

Substituting the value of $r = \frac{d}{12}$:

$$\Rightarrow \frac{d}{s+\frac{d}{12}} + \frac{d}{s-\frac{d}{12}} = 6$$

To simplify, multiply the numerator and denominator by 12:

Alternatively, multiply the entire equation by the least common multiple (LCM) for simplification:

Adjusting the numerator and denominator:

$$\Rightarrow \frac{d(12)}{12s+d} + \frac{d(12)}{12s-d} = 6$$

Multiplying both sides by $(12s + d)(12s - d)$:

$$\Rightarrow 12d(12s - d) + 12d(12s + d) = 6(144s^2 - d^2)$$

Simplification leads to:

$$144ds - 12d^2 + 144ds + 12d^2 = 6(144s^2 - d^2)$$

$$\Rightarrow 288ds = 864s^2 - 6d^2$$

Rearranging the terms to one side:

$$\Rightarrow 144s^2 - 48ds - d^2 = 0$$

Solving this quadratic equation for s using the quadratic formula:

$$s = \frac{48d+\sqrt{(48d)^2+4(144)(d^2)}}{2\cdot144}$$

$$= d\left(\frac{48+\sqrt{48^2+4\cdot144}}{2\cdot144}\right)$$

Further simplification yields:

$$s = d\left(\frac{1}{6} + \frac{\sqrt{5}}{12}\right)$$

Now, calculate the required value:

$$\frac{d}{s+r} = \frac{d}{\frac{d}{6}+\frac{d\sqrt{5}}{12}+\frac{d}{12}}$$

$$\Rightarrow \frac{1}{\frac{1}{6}+\frac{\sqrt{5}}{12}+\frac{1}{12}} = \frac{1}{\frac{3+\sqrt{5}}{12}} = \frac{12}{3+\sqrt{5}}$$

Rationalizing the denominator:

$$\Rightarrow \frac{12}{3+\sqrt{5}} \cdot \frac{3-\sqrt{5}}{3-\sqrt{5}} = \frac{12(3-\sqrt{5})}{9-5} = \frac{12(3-\sqrt{5})}{4}$$

$$\Rightarrow 3(3-\sqrt{5})$$

Thus, the correct option is (C): $\boxed{3(3-\sqrt{5})}$

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10. There are three persons A B, and C in a room. If a person D joins the room, the average weight of the persons in the room reduces by x kg . Instead of D , if person E joins the room, the average weight of the persons in the room increases by 2 kg x . If the weight of E is 12 kg more than that of D , then the value of x is

(A) 1.5

(B) 0.5

(C) 1

(D) 2

Correct Answer: (C) 1

SOLUTION:

Let S represent the total weight of persons A, B, and C, and $\frac{S}{3}$ their average weight.

When person D joins, the new average is $\frac{S+w_D}{4}$, which is x kg less than the original average:

$\frac{S}{3} - x = \frac{S+w_D}{4}$

Multiplying by 12 gives:

$4S - 12x = 3S + 3w_D$

Simplifying yields:

$S = 12x + 3w_D$

If person E joins instead, the average increases by $2x$ kg:

$\frac{S}{3} + 2x = \frac{S+w_E}{4}$

Multiplying by 12 gives:

$4S + 24x = 3S + 3w_E$

Simplifying yields:

$S = 24x + 3w_E$

Equating the two expressions for S:

$12x + 3w_D = 24x + 3w_E$

Dividing by 3 gives:

$4x + w_D = 8x + w_E$

Given $w_E = w_D + 12$, substitute into the equation:

$4x + w_D = 8x + w_D + 12$

Simplifying to solve for x:

$-4x = 12$

Therefore, $x = 1$.

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11. In 2021, the population was 100000. In 2022 it is decreased by x%. In 2023, it will increase by y%. The population in 2023 is more than that of 2021 and difference between x and y is 10. Minimum population in 2021 was?

Correct Answer: —

SOLUTION:

Given:
The town's population in 2020 was 100,000.
The population decreased by y% from 2020 to 2021 and increased by

x% from 2021 to 2022, where x and y are natural numbers.

Population in 2021: $100000\left(\frac{100-y}{100}\right)$

Population in 2022: $100000\left(\frac{100-y}{100}\right)\left(\frac{100+x}{100}\right)$

Additional Information:

The population in 2022 was greater than in 2020, and the difference between x and y is 10.

Therefore:

$100000\left(\frac{100-y}{100}\right)\left(\frac{100+x}{100}\right) > 100000$, and $x - y = 10$

Simplifying the inequality with $x = y + 10$:

$100000\left(\frac{100-y}{100}\right)\left(\frac{100+(y+10)}{100}\right) > 100000$

$\left(\frac{100-y}{100}\right)\left(\frac{110+y}{100}\right) > 1$

To maximize the population in 2021, we need to maximize y.

The inequality $(100 - y)(110 + y) > 10000$ leads to:

$11000 + 100y - 110y - y^2 > 10000$

$11000 - 10y - y^2 > 10000$

$1000 > y^2 + 10y$

$y^2 + 10y < 1000$

Completing the square:

$(y + 5)^2 - 25 < 1000$

$(y + 5)^2 < 1025$

Since $32^2 = 1024$, the largest integer value for $y + 5$ is 32.

$y + 5 = 32$

$y = 27$

Population in 2021:

$100000 \times \left(\frac{100-27}{100}\right) = 100000 \times \frac{73}{100} = 73000$

The population in 2021 was 73,000.

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12. Anil mixes cocoa with sugar in the ratio 3: 2 to prepare mixture A , and coffee with sugar in the ratio 7: 3 to prepare mixture B . He combines mixtures A and B in the ratio 2: 3 to make a new mixture C . If he mixes C with an equal amount of milk to make a drink, then the percentage of sugar in this drink will be

(A) 16

(B) 24

(C) 17

(D) 21

Correct Answer: (C) 17

SOLUTION:

Mixture A has a volume of $200\,\mathrm{ml}$. This mixture contains $120\,\mathrm{ml}$ of cocoa (60%) and $80\,\mathrm{ml}$ of sugar (40%).

Mixture B has a volume of $300\,\mathrm{ml}$. This mixture contains $210\,\mathrm{ml}$ of coffee (70%) and $90\,\mathrm{ml}$ of sugar (30%).

When mixtures A and B are combined in a $2:3$ ratio, 200 ml of A and 300 ml of B are used. The resulting mixture C has a total volume of $500\,\mathrm{ml}$ $(200+300)$. The total amount of sugar in mixture C is $170\,\mathrm{ml}$ $(80+90)$.

Mixture C is then combined with an equal volume of milk. The final volume is $1000\,\mathrm{ml}$ $(500+500)$, and the sugar content remains $170\,\mathrm{ml}$.

The percentage of sugar in the final mixture is calculated as follows:

$$\frac{170}{1000}\times 100=17\%$$

Correct Option: (C) 17%

13. Rahul, Rakshita and Gurmeet, working together, would have taken more than 7 days to finish a job. On the other hand, Rahul and Gurmeet, working together would have taken less than 15 days to finish the job. However, they all worked together for 6 days, followed by Rakshita, who worked alone for 3 more days to finish the job. If Rakshita had worked alone on the job then the number of days she would have taken to finish the job, cannot be

(A) 16

(B) 21

(C) 17

(D) 20

Correct Answer: (B) 21

SOLUTION:

Let the work rates of Rahul, Rakshita, and Gurmeet be a, b, and c units of work per day, respectively.

Let the total work be W.

According to the problem statement:

- The three individuals worked collaboratively for 6 days.
- Rakshita independently completed the remaining work in an additional 3 days.

The total work done can be expressed as:

$W = 6(a + b + c) + 3b$ --- (1)

Additional information provided:

- The work could not be completed by all three individuals together in 7 days:
  $7(a+b+c) < W$ --- (2)
- Rahul and Gurmeet working together would have completed the work in less than 15 days:
  $15(a+c) > W$ --- (3)

Combining inequalities (2) and (3) yields:
$15(a+c) < W < 7(a+b+c)$ --- (4)

Substitute equation (1) into inequality (4):
$15(a+c) < 6(a+b+c) + 3b < 7(a+b+c)$

Simplify the middle expression:
$6(a+b+c) + 3b = 6a + 6b + 6c + 3b = 6a + 9b + 6c$

Now, analyze the two resulting inequalities:
$15(a+c) < 6a + 9b + 6c$ --- (5)
$6a + 9b + 6c < 7(a+b+c)$ --- (6)

Expand the right side of inequality (6):
$7(a+b+c) = 7a + 7b + 7c$

Rewrite inequality (6) with the expanded term:
$6a + 9b + 6c < 7a + 7b + 7c$
Subtracting terms from both sides results in:
$-a + 2b - c < 0 \implies a + c > 2b$ --- (7)

Simplify inequality (5):

$15(a+c) < 6a + 9b + 6c$

$\Rightarrow 15a + 15c < 6a + 9b + 6c$

$\Rightarrow 9a + 9c < 9b$

$\Rightarrow a + c < b$ --- (8)

The derived inequalities are:

- From (7): $a + c > 2b$
- From (8): $a + c < b$

Inequality (8) contradicts inequality (7), indicating a calculation error occurred previously.
Re-evaluating inequality (5) accurately:

$15(a+c) < 6a + 9b + 6c$

$\Rightarrow 15a + 15c < 6a + 9b + 6c$

$\Rightarrow 9a + 9c < 9b$

$\Rightarrow a + c < b$ This result still contradicts (7), suggesting an earlier sign error.

Let's reconcile the findings from the inequalities. Using the relationships derived from the problem constraints:
From $7(a+b+c) < W$ and $W < 15(a+c)$, we infer:

- $7(a+b+c) < 15(a+c)$

From $W = 6(a+b+c) + 3b$, and $W < 7(a+b+c)$: $6(a+b+c) + 3b < 7(a+b+c)$ $6a + 6b + 6c + 3b < 7a + 7b + 7c$ $6a + 9b + 6c < 7a + 7b + 7c$ $2b < a + c$ --- (7') (Corrected inequality (7))

From $W = 6(a+b+c) + 3b$, and $W > 15(a+c)$: $6(a+b+c) + 3b > 15(a+c)$ $6a + 6b + 6c + 3b > 15a + 15c$ $6a + 9b + 6c > 15a + 15c$ $9b$

$> 9a + 9c$ $b > a + c$ --- (8') (Corrected inequality (8))

Combining the corrected inequalities (7') and (8'):
We have $b > a + c$ and $a + c > 2b$. This implies $b > 2b$, which is only possible if b is negative, which is not feasible for a work rate.

Let's re-examine the inequalities from the prompt statement:
$7(a + b + c) < W$ and $15(a + c) > W$. This implies $7(a + b + c) < 15(a + c)$. $7a + 7b + 7c < 15a + 15c$ $7b < 8a + 8c$.

Using $W = 6(a + b + c) + 3b$: From $W < 7(a + b + c)$: $6(a + b + c) + 3b < 7(a + b + c)$ $6a + 9b + 6c < 7a + 7b + 7c$ $2b < a + c$. From $W > 15(a + c)$: $6(a + b + c) + 3b > 15(a + c)$ $6a + 9b + 6c > 15a + 15c$ $9b > 9a + 9c$ $b > a + c$.

The derived inequalities are $2b < a + c$ and $b > a + c$. These inequalities are contradictory. Let's reconsider the initial problem statement implications.

If $7(a + b + c) < W$, then $W/7 > a + b + c$. If $15(a + c) > W$, then $W/15 < a + c$. If Rakshita worked alone for 3 days to finish, $3b = W - 6(a + b + c)$. So $b = (W - 6(a + b + c))/3 = W/3 - 2(a + b + c)$.

Substitute $a + c$ from $W/15 < a + c$: $b > W/15$. Substitute $a + b + c$ from $W/7 > a + b + c$: $b = W/3 - 2(a + b + c) > W/3 - 2(W/7) = W/3 - 2W/7 = (7W - 6W)/21 = W/21$.

So, $b > W/15$ and $b > W/21$. The tighter lower bound is $b > W/15$.

Now consider the upper bound for b. From $7(a + b + c) < W$, we have $a + b + c < W/7$. From $W = 6(a + b + c) + 3b$, we have $3b = W - 6(a + b + c)$. $3b > W - 6(W/7) = W - 6W/7 = W/7$. $b > W/21$. This confirms the lower bound.

From $15(a + c) > W$, we have $a + c > W/15$. $W = 6a + 6b + 6c + 3b = 6(a + c) + 6b + 3b = 6(a + c) + 9b$. Since $a + c > W/15$, $W = 6(a + c)$

$+ 9b > 6(W/15) + 9b = 2W/5 + 9b$. $W - 2W/5 > 9b$ $3W/5 > 9b$ $b < 3W/45 = W/15$.

So we have $b > W/21$ and $b < W/15$. Therefore, b lies in the range $\frac{W}{21} < b < \frac{W}{15}$. This means the number of days for Rakshita is between 15 and 21.

The correct answer is (B): 15 to 21 days.

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14. The number of coins collected per week by two coin-collectors A and B are in the ratio 3 : 4. If the total number of coins collected by A in 5 weeks is a multiple of 7, and the total number of coins collected by B in 3 weeks is a multiple of 24, then the minimum possible number of coins collected by A in one week is

(A) 20

(B) 42

(C) 66

(D) None of Above

Correct Answer: (B) 42

SOLUTION:

Let the weekly coin collection for A be $3x$ and for B be $4x$.

Given:

- A's coins in 5 weeks $= 5 \times 3x = 15x$. This quantity must be divisible by 7.

- B's coins in 3 weeks $= 3 \times 4x = 12x$. This quantity must be divisible by 24.

For $15x$ to be divisible by 7, x must be a multiple of 7. Let $x = 7k$, where k is a positive integer.

Substitute $x = 7k$ into the second condition:
$12x = 12 \times 7k = 84k$
For $84k$ to be a multiple of 24, k must be selected to satisfy this.

Testing values for k:
If $k = 1$, $84k = 84$. 84 is divisible by 12 but **not** by 24.
If $k = 2$, $84k = 168$. 168 **is** divisible by 24.

The smallest valid value for k is 2, thus $x = 7 \times 2 = 14$.

Therefore, the coins collected by A in one week are $3x = 3 \times 14 =$ **42**.

• • •

15. Gautam and Suhani, working together, can finish a job in 20 days. If Gautam does only 60% of his usual work on a day, Suhani must do 150% of her usual work on that day to exactly make up for it. Then, the number of days required by the faster worker to complete the job working alone is

(A) 30

(B) 36

(C) 70

(D) None of Above

Correct Answer: (B) 36

SOLUTION:

Let **W** represent the total work. Let **g** and **s** be the work efficiencies of Gautam and Suhani, respectively.

According to the problem statement:

$\Rightarrow\ g + s = \frac{W}{20}$ (combined work per day) ……. (i)

Gautam works at 60% of his efficiency ($\frac{3g}{5}$), and Suhani works at 150% of her efficiency ($\frac{3s}{2}$).

Using these adjusted efficiencies, the combined work per day is:

$\Rightarrow\ \frac{3g}{5} + \frac{3s}{2} = \frac{W}{20}$ (combined work per day)

Equating the two expressions for combined work per day:

$\Rightarrow\ g + s = \frac{3g}{5} + \frac{3s}{2}$

Solving for the ratio of their efficiencies:

$\Rightarrow\ \frac{s}{g} = \frac{4}{5}$

This indicates that **Gautam is more efficient** than Suhani.

Substituting the relationship $s = \frac{4g}{5}$ into equation (i):

$\Rightarrow\ g + \frac{4g}{5} = \frac{W}{20}$

Simplifying the left side:

$\Rightarrow\ \frac{9}{5}g = \frac{W}{20}$

Solving for Gautam's individual efficiency:

$\Rightarrow\ g = \frac{W}{36}$

Therefore, Gautam requires 36 days to complete the entire work.

The correct option is **(B) : 36**.

• • •

16. A fruit seller has a stock of mangoes, bananas and apples with at least one fruit of each type. At the beginning of a day, the number of mangoes make up 40% of his stock. That day, he sells half of the mangoes, 96 bananas and 40% of the apples. At the end of the day, he ends up selling 50% of the fruits. The smallest possible total number of fruits in the stock at the beginning of the day is

(A) 100

(B) 240

(C) 340

(D) None of Above

Correct Answer: (C) 340

SOLUTION:

Let S represent the initial total stock of all fruits. Let b denote the number of bananas and a denote the number of apples.

The stock of mangoes is 40% of S, which is $\frac{2S}{5}$.

The total number of fruits sold is the sum of mangoes sold, apples sold, and bananas sold.

This is expressed as $\frac{2S}{10} + 96 + \frac{4a}{10}$, which is given to be equal to $\frac{S}{2}$.

Simplifying the equation gives $\frac{S}{5} + 96 + \frac{2a}{5} = \frac{S}{2}$.

Multiplying by 10 to clear the denominators results in $2S + 960 + 4a = 5S$.

Rearranging the terms, we get $3S = 4a + 960$.

Solving for S yields $S = \frac{4a+960}{3} = \frac{4a}{3} + 320$.

For S to be an integer, a must be divisible by 3. Additionally, the term $\frac{4a}{10}$ implies that a must be divisible by 5.

Therefore, the smallest value for a that satisfies both divisibility conditions (multiples of 3 and 5) is the least common multiple of 3 and 5, which is $a = 15$.

Substituting this value of a into the formula for S:

$S = \frac{4\times15}{3} + 320 = 20 + 320 = 340$

The correct answer is (C): 340

17. Let ΔABC be an isosceles triangle such that AB and AC are of equal length. AD is the altitude from A on BC and BE is the altitude from B on AC. If AD and BE intersect at O such that $\angle AOB$ = 105\degree, then $\frac{AD}{BE}$ equals

(A) sin15°

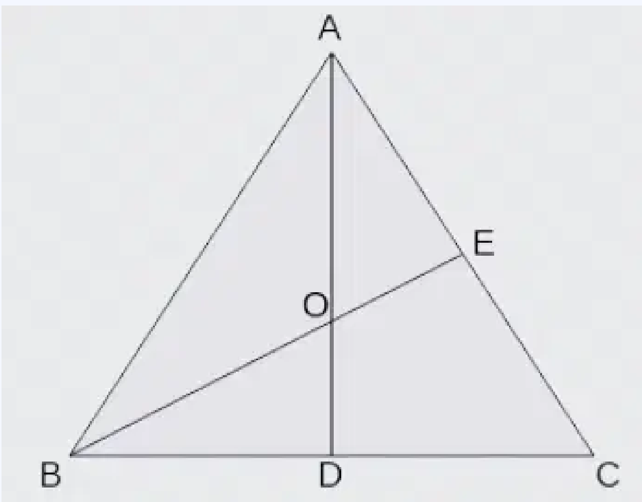
(B) cos15°

(C) 2cos15°

(D) 2sin15°

Correct Answer: (C) 2cos15°

SOLUTION:



Given, $AB = AC$ implies $\angle C = \angle B$ (Equation 1). AD and BE are altitudes, meaning they are perpendicular to their respective sides. $\angle AOB = \angle EOD = 105°$ due to vertically opposite angles. In quadrilateral $DOEC$, the sum of angles is $360°$. Therefore, $\angle C = 360° - 105° - 90° - 90° = 75°$. From Equation 1, $\angle B = 75°$. The area of a triangle can be expressed as $AD \cdot BC = BE \cdot AC$. Rearranging this gives $\frac{AD}{BE} = \frac{AC}{BC}$. Using the sine rule, $\frac{AC}{BC} = \frac{2R\sin B}{2R\sin A}$. Thus, $\frac{AD}{BE} = \frac{\sin 75°}{\sin 30°}$. This simplifies to $\frac{AD}{BE} = 2\sin 75°$, which is equivalent to $\frac{AD}{BE} = 2\cos 15°$.

Correct option: (C) $2\cos 15°$

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18. A rectangle with the largest possible area is drawn inside a semicircle of radius 2 cm. Then, the ratio of the lengths of the largest to the smallest side of this rectangle is

(A) $1:1$

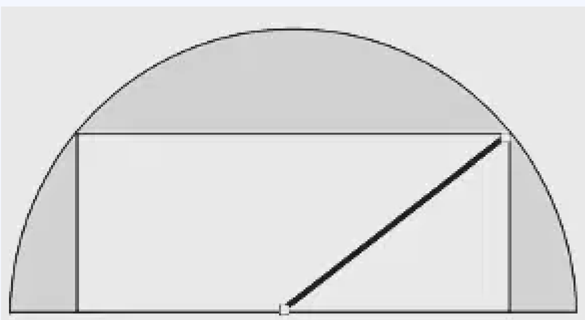
(B) $\sqrt{5}:1$

(C) $\sqrt{2}:1$

(D) $2:1$

Correct Answer: (D) $2:1$

SOLUTION:



Let the length of the rectangle be l and breadth be b.
The radius of the semicircle, half the length of the rectangle, and the breadth form a right-angled triangle.
Applying the Pythagorean theorem:

$$\left(\frac{l}{2}\right)^2 + b^2 = 2^2$$

The area of the rectangle is $l \times b$.

To maximize the area, we can use the AM-GM inequality or set the two squared terms equal:

$$\left(\frac{l}{2}\right)^2 = b^2$$

This yields:

$$\frac{l}{2} = b \Rightarrow l = 2b$$

The ratio of the length to the breadth is therefore:

$$\frac{l}{b} = \frac{2}{1}$$

The correct option is (D): $2:1$

• • •

19. In a regular polygon, any interior angle exceeds the exterior angle by 120 degrees. Then, the number of diagonals of this polygon is

(A) 30

(B) 54

(C) 64

(D) None of Above

Correct Answer: (B) 54

SOLUTION:

The solution begins with established formulas for an n-sided polygon:

- Sum of interior angles $= (n-2) \times 180$
- Sum of exterior angles $= 360°$

The sum of interior angles can also be represented as:

$$(2n-4) \times 90$$

This is equivalent to $(n-2) \times 180$, as shown by:

$$(n-2) \times 180 = (2n-4) \times 90$$

The problem states the difference between the sum of interior angles and exterior angles is:

$$(2n-4) \times 90 - 360 = 120n$$

Simplifying the left side yields:

$$(2n-4) \times 90 - 360 = 180n - 360 - 360 = 180n - 720$$

The equation simplifies to:

$$180n - 720 = 120n$$

Subtracting $120n$ from both sides results in:

$$60n - 720 = 0$$

$$60n = 720$$

$$n = 12$$

With $n = 12$ sides determined, the number of diagonals can be calculated.

The formula for the number of diagonals in an n-sided polygon is:

$$\frac{n(n-3)}{2}$$

Substituting $n = 12$:

$$\frac{12 \times (12-3)}{2} = \frac{12 \times 9}{2} = \frac{108}{2} = 54$$

The polygon has $\boxed{54}$ diagonals.

• • •

20. The value of $1 + \left(1 + \frac{1}{3}\right)\frac{1}{4} + \left(1 + \frac{1}{3} + \frac{1}{9}\right)\frac{1}{16} + \left(1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27}\right)\frac{1}{64}$ +....,is

(A) $\frac{15}{13}$

(B) $\frac{27}{12}$

(C) $\frac{15}{11}$

(D) $\frac{15}{8}$

Correct Answer: (C) $\frac{15}{11}$

SOLUTION:

The problem requires the evaluation of a summation series:

$$1+\left(1+\frac{1}{3}\right)\frac{1}{4}+\left(1+\frac{1}{3}+\frac{1}{9}\right)\frac{1}{16}+\left(1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}\right)\frac{1}{64}+\cdots$$

The expression within the parentheses constitutes a geometric series with first term $a = 1$ and common ratio $r = \frac{1}{3}$, comprising n terms. The sum of a finite geometric series is given by:

$$S_n = \frac{a(1-r^n)}{1-r} \Rightarrow \frac{1-\left(\frac{1}{3}\right)^n}{1-\frac{1}{3}} = \frac{1-\left(\frac{1}{3}\right)^n}{\frac{2}{3}} = \frac{3}{2}\left(1-\left(\frac{1}{3}\right)^n\right)$$

Each of these sums is multiplied by $\frac{1}{4^n}$, resulting in the complete series:

$$1+\sum_{n=1}^{\infty}\left(\frac{3}{2}\left(1-\left(\frac{1}{3}\right)^n\right)\cdot\frac{1}{4^n}\right)$$

The summation is decomposed into two parts:

$$1+\frac{3}{2}\left(\sum_{n=1}^{\infty}\frac{1}{4^n}-\sum_{n=1}^{\infty}\frac{1}{(4\cdot 3)^n}\right) \Rightarrow 1+\frac{3}{2}\left(\sum_{n=1}^{\infty}\frac{1}{4^n}-\sum_{n=1}^{\infty}\frac{1}{12^n}\right)$$

Both of these components are geometric series:

$$\sum_{n=1}^{\infty}\frac{1}{4^n} = \frac{\frac{1}{4}}{1-\frac{1}{4}} = \frac{1}{3}, \quad \sum_{n=1}^{\infty}\frac{1}{12^n} = \frac{\frac{1}{12}}{1-\frac{1}{12}} = \frac{1}{11}$$

Substitution into the expression yields:

$$1+\frac{3}{2}\left(\frac{1}{3}-\frac{1}{11}\right) = 1+\frac{3}{2}\cdot\frac{8}{33} = 1+\frac{24}{66} = \frac{66+24}{66} = \frac{90}{66} = \frac{15}{11}$$

Correction: A review of the calculation indicates an error. The correct step is:

$$1+\frac{3}{2}\left(\frac{1}{3}-\frac{1}{11}\right) = 1+\frac{3}{2}\cdot\frac{11-3}{33} = 1+\frac{3}{2}\cdot\frac{8}{33} = 1+\frac{24}{66}$$

The subsequent simplification was:

$$1 + \frac{24}{66} = \frac{66}{66} + \frac{24}{66} = \frac{90}{66} = \frac{15}{11}$$

✅ **Final Answer:** $\boxed{\frac{15}{11}}$

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21. Let $a_n = 46 + 8n$ and $b_n = 98 + 4n$ be two sequences for natural numbers $n \leq 100$. Then, the sum of all terms common to both the sequences is

(A) 14900

(B) 15000

(C) 14798

(D) 14602

Correct Answer: (A) 14900

SOLUTION:

To determine the sum of terms shared by the sequences $a_n = 46 + 8n$ and $b_n = 98 + 4n$ for $n \leq 100$, we equate the general terms: $46 + 8n = 98 + 4m$.

This simplifies to $8n - 4m = 52$, which further reduces to $2n - m = 13$, or $m = 2n - 13$.

Given that n is a natural number such that $1 \leq n \leq 100$, we find the values of n for which m is also a natural number. The condition for m is $1 \leq m \leq 100$. Substituting $m = 2n - 13$, we get $1 \leq 2n - 13 \leq 100$. Solving this inequality for n, we find $14 \leq 2n \leq 113$, which means $7 \leq n \leq 56$.

The common terms are generated by $a_n = 46 + 8n$ for n ranging from 7 to 56.

The first common term is $a_7 = 46 + 8 \cdot 7 = 102$.

The last common term is $a_{56} = 46 + 8 \cdot 56 = 494$.

The total number of common terms is $56 - 7 + 1 = 50$.

The sum of these common terms, forming an arithmetic sequence, is calculated as $S = \frac{\text{number of terms}}{2}(\text{first term} + \text{last term}) = \frac{50}{2}(102 + 494)$ $= 25 \cdot 596 = 14900$.

The sum of all terms common to both sequences is $\boxed{14900}$.

• • •

22. Suppose $f(x, y)$ is a real-valued function such that $f(3x + 2y, 2x - 5y)$ $= 19x$, for all real numbers x and y . The value of x for which $f(x, 2x)$ $= 27$, is

(A) 3

(B) 4

(C) 42

(D) None of Above

Correct Answer: (A) 3

SOLUTION:

Given:
$f(3x + 2y,\ 2x - 5y) = 19x$
$f(x,\ 2x) = 27$

Let $a = 3x + 2y$ and $b = 2x - 5y$. This means $f(a,\ b) = 19x$.
We are also given $f(x,\ 2x) = 27$.

To match $f(a,\ b)$ with the form $f(x,\ 2x)$, we set the arguments such

that $a = kx$ and $b = 2kx$ for some constant k.
Specifically, we consider the case where $a = \frac{19}{9}x$ and $b = \frac{38}{9}x$, which implies $b = 2a$.

We verify this transformation by solving for y in terms of x using $3x + 2y = \frac{19}{9}x$:
$2y = \frac{19}{9}x - 3x = \frac{19-27}{9}x = -\frac{8}{9}x$
$y = -\frac{4}{9}x$

Substituting this y into $2x - 5y$:
$2x - 5\left(-\frac{4}{9}x\right) = 2x + \frac{20}{9}x = \frac{18+20}{9}x = \frac{38}{9}x$

This confirms that if the first argument is $\frac{19}{9}x$, the second argument is $\frac{38}{9}x$, and the function value is $19x$.
Let $u = \frac{19}{9}x$. Then $x = \frac{9}{19}u$. The relationship $b = 2a$ implies $f(a,\ 2a) = 19x$.
Substituting u for a and x in terms of u:
$f(u,\ 2u) = 19\left(\frac{9}{19}u\right) = 9u$

We are given $f(x,\ 2x) = 27$. From our derived general form, $f(x,\ 2x) = 9x$.
Equating these two expressions for $f(x,\ 2x)$:
$9x = 27 \Rightarrow x = 3$

Therefore, the correct option is (A): 3.