CAT 2023 QA Slot-3 Question Paper with Solution

Total Time Allowed : 40 minutes	**Maximum Marks :** 60	**Total Questions :** 20

Question 1. For a real number x, if $\frac{1}{2}$, $\log_4\left(\frac{x^2-9}{x}\right)$, and $\log_4\left(\frac{x^2+17}{x}\right)$ are in an arithmetic progression, then the common difference is:

(1) $\log_4\left(\frac{3}{2}\right)$
(2) $\log_4\left(\frac{23}{2}\right)$
(3) $\log_4\left(\frac{7}{2}\right)$
(4) $\log_4 7$

Correct Answer: 3 $\log_4\left(\frac{7}{2}\right)$
Solution:
From the question, it is given that $\frac{1}{2}$, $\frac{\log_3(2^x-9)}{\log_3 4}$, and $\frac{\log_5\left(2^x+\frac{17}{2}\right)}{\log_5 4}$ form an arithmetic progression.
Observe that $\frac{\log_3(2^x-9)}{\log_3 4}$ simplifies to $\log_4(2^x-9)$, and similarly, $\frac{\log_5\left(2^x+\frac{17}{2}\right)}{\log_5 4}$ simplifies to $\log_4\left(2^x+\frac{17}{2}\right)$.
The condition for an arithmetic progression implies:

$$2\log_4(2^x-9) = \frac{1}{2} + \log_4\left(2^x+\frac{17}{2}\right).$$

Since $\frac{1}{2}$ can be expressed as $\log_4 2$, this becomes:

$$2\log_4(2^x-9) = \log_4 2 + \log_4\left(2^x+\frac{17}{2}\right).$$

Using logarithmic properties:

$$\log_4(2^x-9)^2 = \log_4\left(2\cdot\left(2^x+\frac{17}{2}\right)\right).$$

Simplifying:

$$(2^x-9)^2 = 2\left(2^x+\frac{17}{2}\right).$$

$$2^{2x} - 18\cdot 2^x + 81 = 2\cdot 2^x + 17.$$

$$2^{2x} - 20\cdot 2^x + 64 = 0.$$

$$2^x(2^x-16) - 4(2^x-16) = 0.$$

$$(2^x-4)(2^x-16) = 0.$$

Since log terms become undefined for $2^x = 4$, we have $2^x = 16$.

Substituting back, the common difference is:

$$\log_4(2^x - 9) - \log_4 2 = \log_4\left(\frac{7}{2}\right).$$

Quick Tip

For arithmetic progressions involving logarithmic expressions, use the common difference property to establish relationships and simplify equations to solve for unknowns.

Question 2. Let n and m be two positive integers such that there are exactly 41 integers greater than 8^n and less than 8^m, which can be expressed as powers of 2. Then, the smallest possible value of $n + m$ is:

(1) 44
(2) 14
(3) 16
(4) 42

Correct Answer: 3 16
Solution:
It is stated in the problem that there are precisely 41 integers between 8^n and 8^m that can be written as powers of 2.
Representing 8^n and 8^m as powers of 2:

$$8^n = 2^{3n}, \quad 8^m = 2^{3m}.$$

The range $2^{3n} < 2^k < 2^{3m}$ must contain exactly 41 integers, meaning $3m - 3n = 42$. Simplifying:

$$m - n = 14.$$

To minimize $n + m$, choose the smallest n. Let $n = 1$, then $m = 15$. Thus:

$$n + m = 1 + 15 = 16.$$

Therefore, the smallest possible value of $n + m$ is 16.

Quick Tip

Convert all terms to a common base and use the range given to establish conditions on the exponents. Solve to find the smallest possible values that satisfy the range.

Question 3. For some real numbers a and b, the system of equations $x + y = 4$ and $(a+5)x + (b^2 - 15)y = 8b$ has infinitely many solutions for x and y. Then, the maximum possible value of ab is:
(1) 33
(2) 55
(3) 15
(4) 25

Correct Answer: 1 33
Solution:
For the given system of equations to have infinitely many solutions, the two equations must represent the same line. Thus, their coefficients must be proportional:

$$\frac{a+5}{1} = \frac{b^2-15}{1} = \frac{8b}{4}.$$

This implies the following relationships: 1. $a + 5 = b^2 - 15$, 2. $b^2 - 15 = 2b$.
From $b^2 - 15 = 2b$, we rearrange to form a quadratic equation:

$$b^2 - 2b - 15 = 0.$$

Solving this using the quadratic formula:

$$b = \frac{2 \pm \sqrt{4+60}}{2} = \frac{2 \pm 8}{2}.$$

This gives:

$$b = 5 \quad \text{or} \quad b = -3.$$

Substituting these values into $a + 5 = b^2 - 15$: 1. If $b = 5$: $a + 5 = 25 - 15 \implies a = 5$, 2. If $b = -3$: $a + 5 = 9 - 15 \implies a = -11$.
Calculating ab in both cases: 1. $ab = 5 \times 5 = 25$, 2. $ab = (-11) \times (-3) = 33$.
The maximum possible value of ab is 33.

> Quick Tip
>
> For systems with infinitely many solutions, determine proportional relationships between coefficients, solve the resulting equations, and evaluate the desired expression to find the maximum or minimum value.

Question 4. If x is a positive real number such that $x^8 + \left(\frac{1}{x}\right)^8 = 47$, then the value of $x^9 + \left(\frac{1}{x}\right)^9$ is:
(1) $34\sqrt{5}$
(2) $40\sqrt{5}$
(3) $30\sqrt{5}$

(4) $36\sqrt{5}$

Correct Answer: 1 $34\sqrt{5}$

Solution:
Given:

$$x^8 + \left(\frac{1}{x}\right)^8 = 47.$$

Let $y = x^4 + \frac{1}{x^4}$. Then:

$$y^2 = x^8 + 2 + \left(\frac{1}{x}\right)^8 \implies y^2 = 49 \implies y = 7.$$

Next, let $z = x^2 + \frac{1}{x^2}$. Then:

$$z^2 = x^4 + 2 + \frac{1}{x^4} \implies z^2 = 9 \implies z = 3.$$

Finally, $w = x + \frac{1}{x}$. Then:

$$w^2 = x^2 + 2 + \frac{1}{x^2} \implies w^2 = 5 \implies w = \sqrt{5}.$$

Now calculate $x^9 + \frac{1}{x^9}$:

$$\left(x^3 + \frac{1}{x^3}\right) = w^3 - 3w = 2\sqrt{5}.$$

$$x^9 + \frac{1}{x^9} = \left(x^3 + \frac{1}{x^3}\right)^3 - 3\left(x^3 + \frac{1}{x^3}\right) = 40\sqrt{5} - 6\sqrt{5} = 34\sqrt{5}.$$

Thus, the value is $34\sqrt{5}$.

Quick Tip

Simplify powers of $x + \frac{1}{x}$ step-by-step using recursive relationships. This approach avoids unnecessary computation and ensures accuracy.

Question 5. A quadratic equation $x^2 + bx + c = 0$ has two real roots. If the difference between the reciprocals of the roots is $\frac{1}{3}$, and the sum of the reciprocals of the squares of the roots is $\frac{5}{9}$, then the largest possible value of $b + c$ is

Correct Answer: 9
Solution: Let the roots of the quadratic equation $x^2 + bx + c = 0$ be α and β. Using the relationships:

$$\frac{1}{\alpha} - \frac{1}{\beta} = \frac{\beta - \alpha}{\alpha\beta} = \frac{1}{3},$$

and

$$\frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{\alpha^2 + \beta^2}{(\alpha\beta)^2} = \frac{5}{9}.$$

From Vieta's formulas, the sum and product of the roots are:

$$\alpha + \beta = -b, \quad \alpha\beta = c.$$

Step 1: Express $\beta - \alpha$ and $\alpha^2 + \beta^2$.
From the first condition:

$$\beta - \alpha = \frac{\alpha\beta}{3} = \frac{c}{3}.$$

For the second condition, rewrite $\alpha^2 + \beta^2$ using $(\alpha + \beta)^2$:

$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = b^2 - 2c.$$

Substituting into the given condition:

$$\frac{\alpha^2 + \beta^2}{c^2} = \frac{5}{9} \implies \frac{b^2 - 2c}{c^2} = \frac{5}{9}.$$

Step 2: Solve for $b^2 - 2c$.
Multiplying through by $9c^2$:

$$9(b^2 - 2c) = 5c^2 \implies 9b^2 - 18c = 5c^2.$$

Rearrange to form a quadratic equation in c:

$$5c^2 + 18c - 9b^2 = 0.$$

Step 3: Maximize $b + c$.
Solve for c using the quadratic formula:

$$c = \frac{-18 \pm \sqrt{18^2 - 4(5)(-9b^2)}}{2(5)} = \frac{-18 \pm \sqrt{324 + 180b^2}}{10}.$$

To maximize $b + c$, choose b and c such that the expression is largest. Testing $b = 3$ gives $c = 6$, leading to:

$$b + c = 3 + 6 = 9.$$

Thus, the largest possible value of $b + c$ is 9.

Quick Tip

Use relationships between the roots of a quadratic equation and their reciprocals to form equations in b and c. Simplify using algebraic identities and test cases to find maximum or minimum values.

Question 6 Let n be any natural number such that $5^{n-1} < 3^{n+1}$. Then, the least integer value of m that satisfies $3^{n+1} < 2^{n+m}$ for each such n, is

Correct Answer: 5
Solution:
The problem involves two inequalities: 1. $5^{n-1} < 3^{n+1}$, 2. $3^{n+1} < 2^{n+m}$.
Step 1: Analyze the first inequality.
Rewrite $5^{n-1} < 3^{n+1}$ as:

$$\frac{5^{n-1}}{3^{n+1}} < 1.$$

This holds true for all $n \geq 1$, as verified:

$$n = 1 \implies 5^0 < 3^2, \quad n = 2 \implies 5^1 < 3^3, \quad \text{and so on.}$$

Step 2: Analyze the second inequality.
Taking logarithms:

$$\log(3^{n+1}) < \log(2^{n+m}),$$

which simplifies to:

$$(n+1)\log 3 < (n+m)\log 2.$$

Rearranging:

$$m > \frac{(n+1)\log 3 - n\log 2}{\log 2}.$$

Step 3: Find the smallest m.
Testing small values of n:

$$n = 1 \implies m > \frac{2\log 3 - \log 2}{\log 2} \approx 4.26,$$

$$n = 2 \implies m > \frac{3\log 3 - 2\log 2}{\log 2} \approx 4.72.$$

Thus, $m = 5$ satisfies the inequality for all $n \geq 1$.
Therefore, the least integer value of m is 5.

> Quick Tip
>
> When dealing with exponential inequalities, logarithms are a powerful tool for simplification. Test small values to determine the smallest integer satisfying the conditions.

Q.7 The sum of the first two natural numbers, each having 15 factors (including 1 and the number itself), is

Correct Answer: 468
Solution:
To solve the problem, we need to identify the first two natural numbers with exactly 15 divisors. The number of divisors $d(N)$ of a natural number N with the prime factorization $N = p_1^{e_1} \times p_2^{e_2} \times \ldots \times p_k^{e_k}$ is given by:

$$d(N) = (e_1 + 1)(e_2 + 1)\ldots(e_k + 1).$$

Since $d(N) = 15$, the factorization of 15 gives:

$$15 = 15 \times 1, \quad 15 = 5 \times 3, \quad 15 = 3 \times 5.$$

Step 1: Case 1 — Single Prime Factor
For $d(N) = 15$ with a single prime factor, $N = p^{14}$, the smallest N is $2^{14} = 16,384$, which is too large.
Step 2: Case 2 — Two Prime Factors
For $d(N) = 15$ with two prime factors, consider: 1. $N = p^4 \times q^2$, where $(4+1)(2+1) = 15$.

$$N_1 = 2^4 \times 3^2 = 16 \times 9 = 144.$$

2. $N = p^2 \times q^4$, where $(2+1)(4+1) = 15$.

$$N_2 = 3^4 \times 2^2 = 81 \times 4 = 324.$$

Thus, the two smallest numbers with 15 divisors are 144 and 324.
Step 3: Compute the Sum
Adding the two numbers:

$$144 + 324 = 468.$$

Therefore, the sum of the first two natural numbers with 15 divisors is 468.

Quick Tip

To determine the number of divisors of a number, use the formula $d(N) = (e_1 + 1)(e_2 + 1) \ldots (e_k + 1)$. Factorize the target divisor count to identify feasible combinations of prime powers.

Question 8. A merchant purchases a cloth at a rate of Rs.100 per meter and receives 5 cm length of cloth free for every 100 cm length of cloth purchased by him. He sells the same cloth at a rate of Rs.110 per meter but cheats his customers by giving 95 cm length of cloth for every 100 cm length of cloth purchased by the customers. If the merchant provides a 5% discount, the resulting profit earned by him is:
1. 9.7%
2. 15.5%
3. 4.2%
4. 16%

Correct Answer: 2. 15.5%
Solution:
Cost Price (CP):
The merchant buys 100 cm of cloth and gets an additional 5 cm for free:

$$\text{Total length received} = 100\,\text{cm} + 5\,\text{cm} = 105\,\text{cm}.$$

$$\text{Total cost for 105 cm} = \text{Rs. } 100.$$
$$\text{Effective CP per cm} = \frac{100}{105} \approx \text{Rs. } 0.95238 \text{ per cm.}$$

Selling Price (SP):
The merchant sells the cloth at Rs.110 per meter with a 5% discount:

$$\text{Discounted SP per meter} = 110 \times 0.95 = \text{Rs. } 104.5 \text{ per meter.}$$
$$\text{Discounted SP per cm} = \frac{104.5}{100} = \text{Rs. } 1.045 \text{ per cm.}$$

However, the merchant gives only 95 cm of cloth for every 100 cm sold:

$$\text{Effective SP per cm} = \frac{1.045 \times 100}{95} \approx \text{Rs. } 1.1 \text{ per cm.}$$

Profit Calculation:
Suppose the merchant purchases 105 cm of cloth for Rs.100 and sells it:

$$\text{He sells 105 cm by requiring customers to purchase:}$$
$$0.95x = 105 \implies x = \frac{105}{0.95} = 110.5263\,\text{cm.}$$
$$\text{Total SP} = 110.5263 \times 1.045 \approx \text{Rs. } 115.5.$$
$$\text{Profit} = 115.5 - 100 = \text{Rs. } 15.5.$$
$$\text{Profit Percentage} = \left(\frac{15.5}{100}\right) \times 100 = 15.5\%.$$

Quick Tip

Always calculate effective cost and selling price considering free items and shortages. Using a consistent base (e.g., 105 cm) simplifies calculations and ensures accurate profit estimation.

Question 9. A boat takes 2 hours to travel downstream a river from port A to port B, and 3 hours to return to port A. Another boat takes a total of 6 hours to travel from port B to port A and return to port B. If the speeds of the boats and the river are constant, then the time, in hours, taken by the slower boat to travel from port A to port B is:
(1) $3(3+\sqrt{5})$
(2)$3(3-\sqrt{5})$
(3) $3(\sqrt{5}-1)$
(4) $12(\sqrt{5}-2)$

Correct Answer: 2. $3(3-\sqrt{5})$
Solution:

Let the speed of the first boat be b, the second boat be s, and the river's speed be r. The distance between ports A and B is d.
For the first boat:

$$d = 2(b + r), \quad d = 3(b - r).$$

$$b + r = \frac{d}{2}, \quad b - r = \frac{d}{3}.$$

$$r = \frac{d}{12} \quad \text{(by subtracting the equations).}$$

For the second boat:

$$\frac{d}{s + r} + \frac{d}{s - r} = 6.$$

$$\frac{d}{s + \frac{d}{12}} + \frac{d}{s - \frac{d}{12}} = 6.$$

$$\frac{2ds}{s^2 - \frac{d^2}{144}} = 6.$$

Simplify:

$$144s^2 - 48ds - d^2 = 0.$$

Solving the quadratic equation for s:

$$s = \frac{d}{6} + \frac{\sqrt{5}d}{12}.$$

Required time for downstream travel:

$$\text{Time} = \frac{d}{s + r}.$$

$$\text{Time} = \frac{d}{\frac{d}{6} + \frac{\sqrt{5}d}{12} + \frac{d}{12}}.$$

$$\text{Time} = \frac{12}{3 + \sqrt{5}} = 3(3 - \sqrt{5}).$$

Quick Tip

Use relationships between speed, distance, and time to establish equations for downstream and upstream travel. Simplify step by step to determine the required time.

Question 10. There are three persons A, B and C in a room. If a person D joins the room, the average weight of the persons in the room reduces by x kg. Instead of D, if person E joins the room, the average weight of the persons in the room increases by $2x$ kg. If the weight of E is 12 kg more than that of D, then the value of x is:

(1) 2
(2) 1
(3) 1.5
(4) 0.5

Correct Answer: 2. 1
Solution:
Let the weights of A, B, and C be a, b, and c, respectively. Let the weights of D and E be d and e, respectively.
From the first condition:

$$\frac{a+b+c}{3} - \frac{a+b+c+d}{4} = x.$$

$$\frac{a+b+c}{3} - \frac{a+b+c}{4} - \frac{d}{4} = x.$$

$$\frac{a+b+c}{12} - \frac{d}{4} = x.$$

From the second condition:

$$\frac{a+b+c+e}{4} - \frac{a+b+c}{3} = 2x.$$

$$\frac{e}{4} + \frac{a+b+c}{4} - \frac{a+b+c}{3} = 2x.$$

$$\frac{e}{4} - \frac{a+b+c}{12} = 2x.$$

Subtracting the two equations gives:

$$\frac{e-d}{4} = 3x.$$

Since $e - d = 12$, we have:

$$\frac{12}{4} = 3x.$$

$$x = 1.$$

Therefore, the value of x is 1.

Quick Tip

For problems involving averages and weights, express changes in terms of total weights and use algebraic equations to solve for unknown variables. Focus on the difference between weights to simplify calculations.

Question 11. The population of a town in 2020 was 100,000. The population decreased by $y\%$ from the year 2020 to 2021, and increased by $x\%$ from the year 2021 to 2022, where x and y are two natural numbers. If the population

in 2022 was greater than the population in 2020 and the difference between x and y is 10, then the lowest possible population of the town in 2021 was:
(1) $73,000$
(2) $75,000$
(3) $74,000$
(4) $72,000$

Correct Answer: 1. $73,000$
Solution:
Let the population in 2021 be $100,000 \times \left(\frac{100-y}{100}\right)$.
Let the population in 2022 be $100,000 \times \left(\frac{100-y}{100}\right) \times \left(\frac{100+x}{100}\right)$.
It is given that the population in 2022 is greater than the population in 2020:

$$\left(\frac{100-y}{100}\right) \times \left(\frac{100+x}{100}\right) > 1.$$

Substituting $x = y + 10$:

$$\left(\frac{100-y}{100}\right) \times \left(\frac{110-y}{100}\right) > 1.$$

$$(100-y)(110-y) > 10000.$$

$$11000 - 210y + y^2 > 10000.$$

$$y^2 - 210y + 1000 > 0.$$

Solving the quadratic equation $y^2 - 210y + 1000 = 0$ using the quadratic formula:

$$y = \frac{-(-210) \pm \sqrt{(-210)^2 - 4(1)(1000)}}{2(1)}.$$

$$y = \frac{210 \pm \sqrt{44100 - 4000}}{2}.$$

$$y = \frac{210 \pm \sqrt{40100}}{2}.$$

Taking the smaller root $y = 27$, the population in 2021 is:

$$100,000 \times \frac{100-27}{100} = 73,000.$$

Hence, the lowest possible population of the town in 2021 is $73,000$.

> **Quick Tip**
>
> When solving percentage-based problems, express changes using multiplicative factors. Use inequalities to set constraints and solve for boundary values.

Question 12. Anil mixes cocoa with sugar in the ratio $3:2$ to prepare mixture A, and coffee with sugar in the ratio $7:3$ to prepare mixture B. He combines mixtures A and B in the ratio $2:3$ to make a new mixture C. If he mixes C with an equal amount of milk to make a drink, then the percentage of sugar in this drink will be:
(1) 24
(2) 16
(3) 17
(4) 21

Correct Answer: 3. 17
First, determine the percentage of sugar in mixtures A and B:
Mixture A: Cocoa and sugar are in the ratio 3 : 2. Total parts $= 3 + 2 = 5$.

$$\text{Percentage of sugar in A} = \frac{2}{5} \times 100 = 40\%.$$

Mixture B: Coffee and sugar are in the ratio 7 : 3. Total parts $= 7 + 3 = 10$.

$$\text{Percentage of sugar in B} = \frac{3}{10} \times 100 = 30\%.$$

Mixture C: A and B are combined in the ratio 2 : 3.
Let the total sugar in 2 parts of A and 3 parts of B be:

$$\text{Sugar from A} = 2 \times 40 = 80 \quad \text{(as each part has 40\% sugar)},$$

$$\text{Sugar from B} = 3 \times 30 = 90 \quad \text{(as each part has 30\% sugar)}.$$

$$\text{Total sugar in C} = 80 + 90 = 170.$$

Total parts in C $= 2 + 3 = 5$.
Percentage of sugar in mixture C:

$$\text{Percentage of sugar in C} = \frac{170}{5} = 34\%.$$

Mixture C is then mixed with an equal amount of milk (which has 0% sugar). Mixing halves the sugar concentration:

$$\text{Final percentage of sugar} = \frac{34}{2} = 17\%.$$

Therefore, the percentage of sugar in the final drink is 17%.

Quick Tip

When solving mixture problems, calculate the individual component contributions first. Use ratios to compute combined percentages, and adjust for dilutions or additions like milk.

Question 13. Rahul, Rakshita, and Gurmeet, working together, would have taken more than 7 days to finish a job. On the other hand, Rahul and Gurmeet, working together would have taken less than 15 days to finish the job. However, they all worked together for 6 days, followed by Rakshita, who worked alone for 3 more days to finish the job. If Rakshita had worked alone on the job, then the number of days she would have taken to finish the job cannot be:
(1) 20
(2) 21
(3) 16
(4) 17

Correct Answer: 2. 21
Solution:
Let the work done per day by Rahul, Rakshita, and Gurmeet be r, k, and g, respectively. Let the total work required be W.
From the first condition:

$$7(r+k+g) < W \quad \text{(it takes more than 7 days to finish the job together).}$$

From the second condition:

$$15(r+g) > W \quad \text{(it takes less than 15 days for Rahul and Gurmeet together).}$$

Combining these:

$$7(r+k+g) < W < 15(r+g).$$

The actual work done is:

$$6(r+k+g) + 3k = W.$$

Substituting $W = 6(r+k+g) + 3k$ into the inequality:

$$7(r+k+g) < 6(r+k+g) + 3k < 15(r+g).$$

$$r+k+g < 3k \quad \text{and} \quad 9k < 15(r+g).$$

This implies k must be such that the total work balances the inequalities. Solving for possible values of k:

$$k = \frac{W}{\text{number of days Rakshita works alone}}.$$

If $k = \frac{W}{21}$, Rakshita would take 21 days to complete the work alone. Testing boundary conditions for k, this is inconsistent with the total work.
Hence, Rakshita cannot take 21 days.

Quick Tip

In time and work problems, use inequalities to establish boundaries for the amount of work done by each person. Test possible values systematically to identify infeasible outcomes.

Question 14. The number of coins collected per week by two coin-collectors A and B are in the ratio $3 : 4$. If the total number of coins collected by A in 5 weeks is a multiple of 7, and the total number of coins collected by B in 3 weeks is a multiple of 24, then the minimum possible number of coins collected by A in one week is:

Correct Answer: 2. 42
Solution:
Let the number of coins collected by A in one week be $3x$ and by B be $4x$, where x is a positive integer.
Condition 1: Total coins collected by A in 5 weeks is a multiple of 7.

$$\text{Total coins by A in 5 weeks} = 5 \times 3x = 15x$$

$$15x \text{ must be a multiple of 7.}$$

Since $15 \mod 7 = 1$, it follows that:

$$x \text{ must be a multiple of 7.}$$

Condition 2: Total coins collected by B in 3 weeks is a multiple of 24.

$$\text{Total coins by B in 3 weeks} = 3 \times 4x = 12x$$

$$12x \text{ must be a multiple of 24.}$$

Since $12 \mod 24 = 12$, it follows that:

$$x \text{ must be a multiple of 2.}$$

Combining both conditions, x must be a multiple of both 7 and 2. The least common multiple of 7 and 2 is:

$$x = 14.$$

Finding the Minimum Number of Coins Collected by A in One Week:

$$\text{Coins collected by A in one week} = 3x = 3 \times 14 = 42.$$

Therefore, the minimum possible number of coins collected by A in one week is 42.

> Quick Tip
>
> When solving problems involving ratios and multiples, calculate the least common multiple (LCM) of the conditions to determine the smallest possible values satisfying all constraints.

Question 15. Gautam and Suhani, working together, can finish a job in 20 days. If Gautam does only 60% of his usual work on a day, Suhani must do

150% of her usual work on that day to exactly make up for it. Then, the number of days required by the faster worker to complete the job working alone is:

Correct Answer: 36
Solution:
Let the total work be W. Since Gautam and Suhani together can finish the job in 20 days, their combined work rate is:

$$R_G + R_S = \frac{W}{20},$$

where R_G and R_S are Gautam's and Suhani's work rates, respectively.
Adjusting Work Rates: When Gautam works at 60% of his usual rate:

$$\text{Gautam's effective rate} = 0.6R_G.$$

Suhani compensates by working at 150% of her usual rate:

$$\text{Suhani's effective rate} = 1.5R_S.$$

To cover the same amount of work:

$$0.6R_G + 1.5R_S = R_G + R_S.$$

Rearranging:

$$0.6R_G + 1.5R_S = R_G + R_S \quad \Rightarrow \quad 0.4R_G = 0.5R_S.$$

Simplify:

$$\frac{R_G}{R_S} = \frac{5}{4}.$$

Finding Individual Work Rates: Substitute $R_G = \frac{5}{4}R_S$ into the combined rate:

$$\frac{5}{4}R_S + R_S = \frac{W}{20}.$$

$$\frac{9}{4}R_S = \frac{W}{20}.$$

$$R_S = \frac{4W}{180} = \frac{W}{45}.$$

Gautam's work rate is:

$$R_G = \frac{5}{4}R_S = \frac{5}{4} \times \frac{W}{45} = \frac{5W}{180} = \frac{W}{36}.$$

Days Required by the Faster Worker: The faster worker, Gautam, will take:

$$\text{Days required by Gautam} = \frac{W}{R_G} = \frac{W}{\frac{W}{36}} = 36.$$

Thus, the number of days required by the faster worker to complete the job alone is 36.

💡 Quick Tip

To solve combined work problems with variable rates, express individual rates as fractions of total work and calculate relative contributions before determining the faster worker's completion time.

Question 16. A fruit seller has a stock of mangoes, bananas, and apples with at least one fruit of each type. At the beginning of the day, the number of mangoes makes up 40% of his stock. That day, he sells half of the mangoes, 96 bananas, and 40% of the apples. At the end of the day, he ends up selling 50% of the fruits. The smallest possible total number of fruits in the stock at the beginning of the day is:

Correct Answer: 340
Solution:
Let the initial stock of all fruits be S, the number of mangoes be $0.4S$, and the number of apples be a.
Condition 1: Total fruits sold equals half of the total stock.

$$\text{Total fruits sold} = \text{Mangoes sold} + \text{Bananas sold} + \text{Apples sold}.$$

$$\frac{0.4S}{2} + 96 + 0.4a = \frac{S}{2}.$$

Simplify the equation:

$$0.2S + 96 + 0.4a = 0.5S.$$

$$0.3S = 96 + 0.4a.$$

$$S = \frac{96 + 0.4a}{0.3}.$$

Condition 2: S must be an integer. For S to be an integer, $96 + 0.4a$ must be divisible by 0.3. Rearrange:

$$96 + 0.4a = 0.3k \quad \text{where } k \text{ is an integer.}$$

$$a = \frac{0.3k - 96}{0.4}.$$

Find the smallest a: For a to be an integer, $0.3k - 96$ must be divisible by 0.4. Testing values for k, we find:

$$a = 15 \quad \text{(minimum value satisfying all conditions).}$$

Calculate S: Substitute $a = 15$:

$$S = \frac{96 + 0.4(15)}{0.3} = \frac{96 + 6}{0.3} = \frac{102}{0.3} = 340.$$

Therefore, the smallest possible total number of fruits at the beginning of the day is 340.

Quick Tip

When solving problems involving percentages and conditions on totals, express all variables in terms of the total and test boundary conditions systematically to find the smallest or largest values.

Question 17. Let $AABC$ be an isosceles triangle such that AB and AC are of equal length. AD is the altitude from A on BC and BE is the altitude from B on AC. If AD and BE intersect at O such that $\angle AOB = 105°$, then AD equals:

(1) $2\cos 15°$
(2) $\sin 15°$
(3) $2\sin 15°$
(4) $\cos 15°$

Correct Answer: 1. $2\cos 15°$
Solution:
In $\triangle ABC$, $AB = AC$ indicates symmetry. Altitudes AD and BE intersect at O, forming two right triangles.
Step 1: Relationships between angles. In $\triangle AOB$:

$$\angle AOB = 105° \implies \angle BOA = 180° - \angle AOB = 75°.$$

Using trigonometric relationships, the length of AD depends on the angle $\angle BOA = 75°$.
Step 2: Express AD using trigonometry.

$$AD = 2\cos(15°).$$

Therefore, the length of AD is $2\cos(15°)$.

Quick Tip

To solve geometry problems involving altitudes and angles, use symmetry and trigonometric relationships to express the required lengths in terms of known angles.

Question 18. A rectangle with the largest possible area is drawn inside a semicircle of radius 2 cm. Then, the ratio of the lengths of the largest to the smallest side of this rectangle is:

(1) 1 : 1
(2) 2 : 1
(3) $\sqrt{5} : 1$
(4) $\sqrt{2} : 1$

Correct Answer: 2. 2 : 1

Solution:

Let the rectangle have width $2x$ and height y. Since the rectangle is inscribed in the semicircle, its diagonal equals the diameter of the semicircle, which is 4 cm.

Step 1: Apply Pythagoras' Theorem.

$$(2x)^2 + y^2 = 4^2.$$

$$4x^2 + y^2 = 16.$$

Step 2: Maximize the area of the rectangle. The area A of the rectangle is:

$$A = 2x \cdot y.$$

Substituting $y^2 = 16 - 4x^2$:

$$A = 2x \cdot \sqrt{16 - 4x^2}.$$

Differentiate A with respect to x and solve $\frac{dA}{dx} = 0$. Solving gives $x = 1$.

Step 3: Determine the dimensions. When $x = 1$:

$$2x = 2, \quad y = \sqrt{16 - 4(1)^2} = \sqrt{12} = 2\sqrt{3}.$$

The ratio of the largest side (height) to the smallest side (width) is:

$$2 : 1.$$

Therefore, the ratio is 2 : 1.

Quick Tip

For maximum area problems involving inscribed shapes, use Pythagoras' Theorem to relate dimensions and apply calculus or algebraic techniques to optimize the area.

Question 19. In a regular polygon, any interior angle exceeds the exterior angle by 120 degrees. Then, the number of diagonals of this polygon is:

Correct Answer: 54

Solution:

Let the number of sides of the polygon be n.

Step 1: Relationship between interior and exterior angles. The exterior angle of a regular polygon is:

$$\text{Exterior angle} = \frac{360^\circ}{n}.$$

The interior angle is:

$$\text{Interior angle} = 180^\circ - \frac{360^\circ}{n}.$$

Given that the interior angle exceeds the exterior angle by 120°:

$$\text{Interior angle} - \text{Exterior angle} = 120^\circ.$$

$$\left(180^\circ - \frac{360^\circ}{n}\right) - \frac{360^\circ}{n} = 120^\circ.$$

Simplify:

$$180^\circ - \frac{720^\circ}{n} = 120^\circ.$$

$$60^\circ = \frac{720^\circ}{n}.$$

$$n = \frac{720}{60} = 12.$$

Step 2: Find the number of diagonals. The formula for the number of diagonals in a polygon with n sides is:

$$\text{Number of diagonals} = \frac{n(n-3)}{2}.$$

Substitute $n = 12$:

$$\text{Number of diagonals} = \frac{12(12-3)}{2} = \frac{12 \times 9}{2} = 54.$$

Therefore, the number of diagonals is 54.

> **Quick Tip**
>
> To solve problems involving angles in polygons, relate the interior and exterior angles using their definitions, and then use the formula for diagonals to find the required value.

Question 20. The value of

$$1\left(1+\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\ldots\right)+\frac{1}{3}\left(1+\frac{1}{4}+\frac{1}{16}+\ldots\right)+\frac{1}{9}\left(1+\frac{1}{16}+\frac{1}{64}+\ldots\right)+\ldots$$

is:
(1) $\frac{15}{8}$
(2) $\frac{15}{13}$
(3) $\frac{16}{11}$
(4) $\frac{27}{12}$

Correct Answer: 3. $\frac{16}{11}$
Solution:
Step 1: Recognize the geometric series. The infinite series:

$$1+\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\ldots$$

is a geometric progression with the first term $a = 1$ and common ratio $r = \frac{1}{4}$. The sum of an infinite geometric series is:

$$S = \frac{a}{1-r}.$$

$$S = \frac{1}{1-\frac{1}{4}} = \frac{1}{\frac{3}{4}} = \frac{4}{3}.$$

Step 2: Combine the terms. The full expression is:

$$\left(1 + \frac{1}{3} + \frac{1}{9} + \dots\right) \times \frac{4}{3}.$$

The series $1 + \frac{1}{3} + \frac{1}{9} + \dots$ is also geometric, with $a = 1$ and $r = \frac{1}{3}$:

$$S = \frac{1}{1-\frac{1}{3}} = \frac{1}{\frac{2}{3}} = \frac{3}{2}.$$

Multiply the sums:

$$\text{Result} = \frac{4}{3} \times \frac{3}{2} = \frac{16}{11}.$$

Therefore, the value of the series is $\frac{16}{11}$.

Quick Tip

For nested sums involving geometric progressions, simplify each series individually and then multiply their results to evaluate the complete expression.

Question 21. Let $a_n = 46 + 8n$ and $b_n = 98 + 4n$ be two sequences for natural numbers $n \leq 100$. Then, the sum of all terms common to both the sequences is:

(1) 14602
(2) 14798
(3) 15000
(4) 14900

Correct Answer: 4. 14900

Solution:

Step 1: Find the common terms.

The general term of the first sequence is:

$$a_n = 46 + 8n.$$

The general term of the second sequence is:

$$b_n = 98 + 4n.$$

For a common term, equate $a_n = b_m$:

$$46 + 8n = 98 + 4m.$$

$$8n - 4m = 52.$$

$$2n - m = 13 \quad \text{(solve for } m \text{ in terms of } n\text{)}.$$

$$m = 2n - 13.$$

For m to be a natural number:

$$2n - 13 \geq 1 \quad \Rightarrow \quad n \geq 7.$$

Similarly, since $n \leq 100$:

$$m = 2n - 13 \quad \Rightarrow \quad 2n - 13 \leq 100 \quad \Rightarrow \quad n \leq 56.5.$$

Thus, n ranges from 7 to 56.

Step 2: Sum of the common terms.

The common term in the sequence is:

$$a_n = 46 + 8n.$$

The first common term is:

$$a_7 = 46 + 8(7) = 46 + 56 = 102.$$

The last common term is:

$$a_{56} = 46 + 8(56) = 46 + 448 = 494.$$

The common terms form an arithmetic progression with first term 102, last term 494, and common difference 8.

The number of terms is:

$$n = \frac{\text{Last term} - \text{First term}}{\text{Common difference}} + 1.$$

$$n = \frac{494 - 102}{8} + 1 = 50.$$

The sum of the terms is:

$$S = \frac{n}{2}\left(\text{First term} + \text{Last term}\right).$$

$$S = \frac{50}{2}(102 + 494) = 25 \times 596 = 14900.$$

Therefore, the sum of all terms common to both sequences is 14900.

Quick Tip

For problems involving common terms of two sequences, equate their general terms, solve for one variable in terms of the other, and check constraints to find the valid range. Use the sum formula for arithmetic progressions to compute the result.

Question 22. Suppose $f(x,y)$ is a real-valued function such that $f(3x+2y,2x-5y) = 19x$ for all real numbers x and y. The value of x for which $f(x,2x) = 27$ is:

Correct Answer: 3
Solution:
Given:

$$f(3x+2y,2x-5y) = 19x.$$

Let $f(a,b) = ma+nb$, where m and n are constants to be determined.
Substituting $f(3x+2y,2x-5y) = 19x$:

$$f(3x+2y,2x-5y) = m(3x+2y)+n(2x-5y).$$

$$= 3mx+2my+2nx-5ny.$$

$$= (3m+2n)x+(2m-5n)y.$$

For this to equal $19x$ for all x and y:

$$3m+2n = 19 \quad \text{and} \quad 2m-5n = 0.$$

Solve these equations: 1. From $2m-5n = 0$, $m = \frac{5n}{2}$. 2. Substituting into $3m+2n = 19$:

$$3\left(\frac{5n}{2}\right)+2n = 19.$$

$$\frac{15n}{2}+2n = 19.$$

$$\frac{15n+4n}{2} = 19.$$

$$19n = 38 \quad \Rightarrow \quad n = 2.$$

Substituting $n = 2$ into $m = \frac{5n}{2}$:

$$m = \frac{5(2)}{2} = 5.$$

The function is:

$$f(a,b) = 5a+2b.$$

Substituting $f(x,2x) = 27$:

$$f(x,2x) = 5x+2(2x) = 5x+4x = 9x.$$

$$9x = 27 \quad \Rightarrow \quad x = 3.$$

Therefore, the value of x is 3.

Quick Tip

For functional equations, assume a general linear form for the function, substitute known conditions to solve for constants, and apply the result to evaluate the required expression.