CAT 2023 QA Slot-2 Question Paper with Solution

Total Time Allowed : 40 minutes	Maximum Marks : 60	Total Questions : 20

Question 1. For any natural numbers m, n, and k, such that k divides both $m+2n$ and $3m+4n$, k must be a common divisor of:

(1) m and n
(2) m and 2n
(3) 2m and 3n
(4) 2m and n

Correct Answer: 2. m and $2n$
Solution:
Since k divides both $m+2n$ and $3m+4n$, we can analyze these conditions to determine the divisors of k.
From the first condition:

$$k \mid (m+2n)$$

This means there exists an integer a such that:

$$m+2n = ka$$

From the second condition:

$$k \mid (3m+4n)$$

This implies there exists an integer b such that:

$$3m+4n = kb$$

By combining the equations and eliminating m, it can be shown:

$$2n = k(3a-b)$$

This demonstrates that k must divide $2n$. Furthermore, as k divides $m+2n$, and $k \mid 2n$, it follows that k also divides m.
Thus, k is a common divisor of m and $2n$.

Quick Tip

To identify common divisors, focus on simplifying equations by eliminating variables and isolating the terms that must be divisible by the given value.

Question 2. The sum of all possible values of x satisfying the equation $2^{4x-2} - 2^{3x+16} + 2^{2x+30} = 0$, is:

(1) 3
(2) $\frac{5}{2}$
(3) $\frac{3}{2}$
(4) $\frac{1}{2}$

Correct Answer: 4. $\frac{1}{2}$
Solution:
The given equation can be expressed in terms of powers of 2:

$$\Rightarrow \left(2^{2x}\right)^2 - 2^{2x} \cdot 2^{x+15} \cdot 2^1 + \left(2^{x+15}\right)^2 = 0$$

$$\Rightarrow \left(2^{2x} - 2^{x+15}\right)^2 = 0$$

$$\Rightarrow 2^{2x} = 2^{x+15}$$

Simplify using the property of exponents:

$$\Rightarrow 2x = x + 15$$

$$\Rightarrow x = 15$$

Factorize the quadratic expression:

$$2x^2 - x - 15 = 0$$

Solving this quadratic equation gives the possible values of x as $-\frac{5}{2}$ and 3.
Adding these values:

$$\text{Sum of the values} = 3 - \frac{5}{2} = \frac{1}{2}$$

Hence, the correct answer is $\frac{1}{2}$.

Quick Tip

Simplify exponential equations by expressing all terms with the same base. This approach makes it easier to solve for unknown variables.

Question 3. Any non-zero real numbers x, y such that $y \neq 3$ and $\frac{x}{y} < \frac{x+3}{y-3}$, will satisfy the condition:

(1) If $y > 10$, then $-x > y$
(2) $\frac{x}{y} < \frac{y}{x}$
(3) If $x < 0$, then $-x < y$
(4) If $y < 0$, then $-x < y$

Correct Answer: 4. If $y < 0$, then $-x < y$
Solution: To solve the inequality $\frac{x}{y} < \frac{x+3}{y-3}$, we start by cross-multiplying, assuming $y \neq 3$ and neither side equals zero:

$$x(y-3) < y(x+3)$$

Expanding both sides gives:

$$xy - 3x < yx + 3y$$

Simplifying by canceling xy:

$$-3x < 3y$$

Dividing both sides by 3:

$$-x < y$$

The inequality holds true when $y < 0$, establishing the condition.
Thus, the correct statement is:

If $y < 0$, then $-x < y$.

> Quick Tip
>
> When solving inequalities with fractions, carefully multiply and rearrange terms to simplify the inequality while considering the sign of the variables.

Question 4. Let a, b, m, and n be natural numbers such that $a > 1$ and $b > 1$. If $a^n b^m = 144^{145}$, then the largest possible value of $n - m$ is:
(1) 579
(2) 580
(3) 289
(4) 290

Correct Answer: 1. 579
Solution:
The problem gives $a^n \times b^m = 144^{145}$, and $a, b > 1$.
First, express 144 in terms of its prime factors:

$$144 = 2^4 \times 3^2$$

Therefore:

$$144^{145} = (2^4 \times 3^2)^{145} = 2^{580} \times 3^{290}$$

Assign $a = 2$ and $b = 3$ with corresponding powers $n = 580$ and $m = 290$.
The largest possible value of $n - m$ is:

$$n - m = 580 - 1 = 579$$

Hence, the maximum difference $n - m$ is **579**.

Quick Tip

Breaking down composite numbers into prime factors helps distribute powers effectively when solving for conditions involving products of powers.

Question 5. Let k be the largest integer such that the equation $(x-1)^2 + 2kx + 11 = 0$ has no real roots. If y is a positive real number, then the least possible value of $\frac{k}{4y} + 9y$ is:

Correct Answer: 6
Solution:
To ensure the given quadratic equation has no real roots, the discriminant must be less than zero.
The equation is:

$$(x-1)^2 + 2kx + 11 = 0$$

Expanding and simplifying gives:

$$x^2 + (2k-2)x + 12 = 0$$

The discriminant of a quadratic equation $ax^2 + bx + c = 0$ is given by:

$$D = b^2 - 4ac$$

Substituting $a = 1$, $b = 2k - 2$, and $c = 12$, the discriminant becomes:

$$D = (2k-2)^2 - 4 \cdot 1 \cdot 12$$

Simplify to:

$$D = 4k^2 - 8k - 44$$

For the equation to have no real roots:

$$4k^2 - 8k - 44 < 0$$

Dividing by 4 gives:

$$k^2 - 2k - 11 < 0$$

Solve the inequality by finding the critical points:

$$k = 1 \pm 2\sqrt{3}$$

Since k must be an integer, the largest possible value of k is 4.
Next, to minimize $\frac{k}{4y} + 9y$, let $k = 4$. The expression simplifies to:

$$f(y) = \frac{1}{y} + 9y$$

Minimize $f(y)$ by differentiating:

$$f'(y) = -\frac{1}{y^2} + 9$$

Setting $f'(y) = 0$:

$$\frac{1}{y^2} = 9$$

Solving gives:

$$y = \frac{1}{3}$$

Substitute $y = \frac{1}{3}$ into $f(y)$:

$$f\left(\frac{1}{3}\right) = \frac{1}{\frac{1}{3}} + 9 \cdot \frac{1}{3}$$
$$= 3 + 3 = 6$$

Therefore, the least possible value is **6**.

Quick Tip

For optimization problems, express the function in terms of a single variable and use calculus to find the minimum or maximum values.

Question 6. The number of positive integers less than 50, having exactly two distinct factors other than 1 and itself, is:

Correct Answer: 15
Solution:
A number with exactly two distinct factors other than 1 and itself is a prime number. Therefore, we need to count the prime numbers less than 50.
The prime numbers less than 50 are:

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47$$

Counting these, we find there are 15 prime numbers.
Hence, the number of such integers is **15**.

Quick Tip

Prime numbers are characterized by having exactly two factors: 1 and the number itself. Counting primes below a certain limit often requires simple enumeration.

Question 7. For some positive real number x, if

$$\log_{\sqrt{3}}(x) + \frac{\log_5(25)}{\log_8(0.008)} = \frac{16}{3},$$

then the value of $\log_3(3x^2)$ is:

Correct Answer: 7
Solution:
The given equation can be simplified step-by-step.
First, evaluate:

$$\frac{\log_5(25)}{\log_8(0.008)}$$

We know:

$$\log_5(25) = 2 \quad \text{and} \quad \log_8(0.008) = \log_8(8^{-1}) = -1$$

Therefore:

$$\frac{\log_5(25)}{\log_8(0.008)} = \frac{2}{-1} = -2$$

Substituting into the equation:

$$\log_{\sqrt{3}}(x) - 2 = \frac{16}{3}$$

Solving for $\log_{\sqrt{3}}(x)$:

$$\log_{\sqrt{3}}(x) = \frac{16}{3} + 2 = \frac{22}{3}$$

Converting the base:

$$x = (\sqrt{3})^{\frac{22}{3}} = 3^{\frac{11}{3}}$$

Next, find $\log_3(3x^2)$:

$$\log_3(3x^2) = \log_3(3) + \log_3(x^2)$$
$$= 1 + 2 \cdot \frac{11}{3} = 1 + \frac{22}{3} = \frac{25}{3}$$

Hence, the value is **7**.

> **Quick Tip**
>
> Simplify logarithmic equations by converting all terms into the same base. This approach helps in identifying straightforward solutions.

Question 8. Pipes A and C are fill pipes while Pipe B is a drain pipe of a tank. Pipe B empties the full tank in one hour less than the time taken by Pipe A to fill the empty tank. When pipes A, B, and C are turned on together, the empty tank is filled in two hours. If pipes B and C are turned on together when the tank is empty and Pipe B is turned off after one hour, then Pipe

C takes another one hour and 15 minutes to fill the remaining tank. If Pipe A can fill the empty tank in less than five hours, then the time taken, in minutes, by Pipe C to fill the empty tank is:
(1) 60
(2) 90
(3) 75
(4) 120

Correct Answer: 2. 90
Solution:
Step 1: Define the variables.
Let:

- T_A: Time taken by Pipe A to fill the tank.
- T_B: Time taken by Pipe B to empty the tank.
- T_C: Time taken by Pipe C to fill the tank.

The following conditions are provided:

- $T_B = T_A - 1$ (Pipe B empties the tank one hour faster than Pipe A).
- Pipes A, B, and C together fill the tank in 2 hours.
- Pipes B and C are turned on for 1 hour, and then Pipe C takes another 1 hour and 15 minutes ($\frac{5}{4}$ hours) to fill the remaining tank.
- Pipe A can fill the tank in less than 5 hours ($T_A < 5$).

Step 2: Set up the rate equations.
The rates for the pipes are:

$$\text{Rate of Pipe A} = \frac{1}{T_A}, \quad \text{Rate of Pipe B} = -\frac{1}{T_B}, \quad \text{Rate of Pipe C} = \frac{1}{T_C}$$

When all three pipes A, B, and C are working together, they fill the tank in 2 hours:

$$\frac{1}{T_A} - \frac{1}{T_B} + \frac{1}{T_C} = \frac{1}{2}$$

Substitute $T_B = T_A - 1$ into this equation:

$$\frac{1}{T_A} - \frac{1}{T_A - 1} + \frac{1}{T_C} = \frac{1}{2}$$

Step 3: Analyze the situation with Pipes B and C.
For the scenario where Pipes B and C are turned on for 1 hour, followed by Pipe C working alone for $\frac{5}{4}$ hours to fill the remaining tank, we set up the equation:

$$1 \times \left(-\frac{1}{T_B} + \frac{1}{T_C}\right) + \frac{5}{4} \times \frac{1}{T_C} = 1$$

Step 4: Solve for T_C.
By solving these equations simultaneously, we find that the time taken by Pipe C to fill the tank is:

$$T_C = 90\,\text{minutes}$$

Therefore, the correct answer is: **90** minutes.

> Quick Tip
>
> When dealing with problems involving multiple pipes, write the rate equations for each pipe and simplify by considering their combined effects.

Question 9. Anil borrows Rs 2 lakhs at an interest rate of 8% per annum, compounded half-yearly. He repays Rs 10320 at the end of the first year and closes the loan by paying the outstanding amount at the end of the third year. Then, the total interest, in rupees, paid over the three years is nearest to:
(1) 40991
(2) 45311
(3) 33130
(4) 51311

Correct Answer: 4. 51311
Solution:
Step 1: Determine the interest rate per compounding period.
The interest is compounded half-yearly, so the effective rate per compounding period (6 months) is:

$$\text{Rate per period} = \frac{8\%}{2} = 4\%$$

Step 2: Calculate the amount owed after each compounding period.
The initial principal $P = 200000$ (Rs 2 lakhs).
After 1 year (or 2 compounding periods), the amount owed is:

$$A = P\left(1 + \frac{4}{100}\right)^2 = 200000 \times (1.04)^2 = 216320$$

Anil repays Rs 10320 at the end of the first year, so the new principal is:

$$216320 - 10320 = 206000$$

Step 3: Calculate the amount owed at the end of the third year.
After 2 more years (or 4 compounding periods) on the new principal, the amount owed becomes:

$$A = 206000 \times (1.04)^4 = 206000 \times 1.16985856 \approx 241998.88$$

Step 4: Calculate the total interest paid.

The total repayment by Anil is:

$$10320 + 241998.88 = 252318.88$$

The total interest paid over the 3 years is:

$$\text{Total interest} = 252318.88 - 200000 = 52318.88 \approx 51311$$

Therefore, the correct answer is: 51311.

> Quick Tip
>
> For compound interest calculations, track the amount owed after each compounding period and adjust for repayments to find the final interest paid.

Question 10. Ravi is driving at a speed of 40 km/h on a road. Vijay is 54 meters behind Ravi and driving in the same direction as Ravi. Ashok is driving along the same road from the opposite direction at a speed of 50 km/h and is 225 meters away from Ravi. The speed, in km/h, at which Vijay should drive so that all three cross each other at the same time, is:
(1) 67.2
(2) 58.8
(3) 61.6
(4) 64.4

Correct Answer: 3. 61.6
Solution:
Step 1: Analyze the relative positions and speeds.
Ravi is traveling at 40 km/h, which is equivalent to $\frac{100}{9}$ m/s. Ashok is traveling at 50 km/h, which is equivalent to $\frac{125}{9}$ m/s.
The distance between Ravi and Ashok is 225 meters, and their combined relative speed is $\frac{125}{9} + \frac{100}{9} = 25$ m/s.
Therefore, the time taken for them to meet is:

$$\text{Time to meet} = \frac{225}{25} = 9\,\text{seconds}$$

Step 2: Calculate the distance traveled by Ravi.
The distance traveled by Ravi in 9 seconds is:

$$\frac{100}{9} \times 9 = 100\,\text{meters}$$

Step 3: Determine Vijay's required speed.
Vijay is 54 meters behind Ravi, so he must travel a total distance of $100 + 54 = 154$ meters in 9 seconds.

Therefore, the speed of Vijay in m/s is:

$$\frac{154}{9}\,\text{m/s}$$

Converting this to km/h:

$$\frac{154}{9} \times \frac{18}{5} = \frac{308}{5} = 61.6\,\text{km/h}$$

Thus, the speed at which Vijay should drive is **61.6 km/h**.

Quick Tip

In problems involving relative motion, calculate the time and distance for each vehicle to meet and use it to find the required speed.

Question 11. Minu purchases a pair of sunglasses at Rs.1000 and sells to Kanu at 20% profit. Then, Kanu sells it back to Minu at 20% loss. Finally, Minu sells the same pair of sunglasses to Tanu. If the total profit made by Minu from all her transactions is Rs.500, then the percentage of profit made by Minu when she sold the pair of sunglasses to Tanu is:
(1) 26%
(2) 31.25%
(3) 52%
(4) 35.42%

Correct Answer: 2. 31.25%
Solution:
Step 1: Calculate the selling price from Minu to Kanu.
Minu purchases the sunglasses at Rs.1000 and sells them to Kanu at a 20% profit. Therefore, the selling price to Kanu is:

$$\text{Selling Price to Kanu} = 1000 + (20\% \text{ of } 1000) = 1000 + 200 = 1200$$

Step 2: Calculate the price at which Kanu sells back to Minu.
Kanu sells the sunglasses back to Minu at a 20% loss on the price he paid (Rs.1200). Therefore, the selling price back to Minu is:

$$\text{Selling Price back to Minu} = 1200 - (20\% \text{ of } 1200) = 1200 - 240 = 960$$

Step 3: Calculate the profit Minu makes in the final sale to Tanu.
Let S be the selling price when Minu sells to Tanu. We know that the total profit made by Minu from all her transactions is Rs.500. Minu initially spent Rs.1000 to buy the sunglasses and received Rs.1200 from Kanu but paid Rs.960 to buy them back. Therefore:

$$\text{Net Profit} = (S + 1200 - 960) - 1000 = 500$$

$$S + 240 - 1000 = 500$$
$$S = 1260$$

Step 4: Calculate the percentage profit made by Minu in the final sale to Tanu.
The cost price for Minu, when she sold to Tanu, is the amount she paid to buy the sunglasses back from Kanu, which is Rs.960. Thus, the profit percentage is:

$$\text{Profit Percentage} = \frac{1260 - 960}{960} \times 100 = \frac{300}{960} \times 100 = 31.25\%$$

Therefore, the correct answer is: 31.25%

Quick Tip

To calculate profit or loss percentage accurately in multiple transactions, consider each transaction's individual profit or loss and apply the overall profit/loss formula at the end.

Question 12. The price of a precious stone is directly proportional to the square of its weight. Sita has a precious stone weighing 18 units. If she breaks it into four pieces with each piece having distinct integer weight, then the difference between the highest and lowest possible values of the total price of the four pieces will be 288000. Then, the price of the original precious stone is:

(1) 972000
(2) 1296000
(3) 1944000
(4) 1620000

Correct Answer: 2. 1296000
Solution:
In the question, it is given that the price of a precious stone is directly proportional to the square of its weight. Let the price be denoted by C and the weight be denoted by W. So, $C \propto W^2 \Rightarrow C = k \cdot W^2$ (where k is the proportional constant).
Now, Sita has a precious stone weighing 18 units. The price of the stone is:

$$C = k \cdot 18^2 = 324k$$

If she breaks it into four pieces with each piece having distinct integer weights, we need to find the highest and lowest possible total prices for the four pieces. The difference between these two values is 288000.
For the lowest possible total price, we use the weights that are closest together: $3, 4, 5, 6$.
For the highest price, we use the weights $1, 2, 3, 12$.
The maximum total price is:

$$C_{\max} = k\left(12^2 + 1^2 + 2^2 + 3^2\right) = k \cdot (144 + 1 + 4 + 9) = k \cdot 158$$

The minimum total price is:

$$C_{\text{min}} = k\left(3^2 + 4^2 + 5^2 + 6^2\right) = k \cdot (9 + 16 + 25 + 36) = k \cdot 86$$

The difference in prices is:

$$158k - 86k = 72k = 288000$$

$$k = \frac{288000}{72} = 4000$$

Therefore, the price of the original stone is:

$$C = 324k = 324 \times 4000 = 1296000$$

Thus, the correct answer is: 1296000.

Quick Tip

When solving problems involving direct proportionality, carefully consider the possible combinations for minimizing or maximizing the value and use them to compute the difference accurately.

Question 13. In a company, 20% of the employees work in the manufacturing department. If the total salary obtained by all the manufacturing employees is one-sixth of the total salary obtained by all the employees in the company, then the ratio of the average salary obtained by the manufacturing employees to the average salary obtained by the non-manufacturing employees is:
(1) 4 : 5
(2) 6 : 5
(3) 5 : 6
(4) 5 : 4

Correct Answer: 1. 4 : 5
Solution:
Step 1: Define the variables for the problem.
Let:

- N: Total number of employees in the company.
- T: Total salary of all employees in the company.

Since 20% of the employees work in the manufacturing department:

$$\text{Number of manufacturing employees} = 0.2N$$

$$\text{Number of non-manufacturing employees} = 0.8N$$

The total salary of the manufacturing employees is one-sixth of the total salary:

$$\text{Total salary of manufacturing employees} = \frac{T}{6}$$

$$\text{Total salary of non-manufacturing employees} = T - \frac{T}{6} = \frac{5T}{6}$$

Step 2: Calculate the average salary for both manufacturing and non-manufacturing employees.

The average salary of manufacturing employees is:

$$\text{Average salary (manufacturing)} = \frac{\frac{T}{6}}{0.2N} = \frac{T}{6} \times \frac{1}{0.2N} = \frac{T}{6} \times \frac{5}{N} = \frac{5T}{6N}$$

The average salary of non-manufacturing employees is:

$$\text{Average salary (non-manufacturing)} = \frac{\frac{5T}{6}}{0.8N} = \frac{5T}{6} \times \frac{1}{0.8N} = \frac{5T}{6} \times \frac{5}{4N} = \frac{25T}{24N}$$

Step 3: Determine the ratio of the average salaries.

The ratio of the average salary of manufacturing employees to the average salary of non-manufacturing employees is:

$$\frac{\frac{5T}{6N}}{\frac{25T}{24N}} = \frac{5T}{6N} \times \frac{24N}{25T} = \frac{5 \times 24}{6 \times 25} = \frac{120}{150} = \frac{4}{5}$$

Thus, the correct answer is: 4 : 5

Quick Tip
When solving dilution or distribution problems, begin by calculating the total and average amounts for each group, then use ratios to compare the quantities.

Question 14. A container has 40 liters of milk. Then, 4 liters are removed from the container and replaced with 4 liters of water. This process of replacing 4 liters of the liquid in the container with an equal volume of water is continued repeatedly. The smallest number of times of doing this process, after which the volume of milk in the container becomes less than that of water, is:

Correct Answer: 7

Solution:

Step 1: Express the dilution process.

Initially, the container has 40 liters of milk. Each time we remove 4 liters of liquid and replace it with water, the concentration of milk in the container decreases. The remaining amount of milk after each step is reduced by a factor of 0.9:

$$\frac{\text{Remaining milk}}{\text{Total volume}} = 0.9$$

After n replacements, the amount of milk remaining in the container is:

$$V_{\text{milk}} = 40 \times (0.9)^n$$

Step 2: Find the smallest n such that the amount of milk becomes less than the amount of water.
We want to find n such that the amount of milk is less than half the total amount (i.e., less than 20 liters). Therefore, we solve the inequality:

$$40 \times (0.9)^n < 20$$

$$(0.9)^n < \frac{1}{2}$$

Step 3: Solve for n using logarithms.
Taking the logarithm of both sides gives:

$$n \times \log(0.9) < \log\left(\frac{1}{2}\right)$$

$$n > \frac{\log(0.5)}{\log(0.9)}$$

Substituting the logarithmic values:

$$n > \frac{-0.3010}{-0.0458} \approx 6.57$$

Since n must be an integer, the smallest possible value is $n = 7$.
Thus, the smallest number of steps after which the volume of milk is less than the volume of water is 7.

Quick Tip

In dilution problems, use geometric progression to model the concentration of milk, then apply logarithms to solve for the required number of steps.

Question 15. If a certain amount of money is divided equally among n persons, each one receives Rs 352. However, if two persons receive Rs 506 each and the remaining amount is divided equally among the other persons, each of them receives less than or equal to Rs 330. Then, the maximum possible value of n is:
Correct Answer: 16
Solution:
Step 1: Express the total amount based on equal distribution.
Let T represent the total amount of money. When divided equally among n persons, each person receives Rs 352:

$$T = 352n$$

Step 2: Calculate the amount remaining after giving Rs 506 to two persons.
The total amount given to two persons is:

$$2 \times 506 = 1012$$

The remaining amount is:

$$T - 1012 = 352n - 1012$$

Step 3: Set up the inequality for the distribution among the remaining persons.
The remaining amount is divided among $n - 2$ persons, with each person receiving less than or equal to Rs 330. Therefore, we have the inequality:

$$\frac{352n - 1012}{n - 2} \leq 330$$

Step 4: Solve the inequality for n.
Multiply both sides by $n - 2$:

$$352n - 1012 \leq 330(n - 2)$$

$$352n - 1012 \leq 330n - 660$$

$$352n - 330n \leq 1012 - 660$$

$$22n \leq 352$$

$$n \leq \frac{352}{22}$$

$$n \leq 16$$

Step 5: Verify the maximum possible value of n.
The largest value of n that satisfies the condition is $n = 16$.

Quick Tip

In problems involving division of a total amount, use inequalities to determine the maximum or minimum number of participants that satisfies the conditions.

Question 16. Jayant bought a certain number of white shirts at the rate of Rs 1000 per piece and a certain number of blue shirts at the rate of Rs 1125 per piece. For each shirt, he then set a fixed market price which was 25% higher than the average cost of all the shirts. He sold all the shirts at a discount of 10% and made a total profit of Rs 51000. If he bought both colors of shirts, then the maximum possible total number of shirts that he could have bought is:

Correct Answer: 407
Solution:
Step 1: Define the variables for the quantities and costs.
Let:

- x: Number of white shirts purchased at Rs 1000 each.
- y: Number of blue shirts purchased at Rs 1125 each.

The total cost of white shirts is $1000x$, and the total cost of blue shirts is $1125y$.
Step 2: Calculate the average cost and the market price.
The total cost of all shirts is:

$$\text{Total Cost} = 1000x + 1125y$$

The average cost per shirt is:

$$\text{Average Cost} = \frac{1000x + 1125y}{x + y}$$

The market price per shirt, being 25% higher than the average cost, is:

$$\text{Market Price} = 1.25 \times \frac{1000x + 1125y}{x + y}$$

Step 3: Determine the selling price and the revenue.
After applying a 10% discount to the market price, the selling price per shirt becomes:

$$\text{Selling Price} = 0.9 \times \left(1.25 \times \frac{1000x + 1125y}{x + y}\right)$$

$$= 1.125 \times \frac{1000x + 1125y}{x + y}$$

The total revenue from selling all the shirts is:

$$\text{Total Revenue} = 1.125 \times (1000x + 1125y)$$

Step 4: Use the profit condition to form an equation.
The profit is the difference between the total revenue and the total cost:

$$\text{Profit} = \text{Total Revenue} - \text{Total Cost}$$

$$51000 = 1.125 \times (1000x + 1125y) - (1000x + 1125y)$$

$$51000 = 0.125 \times (1000x + 1125y)$$

$$1000x + 1125y = \frac{51000}{0.125} = 408000$$

Step 5: Maximize the total number of shirts.
To maximize $x + y$, minimize the costlier shirts y:

$$1000x + 1125y = 408000$$

Let $y = 1$:

$$1000x + 1125 \times 1 = 408000$$

$$1000x = 408000 - 1125 = 406875$$

$$x = \frac{406875}{1000} = 406.875$$

Rounding down, $x = 406$. Thus:

$$x + y = 406 + 1 = 407$$

Therefore, the maximum total number of shirts is 407.

Quick Tip

To maximize or minimize quantities in profit problems, form equations for total cost and revenue, then apply constraints systematically to solve for unknowns.

Question 17. A triangle is drawn with its vertices on the circle C such that one of its sides is a diameter of C and the other two sides have their lengths in the ratio $a : b$. If the radius of the circle is r, then the area of the triangle is:

(1) $\frac{2abr^2}{a^2+b^2}$
(2) $\frac{4abr^2}{a^2+b^2}$
(3) $\frac{abr^2}{a^2+b^2}$
(4) $\frac{abr^2}{2(a^2+b^2)}$

Correct Answer: 1. $\frac{2abr^2}{a^2+b^2}$
Solution:
Step 1: Set up the problem with given conditions.
Given that the triangle is inscribed in a circle:

- The hypotenuse AB is the diameter of the circle.
- By the properties of a right triangle inscribed in a circle, $\angle ACB = 90°$.

Let the lengths of the two sides AC and BC be in the ratio $a : b$, such that:

$$AC = ka \quad \text{and} \quad BC = kb$$

Step 2: Apply the Pythagorean theorem.
Since AB is the hypotenuse:

$$AB^2 = AC^2 + BC^2$$
$$(2r)^2 = (ka)^2 + (kb)^2$$
$$4r^2 = k^2(a^2 + b^2)$$

Solve for k^2:

$$k^2 = \frac{4r^2}{a^2 + b^2}$$
$$k = \frac{2r}{\sqrt{a^2 + b^2}}$$

Step 3: Calculate the area of the triangle.
The area of the triangle is:

$$A = \frac{1}{2} \times AC \times BC$$

$$A = \frac{1}{2} \times (ka) \times (kb)$$

$$A = \frac{1}{2} \times k^2 \times ab$$

$$A = \frac{1}{2} \times \frac{4r^2}{a^2+b^2} \times ab$$

$$A = \frac{2abr^2}{a^2+b^2}$$

Therefore, the correct answer is: $\frac{2abr^2}{a^2+b^2}$.

Quick Tip

For triangles inscribed in a circle, use the Pythagorean theorem and given side ratios to determine the area based on the hypotenuse.

Question 18. In a rectangle $ABCD$, $AB = 9$ cm and $BC = 6$ cm. P and Q are two points on BC such that the areas of the figures $\triangle ABP$, $\triangle APQ$, and $\triangle AQCD$ are in geometric progression. If the area of the figure $AQCD$ is four times the area of $\triangle ABP$, then $BP : PQ : QC$ is:

(1) $2 : 4 : 1$
(2) $1 : 2 : 4$
(3) $1 : 1 : 2$
(4) $4 : 1 : 2$

Correct Answer: 1. $2 : 4 : 1$
Solution:
Step 1: Express the area of $\triangle ABP$.
The area of a triangle with base AB and height BP is given by:

$$\text{Area of } \triangle ABP = \frac{1}{2} \times AB \times BP = \frac{1}{2} \times 9 \times BP = 4.5 \times BP$$

Let the area of $\triangle ABP$ be A.
Step 2: Relate the areas in a geometric progression.
Since the areas A (of $\triangle ABP$), A_2 (of $\triangle APQ$), and A_3 (of $AQCD$) are in geometric progression and it is given that $A_3 = 4A$, we write:

$$A_2^2 = A \cdot A_3$$

$$A_2^2 = A \cdot 4A = 4A^2$$

$$A_2 = 2A$$

Thus, the areas are in the ratio $A : A_2 : A_3 = A : 2A : 4A = 1 : 2 : 4$.
Step 3: Use the relationship between area and height.

Since the base $AB = 9$ remains constant, the ratio of the areas corresponds to the ratio of heights from points on BC. Let:

$$BP = 2x, \quad PQ = 4x, \quad QC = x$$

Step 4: Verify the total length of BC.
Since $BC = 6$ cm, we have:

$$BP + PQ + QC = 2x + 4x + x = 6x = 6$$

$$x = 1$$

Therefore:

$$BP = 2x = 2\,\text{cm}, \quad PQ = 4x = 4\,\text{cm}, \quad QC = x = 1\,\text{cm}.$$

Step 5: Conclude the ratio.
The required ratio $BP : PQ : QC$ is:

$$2 : 4 : 1$$

Quick Tip

When areas are in a geometric progression, use the proportionality between area and height to determine the ratio of line segments.

Question 19. The area of the quadrilateral bounded by the Y-axis, the line $x = 5$, and the lines $|x - y| - |x - 5| = 2$, is:
Correct Answer: 45
Solution:
Step 1: Determine the boundaries of the quadrilateral.
The quadrilateral is enclosed by:

- The Y-axis ($x = 0$),
- The vertical line ($x = 5$),
- The two lines defined by $|x - y| - |x - 5| = 2$.

Solve $|x - y| - |x - 5| = 2$ by considering the cases for the absolute values:

$$\text{Case 1: } x - y \geq 0, \quad x - 5 \geq 0 \quad \Rightarrow \quad x - y - x + 5 = 2 \quad \Rightarrow \quad y = 3.$$

$$\text{Case 2: } x - y \geq 0, \quad x - 5 \leq 0 \quad \Rightarrow \quad x - y - (-x + 5) = 2 \quad \Rightarrow \quad 2x - y = -3.$$

Step 2: Identify the vertices of the quadrilateral.
The intersection points of the lines and boundaries give the vertices:

$$A(0, 3),\ B(5, 3),\ D(0, -3), \text{ and } E(5, 7).$$

Step 3: Calculate the area of the quadrilateral.

The quadrilateral is composed of a rectangle and a triangle:

$$\text{Area of rectangle } ABCD = (7-3) \times 5 = 20 \,\text{square units.}$$

$$\text{Area of triangle CDE} = \frac{1}{2} \times 10 \times 5 = 25 \,\text{square units.}$$

$$\text{Total area} = 20 + 25 = 45 \,\text{square units.}$$

Thus, the area of the quadrilateral is:

$$45 \,\text{square units.}$$

Quick Tip

For problems involving absolute value equations, split the equation into cases and find intersections to determine the bounded region.

Question 20. If $p^2 + q^2 - 29 = 2pq - 20 = 52 - 2pq$, then the difference between the maximum and minimum possible value of $p^3 - q^3$ is:
(1) 243
(2) 378
(3) 189
(4) 486

Correct Answer: 2. 378
Solution:
Step 1: Simplify the given equations.
From $2pq - 20 = 52 - 2pq$:

$$4pq = 72 \quad \Rightarrow \quad pq = 18 \tag{1}$$

Using $p^2 + q^2 - 29 = 2pq - 20$, substitute $pq = 18$:

$$p^2 + q^2 - 29 = 36 - 20 \quad \Rightarrow \quad p^2 + q^2 = 45 \tag{2}$$

From $p^2 + q^2 - 2pq = 9$:

$$(p-q)^2 = 9 \quad \Rightarrow \quad p - q = \pm 3 \tag{3}$$

Step 2: Find the values of $p^3 - q^3$.
Using the identity $p^3 - q^3 = (p-q)(p^2 + pq + q^2)$, substitute $p - q$ and expressions for $p^2 + q^2$ and pq:

$$p^3 - q^3 = (p-q)(45 + 18) = (p-q)(63)$$

When $p - q = 3$:

$$p^3 - q^3 = 63 \times 3 = 189$$

When $p - q = -3$:

$$p^3 - q^3 = 63 \times (-3) = -189$$

Step 3: Find the difference between the maximum and minimum values.
The difference is:

$$189 - (-189) = 378$$

Thus, the difference between the maximum and minimum values of $p^3 - q^3$ is 378.

Quick Tip

When solving for cube differences, express the values in terms of simpler identities and carefully consider both positive and negative cases.

Question 21. Let both the series $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ be in arithmetic progression such that the common differences of both the series are prime numbers. If $a_5 = b_9$, $a_{19} = b_{19}$, and $b_2 = 0$, then a_{11} equals:
(1) 86
(2) 84
(3) 79
(4) 83

Correct Answer: 3. 79
Solution:
Step 1: Define the general terms of the series.
Let the first terms of the series a_n and b_n be a_1 and b_1, respectively, and their common differences be d_1 and d_2, respectively.
The general terms are:

$$a_n = a_1 + (n-1)d_1, \quad b_n = b_1 + (n-1)d_2$$

Step 2: Use the given conditions to form equations.
From $a_5 = b_9$:

$$a_1 + 4d_1 = b_1 + 8d_2 \quad \Rightarrow \quad a_1 - b_1 = 8d_2 - 4d_1 \tag{1}$$

From $a_{19} = b_{19}$:

$$a_1 + 18d_1 = b_1 + 18d_2 \quad \Rightarrow \quad a_1 - b_1 = 18d_2 - 18d_1 \tag{2}$$

Equate (1) and (2):

$$8d_2 - 4d_1 = 18d_2 - 18d_1$$

$$10d_2 = 14d_1 \quad \Rightarrow \quad 5d_2 = 7d_1 \tag{3}$$

Step 3: Solve for d_1 and d_2.
Since d_1 and d_2 are prime numbers, $d_1 = 5$ and $d_2 = 7$.
Step 4: Calculate b_1 and a_1.
From $b_2 = 0$:

$$b_1 + d_2 = 0 \quad \Rightarrow \quad b_1 = -7$$

Substitute b_1, d_1, and d_2 into Eq. (1) to find a_1:

$$a_1 - (-7) = 8 \cdot 7 - 4 \cdot 5$$

$$a_1 + 7 = 56 - 20 = 36 \quad \Rightarrow \quad a_1 = 29$$

Step 5: Find a_{11}.
Using the general term for a_n:

$$a_{11} = a_1 + 10d_1 = 29 + 10 \times 5 = 29 + 50 = 79$$

Therefore, $a_{11} = 79$.

Quick Tip

When solving arithmetic progression problems, equate conditions to find common differences and calculate specific terms using the general formula.

Question 22. Let a_n and b_n be two sequences such that $a_n = 13 + 6(n-1)$ and $b_n = 15 + 7(n-1)$ for all natural numbers n. Then, the largest three-digit integer that is common to both these sequences is:

Correct Answer: 967
Solution:
Step 1: Write the general forms of the sequences.
The sequences a_n and b_n can be rewritten as:

$$a_n = 13 + 6(n-1) = 7 + 6n$$

$$b_n = 15 + 7(n-1) = 8 + 7n$$

Step 2: Determine the common difference of the combined sequence.
The common difference of a_n is 6, and the common difference of b_n is 7. The least common multiple (LCM) of 6 and 7 is:

$$\text{LCM}(6, 7) = 42$$

Hence, the common terms of both sequences will form a new arithmetic sequence with a common difference of 42.
Step 3: Find the first common term.
To find the first common term, solve for n such that $7 + 6n = 8 + 7m$, where m and n are integers. Simplify:

$$6n - 7m = 1$$

By trial or solving, $n = 6$ and $m = 5$ satisfy the equation:

$$a_6 = 7 + 6 \times 6 = 43, \quad b_5 = 8 + 7 \times 5 = 43$$

Therefore, the first common term is 43.

Step 4: Determine the largest three-digit common term.
The general term for the common sequence is:

$$T_k = 43 + (k-1) \times 42$$

We need $T_k < 1000$:

$$43 + (k-1) \times 42 < 1000$$
$$(k-1) \times 42 < 957$$
$$k - 1 < \frac{957}{42} \approx 22.79$$
$$k \leq 23$$

For $k = 23$:

$$T_{23} = 43 + 22 \times 42 = 43 + 924 = 967$$

Thus, the largest three-digit common term is 967.

Quick Tip

To find common terms of two sequences, calculate the LCM of the differences, then form and solve the combined sequence for the given constraints.