

AP EDCET 2025 Maths Question Paper with Solution

Time Allowed :120 Minutes	Maximum Marks :150	Total Questions :150
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General Instructions

Read the following instructions very carefully and strictly follow them:

1. The duration of the test is **3 hours**.
2. Each question carries **1 mark**. There is **no negative marking**.
3. The medium of the question paper is **English** and **Telugu/Urdu** (as opted by the candidate).
4. Candidates must report at the test center **at least 90 minutes** before the commencement of the examination.
5. Candidates must carry:
 - **Filled-in Online Application Form** (printout)
 - **Valid Photo ID Proof** (Aadhaar, Passport, PAN, Driving Licence, etc.)
6. Rough work must be done only on the provided rough sheets. Additional sheets will not be provided.
7. Use of **calculators, mobile phones, smart watches, or any electronic devices** is strictly prohibited.
8. Follow the invigilator's instructions carefully. Any malpractice will result in **disqualification**.

1 General English

Read the passage and answer questions based on the passage: Moral education centres upon this conception of the school as a mode of social life, that the best and the deepest moral training is precisely that which one gets through having to enter into proper relations with others in a unity of work and thought. The present educational systems, so far as they destroy or neglect this unity, render it difficult or impossible to get any genuine, regular moral training.

1. According to the writer, the school is a form of _____.

- (A) individual existence
(B) religious life
(C) spiritual life
(D) social life

Correct Answer: (D) social life

Solution:

The first sentence of the passage explicitly states, "Moral education centres upon this conception of the school as a mode of social life..."

This directly supports the idea that the writer considers the school a form of social life.

Therefore, option (D) is the correct answer.

Quick Tip

In reading comprehension, the first sentence often introduces the main topic or the author's primary argument. Always analyze it carefully for direct answers.

2. Moral training, according to the passage, effects _____.

- (A) a unity of work and thought
- (B) entering into proper relations with oneself
- (C) the skills of work
- (D) the skills of thought

Correct Answer: (A) a unity of work and thought

Solution:

The passage states, "...the best and the deepest moral training is precisely that which one gets through having to enter into proper relations with others in a unity of work and thought."

This sentence clearly establishes that the effect or outcome of the deepest moral training is the achievement of "a unity of work and thought."

The other options are not mentioned as the direct effect of moral training in the text.

Hence, option (A) is correct.

Quick Tip

Look for keywords from the question in the passage. The question asks what moral training "effects," and the passage directly links "moral training" to "a unity of work and thought."

3. The writer's attitude to the present education system can be described as _____.

- (A) Critical
- (B) adulatory
- (C) defensive

(D) indifferent

Correct Answer: (A) Critical

Solution:

The writer claims that present educational systems "destroy or neglect this unity" and "render it difficult or impossible to get any genuine, regular moral training."

The use of strong, negative words like "destroy," "neglect," and "impossible" indicates a disapproving or fault-finding tone.

This tone is best described as "Critical."

"Adulatory" means praising, "defensive" means protecting, and "indifferent" means unconcerned, none of which fit the writer's language.

Quick Tip

An author's attitude is revealed through their choice of words (diction). Pay attention to adjectives and verbs with strong positive or negative connotations.

4. The central concern of the passage is _____.

(A) spiritual education / spiritual training

(B) work education / work training

(C) moral education / moral training

(D) thought education / thought training

Correct Answer: (C) moral education / moral training

Solution:

The passage begins by introducing the topic of "Moral education."

It further elaborates on "the best and the deepest moral training."

The entire passage revolves around the concept of how proper moral training is achieved within the school as a social environment.

Therefore, the central concern is undoubtedly moral education and moral training.

Quick Tip

To find the central theme of a passage, identify the key concept that is consistently discussed from the beginning to the end.

5. Find the word from the passage that is a synonym of accurate.

- (A) Conception
- (B) Neglect
- (C) Precise
- (D) Genuine

Correct Answer: (C) Precise

Solution:

The word "accurate" means correct in all details; exact.

Let's examine the meanings of the options:

- (A) Conception: an idea or a plan.
- (B) Neglect: to fail to give proper care or attention.
- (C) Precise: marked by exactness and accuracy of expression or detail. This is a direct synonym of accurate.
- (D) Genuine: truly what something is said to be; authentic.

The passage uses the word "precisely," the adverb form of "precise." Among the given options, "Precise" is the best synonym for "accurate."

Quick Tip

Synonym questions test your vocabulary. If you are unsure, try substituting each option into a sentence where "accurate" would fit and see which one makes the most sense.

6. Correct the underline word in the sentence below, by substituting with the correct word: I pray for god to give me strength

- (A) regarding
- (B) to
- (C) of
- (D) with

Correct Answer: (B) to

Solution:

The verb "pray" is followed by the preposition "to" when indicating the recipient of the prayer.

The correct grammatical structure is "pray to someone/something."

For example, "People pray to their gods for guidance."

Therefore, "I pray to god to give me strength" is the correct sentence.

Quick Tip

Prepositions are often governed by the verbs they follow. Memorizing common verb-preposition combinations (like "pray to," "listen to," "rely on") is crucial for grammar questions.

7. I think he is _____ honest man.

- (A) the
- (B) an
- (C) a
- (D) no article

Correct Answer: (B) an

Solution:

The choice between the indefinite articles "a" and "an" depends on the sound of the first letter of the word that follows.

The word "honest" begins with the letter 'h', but it is silent. The word starts with a vowel sound ('o').

When a word begins with a vowel sound, the article "an" is used.

Therefore, the correct sentence is "I think he is an honest man."

Quick Tip

The use of 'a' or 'an' is determined by the initial sound of the next word, not the letter itself. For example, it's "an hour" (silent 'h') but "a university" ('y' sound).

8. This is _____ car my friend bought last week.

- (A) the
- (B) an
- (C) a
- (D) no article

Correct Answer: (A) the

Solution:

The sentence is referring to a specific, particular car.

The clause "my friend bought last week" identifies which car is being talked about.

When referring to a specific or previously identified noun, the definite article "the" is used.

Using "a" would imply it is one of many cars, which contradicts the specific information given.

Therefore, "the" is the correct article.

Quick Tip

Use the definite article 'the' when the noun is specific or unique in the context. If the sentence provides identifying information about the noun (like "the man who lives next door" or "the book on the table"), 'the' is almost always correct.

9. Indeed, I am not sure _____ his success.

- (A) of
- (B) regarding
- (C) about
- (D) off

Correct Answer: (A) of

Solution:

The adjective "sure" is followed by the preposition "of" or "about" to indicate what someone is certain of.

While "sure about" is also grammatically correct, "sure of" is a very common and standard collocation, especially in formal contexts.

In the given options, "of" is the most appropriate preposition to complete the phrasal adjective "sure of".

The sentence should be "Indeed, I am not sure of his success."

Quick Tip

Many adjectives are followed by specific prepositions (e.g., "proud of," "interested in," "afraid of"). Learning these "collocations" is key to mastering English prepositions.

10. They saw him fell _____ his horse.

- (A) after
- (B) against
- (C) of
- (D) off

Correct Answer: (D) off

Solution:

The question requires the correct preposition to form a phrasal verb with "fell." The past tense of "fall" is "fell".

The phrasal verb "fall off" means to drop from a higher level or position to a lower one.

This meaning fits the context of someone coming down from a horse.

One "falls off" a horse.

Therefore, the correct preposition is "off."

Quick Tip

Phrasal verbs (a verb + a preposition or adverb) often have meanings that are different from the individual words. "Fall off" specifically means to detach and drop from something.

11. The _____ will meet again, to discuss the matter.

- (A) Comittee
- (B) Commitee
- (C) Committee
- (D) Comitee

Correct Answer: (C) Committee

Solution:

The question tests the correct spelling of the word for a group of people appointed for a specific function.

Let's analyze the options:

- (A) Comittee - Incorrect spelling.
- (B) Commitee - Incorrect spelling.
- (C) Committee - This is the correct spelling. It has a double 'm', a double 't', and a double 'e'.
- (D) Comitee - Incorrect spelling.

Therefore, the correctly spelled word is "Committee".

Quick Tip

A common mnemonic for spelling 'committee' is to remember that it has three sets of double letters: double 'm', double 't', and double 'e'.

12. The Cricket team are in a _____ mood.

- (A) buoyant
- (B) buoyent
- (C) boyant
- (D) buyant

Correct Answer: (A) buoyant

Solution:

The question requires the correctly spelled word that means cheerful and optimistic.

Let's examine the spellings:

- (A) buoyant - This is the correct spelling of the word. It means cheerful or optimistic.
- (B) buoyent - Incorrect spelling.
- (C) boyant - Incorrect spelling.
- (D) buyant - Incorrect spelling.

The correct word to describe a cheerful mood for the team is "buoyant".

Quick Tip

In English, the 'uoy' letter combination is rare but appears in words like 'buoyant'. Associate the word with 'buoy', an object that floats, which can help you remember its meaning (light, cheerful) and spelling.

13. Had he studied hard, he _____ passed.

- (A) has
- (B) had been
- (C) would have
- (D) have been

Correct Answer: (C) would have

Solution:

This sentence is an example of the third conditional, used for hypothetical situations in the past.

The structure of a third conditional sentence is: If + past perfect, ...would have + past participle.

The clause "Had he studied hard" is an inverted form of "If he had studied hard" (past perfect).

Therefore, the main clause must follow the structure "subject + would have + past participle".

The past participle "passed" is already given. The missing part is "would have".

The correct sentence is: "Had he studied hard, he would have passed."

Quick Tip

Recognize the inverted third conditional structure: "Had + subject + past participle...". The main clause will always use "would have," "could have," or "might have" followed by a past participle.

14. I wish I _____ her address.

- (A) know
- (B) have known
- (C) may know
- (D) knew

Correct Answer: (D) knew

Solution:

The expression "I wish..." is used to express a desire for a situation that is not true in the present. This grammatical structure requires the subjunctive mood.

For wishes about the present, the verb that follows "I wish..." should be in the simple past tense.

The simple present "know" is incorrect.

The present perfect "have known" is used for wishes about the past (e.g., "I wish I had known...").

The modal "may know" is incorrect in this structure.

The simple past tense of "know" is "knew". This correctly expresses a current wish for a different reality.

Therefore, the correct sentence is "I wish I knew her address."

Quick Tip

When using "I wish" to talk about a present situation you want to be different, always use the simple past tense for the verb that follows. For example, "I wish I had more time" (but I don't).

15. Incredulous means _____

- (A) obliterate
- (B) skeptical
- (C) practical
- (D) insane

Correct Answer: (B) skeptical

Solution:

The word "incredulous" means unwilling or unable to believe something.

Let's look at the options:

- (A) obliterate: to destroy utterly; wipe out.
- (B) skeptical: not easily convinced; having doubts or reservations. This is a very close synonym for incredulous.
- (C) practical: concerned with the actual doing or use of something rather than with theory and ideas.
- (D) insane: in a state of mind which prevents normal perception, behaviour, or social interaction; seriously mentally ill.

The best match for the meaning of "incredulous" is "skeptical."

Quick Tip

Don't confuse "incredulous" (unwilling to believe) with "incredible" (hard to believe). An incredulous person would find an incredible story hard to believe.

16. Precipitous means _____

- (A) Very steep
- (B) very deep
- (C) presumption
- (D) nascent

Correct Answer: (A) Very steep

Solution:

The word "precipitous" is an adjective. Its primary meaning is dangerously high or steep.

Let's analyze the options:

- (A) Very steep: This is the direct definition of precipitous.
- (B) very deep: This is a different dimension. The word for this is profound or abyssal.
- (C) presumption: an idea that is taken to be true on the basis of probability. This is a noun.
- (D) nascent: just coming into existence and beginning to display signs of future potential.

Therefore, the correct meaning of "precipitous" is "Very steep".

Quick Tip

The word "precipitous" is related to "precipice," which is a very steep rock face or cliff. This connection can help you remember its meaning.

17. _____ is a synonym for ingenuous.

- (A) Clever
- (B) ingenious
- (C) honest
- (D) Indecent

Correct Answer: (C) honest

Solution:

The word "ingenuous" means to be innocent, naive, and candid, often in a way that shows a lack of worldly experience. It implies sincerity and a lack of deceit.

Let's analyze the options:

- (A) Clever: quick to understand, learn, and devise or apply ideas.
- (B) Ingenious: (of a person) clever, original, and inventive. This is often confused with ingenuous.
- (C) honest: free of deceit; truthful and sincere. This quality is a core part of being ingenuous.
- (D) Indecent: not conforming with generally accepted standards of behavior.

Among the given choices, "honest" is the closest synonym for "ingenuous" as it captures the essence of sincerity and truthfulness.

Quick Tip

Be careful not to confuse "ingenuous" (innocent, naive) with "ingenious" (clever, inventive). They look similar but have very different meanings. The 'u' in ingenuous can remind you of 'unworldly'.

18. Keenness is a synonym for _____

- (A) abjure
- (B) enigma
- (C) skip
- (D) acumen

Correct Answer: (D) acumen

Solution:

"Keenness" refers to the quality of being eager or enthusiastic, or having sharpness or acuteness of mind.

Let's examine the options:

- (A) abjure: to solemnly renounce a belief, cause, or claim.
- (B) enigma: a person or thing that is mysterious, puzzling, or difficult to understand.
- (C) skip: to move along lightly, stepping from one foot to the other with a hop or bounce.
- (D) acumen: the ability to make good judgments and quick decisions, typically in a particular domain. This relates directly to mental sharpness or keenness.

Therefore, "acumen" is the best synonym for the mental aspect of "keenness."

Quick Tip

"Keenness" and "acumen" both describe a sharp intellect. While keenness can also refer to enthusiasm, in a vocabulary context, it often points to mental sharpness, making acumen a strong synonym.

19. Invincible is an antonym for _____

- (A) conquerable
- (B) transparent
- (C) fiasco
- (D) instil

Correct Answer: (A) conquerable

Solution:

The word "invincible" means too powerful to be defeated or overcome.

An antonym is a word with the opposite meaning. So, we are looking for a word that means "able to be defeated."

Let's analyze the options:

- (A) conquerable: able to be overcome or defeated. This is the direct opposite of invincible.
- (B) transparent: allowing light to pass through so that objects behind can be distinctly seen.
- (C) fiasco: a complete failure, especially a ludicrous or humiliating one.

(D) instil: to gradually but firmly establish an idea or attitude in a person's mind.

The correct antonym for "invincible" is "conquerable."

Quick Tip

Breaking down words can help find their meaning. "Invincible" comes from 'in-' (not) and 'vincere' (to conquer). So, it means "not conquerable." The opposite is simply "conquerable."

20. Change the following into a simple sentence: He wanted to play with his friends and so he finished his home work quickly.

(A) As he wanted to play with his friends, he finished his homework quickly

(B) So as to play with his friends, he finished his home work quickly

(C) He wants to play with his friends and they he finished his home work quickly

(D) In order to play with his friends, he finished his home work quickly

Correct Answer: (D) In order to play with his friends, he finished his home work quickly

Solution:

The original sentence is a compound sentence, containing two independent clauses joined by the conjunction "and so".

A simple sentence contains only one independent clause. We need to convert one of the clauses into a phrase.

(A) "As he wanted to play..." is a complex sentence because "As..." creates a subordinate clause.

(B) "So as to play..." is grammatically awkward. The correct phrase is "so as to + verb". The rest of the sentence doesn't flow correctly.

(C) This option changes the tense ("wants") and is grammatically incorrect ("and they he finished").

(D) "In order to play with his friends" is an infinitive phrase of purpose. This leaves "he finished his home work quickly" as the single independent clause. This structure correctly forms a simple sentence.

Quick Tip

To convert a compound or complex sentence into a simple sentence, try turning one of the clauses into a phrase (e.g., prepositional phrase, participial phrase, or infinitive phrase).

21. Change the following sentence into a complex sentence: "Despite several obstacles, she succeeded finally"

- (A) In spite of several obstacles, she succeeded finally
- (B) Notwithstanding several obstacles, she succeeded finally
- (C) Though there were several obstacles, she succeeded finally
- (D) There were several obstacles and still she succeeded finally

Correct Answer: (C) Though there were several obstacles, she succeeded finally

Solution:

The original sentence is a simple sentence, with "Despite several obstacles" acting as a prepositional phrase.

A complex sentence must have one independent clause and at least one dependent (subordinate) clause.

- (A) "In spite of..." is just another prepositional phrase, so this is still a simple sentence.
- (B) "Notwithstanding..." is also a prepositional phrase, resulting in a simple sentence.
- (C) "Though there were several obstacles" is a dependent clause (it cannot stand alone as a sentence). "she succeeded finally" is an independent clause. A dependent clause plus an independent clause makes a complex sentence.
- (D) This is a compound sentence, as it has two independent clauses joined by "and still".

Therefore, option (C) is the only correct transformation into a complex sentence.

Quick Tip

A complex sentence is formed using subordinating conjunctions like 'although', 'though', 'because', 'since', 'while', 'when', 'if', etc. These words introduce a dependent clause.

22. Change the following sentence into passive voice: "I saw him opening the door"

- (A) I saw the door and he was opening it
- (B) Opening the door, he was seen by me
- (C) I saw the door being opened by him
- (D) He was seen by me while opening the door

Correct Answer: (C) I saw the door being opened by him

Solution:

The original sentence is in the active voice: "I (subject) saw (verb) him (object) opening the door (participle phrase modifying the object)".

When the active sentence structure is 'subject + verb + object + present participle', one way to form the passive is 'subject + verb + object + being + past participle'.

Applying this rule:

The first part "I saw" can remain active, as the action of seeing is done by 'I'.

The second part "him opening the door" becomes passive. The object of the participle phrase, "the door," becomes the subject of the passive construction.

The present participle "opening" becomes "being opened".

"him" becomes the agent "by him".

Combining these gives: "I saw the door being opened by him."

Option (D), "He was seen by me while opening the door," is also passive but slightly changes the emphasis and structure. Option (C) is a more direct and standard conversion for this type of sentence.

Quick Tip

For sentences involving verbs of perception (see, hear, watch) followed by an object and a present participle, a common passive structure is to keep the main verb active and make the participial phrase passive: "I saw [object] being [past participle]".

23. Change the following into active voice: "I was kept waiting for two hours by him"

(A) I waited for two hours because of him

(B) He kept me waiting for two hours

(C) He is waiting for me for two hours

(D) I was waiting for two hours for him

Correct Answer: (B) He kept me waiting for two hours

Solution:

The given sentence is in the passive voice.

Subject: I

Verb: was kept waiting

Agent: by him

To change to active voice, the agent ("by him") becomes the subject ("He").

The passive verb ("was kept") becomes the active verb ("kept").

The subject of the passive sentence ("I") becomes the object of the active sentence ("me").

So, "He kept me waiting for two hours." is the correct active form.

Option (A) changes the verb and meaning.

Options (C) and (D) use the wrong tense and structure.

Quick Tip

To convert from passive to active voice, identify the agent (the one performing the action, usually after 'by'). This agent becomes the new subject. Then, change the verb from its passive form (be + past participle) to its active form.

24. Change the following into indirect speech: "I will bring my laptop tomorrow", she said

- (A) She said she would bring my laptop the next day
- (B) She said that she would bring her laptop the next day
- (C) She said she will bring her laptop the next day
- (D) She said she would bring her laptop tomorrow

Correct Answer: (B) She said that she would bring her laptop the next day

Solution:

To convert direct speech to indirect speech, several changes are necessary when the reporting verb ("said") is in the past tense.

1. Tense change: "will bring" (future simple) changes to "would bring" (conditional).
2. Pronoun change: "I" changes to "she" (referring to the speaker). "my" changes to "her".
3. Adverb of time change: "tomorrow" changes to "the next day".
4. The conjunction "that" is often used to introduce the reported speech, although it is sometimes optional.

Applying these rules:

"I will bring my laptop tomorrow" becomes "that she would bring her laptop the next day".

So, the full sentence is "She said that she would bring her laptop the next day."

Option (A) incorrectly uses "my" instead of "her".

Option (C) incorrectly uses "will" instead of "would".

Option (D) incorrectly uses "tomorrow" instead of "the next day".

Quick Tip

Remember the key changes for reported speech: tenses shift back one step (e.g., present to past, will to would), pronouns change according to the speaker/listener, and words indicating time/place change (e.g., now to then, here to there, tomorrow to the next day).

25. Change the following into direct speech: "She said that she loved pizza so much"

- (A) She said, "I love pizza so much"
- (B) She says, "I love pizza so much"
- (C) She said, "I loved pizza so much"
- (D) She says, "I loved pizza so much"

Correct Answer: (A) She said, "I love pizza so much"

Solution:

To convert from indirect to direct speech, we reverse the changes.

The reporting verb is "said" (past tense), so it remains "She said".

The conjunction "that" is removed.

The pronoun "she" refers to the speaker, so it becomes "I" inside the quotation marks.

The verb "loved" is in the past tense. In indirect speech, both simple present ("love") and simple past ("loved") in direct speech would become simple past ("loved"). However, loving pizza is a general preference or a timeless truth, which is typically expressed in the simple present tense. Therefore, "love" is the most likely original verb.

This gives us: She said, "I love pizza so much".

Option (B) incorrectly changes the reporting verb to "says".

Option (C) is possible, but less likely for a general preference.

Option (D) incorrectly changes the reporting verb.

Quick Tip

When converting indirect speech with a past tense verb back to direct speech, consider the context. If the statement is a universal truth or a general habit/preference, the verb in direct speech is usually in the simple present tense.

2 Genral Knowledge

26. The primary function of the Election Commission of India is to:

(A) Formation of laws

(B) Conducting elections

(C) Pass budgets

(D) Managing Parliament sessions

Correct Answer: (B) Conducting elections

Solution:

The Election Commission of India is an autonomous constitutional authority.

Its primary and most important function, as established by the Constitution of India, is the administration of election processes.

This includes conducting elections to the Parliament, state legislatures, and the offices of the President and Vice President.

Formation of laws is the function of the legislature (Parliament/State Assemblies).
Passing budgets is a legislative function.
Managing Parliament sessions is handled by the presiding officers (Speaker, Chairman).
Therefore, the correct answer is conducting elections.

Quick Tip

The names of government bodies often give a clue to their primary function. The "Election Commission" is principally concerned with "elections."

27. The largest freshwater lake in India is:

- (A) Vembanad Lake
- (B) Dal Lake
- (C) Wular Lake
- (D) Chilika Lake

Correct Answer: (C) Wular Lake

Solution:

Let's analyze the given options:

- (A) Vembanad Lake is the longest lake in India, but it is a brackish water lagoon.
- (B) Dal Lake is a famous freshwater lake in Srinagar, but it is not the largest by area.
- (C) Wular Lake, located in the Bandipora district of Jammu and Kashmir, is one of the largest freshwater lakes in Asia and the largest in India by surface area.
- (D) Chilika Lake is the largest coastal lagoon in India and the second largest in the world, but it is a brackish water lake.

Therefore, the largest freshwater lake in India is Wular Lake.

Quick Tip

For geography questions, it's important to distinguish between categories: largest freshwater lake (Wular), largest brackish water lake/lagoon (Chilika), and longest lake (Vembanad).

28. The term "carbon footprint" refers to:

- (A) Growth of forests
- (B) Water pollution levels
- (C) Total greenhouse gas emissions
- (D) Fuel efficiency of vehicles

Correct Answer: (C) Total greenhouse gas emissions

Solution:

A "carbon footprint" is a measure of the total amount of greenhouse gases, primarily carbon dioxide, released into the atmosphere as a result of the activities of a particular individual, organization, or community.

It is a comprehensive measure of environmental impact.

(A) Growth of forests helps reduce the carbon footprint (through carbon sequestration) but is not the definition of it.

(B) Water pollution is a different type of environmental issue.

(D) Fuel efficiency affects the carbon footprint of a vehicle, but it is not the definition of the term itself.

The most accurate definition is the total emission of greenhouse gases.

Quick Tip

Think of a "footprint" as the mark you leave behind. A "carbon footprint" is the environmental mark left behind by your activities, measured in terms of greenhouse gas emissions.

29. India's first mission to study the Sun is:

(A) Aditya-L1

(B) Chandrayaan

(C) Surya-N1

(D) Vikram Lander

Correct Answer: (A) Aditya-L1

Solution:

Let's look at the options in the context of India's space missions:

(A) Aditya-L1 is India's first dedicated solar observatory mission, designed to study the Sun. 'Aditya' is a Sanskrit word for the Sun.

(B) The Chandrayaan program consists of a series of lunar exploration missions. 'Chandra' means Moon.

(C) Surya-N1 is not the name of an official Indian space mission. 'Surya' also means Sun, but this is not the mission's name.

(D) Vikram Lander was part of the Chandrayaan-2 and Chandrayaan-3 lunar missions.

Therefore, India's first solar mission is Aditya-L1.

Quick Tip

In questions about Indian space missions, the names often have Sanskrit origins that give clues to their purpose: 'Chandra' for Moon, 'Aditya' or 'Surya' for Sun, 'Mangal' for Mars.

30. The 'Green Revolution' in India was mainly associated with:

- (A) Irrigation reforms
- (B) Industrial development
- (C) Agricultural productivity
- (D) Environmental protection

Correct Answer: (C) Agricultural productivity

Solution:

The Green Revolution refers to a period in the mid-20th century when new agricultural technologies, such as high-yield variety (HYV) seeds, fertilizers, pesticides, and improved irrigation, were introduced in India.

The primary goal and result of these initiatives were a massive increase in the production of food grains like wheat and rice.

This is best described as an increase in "Agricultural productivity."

While irrigation reforms (A) were a part of the Green Revolution, the overall association is with the resulting productivity.

Industrial development (B) and environmental protection (D) were separate initiatives.

Quick Tip

The 'Green' in Green Revolution refers to crops and agriculture, not environmentalism in the modern sense. Its main focus was on increasing food production to achieve self-sufficiency.

31. Which of the following is not a fundamental right in the Indian Constitution?

- (A) Right to Equality
- (B) Right to Property
- (C) Right to Education
- (D) Right to Freedom

Correct Answer: (B) Right to Property

Solution:

Originally, the Right to Property was a Fundamental Right under Article 31 of the Indian

Constitution.

However, the 44th Amendment Act of 1978 removed the Right to Property from the list of Fundamental Rights.

It was made a legal right under Article 300-A in Part XII of the Constitution.

Right to Equality (Articles 14-18), Right to Education (Article 21-A), and Right to Freedom (Articles 19-22) are still Fundamental Rights.

Therefore, the Right to Property is no longer a fundamental right.

Quick Tip

Remember key constitutional amendments. The 44th Amendment (1978) is significant for removing the Right to Property as a fundamental right, a frequent topic in competitive exams.

32. UNESCO stands for:

- (A) United Nations Educational, Scientific and Cultural Organization
- (B) Universal Science and Culture Organization
- (C) United Nations Environmental, Scientific and Cultural Organization
- (D) Union of National Education, Science and Culture Organisation

Correct Answer: (A) United Nations Educational, Scientific and Cultural Organization

Solution:

UNESCO is a specialized agency of the United Nations (UN).

Its full name is the United Nations Educational, Scientific and Cultural Organization.

It aims to promote world peace and security through international cooperation in education, arts, sciences, and culture.

The other options provide incorrect expansions of the acronym.

Quick Tip

Acronyms of major international organizations like UNESCO, UNICEF, WHO, etc., are common in general knowledge sections. It's useful to memorize the full forms and their primary functions.

33. The Indian Parliament consists of:

- (A) Lok Sabha and President
- (B) Lok Sabha and Rajya Sabha
- (C) Lok Sabha, Rajya Sabha and President
- (D) Rajya Sabha and Supreme Court

Correct Answer: (C) Lok Sabha, Rajya Sabha and President

Solution:

According to Article 79 of the Constitution of India, the Parliament of the Union consists of the President and two Houses.

The two Houses are known as the Council of States (Rajya Sabha) and the House of the People (Lok Sabha).

Although the President is not a member of either House, he is an integral part of the Parliament.

Therefore, the Indian Parliament comprises the Lok Sabha, the Rajya Sabha, and the President.

Quick Tip

A common misconception is that the Parliament only includes the two houses (Lok Sabha and Rajya Sabha). Remember that the President is the third and integral component of the Indian Parliament.

34. The capital of Uttarakhand is:

- (A) Mussoorie
- (B) Nainital
- (C) Dehradun
- (D) Haridwar

Correct Answer: (C) Dehradun

Solution:

The state of Uttarakhand has two designated capitals.

Dehradun is the winter capital and the primary, most well-known administrative capital of the state.

Gairsain is designated as the summer capital.

Since Dehradun is listed as an option and is the primary capital, it is the correct answer. The other cities listed are major towns in Uttarakhand but not the capital.

Quick Tip

Be aware that some Indian states have more than one capital (e.g., winter and summer capitals, or legislative, executive, and judicial capitals). For Uttarakhand, Dehradun is the most commonly cited capital.

35. Which Indian River is known as 'Dakshina Ganga'?

- (A) Krishna
- (B) Godavari
- (C) Kaveri
- (D) Mahanadi

Correct Answer: (B) Godavari

Solution:

The river Godavari is often referred to as the 'Dakshina Ganga' (Ganges of the South).

This is because of its large size and extent, making it the second-longest river in India after the Ganga and the longest river in Peninsular India.

It holds great religious significance for Hindus, similar to the Ganga.

The Kaveri river is sometimes called the 'Ganga of the South' (distinct from Dakshina Ganga), but Godavari's claim to 'Dakshina Ganga' is more established due to its length.

Quick Tip

Distinguish between 'Dakshina Ganga' (Godavari) and 'Ganga of the South' (Kaveri). The former title refers to the Godavari's length and size, while the latter refers to the Kaveri's sacredness in the southern region.

36. 'One Earth, One Family, One Future' was the theme of which recent international summit held in India?

- (A) G7 Summit
- (B) SCO Summit
- (C) G20 Summit
- (D) BRICS Summit

Correct Answer: (C) G20 Summit

Solution:

India held the presidency of the G20 from December 1, 2022, to November 30, 2023.

The summit was held in New Delhi in September 2023.

The theme for India's G20 Presidency was 'Vasudhaiva Kutumbakam', which translates to 'One Earth, One Family, One Future'.

This theme was drawn from the ancient Sanskrit text of the Maha Upanishad.

Therefore, the correct answer is the G20 Summit.

Quick Tip

Themes of major international summits, especially those hosted by India, are very important for general awareness. Connect the Sanskrit phrase 'Vasudhaiva Kutumbakam' with the English theme 'One Earth, One Family, One Future' and the G20 summit.

37. The primary role of NITI Aayog is to:

- (A) Conduct elections
- (B) Formulation of policies and strategies
- (C) Collecting taxes
- (D) Enforcement of laws

Correct Answer: (B) Formulation of policies and strategies

Solution:

NITI Aayog (National Institution for Transforming India) replaced the Planning Commission in 2015.

It serves as the premier policy 'Think Tank' of the Government of India.

Its primary role is to provide both directional and policy inputs, design strategic and long-term policies and programmes for the Government of India.

Conducting elections is the role of the Election Commission. Collecting taxes is done by agencies like the CBDT and CBIC. Law enforcement is the role of the police and other agencies.

Quick Tip

Remember that NITI Aayog is a "think tank," not an executive or financial body. Its main job is to advise and formulate policies, promoting cooperative federalism.

38. The first Indian woman to go into space was:

- (A) Sunita Williams

- (B) Kalpana Chawla
- (C) Ritu Karidhal
- (D) Tessy Thomas

Correct Answer: (B) Kalpana Chawla

Solution:

Kalpana Chawla was an American astronaut of Indian origin.

She was the first woman of Indian descent to go to space.

She first flew on Space Shuttle Columbia in 1997 as a mission specialist and primary robotic arm operator.

Sunita Williams is another American astronaut of Indian descent who has been to space, but Kalpana Chawla was the first.

Ritu Karidhal is a scientist at ISRO, and Tessy Thomas is a scientist at DRDO.

Quick Tip

Distinguish between the first Indian citizen in space (Rakesh Sharma) and the first woman of Indian origin in space (Kalpana Chawla). These "firsts" are common general knowledge questions.

39. Which country recently exited the European Union (Brexit)?

- (A) Germany
- (B) France
- (C) Italy
- (D) United Kingdom

Correct Answer: (D) United Kingdom

Solution:

"Brexit" is a portmanteau of "British exit".

It refers to the withdrawal of the United Kingdom (UK) from the European Union (EU).

Following a referendum in June 2016, the UK formally left the EU on 31 January 2020.

Germany, France, and Italy are founding members and remain key countries within the European Union.

Quick Tip

The term "Brexit" itself provides the answer: "Br" for British and "exit." Understanding the etymology of such terms can be a memory aid.

40. 'Mission Shakti' was related to:

- (A) Women's empowerment
- (B) Anti-satellite missile test
- (C) Cybersecurity program
- (D) Skill development for youth

Correct Answer: (B) Anti-satellite missile test

Solution:

'Mission Shakti' was a military operation conducted by the Defence Research and Development Organisation (DRDO) on 27 March 2019.

In this mission, India successfully tested an anti-satellite (ASAT) weapon, destroying a live satellite in Low Earth Orbit.

This demonstrated India's capability to intercept and destroy satellites in space.

While there is also a "Mission Shakti" program related to women's empowerment in some states, the nationally and internationally recognized 'Mission Shakti' refers to the ASAT test.

Quick Tip

In the context of national defence and technology, 'Shakti' (meaning power/strength) is often used in the names of strategic projects, like the Pokhran nuclear tests ('Operation Shakti') and the ASAT test ('Mission Shakti').

3 Teaching Aptitude

41. The purpose of formative assessment is:

- (A) To assign final grades
- (B) To evaluate teaching staff
- (C) To improve learning while it is happening
- (D) To promote students

Correct Answer: (C) To improve learning while it is happening

Solution:

Formative assessment is a range of formal and informal assessment procedures conducted by

teachers during the learning process.

Its primary purpose is to modify teaching and learning activities to improve student attainment.

It provides ongoing feedback to both students and teachers.

Assigning final grades (A) and promoting students (D) are functions of summative assessment, which evaluates learning at the end of an instructional unit.

Evaluating teaching staff (B) is a different administrative process.

Quick Tip

Think of "formative" assessment as assessment FOR learning (to guide and improve), whereas "summative" assessment is assessment OF learning (to judge and grade).

42. Which learning theory is associated with operant conditioning?

- (A) Piaget's theory
- (B) Skinner's theory
- (C) Maslow's theory
- (D) Gardner's theory

Correct Answer: (B) Skinner's theory

Solution:

Operant conditioning is a method of learning that employs rewards and punishments for behavior.

This theory was developed by the behaviorist B.F. Skinner.

Piaget's theory is about cognitive development in stages.

Maslow's theory is a hierarchy of needs related to motivation.

Gardner's theory is about multiple intelligences.

Therefore, operant conditioning is directly associated with Skinner's theory.

Quick Tip

Associate key concepts with theorists: Skinner with operant conditioning (rewards/punishment), Pavlov with classical conditioning (stimulus-response), Piaget with cognitive stages, and Vygotsky with social learning.

43. A democratic classroom environment encourages:

- (A) Authoritarian discipline
- (B) Passive learners
- (C) Student participation and dialogue
- (D) Teacher-centred teaching

Correct Answer: (C) Student participation and dialogue

Solution:

A democratic classroom is one where students have a say in their learning and the classroom rules.

It is characterized by collaboration, respect, and shared responsibility.

This environment naturally encourages students to actively participate in discussions, ask questions, and engage in dialogue with the teacher and peers.

Authoritarian discipline and teacher-centred teaching are characteristics of an autocratic, not democratic, environment.

A democratic approach aims to create active, not passive, learners.

Quick Tip

The word "democratic" in an educational context implies student involvement, choice, and active participation, contrasting with "autocratic" or "teacher-centered" models.

44. Which one is an example of intrinsic motivation?

- (A) Prize for winning
- (B) Fear of punishment
- (C) Desire to learn
- (D) Parental pressure

Correct Answer: (C) Desire to learn

Solution:

Intrinsic motivation involves doing an activity for its inherent satisfaction rather than for some separable consequence. The motivation comes from within the individual.

Extrinsic motivation involves doing something because you want to earn a reward or avoid punishment.

- (A) Prize for winning is an external reward (extrinsic).
- (B) Fear of punishment is an external consequence to be avoided (extrinsic).

- (C) Desire to learn comes from a person's own curiosity and interest in the subject (intrinsic).
(D) Parental pressure is an external force (extrinsic).

Therefore, the only example of intrinsic motivation is the desire to learn.

Quick Tip

To differentiate between intrinsic and extrinsic motivation, ask: "Is the motivation coming from the joy of the activity itself (intrinsic), or from an external reward or pressure (extrinsic)?"

45. A teacher using Bloom's Taxonomy should aim at which higher-order skill?

- (A) Remembering facts
(B) Applying definitions
(C) Creating new knowledge
(D) Understanding rules

Correct Answer: (C) Creating new knowledge

Solution:

Bloom's Taxonomy is a hierarchical model of cognitive skills. The original and revised versions classify skills from lower-order to higher-order.

The levels of the revised taxonomy are: Remembering, Understanding, Applying, Analyzing, Evaluating, and Creating.

'Remembering facts' and 'Understanding rules' are lower-order thinking skills (LOTS).

'Applying definitions' is in the middle.

The highest-order thinking skills (HOTS) are Analyzing, Evaluating, and Creating.

Among the given options, 'Creating new knowledge' represents the highest level of the taxonomy.

Quick Tip

Memorize the six levels of Bloom's (Revised) Taxonomy in order: Remembering, Understanding, Applying, Analyzing, Evaluating, Creating. The top three are considered higher-order thinking skills.

46. In pedagogy, 'feedback' helps:

- (A) Punish students
- (B) Modify teaching strategies
- (C) Give homework
- (D) Mark attendance

Correct Answer: (B) Modify teaching strategies

Solution:

In pedagogy (the method and practice of teaching), feedback is crucial information about a student's performance or understanding.

Effective feedback serves two main purposes: it helps the student understand what to improve, and it helps the teacher understand the effectiveness of their instruction.

By analyzing student responses and providing feedback, a teacher can identify areas where their teaching methods are not effective and consequently modify their teaching strategies to better meet the students' needs.

Feedback is not for punishment, and while it might relate to homework, its primary pedagogical purpose is improvement, not just assignment.

Quick Tip

Effective feedback in education is a two-way street: it guides the student's learning and informs the teacher's instruction. Its core purpose is to improve the teaching-learning process.

47. Which among these promotes inclusive education?

- (A) Segregated classrooms
- (B) Fixed curriculum
- (C) Adaptive teaching methods
- (D) Competitive grouping

Correct Answer: (C) Adaptive teaching methods

Solution:

Inclusive education means that all students, regardless of any challenges they may have, are placed in age-appropriate general education classes to receive high-quality instruction, interventions, and support that enable them to meet success.

- (A) Segregated classrooms are the opposite of inclusion.
- (B) A fixed curriculum does not cater to the diverse needs of students, which is against the principle of inclusion.
- (C) Adaptive teaching methods (also known as differentiated instruction) involve tailoring instruction to meet individual needs. This is a core principle of inclusive education.

(D) Competitive grouping can often marginalize students with different learning paces or abilities.

Therefore, adaptive teaching methods promote inclusive education.

Quick Tip

Inclusion in education is about flexibility and adaptation. Any option that suggests rigidity (fixed curriculum), separation (segregation), or uniform treatment for all will be contrary to the principles of inclusive education.

48. Which is a barrier to effective classroom communication?

- (A) Use of visual aids
- (B) Clear articulation
- (C) Language mismatch
- (D) Active listening

Correct Answer: (C) Language mismatch

Solution:

Effective communication occurs when the message is received and understood by the receiver as intended by the sender.

A barrier to communication is anything that prevents this from happening.

- (A) Use of visual aids generally enhances communication.
- (B) Clear articulation (speaking clearly) is essential for effective communication.
- (C) A language mismatch, where the teacher and students do not share a common language or have different levels of proficiency, is a significant barrier to understanding.
- (D) Active listening by students is a key component of successful communication, not a barrier.

Thus, language mismatch is a barrier.

Quick Tip

Communication barriers can be physical (noise), semantic (language, jargon), psychological (attitudes, emotions), or cultural. A language mismatch is a classic semantic barrier.

49. A good teacher evaluates students by:

- (A) Comparing them with others
- (B) Judging their behaviour

(C) Identifying strengths and weaknesses

(D) Ignoring slow learners

Correct Answer: (C) Identifying strengths and weaknesses

Solution:

The primary goal of student evaluation in modern pedagogy is diagnostic and developmental.

A good teacher evaluates students to understand their current level of knowledge and skills.

This involves identifying what the students are good at (strengths) and where they need improvement (weaknesses).

This information is then used to guide further instruction and support.

Comparing students (A) can be demotivating. Simply judging behavior (B) is not a holistic evaluation of learning. Ignoring any student (D) is poor teaching practice.

Quick Tip

Modern educational evaluation focuses on individual growth and diagnosis for improvement (criterion-referenced) rather than ranking students against each other (norm-referenced).

50. Effective teaching is primarily about:

(A) Completing the syllabus

(B) Giving maximum homework

(C) Enhancing understanding and engagement

(D) Following textbooks strictly

Correct Answer: (C) Enhancing understanding and engagement

Solution:

Effective teaching is measured by the extent to which students learn and grow.

This is achieved not just by covering content, but by ensuring that students are actively engaged in the learning process and are developing a deep understanding of the subject matter.

(A) Completing the syllabus without ensuring understanding is not effective.

(B) Excessive homework is often counterproductive.

(D) Strictly following a textbook can stifle creativity and fail to cater to students' diverse interests and needs.

Therefore, enhancing student understanding and engagement is the hallmark of effective teaching.

Quick Tip

The focus of effective teaching has shifted from being teacher-centric (completing syllabus, following textbook) to being learner-centric (ensuring student understanding, engagement, and skill development).

4 Mathematics

51. Which of the following differential equations is not first-order and first-degree

- (A) $\frac{dy}{dx} = x + y$
- (B) $(\frac{dy}{dx})^2 + y = 0$
- (C) $\frac{dy}{dx} = x^2 + y^2$
- (D) $\frac{dy}{dx} = 3x + 2$

Correct Answer: (B) $(\frac{dy}{dx})^2 + y = 0$

Solution:

The order of a differential equation is the order of the highest derivative present in the equation.

The degree of a differential equation is the highest power of the highest order derivative in the equation, after the equation has been cleared of radicals and fractions.

Let's analyze the options:

(A) $\frac{dy}{dx} = x + y$: The highest derivative is $\frac{dy}{dx}$ (order 1). Its power is 1. So, it is first-order and first-degree.

(B) $(\frac{dy}{dx})^2 + y = 0$: The highest derivative is $\frac{dy}{dx}$ (order 1). Its power is 2. So, it is first-order and second-degree.

(C) $\frac{dy}{dx} = x^2 + y^2$: The highest derivative is $\frac{dy}{dx}$ (order 1). Its power is 1. So, it is first-order and first-degree.

(D) $\frac{dy}{dx} = 3x + 2$: The highest derivative is $\frac{dy}{dx}$ (order 1). Its power is 1. So, it is first-order and first-degree.

The question asks which is NOT first-order and first-degree. Equation (B) is first-order but second-degree.

Quick Tip

To find the degree, first locate the highest order derivative. Then, identify the exponent of that entire term. For example, in $(y'')^3 + y' = 0$, the order is 2 and the degree is 3.

52. The orthogonal trajectories of $y = ax^n$

(A) $x^2 + 2y^2 = c^2$

(B) $y^2 - x^2 = c^2$

(C) $x^2 + ny^2 = c$

(D) $x^2 - ny^2 = c$

Correct Answer: (C) $x^2 + ny^2 = c$

Solution:

The question provides two families of curves, $y = ax^2$ and $y = ax^n$. This is likely a typo. The correct option (C) corresponds to the family $y = ax^n$, so we will solve for that.

Step 1: Find the differential equation for the given family of curves.

Given family: $y = ax^n$.

Differentiating with respect to x : $\frac{dy}{dx} = nax^{n-1}$.

Step 2: Eliminate the arbitrary constant 'a'.

From the given equation, $a = \frac{y}{x^n}$.

Substitute this into the derivative: $\frac{dy}{dx} = n \left(\frac{y}{x^n} \right) x^{n-1} = \frac{ny}{x}$.

Step 3: Find the differential equation for the orthogonal trajectories.

Replace $\frac{dy}{dx}$ with $-\frac{dx}{dy}$.

$$-\frac{dx}{dy} = \frac{ny}{x}.$$

Step 4: Solve the new differential equation.

Separating the variables: $x dx = -ny dy$.

Integrating both sides: $\int x dx = \int -ny dy$.

$$\frac{x^2}{2} = -n\frac{y^2}{2} + C_1.$$

$$x^2 = -ny^2 + 2C_1.$$

$$x^2 + ny^2 = 2C_1.$$

Let $c = 2C_1$. The equation for the orthogonal trajectories is $x^2 + ny^2 = c$.

Quick Tip

The process for finding orthogonal trajectories is: 1. Differentiate the given family of curves. 2. Eliminate the arbitrary constant to get the differential equation. 3. Replace $\frac{dy}{dx}$ with $-\frac{dx}{dy}$. 4. Solve the new differential equation.

53. What is the integrating factor for the differential equation $\frac{dy}{dx} + \left(\frac{2}{x}\right)y = x^2$

(A) x^2

(B) $\frac{1}{x^2}$

(C) x

(D) $\frac{1}{x}$

Correct Answer: (A) x^2

Solution:

The given differential equation is in the linear form $\frac{dy}{dx} + P(x)y = Q(x)$.

By comparing, we have $P(x) = \frac{2}{x}$ and $Q(x) = x^2$.

The formula for the integrating factor (I.F.) is $e^{\int P(x)dx}$.

Let's calculate the integral of $P(x)$:

$$\int P(x)dx = \int \frac{2}{x}dx = 2 \ln|x| = \ln(x^2).$$

Now, we find the integrating factor:

$$\text{I.F.} = e^{\int P(x)dx} = e^{\ln(x^2)} = x^2.$$

Thus, the integrating factor is x^2 .

Quick Tip

For any linear differential equation of the form $y' + P(x)y = Q(x)$, the integrating factor is always calculated as $e^{\int P(x)dx}$. Remember the identity $e^{\ln(f(x))} = f(x)$.

54. Solution of non-exact differential equation $(1 + xy)ydx + (1 - xy)xdy = 0$ with integral factor $\frac{1}{x^2y^2}$ is

(A) $\frac{1}{xy} - \log\left(\frac{x}{y}\right) = c$

(B) $-\frac{1}{xy} - \log\left(\frac{x}{y}\right) = c$

(C) $-\frac{1}{xy} + \log\left(\frac{x}{y}\right) = c$

(D) $\frac{1}{xy} + \log\left(\frac{x}{y}\right) = c$

Correct Answer: (C) $-\frac{1}{xy} + \log\left(\frac{x}{y}\right) = c$

Solution:

The provided question in the PDF, $(1 + xy)ydx + (1 - xy)dy = 0$, has a typo. To match the keyed answer, the equation should be the standard form $(1 + xy)ydx + (1 - xy)xdy = 0$. We will solve this corrected equation.

Step 1: Multiply the equation by the given integrating factor, $\text{I.F.} = \frac{1}{x^2y^2}$.

$$\frac{(1+xy)y}{x^2y^2}dx + \frac{(1-xy)x}{x^2y^2}dy = 0.$$

$$\left(\frac{y}{x^2y^2} + \frac{xy^2}{x^2y^2}\right)dx + \left(\frac{x}{x^2y^2} - \frac{x^2y}{x^2y^2}\right)dy = 0.$$

$$\left(\frac{1}{x^2y} + \frac{1}{x}\right)dx + \left(\frac{1}{xy^2} - \frac{1}{y}\right)dy = 0.$$

Step 2: This is now an exact differential equation $M'dx + N'dy = 0$, where $M' = \frac{1}{x^2y} + \frac{1}{x}$ and $N' = \frac{1}{xy^2} - \frac{1}{y}$.

The solution is given by $\int M'dx$ (treating y as constant) + $\int$ (terms in N' not containing x) $dy = c$.

Step 3: Calculate the first integral.

$$\int M'dx = \int \left(\frac{1}{x^2y} + \frac{1}{x}\right)dx = \frac{1}{y} \int x^{-2}dx + \int \frac{1}{x}dx = \frac{1}{y}\left(\frac{x^{-1}}{-1}\right) + \ln|x| = -\frac{1}{xy} + \ln|x|.$$

Step 4: Calculate the second integral.

The term in N' not containing x is $-\frac{1}{y}$.

$$\int -\frac{1}{y}dy = -\ln|y|.$$

Step 5: Combine the results to get the general solution.

$$-\frac{1}{xy} + \ln|x| - \ln|y| = c.$$

$$-\frac{1}{xy} + \ln\left(\frac{x}{y}\right) = c.$$

This matches option (C).

Quick Tip

If an exam question provides an integrating factor, it's a strong hint that the original equation is not exact. After multiplying by the I.F., you can verify exactness by checking if $\frac{\partial M'}{\partial y} = \frac{\partial N'}{\partial x}$.

55. Find the general solution of the differential equation $\frac{dy}{dx} = 5x^2 + 2$.

(A) $5x^3 + 6x - 3y + 3C = 0$

(B) $12x - 3y^2 + C = 0$

(C) $10x^3 + 12x - y^2 + C = 0$

(D) $10x^2 - 3y^2 + C = 0$

Correct Answer: (A) $5x^3 + 6x - 3y + 3C = 0$

Solution:

This is a first-order differential equation that can be solved by direct integration (separation of variables).

Step 1: Separate the variables.

$$dy = (5x^2 + 2)dx.$$

Step 2: Integrate both sides of the equation.

$$\int dy = \int (5x^2 + 2)dx.$$

$$y = 5\frac{x^3}{3} + 2x + C.$$

Step 3: Rearrange the solution to match the format of the options.

$$y = \frac{5x^3}{3} + 2x + C.$$

Multiply the entire equation by 3 to eliminate the fraction:

$$3y = 5x^3 + 6x + 3C.$$

Move all terms to one side:

$$5x^3 + 6x - 3y + 3C = 0.$$

This matches option (A) exactly.

Quick Tip

After finding the solution to a differential equation, always check the format of the options. You may need to algebraically manipulate your answer (e.g., by multiplying to clear fractions) to match the correct choice.

56. Solve $p^2 - 7p + 12 = 0$ for p

(A) $(y - 3x - c)(y - 4x - c) = 0$

(B) $(y - 4x - c)(y - 2x - c) = 0$

(C) $(y + 3x - c)(y - 4x - c) = 0$

(D) $(y - 3x - c)(y + 4x - c) = 0$

Correct Answer: (A) $(y - 3x - c)(y - 4x - c) = 0$

Solution:

This is a differential equation of the first order but not of the first degree, which is solvable for p , where $p = \frac{dy}{dx}$.

Step 1: Solve the quadratic equation for p .

$$p^2 - 7p + 12 = 0.$$

Factoring the quadratic, we get:

$$(p - 3)(p - 4) = 0.$$

This gives two possible values for p : $p = 3$ or $p = 4$.

Step 2: Solve the differential equation for each value of p .

Case 1: $p = 3$.

$$\frac{dy}{dx} = 3 \implies dy = 3dx.$$

Integrating gives $y = 3x + c$, which can be written as $y - 3x - c = 0$.

Case 2: $p = 4$.

$$\frac{dy}{dx} = 4 \implies dy = 4dx.$$

Integrating gives $y = 4x + c$, which can be written as $y - 4x - c = 0$.

Step 3: Combine the solutions.

The general solution is the product of the individual solution factors:

$$(y - 3x - c)(y - 4x - c) = 0.$$

Quick Tip

For differential equations that are polynomial in $p = \frac{dy}{dx}$ (type: solvable for p), the method is to find the roots of the polynomial. Each root gives a simpler differential equation. The final general solution is the product of the solutions from each root set to zero.

57. The solution of the differential equation $y = px + \sqrt{4 + p^2}$ is

(A) $(y - Cx)^2 + C^2 = 0$

(B) $(y - Cx)^2 + 4C^2 = 0$

(C) $(y - Cx)^2 - C^2 = 4$

(D) $(y - Cx)^2 - 4C^2 = 0$

Correct Answer: (C) $(y - Cx)^2 - C^2 = 4$

Solution:

The given differential equation is of the form $y = px + f(p)$, where $f(p) = \sqrt{4 + p^2}$.

This is a Clairaut's equation.

The general solution of a Clairaut's equation is obtained by simply replacing the parameter p with an arbitrary constant, say C .

Step 1: Replace p with C .

$$y = Cx + \sqrt{4 + C^2}.$$

Step 2: Manipulate the equation to match the form of the given options.

Isolate the square root term:

$$y - Cx = \sqrt{4 + C^2}.$$

Square both sides:

$$(y - Cx)^2 = 4 + C^2.$$

Rearrange the terms:

$$(y - Cx)^2 - C^2 = 4.$$

This matches option (C).

Quick Tip

Recognize Clairaut's equation form: $y = px + f(p)$. The general solution is found instantly by changing p to an arbitrary constant C , giving $y = Cx + f(C)$. You may also need to find a singular solution by differentiating with respect to p .

58. Which of the following equations is solvable for y

- (A) $x^2 + y^2 = 1$
- (B) $x + y^2 = 1$
- (C) $3x + 2y = 8$
- (D) $x + y = 1$ and $x - y = 1$

Correct Answer: (C) $3x + 2y = 8$

Solution:

The question asks which equation is "solvable for y ". In a mathematical context, this typically means that y can be expressed as a unique, single-valued function of x .

Let's analyze the options:

(A) $x^2 + y^2 = 1 \implies y^2 = 1 - x^2 \implies y = \pm\sqrt{1 - x^2}$. For a given x , there are two possible values for y . This does not represent a single function.

(B) $x + y^2 = 1 \implies y^2 = 1 - x \implies y = \pm\sqrt{1 - x}$. Similar to (A), this gives two values for y and does not represent a single function.

(C) $3x + 2y = 8 \implies 2y = 8 - 3x \implies y = \frac{8-3x}{2}$. For any given x , there is exactly one value for y . This expresses y as a unique function of x .

(D) This is a system of two equations, not a single equation.

Therefore, the equation that is uniquely solvable for y as a single function is (C).

Quick Tip

When a question asks if an equation is "solvable" for a variable like y , check if you can isolate y to get a single, unambiguous expression. Equations with terms like y^2 , $|y|$, or $\sin(y)$ often do not have a unique solution for y . Linear equations in y are always uniquely solvable.

59. What is the general solution of the Clairaut's equation $y = px + p^3$

- (A) $y = Cx + C^3$
- (B) $y = Cx + C^2$
- (C) $y = Cx + 3C^2$
- (D) $y = Cx + C^4$

Correct Answer: (A) $y = Cx + C^3$

Solution:

The given differential equation is $y = px + p^3$.

This equation is in the standard form of a Clairaut's equation, which is $y = px + f(p)$.

Here, $f(p) = p^3$.

The general solution for any Clairaut's equation is obtained by replacing the parameter p with an arbitrary constant, which we can call C .

By substituting C for p in the given equation, we get the general solution:

$$y = Cx + C^3.$$

This directly matches option (A).

Quick Tip

The general solution to a Clairaut's equation $y = px + f(p)$ is always $y = Cx + f(C)$, representing a family of straight lines.

60. The particular integral of $(D^2 + D + 1)y = \sin x$ is

- (A) $\sin x$
- (B) $-\sin x$
- (C) $\cos x$
- (D) $-\cos x$

Correct Answer: (D) $-\cos x$

Solution:

To find the particular integral (P.I.) for an equation of the form $f(D)y = \sin(ax)$, we use the operator method.

$$\text{P.I.} = \frac{1}{f(D)} \sin(ax).$$

The rule is to replace D^2 with $-a^2$. Here, $a = 1$, so we replace D^2 with $-1^2 = -1$.

$$\text{P.I.} = \frac{1}{D^2 + D + 1} \sin x.$$

$$\text{P.I.} = \frac{1}{-1 + D + 1} \sin x.$$

$$\text{P.I.} = \frac{1}{D} \sin x.$$

The operator $\frac{1}{D}$ represents integration with respect to x .

$$\text{P.I.} = \int \sin x \, dx.$$

$$\text{P.I.} = -\cos x.$$

This matches option (D).

Quick Tip

When finding the particular integral for $\sin(ax)$ or $\cos(ax)$, always substitute $D^2 = -a^2$. If the denominator becomes zero, you must use a different method (e.g., $\text{P.I.} = x \frac{1}{f'(D)} \sin(ax)$ or using complex exponentials).

61. The general solution of $(D - 2)(D + 1)^2 y = 0$.

- (A) $c_1 e^{2x} + c_2 e^x + c_3 e^{-x}$
(B) $c_1 e^{2x} + (c_2 + c_3 x) e^{-x}$
(C) $(c_1 \cos x + c_2 \sin x) + c_3 e^{2x}$
(D) $c_1 e^{2x} + (c_2 + c_3 x) e^x$

Correct Answer: (B) $c_1 e^{2x} + (c_2 + c_3 x) e^{-x}$

Solution:

This is a homogeneous linear differential equation with constant coefficients.

Step 1: Write the auxiliary equation by replacing the operator D with a variable m .

$$(m - 2)(m + 1)^2 = 0.$$

Step 2: Find the roots of the auxiliary equation.

The roots are $m - 2 = 0 \implies m_1 = 2$.

And $(m + 1)^2 = 0 \implies m_2 = -1$ and $m_3 = -1$.

We have one real, distinct root ($m = 2$) and one real, repeated root ($m = -1$ with multiplicity 2).

Step 3: Write the general solution based on the roots.

For a real, distinct root m_1 , the solution component is $c_1 e^{m_1 x}$. So we have $c_1 e^{2x}$.

For a real root m_2 repeated twice, the solution component is $(c_2 + c_3 x) e^{m_2 x}$. So we have $(c_2 + c_3 x) e^{-x}$.

Step 4: Combine the components to form the complete general solution.

$$y = c_1 e^{2x} + (c_2 + c_3 x) e^{-x}.$$

This matches option (B).

Quick Tip

For the auxiliary equation: - A distinct real root m gives a solution term ce^{mx} . - A real root m repeated k times gives $(c_1 + c_2 x + \dots + c_k x^{k-1})e^{mx}$. - A pair of complex roots $\alpha \pm i\beta$ gives $e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$.

62. If $m = -2, m = -3$ are root of the auxiliary equation of a second order homogeneous differential equation, then the differential equation is

- (A) $(D^2 + 5D + 6)y = 0$
(B) $(D^2 + 4)y = 0$
(C) $(D^2 - 5D + 6)y = 0$
(D) $(D^2 - 3D + 1)y = 0$

Correct Answer: (A) $(D^2 + 5D + 6)y = 0$

Solution:

We are given the roots of the auxiliary equation and need to find the corresponding differential equation.

Step 1: Construct the factors of the auxiliary equation from its roots.

If the roots are $m_1 = -2$ and $m_2 = -3$, the factors are $(m - m_1)$ and $(m - m_2)$.

The factors are $(m - (-2)) = (m + 2)$ and $(m - (-3)) = (m + 3)$.

Step 2: Form the auxiliary equation by multiplying the factors.

$$(m + 2)(m + 3) = 0.$$

$$m^2 + 3m + 2m + 6 = 0.$$

$$m^2 + 5m + 6 = 0.$$

Step 3: Convert the auxiliary equation back into a differential equation.

Replace the variable m with the differential operator D .

The operator is $D^2 + 5D + 6$.

The corresponding differential equation is $(D^2 + 5D + 6)y = 0$.

This matches option (A).

Quick Tip

Remember the relationship between the roots (m_1, m_2) and coefficients of a quadratic auxiliary equation $m^2 + bm + c = 0$: the sum of roots $m_1 + m_2 = -b$ and the product of roots $m_1 m_2 = c$. Here, sum $= (-2) + (-3) = -5$, so $b = 5$. Product $= (-2)(-3) = 6$, so $c = 6$. This gives $m^2 + 5m + 6 = 0$.

63. The particular integral of $f(D)y = e^{ax}$ if $f(a) \neq 0$.

(A) $\frac{1}{f(a)}e^{-a}$

(B) $\frac{e^{ax}x^m}{\phi(a)m!}$

(C) $\frac{1}{f(a)}e^{ax}$

(D) $\frac{e^{-ax}x^m}{\phi(a)m!}$

Correct Answer: (C) $\frac{1}{f(a)}e^{ax}$

Solution:

This question asks for a standard result for finding the particular integral (P.I.) of a linear differential equation with constant coefficients when the right-hand side is an exponential function.

The P.I. is given by the formula $\text{P.I.} = \frac{1}{f(D)}e^{ax}$.

According to the standard theorem, if $f(a) \neq 0$, the operator D can be directly replaced by the constant a .

Therefore, the particular integral is:

$$\text{P.I.} = \frac{1}{f(a)}e^{ax}.$$

This is the standard formula for the non-failure case. The other options represent incorrect formulas or formulas for the case of failure.

Quick Tip

When finding the P.I. for e^{ax} , the first step is always to check the value of $f(a)$. If it's non-zero, simply replace D with a . If $f(a) = 0$ (case of failure), you must use a different formula, typically involving multiplying by x .

64. If Q is a function of x , then the value of $\frac{1}{D-a}Q$ is

(A) $e^{-ax} \int e^{ax} Q dx$

(B) $e^{ax} \int e^{-ax} Q dx$

(C) $e^{-ax} \int e^{-ax} Q dx$

(D) $e^{ax} \int e^{ax} Q dx$

Correct Answer: (B) $e^{ax} \int e^{-ax} Q dx$

Solution:

This question asks for the formula associated with the operator $\frac{1}{D-a}$.

Let $y = \frac{1}{D-a}Q$.

This implies that $(D - a)y = Q$.

In Leibniz notation, this is $\frac{dy}{dx} - ay = Q(x)$.

This is a first-order linear differential equation. We can solve it using an integrating factor (I.F.).

The I.F. is $e^{\int -a dx} = e^{-ax}$.

Multiplying the equation by the I.F.:

$$e^{-ax} \left(\frac{dy}{dx} - ay \right) = Q(x) e^{-ax}.$$

$$\frac{d}{dx} (y e^{-ax}) = Q(x) e^{-ax}.$$

Integrating both sides with respect to x :

$$y e^{-ax} = \int Q(x) e^{-ax} dx.$$

Solving for y :

$$y = e^{ax} \int Q(x) e^{-ax} dx.$$

Since we defined $y = \frac{1}{D-a}Q$, the formula is $e^{ax} \int e^{-ax} Q dx$. This matches option (B).

Quick Tip

The operator formula for $\frac{1}{D-a}Q$ is effectively the solution formula for the linear first-order DE $y' - ay = Q$. Knowing this connection allows you to derive the formula if you forget it.

65. Find the value $\frac{1}{D^2}e^{4x}$

- (A) $\frac{1}{4}e^{4x}$
- (B) $\frac{1}{16}e^{4x}$
- (C) $\frac{1}{12}e^{4x}$
- (D) $\frac{1}{8}e^{4x}$

Correct Answer: (B) $\frac{1}{16}e^{4x}$

Solution:

We need to evaluate $\frac{1}{D^2}e^{4x}$. This can be done in two ways.

Method 1: Using the formula for exponential functions.

The operator is $f(D) = D^2$. The right-hand side is e^{ax} with $a = 4$.

We check $f(a) = f(4) = 4^2 = 16$.

Since $f(a) \neq 0$, we can use the formula P.I. = $\frac{1}{f(a)}e^{ax}$.

So, $\frac{1}{D^2}e^{4x} = \frac{1}{16}e^{4x}$.

Method 2: Direct integration.

The operator $\frac{1}{D}$ means to integrate. $\frac{1}{D^2}$ means to integrate twice.

First integration: $\frac{1}{D}e^{4x} = \int e^{4x}dx = \frac{1}{4}e^{4x}$.

Second integration: $\frac{1}{D}(\frac{1}{4}e^{4x}) = \int \frac{1}{4}e^{4x}dx = \frac{1}{4} \int e^{4x}dx = \frac{1}{4}(\frac{1}{4}e^{4x}) = \frac{1}{16}e^{4x}$.

Both methods give the same result, which matches option (B).

Quick Tip

While direct integration works, using the operator formula (replace D with 'a' if $f(a) \neq 0$) is much faster and less prone to error for exponential functions.

66. The Wronskian of the differential equation $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = \frac{e^x}{1+e^x}$ **is**

- (A) e^x
- (B) e^{3x}
- (C) e^{2x}
- (D) e^{5x}

Correct Answer: (B) e^{3x}

Solution:

The Wronskian of the solutions to a linear homogeneous differential equation depends only on the homogeneous part of the equation, not the right-hand side.

The homogeneous equation is $y'' - 3y' + 2y = 0$.

Step 1: Find the fundamental set of solutions by solving the auxiliary equation.

The auxiliary equation is $m^2 - 3m + 2 = 0$.

Factoring gives $(m - 1)(m - 2) = 0$.

The roots are $m_1 = 1$ and $m_2 = 2$.

The two linearly independent solutions are $y_1 = e^x$ and $y_2 = e^{2x}$.

Step 2: Calculate the Wronskian, $W(y_1, y_2)$.

The Wronskian is the determinant of the matrix formed by the solutions and their derivatives.

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & e^{2x} \\ \frac{d}{dx}(e^x) & \frac{d}{dx}(e^{2x}) \end{vmatrix}.$$

$$W = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix}.$$

Step 3: Evaluate the determinant.

$$W = (e^x)(2e^{2x}) - (e^{2x})(e^x).$$

$$W = 2e^{3x} - e^{3x} = e^{3x}.$$

This matches option (B).

Quick Tip

The Wronskian is independent of the non-homogeneous term (the right-hand side of the DE). Also, Abel's identity states that for $y'' + P(x)y' + Q(x)y = 0$, the Wronskian is $W(x) = Ce^{-\int P(x)dx}$. Here $P(x) = -3$, so $W = Ce^{-\int -3dx} = Ce^{3x}$.

67. The solution of the differential equation is $\frac{dy}{dx} = \sin(x + y) + \cos(x + y)$

(A) $x = \ln(1 + \tan(\frac{x+y}{2})) + c$

(B) $x = \ln(\tan(x + y)) + c$

(C) $x = \ln(x + y + c)$

(D) $x = \tan(x + y + 1) + c$

Correct Answer: (A) $x = \ln(1 + \tan(\frac{x+y}{2})) + c$

Solution:

Step 1: Use a substitution to simplify the equation. Let $v = x + y$.

Differentiating with respect to x : $\frac{dv}{dx} = 1 + \frac{dy}{dx}$, which means $\frac{dy}{dx} = \frac{dv}{dx} - 1$.

Step 2: Substitute into the original equation.

$$\frac{dv}{dx} - 1 = \sin(v) + \cos(v).$$

$$\frac{dv}{dx} = 1 + \sin(v) + \cos(v).$$

Step 3: Separate the variables.

$$\frac{dv}{1+\cos(v)+\sin(v)} = dx.$$

Step 4: Integrate the left side. Use the half-angle identities: $1 + \cos(v) = 2\cos^2(\frac{v}{2})$ and $\sin(v) = 2\sin(\frac{v}{2})\cos(\frac{v}{2})$.

$$\int \frac{dv}{2\cos^2(\frac{v}{2})+2\sin(\frac{v}{2})\cos(\frac{v}{2})} = \int dx.$$

$$\int \frac{dv}{2\cos(\frac{v}{2})(\cos(\frac{v}{2})+\sin(\frac{v}{2}))} = \int dx.$$

Divide the numerator and denominator by $\cos^2(\frac{v}{2})$ inside the integral:

$$\int \frac{\sec^2(\frac{v}{2})dv}{2(1+\tan(\frac{v}{2}))} = \int dx.$$

Step 5: Use a second substitution. Let $u = 1 + \tan(\frac{v}{2})$.

Then $du = \sec^2(\frac{v}{2}) \cdot \frac{1}{2}dv$. So, $\sec^2(\frac{v}{2})dv = 2du$.

The integral becomes $\int \frac{2du}{2u} = \int \frac{du}{u} = \int dx$, which simplifies to $\int \frac{1}{u}du = \int dx$.

$$\ln|u| = x + c_1.$$

$$\ln|1 + \tan(\frac{v}{2})| = x + c_1.$$

Step 6: Substitute back $v = x + y$ and solve for x .

$$\ln(1 + \tan(\frac{x+y}{2})) = x + c_1.$$

$$x = \ln(1 + \tan(\frac{x+y}{2})) - c_1.$$

Let $c = -c_1$. Then $x = \ln(1 + \tan(\frac{x+y}{2})) + c$.

Quick Tip

For equations of the form $\frac{dy}{dx} = f(ax + by + c)$, the substitution $v = ax + by + c$ almost always works to make the equation separable.

68. Solve $(x^2D^2 - 4xD + 6)y = x^2$ **where** $D = \frac{d}{dx}$

(A) $y = c_1x^2 + c_2x^2 - x^2\log x^2$

(B) $y = c_1x^2 + c_2x^3 - x\log x^2$

(C) $y = c_1x^2 + c_2x^3 - x^2\log x$

(D) $y = c_1x^2 + c_2x^2 - x\log x$

Correct Answer: (C) $y = c_1x^2 + c_2x^3 - x^2\log x$

Solution:

This is a Cauchy-Euler equation. We use the substitution $x = e^z$, so $z = \ln x$.

Let $\theta = \frac{d}{dz}$. Then $xD = \theta$ and $x^2D^2 = \theta(\theta - 1)$.

Step 1: Transform the differential equation.

$$[\theta(\theta - 1) - 4\theta + 6]y = (e^z)^2 = e^{2z}.$$

$$[\theta^2 - \theta - 4\theta + 6]y = e^{2z}.$$

$$[\theta^2 - 5\theta + 6]y = e^{2z}.$$

Step 2: Find the complementary function (C.F.).

The auxiliary equation is $m^2 - 5m + 6 = 0$.

$(m - 2)(m - 3) = 0$, so the roots are $m = 2, 3$.

C.F. in terms of z is $y_c = c_1 e^{2z} + c_2 e^{3z}$.

Step 3: Find the particular integral (P.I.).

$$\text{P.I.} = \frac{1}{\theta^2 - 5\theta + 6} e^{2z} = \frac{1}{(\theta - 2)(\theta - 3)} e^{2z}.$$

This is a case of failure since the operator contains a factor $(\theta - 2)$ and the exponential is e^{2z} . Using the failure case formula: $\text{P.I.} = z \frac{1}{2\theta - 5} e^{2z}$ (where $2\theta - 5$ is the derivative of the denominator).

$$\text{Replace } \theta \text{ with } 2: \text{P.I.} = z \frac{1}{2(2) - 5} e^{2z} = z \frac{1}{-1} e^{2z} = -z e^{2z}.$$

Step 4: Write the general solution and substitute back.

$$\text{The general solution in } z \text{ is } y = y_c + y_p = c_1 e^{2z} + c_2 e^{3z} - z e^{2z}.$$

Substitute $e^z = x$ and $z = \ln x$ (or $\log x$ as in the options).

$$y = c_1 (e^z)^2 + c_2 (e^z)^3 - (\ln x)(e^z)^2.$$

$$y = c_1 x^2 + c_2 x^3 - x^2 \log x.$$

This matches option (C).

Quick Tip

For a Cauchy-Euler equation, always use the substitution $x = e^z$. Remember the operator conversions: $x D = \theta$ and $x^2 D^2 = \theta(\theta - 1)$, where $\theta = d/dz$.

69. The polynomial $2x^2 - 4x + 3$ in terms of Legendre's polynomial is

(A) $\frac{1}{3}(4P_2 - 4P_1 + 11P_0)$

(B) $\frac{1}{3}(4P_2 + 12P_1 + 11P_0)$

(C) $\frac{1}{3}(P_2 - 12P_1 + 11P_0)$

(D) $\frac{1}{3}(4P_2 - 12P_1 + 11P_0)$

Correct Answer: (D) $\frac{1}{3}(4P_2 - 12P_1 + 11P_0)$

Solution:

We need to express the given polynomial as a linear combination of Legendre polynomials.

The first three Legendre polynomials are:

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

Step 1: Express powers of x in terms of Legendre polynomials.

From $P_0(x)$, we have $1 = P_0(x)$.

From $P_1(x)$, we have $x = P_1(x)$.

$$\text{From } P_2(x), \text{ we have } 2P_2(x) = 3x^2 - 1 \implies 3x^2 = 2P_2(x) + 1 \implies x^2 = \frac{2}{3}P_2(x) + \frac{1}{3}P_0(x).$$

Step 2: Substitute these expressions into the given polynomial.

$$2x^2 - 4x + 3 = 2\left(\frac{2}{3}P_2(x) + \frac{1}{3}P_0(x)\right) - 4(P_1(x)) + 3(P_0(x)).$$

Step 3: Simplify the expression by grouping terms with the same Legendre polynomial.

$$= \frac{4}{3}P_2(x) + \frac{2}{3}P_0(x) - 4P_1(x) + 3P_0(x).$$

$$= \frac{4}{3}P_2(x) - 4P_1(x) + \left(\frac{2}{3} + 3\right)P_0(x).$$

$$= \frac{4}{3}P_2(x) - 4P_1(x) + \frac{11}{3}P_0(x).$$

Step 4: Factor out $\frac{1}{3}$ to match the format of the options.

$$= \frac{1}{3}(4P_2(x) - 12P_1(x) + 11P_0(x)).$$

This matches option (D).

Quick Tip

To express a polynomial in terms of Legendre polynomials, first write down the required Legendre polynomials (P_0, P_1, P_2 , etc.). Then, rearrange them to express $1, x, x^2$, etc., in terms of the $P_n(x)$. Finally, substitute these into your polynomial and simplify.

70. The linear differential equation $(x^n D^n + a_1 x^{n-1} D^{n-1} + a_1 x^{n-2} D^{n-2} \dots + a_n)y = X$ **represents**

- (A) Cauchy Differential equation
- (B) Legendre Differential equation
- (C) Gauss Differential equation
- (D) Albert Differential equation

Correct Answer: (A) Cauchy Differential equation

Solution:

The given equation is of the form:

$$a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_{n-1} x \frac{dy}{dx} + a_n y = X(x).$$

This is the standard definition of a linear homogeneous differential equation with variable coefficients, specifically known as the Cauchy-Euler equation, or simply the Cauchy differential equation.

A key characteristic is that the power of the coefficient x^k matches the order of the derivative $\frac{d^k y}{dx^k}$ for each term.

The Legendre and Gauss equations have different, more specific forms.

The Legendre equation is $(1 - x^2)y'' - 2xy' + n(n + 1)y = 0$.

The Gauss equation is the hypergeometric differential equation.

"Albert Differential equation" is not a standard term.

Therefore, the given form represents a Cauchy Differential equation.

Quick Tip

Instantly recognize a Cauchy-Euler equation by checking if each derivative term $y^{(k)}$ is multiplied by x^k . If so, you know to use the substitution $x = e^z$.

71. Find the Cartesian equation of the plane passing through the point (3,2,-3) and the normal to the plane is $4\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$?

(A) $4x - 2y + 5z + 7 = 0$

(B) $3x - 2y - 3z + 1 = 0$

(C) $4x - y + 5z + 7 = 0$

(D) $4x - 2y - z + 7 = 0$

Correct Answer: (A) $4x - 2y + 5z + 7 = 0$

Solution:

The equation of a plane passing through a point (x_0, y_0, z_0) with a normal vector $\vec{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is given by the formula:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

Here, the point is $(x_0, y_0, z_0) = (3, 2, -3)$.

The components of the normal vector are $a = 4$, $b = -2$, and $c = 5$.

Substitute these values into the formula:

$$4(x - 3) - 2(y - 2) + 5(z - (-3)) = 0.$$

Now, expand and simplify the equation:

$$4x - 12 - 2y + 4 + 5(z + 3) = 0.$$

$$4x - 12 - 2y + 4 + 5z + 15 = 0.$$

$$4x - 2y + 5z + (-12 + 4 + 15) = 0.$$

$$4x - 2y + 5z + 7 = 0.$$

This matches option (A).

Quick Tip

The coefficients of x , y , and z in the Cartesian equation of a plane ($ax + by + cz + d = 0$) are the components of the normal vector to that plane. This is a quick way to check your answer or identify the normal vector.

72. There is a rectangular garden of 220 metres $\times$ 70 metres. A path of width 4 metres is built around the garden. What is the area of the path?

(A) $2472m^2$

(B) $2162m^2$

(C) $1836m^2$

(D) $2384m^2$

Correct Answer: (D) $2384m^2$

Solution:

Step 1: Find the dimensions of the inner garden.

Length of the garden, $L_{inner} = 220$ m.

Width of the garden, $W_{inner} = 70$ m.

Step 2: Calculate the area of the inner garden.

$$\text{Area}_{inner} = L_{inner} \times W_{inner} = 220 \times 70 = 15400 \text{ m}^2.$$

Step 3: Find the dimensions of the outer rectangle (garden + path).

The path of 4 m width is on all sides, so the length and width each increase by $2 \times 4 = 8$ m.

Outer Length, $L_{outer} = 220 + 8 = 228$ m.

Outer Width, $W_{outer} = 70 + 8 = 78$ m.

Step 4: Calculate the area of the outer rectangle.

$$\text{Area}_{outer} = L_{outer} \times W_{outer} = 228 \times 78 = 17784 \text{ m}^2.$$

Step 5: Calculate the area of the path.

$$\text{Area of Path} = \text{Area}_{outer} - \text{Area}_{inner}.$$

$$\text{Area of Path} = 17784 - 15400 = 2384 \text{ m}^2.$$

This matches option (D), although the checkmark in the provided PDF is next to this option, it is numerically distinct from the text of the option. Based on the calculation, 2384 is the correct result.

Quick Tip

For problems involving a path around a rectangle, remember to add twice the path's width to both the length and the width to find the outer dimensions. Area of path = (Outer Area) - (Inner Area).

73. Which of the following is not the correct formula for representing a plane

(A) $\vec{r} \cdot \vec{n} = d$

(B) $ax + by + cz = d$

(C) $lx + my + nz = d$

(D) $al + bm + cn = d^2$

Correct Answer: (D) $al + bm + cn = d^2$

Solution:

Let's analyze each option to see if it represents a plane.

(A) $\vec{r} \cdot \vec{n} = d$: This is the standard vector equation of a plane, where $\vec{r}$ is the position vector of any point on the plane, $\vec{n}$ is a normal vector, and d is a constant. This is a correct formula.

(B) $ax + by + cz = d$: This is the general Cartesian equation of a plane. Any linear equation in x, y , and z represents a plane. This is a correct formula.

(C) $lx + my + nz = d$: This is the normal form of the equation of a plane, where l, m, n are the direction cosines of the normal to the plane and d is the perpendicular distance from the origin. This is a correct formula.

(D) $al + bm + cn = d^2$: This is not an equation representing a plane. It does not involve the variables x, y, z which would define the coordinates of points on the plane. It appears to be a relation between constants, not a formula for a plane itself.

Therefore, the formula that does not represent a plane is (D).

Quick Tip

The equation of a plane in 3D must be a linear equation involving the coordinate variables x, y , and z . Any option that does not have these variables cannot be the general equation of a plane.

74. If the plane passes through three collinear points $(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3)$ then which of the following is true

(A) $x_1y_1z_1 + x_2y_2z_2 + x_3y_3z_3 = 0$

(B) $\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0$

(C) $\begin{vmatrix} x_1 \\ y_2 \\ z_3 \end{vmatrix} = 0$

(D) $x_1x_2x_3 + y_1y_2y_3 + z_1z_2z_3 = 0$

Correct Answer: (B) $\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0$

Solution:

The question is ambiguously worded. Three collinear points do not define a unique plane; they define a line, and infinitely many planes can pass through a line.

However, let's analyze the given options. Option (B) represents the scalar triple product of the position vectors of the three points being zero.

This condition, $[\vec{r}_1 \vec{r}_2 \vec{r}_3] = 0$, means that the three position vectors are coplanar.

This implies that the three points (x_1, y_1, z_1) , (x_2, y_2, z_2) , (x_3, y_3, z_3) lie on a plane that also passes through the origin $(0, 0, 0)$.

If three points are collinear and their line also passes through the origin, then this condition will be true. For example, the points $(1, 2, 3)$, $(2, 4, 6)$, and $(3, 6, 9)$ are collinear and lie on a plane through the origin. The determinant of their coordinates will be zero.

Given the options, this is the most plausible intended answer, assuming a special case or a flawed question where "collinear" is being confused with "coplanar with the origin".

Quick Tip

The condition for three vectors $\vec{a}, \vec{b}, \vec{c}$ to be coplanar is that their scalar triple product is zero, i.e., $\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$. This is equivalent to the determinant of their components being zero.

75. Find the equation of the plane passing through the three points $(2, 2, 0)$, $(1, 2, 1)$, and $(-1, 2, -2)$

(A) $(\vec{r} - (2\mathbf{i} + 2\mathbf{j})) \cdot ((-\mathbf{i} + \mathbf{k}) \times (-3\mathbf{i} - 2\mathbf{k})) = 0$

(B) $(\vec{r} - (3\mathbf{i} - 2\mathbf{j})) \cdot ((-\mathbf{i} + \mathbf{k}) \times (2\mathbf{i} - 2\mathbf{k})) = 0$

(C) $(\vec{r} + (2\mathbf{i} + 2\mathbf{j})) \cdot ((-\mathbf{i} - \mathbf{k}) \times (-3\mathbf{i} - 2\mathbf{k})) = 0$

(D) $(\vec{r} - (2\mathbf{i} + 2\mathbf{j})) \cdot ((-\mathbf{i} - \mathbf{k}) \times (3\mathbf{i} + 2\mathbf{k})) = 0$

Correct Answer: (A) $(\vec{r} - (2\mathbf{i} + 2\mathbf{j})) \cdot ((-\mathbf{i} + \mathbf{k}) \times (-3\mathbf{i} - 2\mathbf{k})) = 0$

Solution:

The vector equation of a plane passing through a point with position vector $\vec{a}$ and parallel to two vectors $\vec{b}$ and $\vec{c}$ is $(\vec{r} - \vec{a}) \cdot (\vec{b} \times \vec{c}) = 0$.

Let the three points be $A(2, 2, 0)$, $B(1, 2, 1)$, and $C(-1, 2, -2)$.

We can choose the point A, so the position vector is $\vec{a} = 2\mathbf{i} + 2\mathbf{j} + 0\mathbf{k} = 2\mathbf{i} + 2\mathbf{j}$.

The plane is parallel to the vectors $\vec{AB}$ and $\vec{AC}$.

$$\vec{AB} = (\text{Position vector of B}) - (\text{Position vector of A})$$

$$\vec{AB} = (\mathbf{i} + 2\mathbf{j} + \mathbf{k}) - (2\mathbf{i} + 2\mathbf{j}) = -\mathbf{i} + \mathbf{k}.$$

$$\vec{AC} = (\text{Position vector of C}) - (\text{Position vector of A})$$

$$\vec{AC} = (-\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}) - (2\mathbf{i} + 2\mathbf{j}) = -3\mathbf{i} - 2\mathbf{k}.$$

Substituting these into the formula for the equation of the plane:

$$(\vec{r} - (2\mathbf{i} + 2\mathbf{j})) \cdot ((-\mathbf{i} + \mathbf{k}) \times (-3\mathbf{i} - 2\mathbf{k})) = 0.$$

This exactly matches the expression in option (A). The question asks for the equation in its un-simplified vector form.

Quick Tip

To find the equation of a plane through three points A, B, and C, you need a point on the plane (e.g., A) and a normal vector. The normal vector can be found by taking the cross product of two vectors lying in the plane, such as $\vec{AB}$ and $\vec{AC}$. The equation is then $(\vec{r} - \vec{a}) \cdot (\vec{AB} \times \vec{AC}) = 0$.

76. Two lines are said to be perpendicular if the product of their slope is equal to

- (A) -1
- (B) 0
- (C) 1
- (D) 1/2

Correct Answer: (A) -1

Solution:

This question asks for the fundamental condition for two lines to be perpendicular in a 2D Cartesian coordinate system.

Let the slopes of the two lines be m_1 and m_2 .

The condition for the lines to be perpendicular (orthogonal) to each other is that the product of their slopes is equal to -1.

$$m_1 \times m_2 = -1.$$

This is a standard definition in coordinate geometry, provided neither line is vertical (as a vertical line has an undefined slope).

Therefore, the correct answer is -1.

Quick Tip

Remember the conditions for two lines with slopes m_1 and m_2 : - Parallel: $m_1 = m_2$
- Perpendicular: $m_1 \times m_2 = -1$ This means the slope of a perpendicular line is the "negative reciprocal" of the other.

77. What is the distance of (5, 12) from the origin?

- (A) 5 units
- (B) 8 units
- (C) 12 units
- (D) 13 units

Correct Answer: (D) 13 units

Solution:

The origin is the point $(0, 0)$. We need to find the distance between the point $P(5, 12)$ and the origin $O(0, 0)$.

The distance formula between two points (x_1, y_1) and (x_2, y_2) is:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Here, $(x_1, y_1) = (0, 0)$ and $(x_2, y_2) = (5, 12)$.

Substituting the values:

$$d = \sqrt{(5 - 0)^2 + (12 - 0)^2}.$$

$$d = \sqrt{5^2 + 12^2}.$$

$$d = \sqrt{25 + 144}.$$

$$d = \sqrt{169}.$$

$$d = 13.$$

The distance is 13 units. This is a Pythagorean triple $(5, 12, 13)$.

Quick Tip

The distance of any point (x, y) from the origin is simply $\sqrt{x^2 + y^2}$. Memorizing common Pythagorean triples like $(3, 4, 5)$, $(5, 12, 13)$, $(8, 15, 17)$, and $(7, 24, 25)$ can save time in calculations.

78. Two lines are said to be parallel if the difference of their slope is

- (A) -1
- (B) 0
- (C) 1
- (D) 2

Correct Answer: (B) 0

Solution:

This question asks for a condition for two lines to be parallel, expressed in terms of the difference of their slopes.

Let the slopes of the two lines be m_1 and m_2 .

The condition for the lines to be parallel is that their slopes are equal.

$$m_1 = m_2.$$

The question asks for the value of the difference of their slopes.

If $m_1 = m_2$, then their difference is:

$$m_1 - m_2 = 0.$$

Therefore, two lines are parallel if the difference of their slopes is 0.

Quick Tip

Be sure to read the question carefully. While the condition for parallel lines is "equal slopes" ($m_1 = m_2$), the question asks for the "difference of their slopes" ($m_1 - m_2$), which must be 0.

79. The equation of a straight line that passes through the point (3, 4) and is perpendicular to the line $3x + 2y + 5 = 0$ is

(A) $2x - 3y + 6 = 0$

(B) $2x + 3y + 6 = 0$

(C) $2x - 3y - 6 = 0$

(D) $2x + 3y - 6 = 0$

Correct Answer: (A) $2x - 3y + 6 = 0$

Solution:

Step 1: Find the slope of the given line.

The given line is $3x + 2y + 5 = 0$.

To find the slope, we can rewrite it in the slope-intercept form ($y = mx + c$).

$$2y = -3x - 5.$$

$$y = -\frac{3}{2}x - \frac{5}{2}.$$

The slope of this line, let's call it m_1 , is $-\frac{3}{2}$.

Step 2: Find the slope of the perpendicular line.

The slope of the required line, m_2 , is the negative reciprocal of m_1 .

$$m_2 = -\frac{1}{m_1} = -\frac{1}{(-3/2)} = \frac{2}{3}.$$

Step 3: Use the point-slope form to find the equation of the required line.

The line passes through the point $(x_1, y_1) = (3, 4)$ and has a slope $m_2 = \frac{2}{3}$.

The point-slope form is $y - y_1 = m(x - x_1)$.

$$y - 4 = \frac{2}{3}(x - 3).$$

Step 4: Convert the equation to the general form $Ax + By + C = 0$.

Multiply by 3 to clear the fraction:

$$3(y - 4) = 2(x - 3).$$

$$3y - 12 = 2x - 6.$$

Rearrange the terms:

$$0 = 2x - 3y - 6 + 12.$$

$$2x - 3y + 6 = 0.$$

This matches option (A).

Quick Tip

A shortcut for finding the equation of a line perpendicular to $Ax + By + C = 0$ is to use the form $Bx - Ay + K = 0$. Here, for $3x + 2y + 5 = 0$, the perpendicular line is $2x - 3y + K = 0$. Then, substitute the point (3,4) to find K: $2(3) - 3(4) + K = 0 \implies 6 - 12 + K = 0 \implies K = 6$. The equation is $2x - 3y + 6 = 0$.

80. The angle between the line $\frac{x+1}{2} = \frac{y}{3} = \frac{z-3}{6}$ and the plane $10x + 2y - 11z = 3$

(A) $\cos^{-1}\left(\frac{8}{21}\right)$

(B) $\cos^{-1}\left(\frac{8}{21}\right)$

(C) $\sin^{-1}\left(\frac{8}{21}\right)$

(D) $\sin^{-1}\left(\frac{1}{8}\right)$

Correct Answer: (C) $\sin^{-1}\left(\frac{8}{21}\right)$

Solution:

Let θ be the angle between the line and the plane. The formula is given by:

$$\sin \theta = \frac{|\vec{b} \cdot \vec{n}|}{|\vec{b}| |\vec{n}|}$$

where $\vec{b}$ is the direction vector of the line and $\vec{n}$ is the normal vector to the plane.

Step 1: Identify the direction vector of the line.

From the equation $\frac{x+1}{2} = \frac{y}{3} = \frac{z-3}{6}$, the direction ratios are (2, 3, 6).

So, the direction vector is $\vec{b} = 2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$.

Step 2: Identify the normal vector of the plane.

From the equation $10x + 2y - 11z = 3$, the coefficients of x, y, z are the components of the normal vector.

So, the normal vector is $\vec{n} = 10\mathbf{i} + 2\mathbf{j} - 11\mathbf{k}$.

Step 3: Calculate the dot product $\vec{b} \cdot \vec{n}$.

$$\vec{b} \cdot \vec{n} = (2)(10) + (3)(2) + (6)(-11) = 20 + 6 - 66 = -40.$$

$$|\vec{b} \cdot \vec{n}| = |-40| = 40.$$

Step 4: Calculate the magnitudes of the vectors.

$$|\vec{b}| = \sqrt{2^2 + 3^2 + 6^2} = \sqrt{4 + 9 + 36} = \sqrt{49} = 7.$$

$$|\vec{n}| = \sqrt{10^2 + 2^2 + (-11)^2} = \sqrt{100 + 4 + 121} = \sqrt{225} = 15.$$

Step 5: Substitute the values into the formula for $\sin \theta$.

$$\sin \theta = \frac{40}{7 \times 15} = \frac{40}{105}.$$

Simplify the fraction by dividing the numerator and denominator by 5:

$$\sin \theta = \frac{8}{21}.$$

Step 6: Express the angle θ .

$$\theta = \sin^{-1}\left(\frac{8}{21}\right).$$

This matches option (C).

Quick Tip

Be careful! The angle between a line and a plane uses $\sin \theta$, while the angle between two lines or two planes uses $\cos \theta$. This is because the angle is measured between the line and its projection on the plane, which is complementary to the angle between the line's direction vector and the plane's normal vector.

81. The center-radius form of a sphere is

(A) $x^2 + y^2 + z^2 = r^2$

(B) $(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$

(C) $x^2 + y^2 = r^2$

(D) $(x - a)^2 + (y - b)^2 = r^2$

Correct Answer: (B) $(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$

Solution:

A sphere is defined as the set of all points (x, y, z) in three-dimensional space that are at a constant distance, the radius r , from a fixed point, the center (a, b, c) .

Using the distance formula in 3D, the distance between any point (x, y, z) on the sphere and the center (a, b, c) is:

$$\sqrt{(x - a)^2 + (y - b)^2 + (z - c)^2} = r.$$

Squaring both sides of the equation gives the standard center-radius form:

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2.$$

Option (A) represents a sphere centered at the origin. Option (D) represents a circle in a 2D plane. Option (C) represents a circle centered at the origin in a 2D plane.

Quick Tip

The equation of a sphere is a direct 3D extension of the equation of a circle. The center-radius form for a circle is $(x - h)^2 + (y - k)^2 = r^2$. For a sphere, you simply add the term for the z-coordinate, $(z - c)^2$.

82. Two spheres are said to be in touch externally if (where d =distance between their centres)

(A) $d = r_1 + r_2$

(B) $d = r_1 - r_2$

(C) $r_1 = r_2$

(D) $d = r_1 \cdot r_2$

Correct Answer: (A) $d = r_1 + r_2$

Solution:

Consider two spheres with centers C_1 and C_2 , and radii r_1 and r_2 respectively.

Let d be the distance between their centers, so $d = C_1C_2$.

When the two spheres touch each other externally at a single point, their centers and the point of contact are collinear.

The distance between the centers is the length of the line segment connecting them. This segment is formed by the radius of the first sphere and the radius of the second sphere joined together.

Therefore, the distance between the centers is the sum of their radii.

$d = r_1 + r_2$.

Quick Tip

Visualize the conditions for spheres (or circles): - Touch externally: distance between centers = sum of radii ($d = r_1 + r_2$). - Touch internally: distance between centers = difference of radii ($d = |r_1 - r_2|$). - Intersect: $|r_1 - r_2| < d < r_1 + r_2$. - One contains another: $d < |r_1 - r_2|$.

83. The centre and radius of the circle of $x^2 + y^2 + z^2 + 2x - 2y - 4z - 19 = 0$ is

(A) $(-1, 1, 2), 5$

(B) $(1, -1, 2), 4$

(C) $(-1, 1, 2), -4$

(D) $(1, -1, 2), -4$

Correct Answer: (A) $(-1, 1, 2), 5$

Solution:

The given equation is the general form of a sphere. The question seems to use "circle" mistakenly, which is a common error; it refers to the sphere.

Method 1: Completing the Square.

Rearrange the terms: $(x^2 + 2x) + (y^2 - 2y) + (z^2 - 4z) = 19$.

Complete the square for each variable:

$(x^2 + 2x + 1) + (y^2 - 2y + 1) + (z^2 - 4z + 4) = 19 + 1 + 1 + 4$.

$(x + 1)^2 + (y - 1)^2 + (z - 2)^2 = 25$.

This is in the center-radius form $(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$.

Center $(a, b, c) = (-1, 1, 2)$.

Radius $r = \sqrt{25} = 5$.

Method 2: Using the General Form Formula.

The general form is $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$.

Comparing coefficients: $2u = 2 \implies u = 1$, $2v = -2 \implies v = -1$, $2w = -4 \implies w = -2$, and $d = -19$.

Center is $(-u, -v, -w) = (-1, -(-1), -(-2)) = (-1, 1, 2)$.

Radius is $r = \sqrt{u^2 + v^2 + w^2 - d} = \sqrt{1^2 + (-1)^2 + (-2)^2 - (-19)} = \sqrt{1 + 1 + 4 + 19} = \sqrt{25} = 5$.

Quick Tip

Using the formula for the general form is faster. The center is found by taking half the coefficients of x, y, z and reversing their signs. Then use the radius formula $r = \sqrt{u^2 + v^2 + w^2 - d}$.

84. The general form of the sphere is

(A) $x^2 + y^2 + z^2 + ux + vy + wz - d = 0$

(B) $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz - d = 0$

(C) $x^2 + y^2 + z^2 + ux + vy + wz + d = 0$

(D) $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

Correct Answer: (D) $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

Solution:

The general second-degree equation $ax^2 + by^2 + cz^2 + 2hxy + 2gyz + 2fzx + 2ux + 2vy + 2wz + d = 0$ represents a sphere if and only if:

1. $a = b = c (\neq 0)$
2. $h = g = f = 0$ (no product terms)

If we set $a = b = c = 1$ for simplicity, the equation becomes:

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0.$$

This is the standard general form of the equation of a sphere. The factors of 2 on the linear terms $(2ux, 2vy, 2wz)$ are included by convention to simplify the formulas for the center $(-u, -v, -w)$ and radius.

Quick Tip

The convention of using $2u, 2v, 2w$ in the general form of a sphere or circle is important because it avoids fractions when calculating the center coordinates from the equation.

85. Find the value of α where the angle between the plane $2x - 3y + 6z - 11 = 0$ and the x-axis is $\sin^{-1}(\alpha)$

(A) $\frac{\sqrt{2}}{3}$

(B) $\frac{2}{7}$

(C) $\frac{\sqrt{3}}{2}$

(D) $\frac{3}{7}$

Correct Answer: (B) $\frac{2}{7}$

Solution:

Let θ be the angle between a line and a plane. The formula is $\sin \theta = \frac{|\vec{b} \cdot \vec{n}|}{|\vec{b}| |\vec{n}|}$.

Here, $\vec{n}$ is the normal vector to the plane and $\vec{b}$ is the direction vector of the line.

Step 1: Identify the normal vector to the plane.

From the equation $2x - 3y + 6z - 11 = 0$, the normal vector is $\vec{n} = 2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$.

Step 2: Identify the direction vector of the x-axis.

The direction vector for the x-axis is $\vec{b} = \mathbf{i} = 1\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$.

Step 3: Calculate the dot product $\vec{b} \cdot \vec{n}$.

$$\vec{b} \cdot \vec{n} = (1)(2) + (0)(-3) + (0)(6) = 2.$$

$$\text{So, } |\vec{b} \cdot \vec{n}| = 2.$$

Step 4: Calculate the magnitudes of the vectors.

$$|\vec{n}| = \sqrt{2^2 + (-3)^2 + 6^2} = \sqrt{4 + 9 + 36} = \sqrt{49} = 7.$$

$$|\vec{b}| = \sqrt{1^2 + 0^2 + 0^2} = \sqrt{1} = 1.$$

Step 5: Calculate $\sin \theta$.

$$\sin \theta = \frac{2}{7 \times 1} = \frac{2}{7}.$$

We are given that the angle is $\sin^{-1}(\alpha)$, so $\sin \theta = \alpha$.

$$\text{Therefore, } \alpha = \frac{2}{7}.$$

Quick Tip

The angle between a line and a plane is the complement of the angle between the line's direction vector and the plane's normal vector. This is why the formula uses $\sin \theta$ instead of the $\cos \theta$ used for line-line or plane-plane angles.

86. The total surface area of a cone with a base radius of "r" and a slanted height of "l" is equal to

(A) $\pi r l + 2\pi r^2$

(B) $2\pi r l + 2\pi r^2$

(C) $\pi r l + \pi r^2$

(D) $2\pi r^2$

Correct Answer: (C) $\pi rl + \pi r^2$

Solution:

The total surface area (TSA) of a cone is the sum of its two component areas:

1. The area of its circular base.
2. The area of its curved (lateral) surface.

The formula for the area of the circular base with radius r is:

$$\text{Area}_{\text{base}} = \pi r^2.$$

The formula for the curved surface area (CSA) with radius r and slant height l is:

$$\text{CSA} = \pi rl.$$

The total surface area is the sum of these two areas:

$$\text{TSA} = \text{Area}_{\text{base}} + \text{CSA} = \pi r^2 + \pi rl.$$

This can also be written by factoring out πr : $\text{TSA} = \pi r(r + l)$.

Option (C) matches the formula.

Quick Tip

Always break down Total Surface Area (TSA) problems into the sum of the areas of the individual surfaces. For a cone, it's the flat circular base plus the slanted, curved side.

87. What is the total surface area of a cone with a radius of 7cm and a height of 24cm

(A) 710cm^2

(B) 704cm^2

(C) 700cm^2

(D) 725cm^2

Correct Answer: (B) 704cm^2

Solution:

Given: radius $r = 7$ cm and height $h = 24$ cm.

Step 1: Find the slant height (l).

The radius, height, and slant height of a cone form a right-angled triangle, with l as the hypotenuse.

Using the Pythagorean theorem: $l^2 = r^2 + h^2$.

$$l^2 = 7^2 + 24^2 = 49 + 576 = 625.$$

$$l = \sqrt{625} = 25 \text{ cm.}$$

Step 2: Calculate the Total Surface Area (TSA).

The formula for TSA is $\pi r(r + l)$.

Using $\pi \approx \frac{22}{7}$:

$$\text{TSA} = \frac{22}{7} \times 7 \times (7 + 25).$$

$$\text{TSA} = 22 \times (32).$$

$$\text{TSA} = 704 \text{ cm}^2.$$

Quick Tip

Recognizing Pythagorean triples (like 7, 24, 25) can save you calculation time. When you see two of the numbers, you can immediately find the third without squaring and taking the square root.

88. The curved surface area of a cone is 1980 cm^2 , and its radius is 21cm; what is the slanted height of a cone

(A) 25cm

(B) 38cm

(C) 35cm

(D) 30cm

Correct Answer: (D) 30cm

Solution:

Given: Curved Surface Area (CSA) = 1980 cm^2 and radius $r = 21 \text{ cm}$.

The formula for the curved surface area of a cone is:

$\text{CSA} = \pi r l$, where l is the slant height.

We need to find l . Substitute the given values into the formula:

$$1980 = \frac{22}{7} \times 21 \times l.$$

Simplify the right side:

$$1980 = 22 \times \frac{21}{7} \times l.$$

$$1980 = 22 \times 3 \times l.$$

$$1980 = 66 \times l.$$

Now, solve for l :

$$l = \frac{1980}{66}.$$

$$\text{Since } 66 \times 3 = 198, \text{ we have } l = \frac{198 \times 10}{66} = 3 \times 10 = 30.$$

The slant height is 30 cm.

Quick Tip

When solving for a variable in a formula, isolate it by performing the inverse operations. Here, to undo the multiplication by 66, we divide by 66.

89. Find the equation of the tangent plane to the sphere $x^2 + y^2 + z^2 - 4x + 2y - 6z + 5 = 0$ which is parallel to the plane $3x + 2y - 2z = 0$

(A) $3x + 2y - 2z + 3 + 2\sqrt{17} = 0$

(B) $3x + 2y - 2z + 3 - 2\sqrt{17} = 0$

(C) $3x + 2y - 2z + 2 + 3\sqrt{17} = 0$

(D) $3x - 2y - 2z - 3 - 3\sqrt{17} = 0$

Correct Answer: (C) $3x + 2y - 2z + 2 + 3\sqrt{17} = 0$

Solution:

Step 1: Find the center and radius of the sphere.

The equation is $x^2 + y^2 + z^2 - 4x + 2y - 6z + 5 = 0$.

Center = $(-\frac{-4}{2}, -\frac{2}{2}, -\frac{-6}{2}) = (2, -1, 3)$.

Radius $r = \sqrt{(2)^2 + (-1)^2 + (3)^2 - 5} = \sqrt{4 + 1 + 9 - 5} = \sqrt{9} = 3$.

Step 2: Write the equation of a plane parallel to the given plane.

The given plane is $3x + 2y - 2z = 0$. Any parallel plane will have the form $3x + 2y - 2z + k = 0$.

Step 3: Apply the condition of tangency.

For a plane to be tangent to a sphere, the perpendicular distance from the center of the sphere to the plane must be equal to the radius.

$$\text{Distance} = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}} = r.$$

$$\frac{|3(2) + 2(-1) - 2(3) + k|}{\sqrt{3^2 + 2^2 + (-2)^2}} = 3.$$

$$\frac{|6 - 2 - 6 + k|}{\sqrt{9 + 4 + 4}} = 3.$$

$$\frac{|k - 2|}{\sqrt{17}} = 3.$$

$$|k - 2| = 3\sqrt{17}.$$

Step 4: Solve for k .

This gives two possible values for k :

$$k - 2 = 3\sqrt{17} \implies k = 2 + 3\sqrt{17}.$$

$$k - 2 = -3\sqrt{17} \implies k = 2 - 3\sqrt{17}.$$

Step 5: Write the equations of the two possible tangent planes.

The equations are $3x + 2y - 2z + 2 + 3\sqrt{17} = 0$ and $3x + 2y - 2z + 2 - 3\sqrt{17} = 0$.

Option (C) matches one of these equations.

Quick Tip

For a given sphere and a given direction, there are always two parallel tangent planes, one on each side of the sphere. The only difference in their equations will be the constant term, arising from the $\pm$ in solving the distance formula.

90. A right circular cone height 15cm and base radius 15cm is carved out of a wooden sphere of radius 15cm volume of the remaining wood (in %) is

- (A) 25
- (B) 37.5
- (C) 50
- (D) 75

Correct Answer: (D) 75

Solution:

Step 1: Calculate the volume of the original wooden sphere.

Radius of the sphere, $R = 15$ cm.

Volume of sphere, $V_{\text{sphere}} = \frac{4}{3}\pi R^3 = \frac{4}{3}\pi(15)^3$.

Step 2: Calculate the volume of the cone that is carved out.

Height of the cone, $h = 15$ cm.

Base radius of the cone, $r = 15$ cm.

Volume of cone, $V_{\text{cone}} = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(15)^2(15) = \frac{1}{3}\pi(15)^3$.

Step 3: Calculate the volume of the remaining wood.

$V_{\text{remaining}} = V_{\text{sphere}} - V_{\text{cone}} = \frac{4}{3}\pi(15)^3 - \frac{1}{3}\pi(15)^3$.

$V_{\text{remaining}} = (\frac{4}{3} - \frac{1}{3})\pi(15)^3 = \frac{3}{3}\pi(15)^3 = \pi(15)^3$.

Step 4: Calculate the percentage of the remaining wood.

Percentage remaining = $\frac{V_{\text{remaining}}}{V_{\text{sphere}}} \times 100\%$.

Percentage remaining = $\frac{\pi(15)^3}{\frac{4}{3}\pi(15)^3} \times 100\%$.

The $\pi(15)^3$ terms cancel out.

Percentage remaining = $\frac{1}{4/3} \times 100\% = \frac{3}{4} \times 100\% = 75\%$.

Quick Tip

When dealing with ratios or percentages of geometric volumes, keep the formulas in terms of variables and π as long as possible. The common terms often cancel, simplifying the arithmetic significantly.

91. Let $Z_n = \{0, 1, 2, \dots, n-1\}$ fails to be a group under multiplication modulate because

- (A) Closure property fails
- (B) Closure property holds but not associate
- (C) There is no identity
- (D) There is no inverse for an element '0' of the set

Correct Answer: (D) There is no inverse for an element '0' of the set

Solution:

For a set to be a group under an operation, it must satisfy four properties: closure, associativity, identity, and inverse.

1. Closure: The product of any two elements in Z_n under multiplication modulo n is also in Z_n . So, closure holds.
2. Associativity: Multiplication modulo n is associative. So, associativity holds.
3. Identity: The element '1' acts as the multiplicative identity since $1 \cdot a \equiv a \pmod{n}$ for any $a \in Z_n$. So, an identity element exists (for $n > 1$).
4. Inverse: For every element a in the set, there must exist an inverse a^{-1} such that $a \cdot a^{-1} = 1$.

Consider the element '0' in the set Z_n . For any element $x \in Z_n$, the product is $0 \cdot x \equiv 0 \pmod{n}$.

It is impossible to find an element x such that $0 \cdot x = 1$.

Therefore, the element '0' does not have a multiplicative inverse.

Since not every element has an inverse, Z_n fails to be a group under multiplication modulo n .

Quick Tip

A set forms a group under multiplication modulo n if and only if you consider the set of units, $Z_n^* = \{a \in Z_n \mid \gcd(a, n) = 1\}$. The set $\{0, 1, \dots, n-1\}$ will never form a group under multiplication for $n > 1$ because 0 lacks an inverse.

92. If 'a' is non-identity self-inverse element then O(a) must be

- (A) 5
- (B) 4
- (C) 2
- (D) 1

Correct Answer: (C) 2

Solution:

Let e be the identity element of the group.

The term "self-inverse" means that an element is its own inverse.

Mathematically, this is written as $a = a^{-1}$, which implies $a \cdot a = e$, or $a^2 = e$.

The term "non-identity" means that the element a is not the identity element, i.e., $a \neq e$.

The order of an element a , denoted $O(a)$, is the smallest positive integer k such that $a^k = e$.

From the condition $a^2 = e$, we know that the order of a must be a divisor of 2. So, the order can be either 1 or 2.

If the order were 1, then $a^1 = e$, which would mean $a = e$.

However, the problem states that a is a non-identity element, so $a \neq e$.

Therefore, the order cannot be 1. The only remaining possibility is that the order of a is 2. $O(a) = 2$.

Quick Tip

An element of order 2 is always a non-identity, self-inverse element. For example, in the group of symmetries of a rectangle, the reflection about the horizontal axis is self-inverse.

93. The number of generators of a cyclic group of order 60 is

- (A) 32
- (B) 16
- (C) 8
- (D) 30

Correct Answer: (B) 16

Solution:

The number of generators of a finite cyclic group of order n is given by Euler's totient function, $\phi(n)$.

$\phi(n)$ counts the number of positive integers less than or equal to n that are relatively prime to n .

Here, the order of the group is $n = 60$. We need to calculate $\phi(60)$.

Step 1: Find the prime factorization of 60.

$$60 = 6 \times 10 = (2 \times 3) \times (2 \times 5) = 2^2 \times 3^1 \times 5^1.$$

Step 2: Use the formula for Euler's totient function.

$$\phi(n) = n \left(1 - \frac{1}{p_1}\right) \left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_k}\right), \text{ where } p_1, p_2, \dots \text{ are the distinct prime factors of } n.$$

$$\phi(60) = 60 \left(1 - \frac{1}{2}\right) \left(1 - \frac{1}{3}\right) \left(1 - \frac{1}{5}\right).$$

Step 3: Calculate the value.

$$\phi(60) = 60 \times \frac{1}{2} \times \frac{2}{3} \times \frac{4}{5}.$$

$$\phi(60) = 30 \times \frac{2}{3} \times \frac{4}{5}.$$

$$\phi(60) = 20 \times \frac{4}{5}.$$

$$\phi(60) = 16.$$

Therefore, a cyclic group of order 60 has 16 generators.

Quick Tip

In a cyclic group Z_n , an element k is a generator if and only if $\gcd(k, n) = 1$. Finding the number of generators is the same as finding the number of integers from 1 to $n - 1$ that are coprime to n .

94. Find the order of the non-abelian group

- (A) 4
- (B) 36
- (C) 25
- (D) 49

Correct Answer: (B) 36

Solution:

The question is phrased poorly but implies "Which of the following numbers can be the order of a non-abelian group?". Let's analyze the options based on group theory theorems.

1. A group of order p , where p is a prime number, is always cyclic and therefore abelian.
2. A group of order p^2 , where p is a prime number, is always abelian.

Let's check the given orders:

(A) Order 4: $4 = 2^2$. Since this is of the form p^2 , all groups of order 4 are abelian.

(C) Order 25: $25 = 5^2$. Since this is of the form p^2 , all groups of order 25 are abelian.

(D) Order 49: $49 = 7^2$. Since this is of the form p^2 , all groups of order 49 are abelian.

(B) Order 36: $36 = 2^2 \times 3^2$. This is not of the form p or p^2 . There exist non-abelian groups of order 36. For example, the direct product of the symmetric group S_3 (which is non-abelian and has order 6) and the cyclic group Z_6 gives a non-abelian group of order $6 \times 6 = 36$.

Therefore, among the given options, only 36 can be the order of a non-abelian group.

Quick Tip

Remember these simple rules: any group of prime order is cyclic and abelian. Any group of order p^2 (p is prime) is abelian. The smallest non-abelian group is the symmetric group S_3 , which has order $3! = 6$.

95. Let $G = \{1, -1, i, -i\}$ is multiplication group then

- (A) G is cyclic and 'i' is the only generator
- (B) G is must cyclic
- (C) G is cyclic with two generators 'i' and '-i'
- (D) G is cyclic '1' is the only generator

Correct Answer: (C) G is cyclic with two generators 'i' and '-i'

Solution:

A group is cyclic if there exists an element (a generator) such that every other element in the group can be expressed as a power of it. The order of the group G is 4.

Let's check the powers of each element:

1. Element '1': $1^1 = 1$. It only generates the subgroup $\{1\}$. Not a generator.
2. Element '-1': $(-1)^1 = -1$, $(-1)^2 = 1$. It generates the subgroup $\{-1, 1\}$. Not a generator.
3. Element 'i': $i^1 = i$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$. The powers of 'i' generate all four elements of G . So, 'i' is a generator.
4. Element '-i': $(-i)^1 = -i$, $(-i)^2 = -1$, $(-i)^3 = i$, $(-i)^4 = 1$. The powers of '-i' also generate all four elements of G . So, '-i' is also a generator.

Since the group G has generators (namely 'i' and '-i'), it is a cyclic group.

Analyzing the options:

- (A) False, because '-i' is also a generator.
- (B) True, but not the most complete and accurate description.
- (C) True and provides the complete information that the group is cyclic and specifies both of its generators.
- (D) False, '1' is the identity element, not a generator.

Therefore, option (C) is the best answer.

Quick Tip

For a cyclic group of order n , the number of generators is $\phi(n)$. For $G = \{1, -1, i, -i\}$, the order is $n = 4$. $\phi(4) = 4(1 - 1/2) = 2$. So, we expect exactly two generators.

96. Let $(G, +)$ be a group. A nonempty subset H of G is a subgroup of G if and only if for all $a, b \in H$

- (A) $ab^{-1} \in H$
- (B) $ab \in H$
- (C) $a + (-b) \in H$
- (D) $a + b^{-1} \in H$

Correct Answer: (C) $a + (-b) \in H$

Solution:

This question asks for the one-step subgroup test.

For a group $(G, \cdot)$ with multiplicative notation, the one-step subgroup test states that a non-empty subset H is a subgroup if and only if for all $a, b \in H$, the element $a \cdot b^{-1}$ is also in H . This matches option (A).

However, the group in the question is an additive group, denoted $(G, +)$. We must translate the subgroup test into additive notation.

The operation is addition $(+)$.

The identity element is 0.

The inverse of an element b is denoted by $-b$.

So, the multiplicative condition $a \cdot b^{-1} \in H$ becomes the additive condition $a + (-b) \in H$. This is also commonly written as $a - b \in H$.

Let's check the options:

(A) $ab^{-1} \in H$: This is the test for a multiplicative group.

(B) $ab \in H$: This only shows closure and is insufficient.

(C) $a + (-b) \in H$: This is the correct translation of the one-step subgroup test for an additive group.

(D) $a + b^{-1} \in H$: This incorrectly mixes additive and multiplicative notation.

Quick Tip

Always pay attention to the group's operation. The fundamental concepts are the same, but the notation changes. - Multiplicative group: ab^{-1} - Additive group: $a - b$ or $a + (-b)$

97. Let $O(G) = 12$ and the index of a subgroup of H of G is 3. then order of H is

(A) 2

(B) 4

(C) 6

(D) 12

Correct Answer: (B) 4

Solution:

This problem uses Lagrange's Theorem for finite groups.

Lagrange's Theorem states that the order of a subgroup H divides the order of the group G . The relationship is given by the formula:

$$O(G) = O(H) \times [G : H]$$

where:

$O(G)$ is the order of the group G .

$O(H)$ is the order of the subgroup H .

$[G : H]$ is the index of H in G (which is the number of distinct left or right cosets of H in G).

We are given:

$$O(G) = 12.$$

$$[G : H] = 3.$$

Substitute the known values into the formula:

$$12 = O(H) \times 3.$$

To find the order of H , we solve for $O(H)$:

$$O(H) = \frac{12}{3} = 4.$$

The order of the subgroup H is 4.

Quick Tip

Lagrange's Theorem ($|G| = |H| \cdot [G : H]$) is one of the most fundamental results in finite group theory. It connects the order of a group, a subgroup, and its index.

98. Every finite group G of composite order possesses

(A) improper sub group

(B) proper subgroup

(C) no subgroup

(D) abelian group

Correct Answer: (B) proper subgroup

Solution:

Let's analyze the terms:

- Composite order: The order of the group, $|G| = n$, is a composite number (i.e., not prime and greater than 1).
- Improper subgroups: These are the trivial subgroup $\{e\}$ and the group G itself. Every group has these.
- Proper subgroup: Any subgroup of G other than the improper subgroups.

The question asks what a finite group of composite order must possess.

According to Cauchy's Theorem, if the order of a finite group G is divisible by a prime number p , then G has an element of order p . This element generates a cyclic subgroup of order p .

Let the composite order of G be n . Since n is composite, it must have a prime factor p . For example, if $n = 6$, a prime factor is $p = 2$. If $n = 10$, a prime factor is $p = 5$.

Since p is a prime factor of a composite number n , we must have $1 < p < n$.

By Cauchy's Theorem, since p divides $n = |G|$, there exists a subgroup H of G with order $|H| = p$.

Since $1 < |H| < |G|$, this subgroup H is a proper subgroup.

Therefore, every finite group of composite order must possess a proper subgroup.

Quick Tip

While Lagrange's Theorem says the order of a subgroup must divide the order of the group, it doesn't guarantee a subgroup for every divisor. Cauchy's Theorem provides a partial converse: it guarantees a subgroup for every prime divisor of the group's order.

99. A permutation of the form $\begin{pmatrix} a_1 & a_2 & \dots & a_k \\ a_2 & a_3 & \dots & a_1 \end{pmatrix}$ is called a — of length k .

- (A) permutation
- (B) cycle
- (C) transposition
- (D) matrix

Correct Answer: (B) cycle

Solution:

The given two-line notation for a permutation shows the following mapping:

$$a_1 \rightarrow a_2$$

$$a_2 \rightarrow a_3$$

...

$$a_{k-1} \rightarrow a_k$$

$$a_k \rightarrow a_1$$

This describes a permutation where a set of k elements are cyclically permuted among themselves, while all other elements (if any) are left unchanged.

This specific structure is called a cycle of length k . It is often written in a more compact one-line notation as $(a_1 \ a_2 \ \dots \ a_k)$.

Let's analyze the options:

- (A) Permutation: This is the general term for any rearrangement. While correct, it's not the most specific name for this particular structure.
- (B) Cycle: This is the specific name for this type of permutation.
- (C) Transposition: This is a cycle of length 2, i.e., when $k = 2$. It's a special case of a cycle.
- (D) Matrix: This is a rectangular array of numbers and is unrelated to permutations in this context.

The most accurate and specific term is "cycle".

Quick Tip

Recognize the different notations for permutations. The two-line form is general, while the one-line cycle notation $(a_1 a_2 \dots a_k)$ is used for the specific structure where elements are permuted in a circle.

100. If N and M are normal subgroups of G and $M \cap N = \{e\}$ for $n \in N, m \in M$ then

(A) $nm = mn$

(B) $nm \neq mn$

(C) $nm = 1$

(D) $n = m$

Correct Answer: (A) $nm = mn$

Solution:

We are given that N and M are normal subgroups of G and their intersection is only the identity element, e .

Let's consider the commutator of an element $m \in M$ and an element $n \in N$, which is defined as $mnm^{-1}n^{-1}$.

Step 1: Consider the element mnm^{-1} .

Since N is a normal subgroup, for any $g \in G$ and $n \in N$, we have $gng^{-1} \in N$.

Here, we can take $g = m$. So, $mnm^{-1} \in N$.

Since n^{-1} is also in N (because N is a subgroup), the product $(mnm^{-1})n^{-1}$ must be in N .

Step 2: Consider the element $nm^{-1}n^{-1}$.

Since M is a normal subgroup, for any $g \in G$ and $m \in M$, we have $gmg^{-1} \in M$.

Here, $m^{-1} \in M$. We can take $g = n$. So, $nm^{-1}n^{-1} \in M$.

Since m is also in M , the product $m(nm^{-1}n^{-1})$ must be in M .

Step 3: Combine the results.

The commutator element $mnm^{-1}n^{-1}$ belongs to N (from Step 1) and also to M (from Step 2).

Therefore, the commutator $mnm^{-1}n^{-1}$ must be in the intersection $M \cap N$.

Step 4: Use the given intersection condition.

We are given that $M \cap N = \{e\}$.

So, we must have $mnm^{-1}n^{-1} = e$.

Step 5: Rearrange the equation.

$$mnm^{-1}n^{-1} = e$$

Multiply by n on the right: $mnm^{-1} = n$.

Multiply by m on the right: $mn = nm$.

This shows that any element from M commutes with any element from N .

Quick Tip

The commutator $[m, n] = mn m^{-1} n^{-1}$ is a powerful tool. The fact that $[m, n] = e$ is equivalent to $mn = nm$. If two normal subgroups have a trivial intersection, then their elements commute with each other.

101. If G is abelian and N is normal subgroup of G then G/N is

- (A) non-abelian
- (B) abelian
- (C) cyclic
- (D) non-cyclic

Correct Answer: (B) abelian

Solution:

Let G be an abelian group and N be a normal subgroup of G .

(Note: Every subgroup of an abelian group is normal, so the condition " N is normal" is automatically satisfied.)

The quotient group (or factor group) G/N consists of the cosets of N in G , i.e., $G/N = \{gN \mid g \in G\}$.

The operation in the quotient group is defined as $(aN)(bN) = (ab)N$.

To check if G/N is abelian, we need to verify if $(aN)(bN) = (bN)(aN)$ for all $aN, bN \in G/N$.

Let's evaluate both sides:

Left side: $(aN)(bN) = (ab)N$.

Right side: $(bN)(aN) = (ba)N$.

We need to check if $(ab)N = (ba)N$.

Since G is given to be abelian, we know that $ab = ba$ for all $a, b \in G$.

Therefore, $(ab)N = (ba)N$.

This proves that the quotient group G/N is abelian.

Quick Tip

A key result in group theory is that a quotient group G/N inherits properties from the parent group G . If G is abelian, G/N is abelian. If G is cyclic, G/N is cyclic. However, the reverse is not always true.

102. The order of the symmetric group S_n on n symbols is

- (A) n
- (B) 2^n
- (C) $n!$
- (D) $(n - 1)!$

Correct Answer: (C) $n!$

Solution:

The symmetric group S_n is the group of all possible permutations of a set with n distinct elements.

Let's find the number of such permutations.

Consider a set with n elements $\{1, 2, \dots, n\}$. A permutation is a bijection from this set to itself.

For the first element, there are n choices for where it can be mapped.

For the second element, there are $n - 1$ remaining choices.

For the third element, there are $n - 2$ remaining choices.

...

For the last element, there is only 1 choice left.

By the multiplicative principle, the total number of possible permutations is the product of these choices:

$$n \times (n - 1) \times (n - 2) \times \cdots \times 1.$$

This product is the definition of the factorial of n , denoted as $n!$.

The order of a group is the number of elements in it. Therefore, the order of S_n is $n!$.

Quick Tip

The symmetric group S_n deals with all permutations of n items. The alternating group A_n (the group of even permutations) has order $n!/2$. Don't confuse these two.

103. If N is a normal subgroup of a group G and $a, b \in G$ then

- (A) $NaNb = Nb$
- (B) $aNbN = abN$
- (C) $NaNb = aN$
- (D) $NaNb = Na$

Correct Answer: (B) $aNbN = abN$

Solution:

This question deals with the product of cosets in a group. The set of cosets forms a group (the

quotient group G/N if and only if N is a normal subgroup.

The product of two left cosets aN and bN is defined as the set of all products $x \cdot y$ where $x \in aN$ and $y \in bN$.

Let's show that this set product is equal to the single coset $(ab)N$.

The definition of the product of cosets in the quotient group G/N is $(aN)(bN) = (ab)N$.

The question uses the notation $aNbN$. Let's interpret this as the set product $(aN)(bN)$.
 $(aN)(bN) = a(Nb)N$.

Since N is a normal subgroup, we have $gN = Ng$ for any $g \in G$.

So, $Nb = bN$.

Substituting this into our expression:

$$a(Nb)N = a(bN)N = (ab)(NN).$$

Since N is a subgroup, it is closed under the group operation, so the product of the set N with itself is N (i.e., $NN = N$).

Therefore, $(ab)(NN) = (ab)N$.

So we have shown that $(aN)(bN) = (ab)N$.

The notation in option (B) is unconventional but is intended to represent this standard result.

Quick Tip

The property that N is normal ($gN = Ng$ for all $g \in G$) is the crucial condition that makes the product of cosets well-defined and allows the set of cosets to form a group.

104. A homomorphism $\phi : G \rightarrow \bar{G}$ is an isomorphism if.

- (A) ϕ is one to one
- (B) ϕ is one to one & onto
- (C) ϕ is onto
- (D) ϕ is neither one to one

Correct Answer: (B) ϕ is one to one & onto

Solution:

By definition, a homomorphism is a map between two groups that preserves the group operation.

$$\phi(ab) = \phi(a)\phi(b) \text{ for all } a, b \in G.$$

An isomorphism is a special type of homomorphism that satisfies additional properties, indicating that the two groups have an identical structure.

For a homomorphism $\phi : G \rightarrow \bar{G}$ to be an isomorphism, it must be a bijection.

A map is a bijection if it is both:

1. Injective (one-to-one): Different elements in the domain map to different elements in the codomain. If $\phi(a) = \phi(b)$, then $a = b$.
2. Surjective (onto): Every element in the codomain is the image of at least one element from the domain.

Therefore, an isomorphism is a homomorphism that is both one-to-one and onto.

Quick Tip

Remember the hierarchy of group maps: - Homomorphism: Preserves the operation. - Monomorphism (or embedding): An injective (one-to-one) homomorphism. - Epimorphism: A surjective (onto) homomorphism. - Isomorphism: A bijective (one-to-one and onto) homomorphism.

105. Which one of the following set is not a ring

- (A) set of real numbers
- (B) set of rational numbers
- (C) set of natural numbers
- (D) set of integers

Correct Answer: (C) set of natural numbers

Solution:

A set R with two binary operations, addition (+) and multiplication ($\cdot$), is a ring if:

1. $(R, +)$ is an abelian group.
2. Multiplication is associative.
3. The distributive laws hold: $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$.

Let's check the condition that $(R, +)$ must be an abelian group for each option.

An abelian group must have closure, associativity, an identity element (0), and an inverse for every element.

(A) Set of real numbers $(\mathbb{R}, +)$: Is an abelian group (identity 0, inverse of x is $-x$). So, $\mathbb{R}$ is a ring.

(B) Set of rational numbers $(\mathbb{Q}, +)$: Is an abelian group. So, $\mathbb{Q}$ is a ring.

(D) Set of integers $(\mathbb{Z}, +)$: Is an abelian group. So, $\mathbb{Z}$ is a ring.

(C) Set of natural numbers $(\mathbb{N} = \{1, 2, 3, \dots\} \text{ or } \{0, 1, 2, \dots\})$.

Let's consider $(\mathbb{N}, +)$.

- Identity: The additive identity is 0. If we use the definition $\mathbb{N} = \{1, 2, 3, \dots\}$, it does not have an identity element. If we use $\mathbb{N} = \{0, 1, 2, \dots\}$, it has the identity 0.

- Inverse: For any natural number $n > 0$, its additive inverse is $-n$. However, $-n$ is not a natural number. For example, the inverse of 3 is -3, which is not in $\mathbb{N}$.

Since the natural numbers do not have additive inverses for every element, $(\mathbb{N}, +)$ is not a group.

Because $(\mathbb{N}, +)$ is not an abelian group, the set of natural numbers is not a ring.

Quick Tip

To quickly check if a set is a ring, first verify if it forms an abelian group under addition. The most common reasons for failure are the lack of an additive identity (0) or the lack of additive inverses (negative numbers).

106. If R is a Boolean ring then

(A) $a + a = a, \forall a \in R$

(B) $a + a = 1, \forall a \in R$

(C) $a + a = 0, \forall a \in R$

(D) $a + a = -a, \forall a \in R$

Correct Answer: (C) $a + a = 0, \forall a \in R$

Solution:

A Boolean ring is a ring R with an identity in which every element is idempotent.

The idempotent property means that for every element $a \in R$, we have $a^2 = a \cdot a = a$.

We need to find the property of addition in such a ring.

Consider the element $(a + a)$. Since this is an element of the ring, it must also be idempotent.

So, $(a + a)^2 = a + a$.

Let's expand the left side using the distributive law:

$$\begin{aligned}(a + a)^2 &= (a + a)(a + a) = a(a + a) + a(a + a) \\ &= a^2 + a^2 + a^2 + a^2.\end{aligned}$$

Since every element is idempotent, $a^2 = a$.

So, $a^2 + a^2 + a^2 + a^2 = a + a + a + a$.

Now we equate this with the right side:

$$a + a + a + a = a + a.$$

Let $x = a + a$. The equation is $x + x = x$.

In a ring (which is a group under addition), we can add the inverse of x , which is $-x$, to both sides.

$$(x + x) + (-x) = x + (-x).$$

$$x + (x + (-x)) = 0.$$

$$x + 0 = 0.$$

$$x = 0.$$

Since we defined $x = a + a$, we have shown that $a + a = 0$ for all $a \in R$.

This is also known as having characteristic 2.

Quick Tip

A key property of Boolean rings is that they are always commutative and have characteristic 2 ($a + a = 0$). The idempotency ($a^2 = a$) is the defining feature from which these other properties can be derived.

107. Which one of the following is correct

- (A) every integral domain is a field
- (B) An infinite integral domain is field
- (C) A finite integral domain is a field
- (D) integral domain is a field

Correct Answer: (C) A finite integral domain is a field

Solution:

Let's analyze the relationship between integral domains and fields.

- An integral domain is a commutative ring with identity and no zero divisors.
- A field is a commutative ring with identity in which every non-zero element has a multiplicative inverse.

Every field is an integral domain, but not every integral domain is a field. The key difference is the requirement of multiplicative inverses for all non-zero elements.

Let's evaluate the options:

- (A) False. The set of integers, $\mathbb{Z}$, is an integral domain but not a field (e.g., 2 has no multiplicative inverse in $\mathbb{Z}$).
- (B) False. $\mathbb{Z}$ is also an infinite integral domain that is not a field.
- (C) True. This is a fundamental theorem in ring theory, often called Wedderburn's Little Theorem. It states that every finite integral domain is a field. The proof involves showing that for any non-zero element 'a' in a finite integral domain, the map $x \rightarrow ax$ is a bijection, which implies the existence of an inverse for 'a'.
- (D) This is the same as (A) and is false.

Quick Tip

The key difference between an integral domain and a field is the existence of multiplicative inverses. "Finiteness" is the powerful extra condition that forces a commutative ring with no zero-divisors to have inverses for all its non-zero elements.

108. Number of ideals a field has

- (A) 3
- (B) 1
- (C) 0
- (D) 2

Correct Answer: (D) 2

Solution:

Let F be a field. An ideal I of F is a subring such that for every $r \in F$ and $x \in I$, the product rx is in I .

Let I be an ideal of the field F .

Case 1: The ideal I contains only the additive identity, $I = \{0\}$. This is the trivial ideal and is always present in any ring.

Case 2: The ideal I contains at least one non-zero element.

Let $a \in I$, where $a \neq 0$.

Since F is a field, every non-zero element has a multiplicative inverse. So, a^{-1} exists in F .

By the definition of an ideal, if we take any element from the field (like a^{-1}) and multiply it by an element from the ideal (like a), the result must be in the ideal.

So, $a^{-1} \cdot a$ must be in I .

But $a^{-1} \cdot a = 1$, where 1 is the multiplicative identity. So, $1 \in I$.

Now, if the ideal I contains the identity element 1, then for any element $r \in F$, the product $r \cdot 1 = r$ must also be in I .

This means that every element of F is in I . So, $I = F$.

Thus, the only possible ideals of a field F are the trivial ideal $\{0\}$ and the field F itself. These are two distinct ideals.

Quick Tip

A key theorem states that a commutative ring R with identity is a field if and only if its only ideals are $\{0\}$ and R itself. This property is often used as an alternative definition of a field.

109. If R is a non-zero ring so that $a^2 = a, \forall a \in R$ then the characteristics of R is

- (A) 2
- (B) 3
- (C) 4
- (D) 6

Correct Answer: (A) 2

Solution:

A ring where every element is idempotent ($a^2 = a$) is called a Boolean ring. We need to find its characteristic.

The characteristic of a ring R is the smallest positive integer n such that $na = 0$ for all $a \in R$. If no such integer exists, the characteristic is 0. Note that na means $a + a + \dots + a$ (n times).

Let a be any element in R . Then $a + a$ is also an element in R .

By the idempotent property, we have $(a + a)^2 = (a + a)$.

Using the distributive property to expand the left side:

$$(a + a)^2 = (a + a)(a + a) = a(a + a) + a(a + a) = a^2 + a^2 + a^2 + a^2.$$

Since $a^2 = a$ for all elements, we can substitute this in:

$$a + a + a + a = (a + a).$$

Let $x = a + a$. The equation becomes $x + x = x$.

In the additive group of the ring, we can add the inverse of x , which is $-x$, to both sides:

$$(x + x) + (-x) = x + (-x).$$

$$x + (x + (-x)) = 0.$$

$$x + 0 = 0.$$

$$x = 0.$$

Since $x = a + a$, this means $a + a = 0$ for all $a \in R$.

This is equivalent to $2a = 0$.

The smallest positive integer n for which $na = 0$ for all a is $n = 2$.

Therefore, the characteristic of the ring is 2.

Quick Tip

This is a standard result: every Boolean ring has characteristic 2. The key to the proof is applying the idempotent property ($x^2 = x$) to the element $(a + a)$.

110. If (G, o) is a group of order 24 then G can have a subgroup order

(A) 5

(B) 7

(C) 8

(D) 9

Correct Answer: (C) 8

Solution:

This question applies two fundamental theorems of finite group theory: Lagrange's Theorem and Sylow's Theorems.

1. Lagrange's Theorem: The order of any subgroup of a finite group must divide the order of the group.

The order of the group G is 24.

Let's check which of the given options are divisors of 24.

- Divisors of 24 are 1, 2, 3, 4, 6, 8, 12, 24.

- Option (A) 5: Does not divide 24.

- Option (B) 7: Does not divide 24.

- Option (C) 8: Divides 24 ($24 / 8 = 3$).
- Option (D) 9: Does not divide 24.

Based on Lagrange's Theorem, only a subgroup of order 8 is possible.

2. Sylow's First Theorem: If the order of a group is $|G| = p^k m$ where p is a prime and p does not divide m , then G has a subgroup of order p^k . This is called a Sylow p -subgroup.

The order of G is $24 = 8 \times 3 = 2^3 \times 3^1$.

Here, we can take $p = 2$, $k = 3$, and $m = 3$. Sylow's First Theorem guarantees that G must have a subgroup of order $2^3 = 8$.

So, a group of order 24 must have a subgroup of order 8.

Quick Tip

Lagrange's Theorem gives necessary conditions for subgroup orders (they must be divisors). Sylow's Theorems give sufficient conditions (guaranteeing subgroups of certain prime-power orders).

111. A set A is said to be countable if there exist a function $f : A \rightarrow \mathbb{N}$ such that

- (A) f is bijective
- (B) f is surjective
- (C) f is identity map
- (D) f is additive identity

Correct Answer: (A) f is bijective

Solution:

By definition, a set A is said to be countable if its elements can be put into a one-to-one correspondence with the set of natural numbers $\mathbb{N}$ or a subset of it.

This one-to-one correspondence is mathematically described by a function.

- If the set A is finite, there exists a bijective function from A to a finite subset of $\mathbb{N}$, like $\{1, 2, \dots, n\}$.
- If the set A is countably infinite, there exists a bijective function $f : A \rightarrow \mathbb{N}$.

A bijective function is both injective (one-to-one) and surjective (onto). This ensures that every element in A corresponds to exactly one unique natural number, and every natural number corresponds to exactly one element in A .

A surjective (onto) function alone is not sufficient, as multiple elements of A could map to the same natural number.

Therefore, the existence of a bijective function is the condition that defines a set as being countably infinite, which is a specific case of being countable. In the context of the given

options, this is the best and most precise answer.

Quick Tip

The formal definition of a countable set is one for which there is an injective (one-to-one) function from the set into the natural numbers. However, for countably infinite sets, this is equivalent to the existence of a bijection. Multiple-choice questions often use "bijective" as the key concept for countability.

112. Which of the following are true about Dirichlet function

- (A) Dirichlet function is continuous everywhere on $\mathbb{R}$.
- (B) Dirichlet function is discontinuous everywhere on $\mathbb{R}$.
- (C) Dirichlet function is continuous only at rationals.
- (D) Dirichlet function is discontinuous only at irrationals.

Correct Answer: (B) Dirichlet function is discontinuous everywhere on $\mathbb{R}$.

Solution:

The Dirichlet function, $f(x)$, is defined as:

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

For a function to be continuous at a point c , the limit as x approaches c must exist and be equal to $f(c)$.

Case 1: Let c be any rational number. Then $f(c) = 1$.

However, in any arbitrarily small neighborhood around c , there are infinitely many irrational numbers.

For any sequence of irrational numbers $\{x_n\}$ converging to c , we have $\lim_{n \rightarrow \infty} f(x_n) = \lim_{n \rightarrow \infty} 0 = 0$.

Since this limit (0) is not equal to $f(c)$ (which is 1), the function is discontinuous at every rational number.

Case 2: Let c be any irrational number. Then $f(c) = 0$.

However, in any arbitrarily small neighborhood around c , there are infinitely many rational numbers.

For any sequence of rational numbers $\{x_n\}$ converging to c , we have $\lim_{n \rightarrow \infty} f(x_n) = \lim_{n \rightarrow \infty} 1 = 1$.

Since this limit (1) is not equal to $f(c)$ (which is 0), the function is discontinuous at every irrational number.

Since the function is discontinuous at all rational and all irrational numbers, it is discontinuous everywhere on $\mathbb{R}$.

Quick Tip

The Dirichlet function is the classic example of a function that is nowhere continuous. It relies on the property that both the rational and irrational numbers are dense in the real numbers.

113. Choose the correct option

- (A) Let $a, b \in \mathbb{R}$ and $f: (a, b] \rightarrow \mathbb{R}$ continuous on $(a, b]$ then f is bounded on $(a, b]$.
- (B) Let $a, b \in \mathbb{R}$ and $f: [a, b) \rightarrow \mathbb{R}$ continuous on $[a, b)$ then f is bounded on $[a, b)$.
- (C) Let $a, b \in \mathbb{R}$ and $f: (a, b) \rightarrow \mathbb{R}$ continuous on (a, b) then f is bounded on (a, b) .
- (D) Let $a, b \in \mathbb{R}$ and $f: [a, b] \rightarrow \mathbb{R}$ continuous on $[a, b]$ then f is bounded on $[a, b]$.

Correct Answer: (D) Let $a, b \in \mathbb{R}$ and $f: [a, b] \rightarrow \mathbb{R}$ continuous on $[a, b]$ then f is bounded on $[a, b]$.

Solution:

This question tests the Boundedness Theorem from real analysis.

The Boundedness Theorem states that if a real-valued function f is continuous on a closed and bounded interval $[a, b]$, then f is bounded on that interval.

Let's analyze the options:

- (A) Incorrect. The interval $(a, b]$ is not closed. A counterexample is $f(x) = 1/x$ on $(0, 1]$. This function is continuous on $(0, 1]$ but is not bounded above as $x \rightarrow 0^+$.
- (B) Incorrect. The interval $[a, b)$ is not closed. A counterexample is $f(x) = 1/(1-x)$ on $[0, 1)$. This function is continuous on $[0, 1)$ but is not bounded above as $x \rightarrow 1^-$.
- (C) Incorrect. The interval (a, b) is open. The same counterexamples as above apply. For instance, $f(x) = 1/x$ is continuous but unbounded on $(0, 1)$.
- (D) Correct. This is the precise statement of the Boundedness Theorem. The interval $[a, b]$ is closed and bounded (compact), which guarantees that a continuous function on it will be bounded.

Quick Tip

The conditions for the Boundedness Theorem are very strict. The function must be continuous, and the domain must be a closed and bounded interval (a compact set). If either condition is not met, the function is not guaranteed to be bounded.

114. Let $a, b, c, d \in \mathbb{R}$. Choose the true statement.

- (A) There exists a continuous onto function $f: [a, b] \rightarrow [c, d]$.
- (B) There exists a continuous onto function $f: [a, b] \rightarrow (a, b]$.
- (C) There exists a continuous onto function $f: [a, b] \rightarrow [a, b)$.
- (D) There exists a continuous onto function $f: [a, b] \rightarrow (a, b)$.

Correct Answer: (A) There exists a continuous onto function $f: [a, b] \rightarrow [c, d]$.

Solution:

This question relates to the properties of continuous functions on compact sets.

A fundamental theorem in topology and real analysis states that the continuous image of a compact set is compact.

The interval $[a, b]$ is a compact set in $\mathbb{R}$ (it is closed and bounded).

Therefore, if $f: [a, b] \rightarrow \mathbb{R}$ is a continuous function, its image $f([a, b])$ must also be a compact set.

In $\mathbb{R}$, a compact set is a closed and bounded set.

Let's analyze the images in the options:

(A) $[c, d]$ is a closed and bounded interval, so it is a compact set. It is possible for this to be the image of $[a, b]$. For example, the linear function $f(x) = c + \frac{d-c}{b-a}(x-a)$ maps $[a, b]$ continuously and onto $[c, d]$.

(B) $(a, b]$ is not a closed set, so it is not compact. Therefore, it cannot be the continuous image of a compact set like $[a, b]$.

(C) $[a, b)$ is not a closed set, so it is not compact.

(D) (a, b) is not a closed set, so it is not compact.

Therefore, only option (A) describes a possible outcome.

Quick Tip

Remember the rule: "The continuous image of a compact set is compact." In the context of real numbers, this means that if you have a continuous function defined on a closed interval $[a, b]$, its range must also be a closed interval $[m, M]$.

115. Choose the true statement.

(A) Continuous real valued functions are Lipschitz.

(B) Lipschitz real valued functions are Continuous.

(C) Uniformly continuous real valued functions are Lipschitz.

(D) Continuous real valued functions are uniformly continuous.

Correct Answer: (B) Lipschitz real valued functions are Continuous.

Solution:

Let's analyze the relationships between these types of continuity.

A function f is Lipschitz continuous on a set A if there exists a constant $K > 0$ such that for all $x, y \in A$, $|f(x) - f(y)| \leq K|x - y|$.

(A) False. A continuous function is not necessarily Lipschitz. Counterexample: $f(x) = \sqrt{x}$ on $[0, 1]$ is continuous, but not Lipschitz, as the slope of the secant line near 0 can be arbitrarily

large.

(B) True. If a function is Lipschitz, it is uniformly continuous, and every uniformly continuous function is continuous. To show this directly: For any $\epsilon > 0$, choose $\delta = \epsilon/K$. If $|x - y| < \delta$, then $|f(x) - f(y)| \leq K|x - y| < K\delta = K(\epsilon/K) = \epsilon$. This is the definition of uniform continuity, which implies continuity.

(C) False. A uniformly continuous function is not necessarily Lipschitz. Counterexample: $f(x) = \sqrt{x}$ on $[0, 1]$ is uniformly continuous but not Lipschitz.

(D) False. A continuous function is not necessarily uniformly continuous. This is only guaranteed if the domain is compact (e.g., a closed interval). Counterexample: $f(x) = x^2$ is continuous on $\mathbb{R}$ but not uniformly continuous.

Quick Tip

The hierarchy of function continuity is: Differentiable with bounded derivative $\implies$ Lipschitz continuous $\implies$ Uniformly continuous $\implies$ Continuous. The implications only go in this one direction in general.

116. The value of $\lim_{x \rightarrow \infty} (1 + \frac{1}{x})^x$

(A) $\ln 2$

(B) 1

(C) e

(D) ∞

Correct Answer: (C) e

Solution:

This is one of the standard definitions of the mathematical constant e .

The limit is of the indeterminate form 1^∞ .

To evaluate it, let $y = (1 + \frac{1}{x})^x$.

Take the natural logarithm of both sides:

$$\ln y = \ln \left((1 + \frac{1}{x})^x \right) = x \ln(1 + \frac{1}{x}).$$

Now, find the limit of $\ln y$ as $x \rightarrow \infty$.

$$\lim_{x \rightarrow \infty} \ln y = \lim_{x \rightarrow \infty} x \ln(1 + \frac{1}{x}).$$

This is of the form $\infty \cdot 0$. We can rewrite it as a fraction to use L'Hopital's Rule. Let $t = 1/x$.

As $x \rightarrow \infty$, $t \rightarrow 0^+$.

$$\lim_{t \rightarrow 0^+} \frac{\ln(1+t)}{t}.$$

This is of the form $0/0$. Applying L'Hopital's Rule:

$$\lim_{t \rightarrow 0^+} \frac{\frac{d}{dt}(\ln(1+t))}{\frac{d}{dt}(t)} = \lim_{t \rightarrow 0^+} \frac{\frac{1}{1+t}}{1} = \frac{1}{1+0} = 1.$$

So, we have found that $\lim_{x \rightarrow \infty} \ln y = 1$.

Since $\ln(\lim_{x \rightarrow \infty} y) = 1$, we can find the limit of y by exponentiating:

$$\lim_{x \rightarrow \infty} y = e^1 = e.$$

Quick Tip

Memorize the two fundamental definitions of e : 1. $e = \lim_{n \rightarrow \infty} (1 + \frac{1}{n})^n$ 2. $e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots$. These are frequently tested.

117. The value of $\lim_{x \rightarrow 0} (\tan x \log x)$

- (A) 2
- (B) 1
- (C) 3
- (D) 0

Correct Answer: (D) 0

Solution:

We are evaluating the limit as x approaches 0 from the right side, since $\log x$ is only defined for $x > 0$.

$$\lim_{x \rightarrow 0^+} (\tan x \log x).$$

As $x \rightarrow 0^+$, we have $\tan x \rightarrow 0$ and $\log x \rightarrow -\infty$.

This is an indeterminate form of the type $0 \cdot (-\infty)$.

To apply L'Hopital's Rule, we must rewrite this as a fraction of the form $0/0$ or ∞/∞ .

$$\lim_{x \rightarrow 0^+} (\tan x \log x) = \lim_{x \rightarrow 0^+} \frac{\log x}{\cot x}.$$

Now, as $x \rightarrow 0^+$, $\log x \rightarrow -\infty$ and $\cot x \rightarrow \infty$. This is of the form $-\infty/\infty$.

We can now apply L'Hopital's Rule by differentiating the numerator and the denominator:

$$\lim_{x \rightarrow 0^+} \frac{\frac{d}{dx}(\log x)}{\frac{d}{dx}(\cot x)} = \lim_{x \rightarrow 0^+} \frac{1/x}{-\csc^2 x}.$$

Let's simplify this expression:

$$= \lim_{x \rightarrow 0^+} \frac{-\sin^2 x}{x}.$$

This is still an indeterminate form $0/0$. We can rewrite it as:

$$= \lim_{x \rightarrow 0^+} (-\sin x) \cdot \left(\frac{\sin x}{x} \right).$$

Using the property that the limit of a product is the product of the limits:

$$= (\lim_{x \rightarrow 0^+} -\sin x) \cdot \left(\lim_{x \rightarrow 0^+} \frac{\sin x}{x} \right).$$

The first limit is $-\sin(0) = 0$.

The second limit is a standard result, $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$.

Therefore, the final result is $0 \cdot 1 = 0$.

Quick Tip

When faced with an indeterminate product like $0 \cdot \infty$, rewrite it as a fraction by taking the reciprocal of one of the terms, e.g., $f(x)g(x) = \frac{f(x)}{1/g(x)}$. This sets it up for L'Hopital's Rule.

118. If $f'(x) = \frac{1}{2-x^2}$ Then what would be the sum of the lower and upper bound of $f(1)$, if $f(0) = 1$ and $f(x)$ is defined for $[0, 1]$

(A) 0.5

(B) 1.5

(C) 2.5

(D) 3.5

Correct Answer: (D) 3.5

Solution:

Step 1: Relate $f(1)$ and $f(0)$ using the Fundamental Theorem of Calculus.

$$f(1) - f(0) = \int_0^1 f'(x) dx.$$

$$f(1) = f(0) + \int_0^1 \frac{1}{2-x^2} dx.$$

$$\text{Given } f(0) = 1, \text{ we have } f(1) = 1 + \int_0^1 \frac{1}{2-x^2} dx.$$

Step 2: Find the minimum and maximum values of the integrand $f'(x) = \frac{1}{2-x^2}$ on the interval $[0, 1]$.

The denominator $g(x) = 2 - x^2$ is a downward-opening parabola. On $[0, 1]$, it is a decreasing function.

Its maximum value is at $x = 0$, $g(0) = 2$.

Its minimum value is at $x = 1$, $g(1) = 2 - 1 = 1$.

Since the integrand $f'(x)$ is the reciprocal of $g(x)$, its minimum and maximum values will be inverted.

Minimum of $f'(x)$ on $[0, 1]$ is $\frac{1}{\max(g(x))} = \frac{1}{2}$.

Maximum of $f'(x)$ on $[0, 1]$ is $\frac{1}{\min(g(x))} = \frac{1}{1} = 1$.

Step 3: Use the bounding property of integrals.

$$\min(f') \cdot (b - a) \leq \int_a^b f'(x) dx \leq \max(f') \cdot (b - a).$$

$$\frac{1}{2} \cdot (1 - 0) \leq \int_0^1 \frac{1}{2-x^2} dx \leq 1 \cdot (1 - 0).$$

$$0.5 \leq \int_0^1 \frac{1}{2-x^2} dx \leq 1.$$

Step 4: Find the bounds for $f(1)$.

Add 1 to all parts of the inequality:

$$1 + 0.5 \leq 1 + \int_0^1 \frac{1}{2-x^2} dx \leq 1 + 1.$$

$$1.5 \leq f(1) \leq 2.$$

Step 5: Find the sum of the lower and upper bounds.

Lower bound = 1.5.

Upper bound = 2.
Sum = $1.5 + 2 = 3.5$.

Quick Tip

For an integral $\int_a^b h(x)dx$, if you can find the minimum (m) and maximum (M) values of $h(x)$ on $[a, b]$, you can bound the integral: $m(b - a) \leq \int_a^b h(x)dx \leq M(b - a)$.

119. If f and g belong to $R[a, b]$, then choose the false statement.

- (A) the sum $f + g$ belongs to $R[a, b]$.
- (B) the difference $f - g$ belongs to $R[a, b]$.
- (C) the product $f \cdot g$ belongs to $R[a, b]$.
- (D) the quotient f/g belongs to $R[a, b]$.

Correct Answer: (D) the quotient f/g belongs to $R[a, b]$.

Solution:

The set of Riemann-integrable functions on an interval $[a, b]$ is denoted by $R[a, b]$. We need to check the closure properties of this set under various arithmetic operations.

(A) True. The sum of two Riemann-integrable functions is also Riemann-integrable. $\int(f + g) = \int f + \int g$.

(B) True. The difference of two Riemann-integrable functions is also Riemann-integrable. $\int(f - g) = \int f - \int g$.

(C) True. The product of two Riemann-integrable functions is also Riemann-integrable. (The integral of the product is not the product of the integrals, though).

(D) False. The quotient f/g is not necessarily Riemann-integrable. There are two main problems:

1. If $g(x) = 0$ for some $x \in [a, b]$, the quotient $f(x)/g(x)$ is not even defined at that point.
2. Even if $g(x) \neq 0$, the quotient function f/g might be unbounded, and an unbounded function on a finite interval cannot be Riemann-integrable. For example, let $f(x) = 1$ and $g(x) = x$ on $[0, 1]$. Both are integrable, but $f/g = 1/x$ is unbounded near $x = 0$ and is not Riemann-integrable on $[0, 1]$.

The statement for the quotient is only true if there is an additional condition, such as $|g(x)| \geq c > 0$ for some constant c and all $x \in [a, b]$. Since this condition is not given, the statement is false in general.

Quick Tip

The set of Riemann-integrable functions on an interval forms a vector space and an algebra, meaning it's closed under addition, scalar multiplication, and function multiplication. However, it is not closed under division.

120. For $x \in \mathbb{R}$, $\lim_{n \rightarrow \infty} \left(\frac{1}{n} \sin(nx + n) \right) =$

(A) -1

(B) 0

(C) 1

(D) 2

Correct Answer: (B) 0

Solution:

This limit can be evaluated using the Squeeze Theorem (also known as the Sandwich Theorem).

Step 1: Bound the sine function.

We know that for any real number θ , the sine function is bounded between -1 and 1.

$$-1 \leq \sin(\theta) \leq 1.$$

Applying this to our specific case, for any n and x :

$$-1 \leq \sin(nx + n) \leq 1.$$

Step 2: Multiply the inequality by the term outside the sine function.

The term is $\frac{1}{n}$. Since we are taking the limit as $n \rightarrow \infty$, we can assume n is positive, so the inequality signs do not change.

$$-\frac{1}{n} \leq \frac{1}{n} \sin(nx + n) \leq \frac{1}{n}.$$

Step 3: Apply the limit to all parts of the inequality.

$$\lim_{n \rightarrow \infty} \left(-\frac{1}{n} \right) \leq \lim_{n \rightarrow \infty} \left(\frac{1}{n} \sin(nx + n) \right) \leq \lim_{n \rightarrow \infty} \left(\frac{1}{n} \right).$$

Step 4: Evaluate the limits of the bounding functions.

$$\lim_{n \rightarrow \infty} \left(-\frac{1}{n} \right) = 0.$$

$$\lim_{n \rightarrow \infty} \left(\frac{1}{n} \right) = 0.$$

Step 5: Apply the Squeeze Theorem.

Since the function $\frac{1}{n} \sin(nx + n)$ is "squeezed" between two functions that both approach 0, its limit must also be 0.

$$0 \leq \lim_{n \rightarrow \infty} \left(\frac{1}{n} \sin(nx + n) \right) \leq 0.$$

$$\text{Therefore, } \lim_{n \rightarrow \infty} \left(\frac{1}{n} \sin(nx + n) \right) = 0.$$

Quick Tip

The Squeeze Theorem is extremely useful for limits involving products of a function that goes to zero and a bounded function (like sine or cosine). The result is always zero.

121. Find $\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} =$

(A) $-\frac{1}{2}$

(B) 0

(C) $\frac{1}{2}$

(D) $\frac{1}{4}$

Correct Answer: (D) $\frac{1}{4}$

Solution:

This is an infinite series that can be solved using the method of partial fractions.

Step 1: Decompose the general term into partial fractions.

Let $\frac{1}{n(n+1)(n+2)} = \frac{A}{n} + \frac{B}{n+1} + \frac{C}{n+2}$.

$1 = A(n+1)(n+2) + Bn(n+2) + Cn(n+1)$.

For $n = 0$: $1 = A(1)(2) \implies A = 1/2$.

For $n = -1$: $1 = B(-1)(1) \implies B = -1$.

For $n = -2$: $1 = C(-2)(-1) \implies C = 1/2$.

So, the term is $\frac{1}{2n} - \frac{1}{n+1} + \frac{1}{2(n+2)}$.

Step 2: Rewrite the term to reveal the telescoping nature.

$\frac{1}{2} \left(\frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2} \right) = \frac{1}{2} \left[\left(\frac{1}{n} - \frac{1}{n+1} \right) - \left(\frac{1}{n+1} - \frac{1}{n+2} \right) \right]$.

Step 3: Write out the partial sum S_N .

Let $a_n = \frac{1}{n} - \frac{1}{n+1}$. The series is $\frac{1}{2} \sum_{n=1}^N (a_n - a_{n+1})$.

$S_N = \frac{1}{2} [(a_1 - a_2) + (a_2 - a_3) + \cdots + (a_N - a_{N+1})]$.

This is a telescoping sum which simplifies to $S_N = \frac{1}{2}(a_1 - a_{N+1})$.

Step 4: Calculate a_1 and the limit of a_{N+1} .

$a_1 = \frac{1}{1} - \frac{1}{1+1} = 1 - \frac{1}{2} = \frac{1}{2}$.

$\lim_{N \rightarrow \infty} a_{N+1} = \lim_{N \rightarrow \infty} \left(\frac{1}{N+1} - \frac{1}{N+2} \right) = 0 - 0 = 0$.

Step 5: Find the sum of the infinite series.

Sum $= \lim_{N \rightarrow \infty} S_N = \frac{1}{2}(a_1 - 0) = \frac{1}{2} \left(\frac{1}{2} \right) = \frac{1}{4}$.

Quick Tip

For sums of rational functions of n , always try partial fraction decomposition first. Often, this will reveal a telescoping series where most intermediate terms cancel out.

122. Function f should be _____ on $[a, b]$ according to Rolle's theorem.

- (A) continuous
- (B) non-continuous
- (C) integral
- (D) non-existent

Correct Answer: (A) continuous

Solution:

Rolle's Theorem specifies three conditions that a function f must satisfy to guarantee the existence of a point $c \in (a, b)$ where $f'(c) = 0$.

The conditions are:

1. f must be continuous on the closed interval $[a, b]$.
2. f must be differentiable on the open interval (a, b) .
3. $f(a) = f(b)$.

The question asks for the property the function should have on the closed interval $[a, b]$. According to the first condition of the theorem, the function must be continuous on $[a, b]$.

Therefore, option (A) is the correct answer.

Quick Tip

Remember the three 'C's for Rolle's Theorem: Continuous on the closed interval, Calculable (differentiable) on the open interval, and Corner-points equal ($f(a) = f(b)$).

123. What are/is the conditions to satisfy Lagrange's mean value theorem?

- (A) f is continuous on (a, b)
- (B) f is differentiable on (a, b)
- (C) f is differentiable and continuous on (a, b)
- (D) f is differentiable and non-continuous on (a, b)

Correct Answer: (C) f is differentiable and continuous on (a, b)

Solution:

Lagrange's Mean Value Theorem (MVT) is a generalization of Rolle's Theorem.

It states that if a function f satisfies two conditions, there exists a point $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

The two conditions required for the theorem to hold are:

1. f must be continuous on the closed interval $[a, b]$.
2. f must be differentiable on the open interval (a, b) .

Let's evaluate the given options. There appears to be a typo in the English translation of the options in the source document. The correct combination of conditions is continuity on the closed interval and differentiability on the open interval.

- Option (A) is incorrect as continuity is required on the closed interval $[a, b]$.
- Option (B) is a correct condition, but it is incomplete.
- Option (C) states "differentiable and continuous on (a, b) ". While this is not the precise statement of the theorem (continuity is on $[a, b]$), it is the only option that includes both necessary concepts of differentiability and continuity. Given the key, this is the intended answer, assuming the typo meant "differentiable on (a, b) and continuous on $[a, b]$ ".
- Option (D) is incorrect as continuity is required.

Therefore, based on the standard theorem, the combination of continuity on $[a, b]$ and differentiability on (a, b) is required. Option (C) is the closest, albeit imperfect, choice.

Quick Tip

Lagrange's MVT has two conditions: continuity on the closed interval ' $[]$ ' and differentiability on the open interval ' $()$ '. This distinction is crucial and frequently tested.

124. Check about the continuity of function given by $f(x) = \sqrt{x}$ in interval $[0, 2]$

- (A) Discontinuity everywhere
- (B) can't say about continuity of f
- (C) it is not uniformly continuous
- (D) it is uniformly continuous

Correct Answer: (D) it is uniformly continuous

Solution:

Step 1: Check for standard continuity.

The function $f(x) = \sqrt{x}$ is a standard elementary function. It is known to be continuous on its domain, which is $[0, \infty)$.

Therefore, $f(x) = \sqrt{x}$ is continuous on the interval $[0, 2]$.

Step 2: Check for uniform continuity.

We use the Heine-Cantor theorem, which states that if a function is continuous on a closed and bounded interval (a compact set), then it is also uniformly continuous on that interval.

The interval given is $[0, 2]$. This interval is both closed (it includes its endpoints) and bounded. Since we established in Step 1 that $f(x) = \sqrt{x}$ is continuous on $[0, 2]$, the Heine-Cantor theorem applies.

Therefore, the function $f(x) = \sqrt{x}$ is uniformly continuous on the interval $[0, 2]$.

Quick Tip

A key theorem to remember is: Continuity on a compact set (like a closed and bounded interval $[a,b]$) implies uniform continuity on that set. This is a powerful tool for quickly determining uniform continuity.

125. The value of 'x' for which the function $f(x) = \frac{x^2-3x-4}{x^2+3x-4}$ is not continuous are

- (A) 4 and -1
- (B) 4 and 1
- (C) -4 and 1
- (D) -4 and -1

Correct Answer: (C) -4 and 1

Solution:

The function $f(x)$ is a rational function, which is a ratio of two polynomials.

A rational function is continuous everywhere except at the points where its denominator is equal to zero.

Step 1: Set the denominator equal to zero.

$$x^2 + 3x - 4 = 0.$$

Step 2: Solve the quadratic equation for x .

We can solve this by factoring. We need two numbers that multiply to -4 and add to +3. These numbers are +4 and -1.

$$(x + 4)(x - 1) = 0.$$

Step 3: Find the values of x that make the denominator zero.

The solutions are $x + 4 = 0 \implies x = -4$.

And $x - 1 = 0 \implies x = 1$.

The function is not continuous at $x = -4$ and $x = 1$.

Quick Tip

To find the points of discontinuity for a rational function, always focus on the denominator. The function will be discontinuous at the roots of the denominator polynomial.

126. Find $\|P\|$ for the partition $P = \{1, 4, 5, 8, 13\}$

- (A) 1
- (B) 3
- (C) 5
- (D) 7

Correct Answer: (C) 5

Solution:

The norm (or mesh) of a partition, denoted by $\|P\|$, is defined as the length of the longest subinterval in the partition.

The given partition is $P = \{x_0, x_1, x_2, x_3, x_4\} = \{1, 4, 5, 8, 13\}$.

This partition creates the following subintervals:

$$[x_0, x_1] = [1, 4]$$

$$[x_1, x_2] = [4, 5]$$

$$[x_2, x_3] = [5, 8]$$

$$[x_3, x_4] = [8, 13]$$

Now, we calculate the length of each subinterval, $\Delta x_i = x_i - x_{i-1}$:

$$\Delta x_1 = 4 - 1 = 3.$$

$$\Delta x_2 = 5 - 4 = 1.$$

$$\Delta x_3 = 8 - 5 = 3.$$

$$\Delta x_4 = 13 - 8 = 5.$$

The lengths of the subintervals are $\{3, 1, 3, 5\}$.

The norm $\|P\|$ is the maximum of these lengths.

$$\|P\| = \max\{3, 1, 3, 5\} = 5.$$

Quick Tip

To find the norm of a partition, simply calculate the difference between each adjacent pair of points in the set and then identify the largest of these differences.

127. If $f : [a, b] \rightarrow R$ is integrable on $[a, b]$. then— where M and m are supremum and infimum of f on $[a, b]$.

(A) $m(b - a) > \int_a^b f(x)dx > M(b - a)$

(B) $m(b - a) \geq \int_a^b f(x)dx \geq M(b - a)$

(C) $m(b - a) \leq \int_a^b f(x)dx \leq M(b - a)$

(D) $m(b - a) < \int_a^b f(x)dx < M(b - a)$

Correct Answer: (C) $m(b - a) \leq \int_a^b f(x)dx \leq M(b - a)$

Solution:

This question refers to a fundamental property of the definite integral.

Let f be a Riemann-integrable function on the closed interval $[a, b]$.

Let $m = \inf\{f(x) \mid x \in [a, b]\}$ be the infimum (greatest lower bound) of the function on the interval.

Let $M = \sup\{f(x) \mid x \in [a, b]\}$ be the supremum (least upper bound) of the function on the interval.

For every $x \in [a, b]$, by definition, we have:
 $m \leq f(x) \leq M$.

By the monotonicity property of integrals, if $f(x) \leq g(x)$ on $[a, b]$, then $\int_a^b f(x)dx \leq \int_a^b g(x)dx$. We can integrate the above inequality over the interval $[a, b]$:
 $\int_a^b m dx \leq \int_a^b f(x) dx \leq \int_a^b M dx$.

Since m and M are constants, their integrals are:

$$m \int_a^b 1 dx \leq \int_a^b f(x) dx \leq M \int_a^b 1 dx.$$

$$m[x]_a^b \leq \int_a^b f(x) dx \leq M[x]_a^b.$$

$$m(b-a) \leq \int_a^b f(x) dx \leq M(b-a).$$

This matches option (C).

Quick Tip

This property can be visualized as the area under the curve $y = f(x)$ being bounded by the area of two rectangles: one with height equal to the minimum value of the function and one with height equal to the maximum value.

128. What is the value of the given, $\lim_{x \rightarrow 0}(\frac{2}{x})$

(A) ∞

(B) 0

(C) $1/2$

(D) $3/2$

Correct Answer: (A) ∞

Solution:

We are asked to find the limit of the function $f(x) = \frac{2}{x}$ as x approaches 0.

This is a two-sided limit, so we should consider the limits from the left and the right.

1. Right-hand limit: As x approaches 0 from the positive side ($x \rightarrow 0^+$), x is a small positive number. The fraction $\frac{2}{x}$ becomes a large positive number.

$$\lim_{x \rightarrow 0^+} \frac{2}{x} = +\infty.$$

2. Left-hand limit: As x approaches 0 from the negative side ($x \rightarrow 0^-$), x is a small negative number. The fraction $\frac{2}{x}$ becomes a large negative number.

$$\lim_{x \rightarrow 0^-} \frac{2}{x} = -\infty.$$

Since the right-hand limit $(+\infty)$ is not equal to the left-hand limit $(-\infty)$, the two-sided limit technically does not exist.

However, in the context of multiple-choice questions where ∞ is an option, it often refers to the behavior of the function becoming unboundedly large in magnitude. Sometimes it specifically refers to the right-hand limit by convention. Given that ∞ is the keyed answer, it is interpreted in this context.

Quick Tip

When evaluating limits of the form c/x as $x \rightarrow 0$ (where $c \neq 0$), always consider the sign of x . The limit will be $+\infty$ or $-\infty$ depending on the signs of c and the direction of approach. If the two-sided limit is asked, it does not exist unless the options allow for an unsigned infinity.

129. Find $\int_0^8 x \, dx =$

- (A) 34
- (B) 32
- (C) 21
- (D) 24

Correct Answer: (B) 32

Solution:

We need to evaluate the definite integral of the function $f(x) = x$ from $x = 0$ to $x = 8$.

Step 1: Find the antiderivative (indefinite integral) of the function.

Using the power rule for integration, $\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$.

$$\int x^1 \, dx = \frac{x^{1+1}}{1+1} = \frac{x^2}{2}.$$

Step 2: Apply the Fundamental Theorem of Calculus.

$\int_a^b f(x) \, dx = [F(x)]_a^b = F(b) - F(a)$, where $F(x)$ is the antiderivative of $f(x)$.

$$\int_0^8 x \, dx = \left[\frac{x^2}{2} \right]_0^8.$$

Step 3: Evaluate the antiderivative at the upper and lower limits.

$$= \frac{8^2}{2} - \frac{0^2}{2}.$$

$$= \frac{64}{2} - \frac{0}{2}.$$

$$= 32 - 0 = 32.$$

The value of the definite integral is 32.

Quick Tip

Geometrically, the integral $\int_0^8 x \, dx$ represents the area of a right-angled triangle with base 8 and height 8. The area is $\frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2} \times 8 \times 8 = 32$. This is a quick way to verify the result for simple linear functions.

130. What will be nature of the $f(x) = 10 - 9x + 6x^2 - x^3$ for $x < 1$

- (A) Decreases
- (B) Increases
- (C) Cannot be determined for $x < 1$
- (D) A constant function

Correct Answer: (A) Decreases

Solution:

The nature of a function (whether it is increasing or decreasing) is determined by the sign of its first derivative, $f'(x)$.

- If $f'(x) > 0$, the function is increasing.
- If $f'(x) < 0$, the function is decreasing.

Step 1: Find the first derivative of the function $f(x)$.

$$f(x) = 10 - 9x + 6x^2 - x^3.$$

$$f'(x) = \frac{d}{dx}(10 - 9x + 6x^2 - x^3).$$

$$f'(x) = 0 - 9 + 12x - 3x^2 = -3x^2 + 12x - 9.$$

Step 2: Determine the sign of $f'(x)$ for the given interval $x < 1$.

We can analyze the quadratic $f'(x) = -3x^2 + 12x - 9$.

Let's factor it:

$$f'(x) = -3(x^2 - 4x + 3).$$

$$f'(x) = -3(x - 1)(x - 3).$$

Step 3: Analyze the sign of the factors for $x < 1$.

If $x < 1$:

- The factor $(x - 1)$ will be negative.
- The factor $(x - 3)$ will also be negative.

The product of two negative factors, $(x - 1)(x - 3)$, will be positive.

So, for $x < 1$, the derivative is:

$$f'(x) = -3 \times (\text{a positive number}).$$

Therefore, $f'(x)$ is negative.

Since $f'(x) < 0$ for $x < 1$, the function $f(x)$ is decreasing on this interval.

Quick Tip

To determine if a function is increasing or decreasing on an interval, find the derivative and test its sign. Factoring the derivative helps to easily identify the intervals where it is positive or negative based on its roots.

131. Let V be a vector space over field F then which are of the following is incorrect

- (A) V is an abelian group under addition
- (B) There must be two binary operations in V
- (C) V is closed w.r.t scalar multiplication
- (D) V is an abelian group under multiplication

Correct Answer: (D) V is an abelian group under multiplication

Solution:

Let's review the axioms of a vector space V over a field F .

1. $(V, +)$ is an abelian group. This means V is closed under vector addition, addition is associative and commutative, there's an additive identity (the zero vector), and every vector has an additive inverse. This makes option (A) a correct statement.
2. V is closed under scalar multiplication. For any scalar $\alpha \in F$ and vector $v \in V$, the product αv is in V . This makes option (C) a correct statement.
3. Scalar multiplication distributes over vector addition: $\alpha(u + v) = \alpha u + \alpha v$.
4. Scalar multiplication distributes over scalar addition: $(\alpha + \beta)v = \alpha v + \beta v$.
5. Scalar multiplication is compatible with field multiplication: $\alpha(\beta v) = (\alpha\beta)v$.
6. There is a multiplicative identity for scalar multiplication: $1v = v$, where 1 is the multiplicative identity in F .

The definition involves two operations: vector addition ($+ : V \times V \rightarrow V$) and scalar multiplication ($\cdot : F \times V \rightarrow V$). However, these are not two binary operations *in V *. Vector addition is a binary operation on V , but scalar multiplication is an external operation. The question in option (B) is ambiguously phrased, but the structure does require these two operations.

Let's look at option (D). The axioms of a vector space say nothing about a multiplicative operation between two vectors. In general, vector multiplication is not even defined, let alone required to form an abelian group.

Comparing the options, (A) and (C) are definitively true axioms. (D) is definitively false; a vector space is not defined as a group under multiplication. Therefore, (D) is the incorrect statement.

Quick Tip

A vector space has one internal operation (vector addition, which forms an abelian group) and one external operation (scalar multiplication). It is not a ring or a field, so it doesn't need to be a group under multiplication.

132. If a vector $\mathbf{u} = (1, -2, k)$ in R^3 be a linear combination of vector $\mathbf{v} = (3, 0, -2)$ and $\mathbf{w} = (2, -1, -5)$ then

- (A) $k = -8$
- (B) k can be any real number
- (C) $k = 8$
- (D) $k = -3$

Correct Answer: (A) $k = -8$

Solution:

If vector u is a linear combination of vectors v and w , then there exist scalars a and b such that $u = av + bw$.

Substituting the given vectors:

$$\begin{aligned}(1, -2, k) &= a(3, 0, -2) + b(2, -1, -5). \\(1, -2, k) &= (3a, 0, -2a) + (2b, -b, -5b). \\(1, -2, k) &= (3a + 2b, -b, -2a - 5b).\end{aligned}$$

This gives us a system of three linear equations by equating the components:

- 1) $3a + 2b = 1$
- 2) $-b = -2$
- 3) $-2a - 5b = k$

From equation (2), we can directly solve for b :
 $b = 2$.

Substitute $b = 2$ into equation (1) to solve for a :

$$\begin{aligned}3a + 2(2) &= 1. \\3a + 4 &= 1. \\3a &= -3. \\a &= -1.\end{aligned}$$

Now, substitute the values of $a = -1$ and $b = 2$ into equation (3) to find k :

$$\begin{aligned}k &= -2a - 5b. \\k &= -2(-1) - 5(2). \\k &= 2 - 10. \\k &= -8.\end{aligned}$$

Quick Tip

To check if a vector is a linear combination of others, set up the vector equation $u = c_1v_1 + c_2v_2 + \dots$. This will always lead to a system of linear equations for the unknown scalars c_i .

133. If vector $(x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4), (z_1, z_2, z_3, z_4), (w_1, w_2, w_3, w_4)$ are linearly independent in R^4 if $\alpha, \beta, \gamma, \delta$ are scalars then $\alpha(x_1, x_2, x_3, x_4) + \beta(y_1, y_2, y_3, y_4) + \gamma(z_1, z_2, z_3, z_4) + \delta(w_1, w_2, w_3, w_4) = 0$ then the value of $\alpha + \beta + \gamma + \delta$ is

(A) 4

(B) 0

(C) -4

(D) 2

Correct Answer: (B) 0

Solution:

This question tests the definition of linear independence.

Let the four vectors be $\vec{x}, \vec{y}, \vec{z}, \vec{w}$.

A set of vectors is defined as linearly independent if the only way their linear combination can equal the zero vector is when all the scalar coefficients are zero.

The problem states that the four vectors are linearly independent.

It also gives the equation for a linear combination of these vectors equaling the zero vector:

$$\alpha\vec{x} + \beta\vec{y} + \gamma\vec{z} + \delta\vec{w} = \vec{0}.$$

By the definition of linear independence, this equation can only be true if all the scalars are zero.

Therefore, we must have:

$$\alpha = 0$$

$$\beta = 0$$

$$\gamma = 0$$

$$\delta = 0$$

The question asks for the value of the sum of these scalars:

$$\alpha + \beta + \gamma + \delta = 0 + 0 + 0 + 0 = 0.$$

Quick Tip

The definition is the key: A set of vectors $\{v_1, \dots, v_k\}$ is linearly independent if the equation $c_1v_1 + \dots + c_kv_k = 0$ implies that $c_1 = c_2 = \dots = c_k = 0$.

134. Let $S = \{(-3, 5, 2), (0, 2, -2), (8, -1, 1/2), (-4, 1, 0)\}$ be a subset of a vector space $\mathbb{R}^3$ then

- (A) S is linearly independent since it has finite vectors
- (B) S is linearly independent since it has more than three vectors
- (C) S is linearly dependent since it has finite vectors
- (D) S is linearly dependent since it has more than three vectors

Correct Answer: (D) S is linearly dependent since it has more than three vectors

Solution:

This question deals with the concept of linear dependence and the dimension of a vector space.

The vector space is $\mathbb{R}^3$. The dimension of $\mathbb{R}^3$ is 3.

A fundamental theorem in linear algebra states that any set of vectors in an n-dimensional vector space containing more than n vectors must be linearly dependent.

Here, our vector space is $\mathbb{R}^3$, so $n = 3$.

The set S contains 4 vectors: $(-3, 5, 2), (0, 2, -2), (8, -1, 1/2), (-4, 1, 0)$.

The number of vectors in the set S is 4.

Since the number of vectors (4) is greater than the dimension of the vector space (3), the set S must be linearly dependent.

Let's evaluate the options:

(A) and (B) are incorrect because the set is linearly dependent.

(C) The reason "since it has finite vectors" is incorrect. A set with fewer than 3 vectors could be independent.

(D) The reason "since it has more than three vectors" is the correct justification based on the theorem. A set of 4 vectors in $\mathbb{R}^3$ is always linearly dependent.

Quick Tip

In an n-dimensional vector space, remember these two rules: 1. Any set with more than n vectors is linearly dependent. 2. Any set with fewer than n vectors cannot span the space.

135. Which one of the following sets of vectors in $\mathbb{R}^3$ are linearly independent

- (A) $\{(0, 0, 1), (0, 1, 0), (0, 1, 1)\}$
- (B) $\{(0, 1, 0), (1, 0, 1), (1, 1, 0)\}$
- (C) $\{(1, 0, 0), (0, 1, 0), (0, 0, 0)\}$
- (D) $\{(1, 0, 0), (0, 1, 0), (1, 1, 0)\}$

Correct Answer: (B) $\{(0, 1, 0), (1, 0, 1), (1, 1, 0)\}$

Solution:

A set of vectors is linearly independent if no vector in the set can be written as a linear combination of the others. For a set of three vectors in $\mathbb{R}^3$, this is equivalent to the determinant of the matrix formed by these vectors being non-zero.

(A) $\{(0, 0, 1), (0, 1, 0), (0, 1, 1)\}$. Notice that $(0, 1, 1) = (0, 1, 0) + (0, 0, 1)$. The third vector is a sum of the first two, so the set is linearly dependent. Determinant = $\begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 0$.

(C) $\{(1, 0, 0), (0, 1, 0), (0, 0, 0)\}$. Any set containing the zero vector is always linearly dependent, because $c_1(1, 0, 0) + c_2(0, 1, 0) + c_3(0, 0, 0) = (0, 0, 0)$ can be satisfied with a non-zero c_3 (e.g., $c_3 = 1$) and $c_1 = c_2 = 0$.

(D) $\{(1, 0, 0), (0, 1, 0), (1, 1, 0)\}$. Notice that $(1, 1, 0) = (1, 0, 0) + (0, 1, 0)$. The third vector is a sum of the first two, so the set is linearly dependent. Determinant = $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{vmatrix} = 0$.

(B) $\{(0, 1, 0), (1, 0, 1), (1, 1, 0)\}$. Let's check the determinant of the matrix formed by these vectors.

$$D = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}.$$

Expanding along the first row:

$$D = 0 \cdots - 1 \cdot \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} + 0 \cdots = -1((1)(0) - (1)(1)) = -1(-1) = 1.$$

Since the determinant is $1 \neq 0$, the vectors are linearly independent.

Quick Tip

For a set of n vectors in $\mathbb{R}^n$, the quickest way to check for linear independence is to form a matrix with the vectors as rows (or columns) and calculate its determinant. If the determinant is non-zero, the vectors are linearly independent.

136. Which of the following is not a linear transformation

(A) $T : V_3(R) \rightarrow V_3(R)$ is defined by $T(x, y, z) = (2x - 3y, 7y + 2z)$

(B) $T : V_3(R) \rightarrow V_2(R)$ is defined by $T(x, y, z) = (z, 2x + y)$

(C) $T : V_3(R) \rightarrow V_2(R)$ is defined by $T(x, y, z) = (x^2, y)$

(D) $T : V_3(R) \rightarrow V_3(R)$ is defined by $T(x, y, z) = (y, -z, -x)$

Correct Answer: (C) $T : V_3(R) \rightarrow V_2(R)$ is defined by $T(x, y, z) = (x^2, y)$

Solution:

A transformation $T : V \rightarrow W$ is linear if it satisfies two conditions for all vectors $u, v \in V$ and scalar c :

1. Additivity: $T(u + v) = T(u) + T(v)$

2. Homogeneity: $T(cu) = cT(u)$

A transformation is not linear if it fails either of these conditions. A quick test is to check if $T(\vec{0}) = \vec{0}$. If not, it's not linear. Another quick check is for non-linear terms like powers (x^2), products of variables (xy), or constants added.

(A) $T(x, y, z) = (2x - 3y, 7y + 2z)$. The domain and codomain are mismatched in the text (V_3 to V_3 but output has 2 components), but assuming it's $T : V_3 \rightarrow V_2$, all components are linear combinations of the input variables. This is a linear transformation.

(B) $T(x, y, z) = (z, 2x + y)$. All components are linear combinations. This is a linear transformation.

(D) $T(x, y, z) = (y, -z, -x)$. All components are linear combinations. This is a linear transformation.

(C) $T(x, y, z) = (x^2, y)$. The first component contains a non-linear term, x^2 . Let's test the homogeneity condition.

Let $u = (x, y, z)$ and c be a scalar.

$$T(cu) = T(cx, cy, cz) = ((cx)^2, cy) = (c^2x^2, cy).$$

$$cT(u) = c(x^2, y) = (cx^2, cy).$$

Since $T(cu) = (c^2x^2, cy)$ is not equal to $cT(u) = (cx^2, cy)$ for all c , the transformation is not linear.

Quick Tip

A transformation is linear if and only if each component of the output vector is a linear combination of the components of the input vector. Look for any non-linear terms like squares, square roots, products of variables, absolute values, or added constants.

137. Let W be a subset of an n -dimensional vector space V . then which one of the following is not possible

(A) $\dim W = \dim V$

(B) $\dim W < \dim V$

(C) $\dim W > \dim V$

(D) $\dim W = 0$

Correct Answer: (C) $\dim W > \dim V$

Solution:

The question specifies that W is a subset of V , but for the concept of dimension to apply, we must assume W is a subspace of V .

A fundamental theorem of linear algebra states that if W is a subspace of a finite-dimensional vector space V , then the dimension of W is less than or equal to the dimension of V .

Mathematically, $\dim(W) \leq \dim(V)$.

Let's analyze the given options based on this theorem:

(A) $\dim(W) = \dim(V)$: This is possible. It occurs if and only if $W = V$.

- (C) $\dim(W) < \dim(V)$: This is possible. It occurs if W is a proper subspace of V (i.e., $W \neq V$).
- (D) $\dim(W) = 0$: This is possible. It occurs if W is the trivial subspace containing only the zero vector, $W = \{\vec{0}\}$.
- (B) $\dim(W) > \dim(V)$: This is not possible. A subspace cannot have a larger dimension than the space it is contained within. The dimension represents the maximum number of linearly independent vectors, and you cannot have more linearly independent vectors in a subspace than in the entire space.

Therefore, the statement that is not possible is $\dim(W) > \dim(V)$.

Quick Tip

Think of dimension as the number of "degrees of freedom" or independent directions. A subspace, being a part of the larger space, cannot have more degrees of freedom than the entire space.

138. Let U and V be two vector spaces over the same field $\mathbb{R}$. if $T: U \rightarrow V$ be a linear map the which one of the following is incorrect

- (A) $T(u_1 - u_2) = T(u_1) - T(u_2), u_1, u_2 \in U$
- (B) $T(u_1 + u_2) = T(u_1) + T(u_2), u_1, u_2 \in U$
- (C) $T(u_1 \cdot u_2) = T(u_1) \cdot T(u_2)$
- (D) $T(3u_1 + 2u_2) = 3T(u_1) + 2T(u_2)$
- Correct Answer:** (C) $T(u_1 \cdot u_2) = T(u_1) \cdot T(u_2)$

Solution:

The definition of a linear map (or linear transformation) $T: U \rightarrow V$ requires two properties:

1. Additivity: $T(u_1 + u_2) = T(u_1) + T(u_2)$ for all $u_1, u_2 \in U$.
2. Homogeneity: $T(cu) = cT(u)$ for any scalar c and vector $u \in U$.

These two properties can be combined into a single condition: $T(c_1u_1 + c_2u_2) = c_1T(u_1) + c_2T(u_2)$.

Let's evaluate the options:

(B) $T(u_1 + u_2) = T(u_1) + T(u_2)$: This is the additivity property, so it is a correct statement about linear maps.

(A) $T(u_1 - u_2) = T(u_1 - u_2) = T(u_1 + (-1)u_2) = T(u_1) + T((-1)u_2) = T(u_1) + (-1)T(u_2) = T(u_1) - T(u_2)$. This is a direct consequence of the two basic properties and is correct.

(D) $T(3u_1 + 2u_2) = 3T(u_1) + 2T(u_2)$: This is a specific case of the combined property $T(c_1u_1 + c_2u_2) = c_1T(u_1) + c_2T(u_2)$ with $c_1 = 3$ and $c_2 = 2$. This is correct.

(C) $T(u_1 \cdot u_2) = T(u_1) \cdot T(u_2)$: This describes a property related to multiplication. In a general vector space, the product of two vectors, $u_1 \cdot u_2$, is not defined. Linear transformations preserve vector addition and scalar multiplication, not vector multiplication. Therefore, this statement is incorrect and not a property of linear maps.

Quick Tip

Linear maps preserve the two fundamental operations of a vector space: vector addition and scalar multiplication. They do not, in general, preserve any kind of vector product.

139. Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $T(1,2)=(2,3)$ and $T(0,1)=(1,4)$ then

- (A) T is not linear
- (B) T is linear but not unique
- (C) T is linear and unique
- (D) T may or may not be linear

Correct Answer: (C) T is linear and unique

Solution:

A key theorem in linear algebra states that a linear transformation is uniquely determined by its action on a basis of the domain vector space.

Step 1: Check if the vectors on which T is defined form a basis for the domain, $\mathbb{R}^2$.

The vectors are $v_1 = (1, 2)$ and $v_2 = (0, 1)$.

The dimension of $\mathbb{R}^2$ is 2. We have 2 vectors, so we just need to check if they are linearly independent.

We can form a matrix and find its determinant: $\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = (1)(1) - (2)(0) = 1$.

Since the determinant is non-zero, the vectors are linearly independent and thus form a basis for $\mathbb{R}^2$.

Step 2: Apply the theorem.

Since the action of T is defined on a basis of the domain $\mathbb{R}^2$, this is sufficient to uniquely determine a linear transformation T .

Any vector (x, y) in $\mathbb{R}^2$ can be written as a unique linear combination of the basis vectors, $(x, y) = c_1(1, 2) + c_2(0, 1)$.

The action of T on (x, y) would then be $T(x, y) = c_1T(1, 2) + c_2T(0, 1) = c_1(2, 3) + c_2(1, 4)$, which is a unique vector.

For example, to find the formula for $T(x, y)$:

$$(x, y) = c_1(1, 2) + c_2(0, 1) \implies x = c_1, y = 2c_1 + c_2 \implies c_1 = x, c_2 = y - 2x.$$

$$T(x, y) = x(2, 3) + (y - 2x)(1, 4) = (2x + y - 2x, 3x + 4y - 8x) = (y, 4y - 5x).$$

This is a valid linear transformation, and it is uniquely defined.

Therefore, T is linear and unique.

Quick Tip

If you are given the values of a transformation T on a set of vectors, first check if those vectors form a basis for the domain. If they do, then T is a uniquely defined linear transformation.

140. Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $T(1,1)=(2,0)$ and $T(0,1)=(1,-1)$ then

- (A) $T(5,4)=(5,4)$
- (B) $T(5,4)=(1,0)$
- (C) $T(5,4)=(10,8)$
- (D) $T(5,4)=(9,1)$

Correct Answer: (D) $T(5,4)=(9,1)$

Solution:

Step 1: Verify that the given vectors form a basis for $\mathbb{R}^2$.

The vectors are $v_1 = (1, 1)$ and $v_2 = (0, 1)$. The determinant of the matrix they form is $\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$. They are linearly independent and form a basis. Thus, the linear transformation T is uniquely defined.

Step 2: Express the input vector $(5,4)$ as a linear combination of the basis vectors.

Let $(5, 4) = a(1, 1) + b(0, 1)$.

$(5, 4) = (a, a + b)$.

Equating components:

$a = 5$.

$a + b = 4 \implies 5 + b = 4 \implies b = -1$.

So, $(5, 4) = 5(1, 1) - 1(0, 1)$.

Step 3: Use the linearity property of T to find $T(5,4)$.

$T(5, 4) = T(5(1, 1) - 1(0, 1))$.

Since T is linear, $T(c_1v_1 + c_2v_2) = c_1T(v_1) + c_2T(v_2)$.

$T(5, 4) = 5 \cdot T(1, 1) - 1 \cdot T(0, 1)$.

Step 4: Substitute the given values of $T(1,1)$ and $T(0,1)$.

$T(5, 4) = 5(2, 0) - 1(1, -1)$.

$T(5, 4) = (10, 0) - (1, -1)$.

$T(5, 4) = (10 - 1, 0 - (-1))$.

$T(5, 4) = (9, 1)$.

Quick Tip

To find the image of a vector under a linear transformation, first express the input vector as a linear combination of the basis vectors for which the transformation is defined. Then, apply the transformation using the linearity property.

141. Let $T: U \rightarrow V$ be a linear map the

- (A) $R(T)$ is a subspace of U and $N(T)$ is a subspace of V
- (B) $N(T)$ is a subspace of U but $R(T)$ is not a subspace of V
- (C) $R(T)$ is a subspace of V and $N(T)$ is a subspace of U
- (D) $R(T)$ is a subspace of V but $N(T)$ is not a subspace of U

Correct Answer: (C) $R(T)$ is a subspace of V and $N(T)$ is a subspace of U

Solution:

This question tests the definitions of the range (image) and kernel (null space) of a linear map.

Let $T: U \rightarrow V$ be a linear map.

1. The Range of T , denoted $R(T)$ or $Im(T)$, is the set of all possible outputs of the transformation.

$$R(T) = \{T(u) \mid u \in U\}.$$

Since all the output vectors $T(u)$ are elements of V , the range $R(T)$ is a subset of the codomain V . It can be proven that $R(T)$ is a subspace of V .

2. The Kernel of T , denoted $N(T)$ or $Ker(T)$, is the set of all vectors in the domain U that map to the zero vector in V .

$$N(T) = \{u \in U \mid T(u) = \vec{0}_V\}.$$

By definition, the kernel is a set of vectors from the domain U . It can be proven that $N(T)$ is a subspace of U .

Therefore, $R(T)$ is a subspace of V and $N(T)$ is a subspace of U . This matches option (C).

Quick Tip

Remember the locations of the key subspaces: The kernel/null space lives in the domain (the "input" space), while the range/image/column space lives in the codomain (the "output" space).

142. Let $T: R^3 \rightarrow R^3$ be a linear map defined by $T(x_1, x_2, x_3) = (x_1, x_2, 0)$ then rank of T is

- (A) 0
- (B) 1
- (C) 2
- (D) 3

Correct Answer: (C) 2

Solution:

The rank of a linear transformation T is the dimension of its range (or image), denoted

$\text{rank}(T) = \dim(R(T))$.

The range $R(T)$ is the set of all possible output vectors.

The transformation is given by $T(x_1, x_2, x_3) = (x_1, x_2, 0)$.

Any output vector is of the form $(x_1, x_2, 0)$, which can be written as a linear combination:

$$(x_1, x_2, 0) = x_1(1, 0, 0) + x_2(0, 1, 0).$$

This shows that every vector in the range of T is a linear combination of the two vectors $v_1 = (1, 0, 0)$ and $v_2 = (0, 1, 0)$.

These two vectors span the range of T .

The vectors v_1 and v_2 are clearly linearly independent.

Since the range is spanned by two linearly independent vectors, the dimension of the range is 2.

Therefore, the rank of T is 2.

Alternatively, using the Rank-Nullity Theorem:

Nullity is the dimension of the kernel. The kernel is the set of vectors (x_1, x_2, x_3) such that $T(x_1, x_2, x_3) = (0, 0, 0)$.

$(x_1, x_2, 0) = (0, 0, 0)$ implies $x_1 = 0$ and $x_2 = 0$. x_3 can be any real number.

So, a vector in the kernel is of the form $(0, 0, x_3) = x_3(0, 0, 1)$.

The kernel is spanned by the single vector $(0, 0, 1)$, so the nullity is 1.

Rank-Nullity Theorem: $\text{rank}(T) + \text{nullity}(T) = \dim(\text{domain})$.

$$\text{rank}(T) + 1 = 3.$$

$$\text{rank}(T) = 2.$$

Quick Tip

The rank of a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ is the number of linearly independent vectors that span the image. Geometrically, $T(x_1, x_2, x_3) = (x_1, x_2, 0)$ is a projection onto the xy -plane, which is a 2-dimensional space.

143. Let $T: U \rightarrow V$ be a linear map U is finite-dimensional vector space then which one of the following is correct

(A) $\dim R(T) + \dim V = \dim U$

(B) $\dim R(T) + \dim N(T) = \dim V$

(C) $\dim R(T) + \dim U = \dim N(T)$

(D) $\dim R(T) + \dim N(T) = \dim U$

Correct Answer: (D) $\dim R(T) + \dim N(T) = \dim U$

Solution:

This question asks for the statement of the Rank-Nullity Theorem (also known as the Fundamental Theorem of Linear Maps).

Let $T : U \rightarrow V$ be a linear map, where U is a finite-dimensional vector space.

The theorem states that the dimension of the domain (U) is equal to the sum of the dimension of the range ($R(T)$) and the dimension of the kernel ($N(T)$).

The dimension of the range is called the rank of T .

$$\text{rank}(T) = \dim(R(T)).$$

The dimension of the kernel is called the nullity of T .

$$\text{nullity}(T) = \dim(N(T)).$$

So, the theorem can be written as:

$$\text{rank}(T) + \text{nullity}(T) = \dim(U).$$

Substituting the full names:

$$\dim(R(T)) + \dim(N(T)) = \dim(U).$$

This exactly matches the statement in option (D).

Quick Tip

Remember the Rank-Nullity Theorem as "rank + nullity = dimension of the domain". The dimension of the codomain (V) is not directly part of the main formula, although it does bound the rank ($\text{rank}(T) \leq \dim(V)$).

144. The minimum and maximum eigenvalues of the matrix $\begin{pmatrix} 1 & 5 & 1 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}$ are -2 and 6, respectively. What is the other eigenvalue

(A) 5

(B) 3

(C) 1

(D) -1

Correct Answer: (B) 3

Solution:

A fundamental property of a square matrix is that the sum of its eigenvalues is equal to the trace of the matrix.

The trace of a matrix is the sum of the elements on its main diagonal.

Let the given matrix be A . $A = \begin{pmatrix} 1 & 5 & 1 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}$.

Step 1: Calculate the trace of matrix A.

$$\text{Trace}(A) = 1 + 5 + 1 = 7.$$

Step 2: Use the property relating trace and eigenvalues.

Let the eigenvalues of the 3x3 matrix be λ_1, λ_2 , and λ_3 .

$$\text{Sum of eigenvalues} = \lambda_1 + \lambda_2 + \lambda_3 = \text{Trace}(A).$$

Step 3: Substitute the known values.

We are given the minimum and maximum eigenvalues are -2 and 6. Let $\lambda_1 = -2$ and $\lambda_2 = 6$.

Let the third eigenvalue be λ_3 .

$$-2 + 6 + \lambda_3 = 7.$$

$$4 + \lambda_3 = 7.$$

Step 4: Solve for the other eigenvalue, λ_3 .

$$\lambda_3 = 7 - 4 = 3.$$

The other eigenvalue is 3.

Quick Tip

For any square matrix, remember these two important properties: 1. Sum of eigenvalues = Trace of the matrix. 2. Product of eigenvalues = Determinant of the matrix. These are extremely useful for finding an unknown eigenvalue when others are given.

145. The characteristic equation of matrix $A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$ is

(A) $\lambda^3 + 6\lambda^2 + 9\lambda + 4 = 0$

(B) $\lambda^3 - 6\lambda^2 - 9\lambda + 4 = 0$

(C) $\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$

(D) $\lambda^3 - 6\lambda^2 + 12\lambda - 4 = 0$

Correct Answer: (C) $\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$

Solution:

For a 3x3 matrix, the characteristic equation is given by $\det(A - \lambda I) = 0$.

A shortcut formula for the characteristic equation of a 3x3 matrix is:

$$\lambda^3 - (\text{trace}(A))\lambda^2 + (M_{11} + M_{22} + M_{33})\lambda - \det(A) = 0, \text{ where } M_{ii} \text{ are the principal minors.}$$

Step 1: Calculate the trace of A.

$$\text{trace}(A) = 2 + 2 + 2 = 6.$$

Step 2: Calculate the sum of the principal minors.

$$M_{11} = \det \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = (2)(2) - (-1)(-1) = 4 - 1 = 3.$$

$$M_{22} = \det \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = (2)(2) - (1)(1) = 4 - 1 = 3.$$

$$M_{33} = \det \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = (2)(2) - (-1)(-1) = 4 - 1 = 3.$$

$$\text{Sum of minors} = 3 + 3 + 3 = 9.$$

Step 3: Calculate the determinant of A.

$$\det(A) = 2(M_{11}) - (-1) \det \begin{pmatrix} -1 & -1 \\ 1 & 2 \end{pmatrix} + 1 \det \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}.$$

$$\det(A) = 2(3) + 1((-1)(2) - (-1)(1)) + 1((-1)(-1) - (2)(1)).$$

$$\det(A) = 6 + 1(-2 + 1) + 1(1 - 2).$$

$$\det(A) = 6 + (-1) + (-1) = 4.$$

Step 4: Substitute these values into the characteristic equation formula.

$$\lambda^3 - (6)\lambda^2 + (9)\lambda - (4) = 0.$$

$$\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0.$$

This matches option (C).

Quick Tip

Using the shortcut formula for the 3x3 characteristic equation ($\lambda^3 - \text{tr}(A)\lambda^2 + \text{sum of principal minors}\lambda - \det(A) = 0$) is often faster and less error-prone than directly computing $\det(A - \lambda I)$.

146. In an inner product space $V(F)$ for $a \in F, \alpha, \beta, \gamma \in V, (\alpha, a\beta + a\gamma)$

(A) $a(\alpha, \beta) + a(\alpha, \gamma)$

(B) $\bar{a}(\beta, \alpha) + \bar{a}(\gamma, \alpha)$

(C) $\bar{a}(\alpha, \beta) + \bar{a}(\alpha, \gamma)$

(D) $\bar{a}(\alpha, \beta) + \bar{a}(\alpha, \gamma)$

Correct Answer: (C) $\bar{a}(\alpha, \beta) + \bar{a}(\alpha, \gamma)$

Solution:

This question tests the properties of an inner product, particularly when the field F is the complex numbers $\mathbb{C}$. The notation $(\cdot, \cdot)$ denotes the inner product.

The axioms for an inner product are:

1. Linearity in the first argument: $(\alpha_1 + \alpha_2, \beta) = (\alpha_1, \beta) + (\alpha_2, \beta)$ and $(c\alpha, \beta) = c(\alpha, \beta)$.
2. Conjugate symmetry: $(\alpha, \beta) = \overline{(\beta, \alpha)}$.
3. Positive-definiteness: $(\alpha, \alpha) \geq 0$ and $(\alpha, \alpha) = 0$ iff $\alpha = 0$.

From these axioms, we can derive the property for the second argument. This property is called conjugate linearity or sesquilinearity.

Let's find the property for $(\alpha, c\beta)$:

Using conjugate symmetry: $(\alpha, c\beta) = \overline{(c\beta, \alpha)}$.

Using linearity in the first argument: $\overline{(c\beta, \alpha)} = \overline{c(\beta, \alpha)}$.

Using properties of complex conjugates: $\overline{c(\beta, \alpha)} = \bar{c}(\beta, \alpha)$.

Using conjugate symmetry again: $\bar{c}(\beta, \alpha) = \bar{c}(\alpha, \beta)$.

So, $(\alpha, c\beta) = \bar{c}(\alpha, \beta)$. Scalars come out of the second argument as their conjugate.

Now let's apply this to the given expression: $(\alpha, a\beta + a\gamma)$.

First, linearity in the second argument (derived from axioms): $(\alpha, \beta + \gamma) = (\alpha, \beta) + (\alpha, \gamma)$.

So, $(\alpha, a\beta + a\gamma) = (\alpha, a\beta) + (\alpha, a\gamma)$.

Now, pull the scalar 'a' out of the second argument from each term:

$(\alpha, a\beta) = \bar{a}(\alpha, \beta)$.

$(\alpha, a\gamma) = \bar{a}(\alpha, \gamma)$.

Combining these, we get:

$(\alpha, a\beta + a\gamma) = \bar{a}(\alpha, \beta) + \bar{a}(\alpha, \gamma)$.

This matches option (C). Note that options C and D are identical in the PDF.

Quick Tip

In a complex inner product space, remember the rule of thumb: scalars come out of the first argument "clean" ($c\alpha$) but come out of the second argument "conjugated" ($\bar{c}\beta$).

147. If α, β are orthogonal unit vectors, then $\|\alpha - \beta\| =$

(A) 0

(B) 1

(C) $\sqrt{2}$

(D) 2

Correct Answer: (C) $\sqrt{2}$

Solution:

We are given two vectors, α and β , with the following properties:

1. They are unit vectors: This means their norm (length) is 1.

$\|\alpha\| = 1$ and $\|\beta\| = 1$.

2. They are orthogonal: This means their inner product is 0.

$(\alpha, \beta) = 0$.

We need to find the norm of their difference, $\|\alpha - \beta\|$.

The norm is related to the inner product by the formula $\|\vec{v}\|^2 = (\vec{v}, \vec{v})$.

Let's calculate $\|\alpha - \beta\|^2$:

$\|\alpha - \beta\|^2 = (\alpha - \beta, \alpha - \beta)$.

Using the properties of the inner product:

$= (\alpha, \alpha - \beta) - (\beta, \alpha - \beta)$.

$$= (\alpha, \alpha) - (\alpha, \beta) - (\beta, \alpha) + (\beta, \beta).$$

Now substitute the known values:

$$(\alpha, \alpha) = \|\alpha\|^2 = 1^2 = 1.$$

$$(\beta, \beta) = \|\beta\|^2 = 1^2 = 1.$$

$$(\alpha, \beta) = 0.$$

From conjugate symmetry, $(\beta, \alpha) = \overline{(\alpha, \beta)} = \overline{0} = 0$.

Substituting these into the expression:

$$\|\alpha - \beta\|^2 = 1 - 0 - 0 + 1 = 2.$$

Now, take the square root to find the norm:

$$\|\alpha - \beta\| = \sqrt{2}.$$

Quick Tip

This result is the Pythagorean theorem in a vector space. If α and β are orthogonal, then $\|\alpha - \beta\|^2 = \|\alpha\|^2 + \|\beta\|^2$. If they are also unit vectors, this simplifies to $1^2 + 1^2 = 2$.

148. If $\alpha = (-1, 0, 1)$, $\beta = (2, 0, -2)$ then $\|\alpha + \beta\| =$

(A) 0

(B) 1

(C) 2

(D) $\sqrt{2}$

Correct Answer: (D) $\sqrt{2}$

Solution:

Step 1: Calculate the sum of the vectors $\alpha + \beta$.

$$\alpha + \beta = (-1, 0, 1) + (2, 0, -2).$$

$$\alpha + \beta = (-1 + 2, 0 + 0, 1 - 2).$$

$$\alpha + \beta = (1, 0, -1).$$

Step 2: Calculate the norm (magnitude) of the resulting vector.

The norm of a vector $\vec{v} = (x, y, z)$ in $\mathbb{R}^3$ with the standard Euclidean inner product is given by

$$\|\vec{v}\| = \sqrt{x^2 + y^2 + z^2}.$$

Let $\vec{v} = \alpha + \beta = (1, 0, -1)$.

$$\|\alpha + \beta\| = \sqrt{1^2 + 0^2 + (-1)^2}.$$

$$\|\alpha + \beta\| = \sqrt{1 + 0 + 1}.$$

$$\|\alpha + \beta\| = \sqrt{2}.$$

Quick Tip

Vector operations are performed component-wise. To find the norm of a sum of vectors, first compute the sum vector by adding the corresponding components, then compute the norm of the resulting vector.

149. The vector of unit length orthogonal to $\alpha = (2, -1, 6)$ in $V_3(R)$ w.r.to standard inner product is

(A) $(\frac{2}{3}, -\frac{2}{3}, -\frac{1}{3})$

(B) $(2, -2, -1)$

(C) $(1, 1, 1)$

(D) $(0, 0, 0)$

Correct Answer: (A) $(\frac{2}{3}, -\frac{2}{3}, -\frac{1}{3})$

Solution:

The question asks for a unit vector that is orthogonal to the given vector $\alpha = (2, -1, 6)$. Let the unknown vector be $v = (x, y, z)$.

Two vectors are orthogonal if their inner product (dot product) is zero. So, we must have $(\alpha, v) = 0$.

Let's check this condition for the vector part of each option.

(B) Let $v = (2, -2, -1)$.

$(\alpha, v) = (2)(2) + (-1)(-2) + (6)(-1) = 4 + 2 - 6 = 0$. This vector is orthogonal.

(C) Let $v = (1, 1, 1)$.

$(\alpha, v) = (2)(1) + (-1)(1) + (6)(1) = 2 - 1 + 6 = 7 \neq 0$. Not orthogonal.

(D) The zero vector is orthogonal to every vector, but it is not a unit vector.

Now we need to check if the orthogonal vector from the options is a unit vector. A vector is a unit vector if its norm is 1.

Let's check the norm of the vector from option (B), which is $(2, -2, -1)$.

$\|(2, -2, -1)\| = \sqrt{2^2 + (-2)^2 + (-1)^2} = \sqrt{4 + 4 + 1} = \sqrt{9} = 3$.

This is not a unit vector.

To make it a unit vector, we must divide it by its norm:

$\hat{v} = \frac{v}{\|v\|} = \frac{(2, -2, -1)}{3} = (\frac{2}{3}, -\frac{2}{3}, -\frac{1}{3})$.

This resulting vector is the unit vector in the direction of $(2, -2, -1)$, and it is orthogonal to α . This matches option (A).

Quick Tip

To solve "find a unit vector orthogonal to X ", it's often easiest to test the orthogonality condition (dot product = 0) on the non-normalized vectors in the options first. Then, check if the correct option is properly normalized (divided by its length).

150. . let W be a subspace of a finite-dimensional vector space $V(F)$ then $\dim(V/W) =$

- (A) $\dim W - \dim V$
- (B) $\dim V - \dim W$
- (C) $\dim V + \dim W$
- (D) 0

Correct Answer: (B) $\dim V - \dim W$

Solution:

This question asks for the dimension of a quotient space (or factor space) V/W .

Let V be a finite-dimensional vector space and W be a subspace of V .

The quotient space V/W is the set of all cosets of W in V , i.e., $V/W = \{v + W \mid v \in V\}$.

A fundamental theorem in linear algebra relates the dimensions of V , W , and the quotient space V/W .

The theorem states:

$$\dim(V) = \dim(W) + \dim(V/W).$$

This is analogous to Lagrange's theorem for groups, $O(G) = O(H) \cdot [G : H]$, and the Rank-Nullity theorem, $\dim(\text{Domain}) = \text{rank} + \text{nullity}$.

To find the dimension of the quotient space, we rearrange the formula:

$$\dim(V/W) = \dim(V) - \dim(W).$$

This matches option (B).

Quick Tip

Remember this fundamental dimension formula: $\dim(V/W) = \dim(V) - \dim(W)$. It intuitively means that the number of "independent directions" in the quotient space is what's left after you "collapse" the directions contained within the subspace W .