JEE Main (9 April Shift 1)

Mathematics

Question 1. Let the line L intersect the lines:

$$x-2=-y=z-1, \quad 2(x+1)=2(y-1)=z+1$$

and be parallel to the line:

$$\frac{x-2}{3}=\frac{y-1}{1}=\frac{z-2}{2}.$$

Then which of the following points lies on L?

1. $\left(\frac{1}{3},-1,1\right)$
2. $\left(-\frac{1}{3},1,-1\right)$
3. $\left(-\frac{1}{3},-1,-1\right)$
4. $\left(-\frac{1}{3},-1,1\right)$

Correct Answer: (2)
Solution:
Consider the given lines:

$$L_1 : x-2=-y=z-1$$

$$L_2 : (x+1)=2(y-1)=z+1$$

The line L is parallel to:

$$\frac{x-2}{3}=\frac{y-1}{1}=\frac{z-2}{2}$$

This indicates that the direction ratios (dr's) of line L are $(3,1,2)$.
To find the coordinates of the intersection point M of lines L_1 and L_2, we parameterize them using λ and μ respectively:

$$L_1 : x=2+\lambda, \quad y=-\lambda, \quad z=1+\lambda$$

$$L_2 : x=-\frac{1}{2}\mu, \quad y=\frac{\mu}{2}+1, \quad z=\mu+1$$

Equating the coordinates: 1. $2+\lambda=-\frac{1}{2}\mu$ 2. $-\lambda=\frac{\mu}{2}+1$ 3. $1+\lambda=\mu+1$
From equation (3):

$$\lambda=\mu$$

Substituting $\lambda=\mu$ in equations (1) and (2):

$$2+\lambda=-\frac{1}{2}\lambda \implies \frac{3}{2}\lambda=-2 \implies \lambda=-\frac{4}{3}, \quad \mu=-\frac{4}{3}$$

Using these values, the coordinates of point M are:

$$M\,(2+\lambda, -\lambda, 1+\lambda) = \left(2-\frac{4}{3}, \frac{4}{3}, 1-\frac{4}{3}\right) = \left(\frac{2}{3}, \frac{4}{3}, -\frac{1}{3}\right)$$

The equation of the line through M parallel to $(3,1,2)$ is:

$$\frac{x-\frac{2}{3}}{3} = \frac{y-\frac{4}{3}}{1} = \frac{z+\frac{1}{3}}{2} = k$$

For any point on this line:

$$x = \frac{2}{3}+3k, \quad y = \frac{4}{3}+k, \quad z = -\frac{1}{3}+2k$$

Substituting each option: 1. $\left(-\frac{1}{3},1,1\right)$ does not satisfy the equation. 2. $\left(-1,-\frac{1}{3},1\right)$ satisfies for $k=-\frac{1}{3}$. 3. $\left(1,-\frac{1}{3},-1\right)$ does not satisfy. 4. $(-1,3,-1)$ does not satisfy. Thus, the correct point is:

$$\left(-1,-\frac{1}{3},1\right)$$

Quick Tip

To find points on a line in 3D geometry, use parametric equations and systematically verify each option for consistency with the given line equations.

Q.2 The parabola $y^2 = 4x$ divides the area of the circle $x^2+y^2=5$ in two parts. The area of the smaller part is equal to:

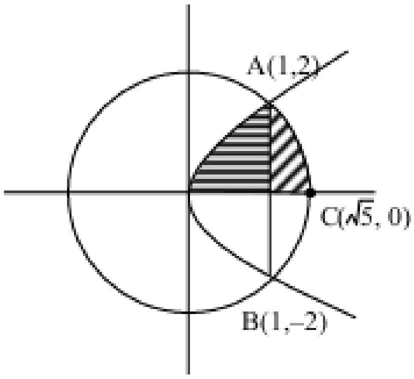
1. $\frac{2}{3}+5\sin^{-1}\left(\frac{2}{\sqrt{5}}\right)$
2. $\frac{1}{3}+5\sin^{-1}\left(\frac{2}{\sqrt{5}}\right)$
3. $\frac{1}{3}+\sqrt{5}\sin^{-1}\left(\frac{2}{\sqrt{5}}\right)$
4. $\frac{2}{3}+\sqrt{5}\sin^{-1}\left(\frac{2}{\sqrt{5}}\right)$

Correct Answer: (1)
Solution:

Given equations:

$$y^2 = 4x \quad \text{(parabola)}$$
$$x^2+y^2 = 5 \quad \text{(circle)}$$

The parabola divides the circle into two regions, and we need to find the area of the smaller part.



Finding Points of Intersection: To find the intersection points, substitute $y^2 = 4x$ into $x^2 + y^2 = 5$:

$$x^2 + 4x = 5 \implies x^2 + 4x - 5 = 0$$

Solving this quadratic equation:

$$x = \frac{-4 \pm \sqrt{16+20}}{2} = \frac{-4 \pm \sqrt{36}}{2} = \frac{-4 \pm 6}{2}$$

$$x = 1 \quad \text{or} \quad x = -5$$

Since $x = -5$ is not within the domain of the parabola, we have $x = 1$. The corresponding points are $(1, 2)$ and $(1, -2)$.

Area Calculation: The area A_1 of the region bounded by the parabola and the circle in the first quadrant is given by:

$$A_1 = \int_0^1 \sqrt{4x}\,dx + \int_1^{\sqrt{5}} \sqrt{5 - x^2}\,dx$$

1. Area under the parabola:

$$\int_0^1 \sqrt{4x}\,dx = \int_0^1 2\sqrt{x}\,dx = \left[\frac{4}{3}x^{3/2}\right]_0^1 = \frac{4}{3}$$

2. Area under the circle:

$$\int_1^{\sqrt{5}} \sqrt{5 - x^2}\,dx$$

Using trigonometric substitution $x = \sqrt{5}\sin\theta$, $dx = \sqrt{5}\cos\theta\,d\theta$, where θ varies from $\sin^{-1}\left(\frac{1}{\sqrt{5}}\right)$ to $\frac{\pi}{2}$. After integration, we find:

$$\int_1^{\sqrt{5}} \sqrt{5 - x^2}\,dx = \frac{5}{2}\left(\sin^{-1}\left(\frac{2}{\sqrt{5}}\right)\right)$$

Total Area of the Smaller Region: The required area is twice A_1:

$$\text{Required Area} = 2\left(\frac{4}{3} + \frac{5}{2}\sin^{-1}\left(\frac{2}{\sqrt{5}}\right)\right) = \frac{2}{3} + \frac{5}{2}\sin^{-1}\left(\frac{2}{\sqrt{5}}\right)$$

Final Answer: (1).

Quick Tip

For problems involving intersections of curves, identify symmetry and simplify using geometry formulas like segment area and parabola sector area.

Q.3 The solution curve of the differential equation

$$2y\frac{dy}{dx} + 3 = 5\frac{dy}{dx},$$

passing through the point $(0, 1)$, is a conic whose vertex lies on the line:

1. $2x + 3y = 9$
2. $2x + 3y = -9$
3. $2x + 3y = -6$
4. $2x + 3y = 6$

Correct Answer: (1)
Solution:

Given the differential equation:

$$2y\frac{dy}{dx} + 3 = 5\frac{dy}{dx}$$

Rearranging terms:

$$(2y - 5)\frac{dy}{dx} = -3$$

Separating variables:

$$(2y - 5)dy = -3dx$$

Integrating both sides:

$$\int (2y - 5)dy = \int -3dx$$

Calculating the integrals:

$$\frac{2y^2}{2} - 5y = -3x + \lambda \implies y^2 - 5y = -3x + \lambda$$

To find the value of λ, we use the point $(0, 1)$:

$$1^2 - 5 \times 1 = -3 \times 0 + \lambda \implies \lambda = -4$$

Thus, the equation of the curve becomes:

$$y^2 - 5y = -3x - 4 \implies \left(y - \frac{5}{2}\right)^2 = -3\left(x - \frac{3}{4}\right)$$

This represents a parabola with vertex at:

$$\left(\frac{3}{4}, \frac{5}{2}\right)$$

The vertex lies on the line:

$$2x + 3y = 9$$

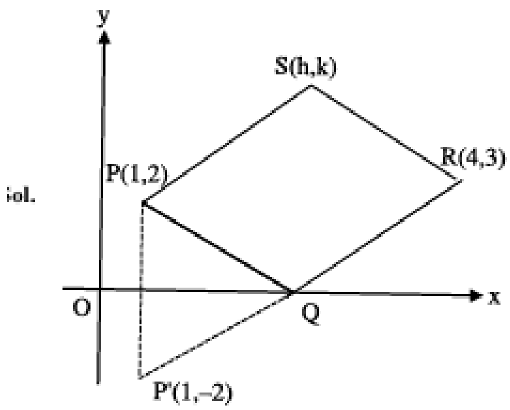
Quick Tip

When solving differential equations that form conics, convert to a standard form and identify key parameters like vertices or centers by completing the square.

Question 4.
A ray of light coming from the point $P(1, 2)$ gets reflected from the point Q on the x-axis and then passes through the point $R(4, 3)$. If the point $S(h, k)$ is such that $PQRS$ is a parallelogram, then hk^2 is equal to:

1. 80
2. 90
3. 60
4. 70

Correct Answer: (4)
Solution:
Given:



$$P(1, 2), \quad R(4, 3)$$

The image of point P with respect to the x-axis is $P'(1, -2)$.

Finding the Equation of Line PR: The equation of the line passing through points $P'(1, -2)$ and $R(4, 3)$ is given by:

$$y - (-2) = \frac{3 - (-2)}{4 - 1}(x - 1)$$

Simplifying:

$$y + 2 = \frac{5}{3}(x - 1)$$

Rearranging:

$$y = \frac{5}{3}x - \frac{11}{3}$$

Finding Point of Intersection Q with the x-axis: At the x-axis, $y = 0$:

$$0 = \frac{5}{3}x - \frac{11}{3} \implies x = \frac{11}{5}$$

Thus, $Q = \left(\frac{11}{5}, 0\right)$.

Condition for $PQRS$ to be a Parallelogram: For $PQRS$ to be a parallelogram, its diagonals must bisect each other. Therefore, the midpoint of PR must coincide with the midpoint of QS.

Midpoint Calculations: The midpoint of PR is:

$$\left(\frac{1+4}{2}, \frac{2+3}{2}\right) = \left(\frac{5}{2}, \frac{5}{2}\right)$$

The midpoint of QS is:

$$\left(\frac{\frac{11}{5} + h}{2}, \frac{0 + k}{2}\right)$$

Equating the midpoints:

$$\frac{5}{2} = \frac{\frac{11}{5} + h}{2} \quad \text{and} \quad \frac{5}{2} = \frac{k}{2}$$

Solving these equations:

$$\frac{11}{5} + h = 5 \implies h = \frac{14}{5}, \quad k = 5$$

Calculating hk^2:

$$hk^2 = \frac{14}{5} \times 5^2 = \frac{14}{5} \times 25 = 70$$

Quick Tip

For problems involving reflections and parallelograms, use symmetry properties and midpoint conditions to simplify calculations.

Question 5.

Let $\lambda, \mu \in R$. If the system of equations

$$3x + 5y + \lambda z = 3,$$

$$7x + 11y - 9z = 2,$$
$$97x + 155y - 189z = \mu$$

has infinitely many solutions, then $\mu + 2\lambda$ is equal to:

1. 25
2. 24
3. 27
4. 22

Correct Answer: (1)
Solution:

Given the system of linear equations:
Subtract appropriate multiples of rows to equate terms and isolate variables.

$$3x + 5y + \lambda z = 3$$
$$7x + 11y - 9z = 2$$
$$97x + 155y - 189z = \mu$$
$$93x + 155y + 31z = 93$$
$$97x + 155y - 189z = \mu$$

$$-4x + (31\lambda + 189)z = 93 - \mu$$
$$1085x + 1705y - 1395z = 310$$
$$1067x + 1705y - 2079z = 11\mu$$

$$18x + 684z = 310 - 11\mu$$
$$-36x + 9(31\lambda + 189)z = 9(93 - \mu)$$
$$36x + 1368z = 2(310 - 11\mu)$$
$$(279\lambda + 3069)z = 1457 - 31\mu$$

Similarly, equating further coefficients gives:

$$\lambda = \frac{-3069}{279}, \quad \mu = \frac{1457}{31}$$

Then:

$$\mu + 2\lambda = \frac{1457}{31} + 2\left(\frac{-3069}{279}\right) = 25$$

💡 Quick Tip

For systems with infinitely many solutions, ensure the determinant condition is met, and check consistency for augmented equations.

Q.6 The coefficient of x^{70} in

$$x^2(1+x)^{98} + x^3(1+x)^{97} + x^4(1+x)^{96} + \cdots + x^{54}(1+x)^{46}$$

is ${}^{99}C_p - {}^{46}C_q$.
Then a possible value to $p+q$ is:

(1) 55 (2) 61 (3) 68 (4) 83

Ans. (4)
Solution:
Given:

$$x^2(1+x)^{98} + x^3(1+x)^{97} + x^4(1+x)^{96} + \ldots + x^{54}(1+x)^{46}$$

$$x^{54}(1+x)^{46}$$

$$\text{Coeff. of } x^{70} : \binom{98}{C_{68}} + \binom{97}{C_{67}} + \binom{96}{C_{66}} + \ldots$$

$$\binom{47}{C_{17}} + \binom{46}{C_{16}}$$

$$= \binom{46}{C_{30}} + \binom{47}{C_{30}} + \ldots \binom{98}{C_{30}}$$

$$= \left(\binom{46}{C_{31}} + \binom{46}{C_{30}}\right) + \binom{47}{C_{30}} + \ldots \binom{98}{C_{30}} - \binom{46}{C_{31}}$$

$$\binom{47}{C_{31}} + \ldots \binom{98}{C_{31}} - \binom{46}{C_{31}}$$

$$= \binom{99}{C_{31}} - \binom{99}{C_p} - \binom{46}{C_q}$$

Possible values of $(p+q)$ are $62, 83, 99, 46$

$$\Rightarrow p + q = 83$$

💡 Quick Tip

Utilize combinatorial identities and expansions to simplify expressions and find relationships between terms.

Question 7.
Let

$$\int \frac{2 - \tan x}{3 + \tan x}\, dx = \frac{1}{2}\left(\alpha x + \log_e |\beta \sin x + \gamma \cos x|\right) + C,$$

where C is the constant of integration. Then $\alpha + \frac{\gamma}{\beta}$ is equal to:

1. 3
2. 1
3. 4
4. 7

Correct Answer: (3)
Solution:

$$\int \frac{2-\tan x}{3+\tan x}\,dx = \int \frac{2\cos x - \sin x}{3\cos x + \sin x}\,dx$$

Let:

$$2\cos x - \sin x = A(3\cos x + \sin x) + B(\cos x - 3\sin x)$$

Equating coefficients:

$$3A + B = 2, \quad A - 3B = -1$$

Solving these equations:

$$A = \frac{1}{2}, \quad B = \frac{1}{2}$$

Thus:

$$\int \frac{2\cos x - \sin x}{3\cos x + \sin x}\,dx = \int \frac{x}{2} + \frac{1}{2}\ln|3\cos x + \sin x| + C$$

Simplify:

$$= \frac{1}{2}\left(x + \ln|3\cos x + \sin x|\right) + C$$

Rewriting in the given form:

$$= \frac{1}{2}\left(\alpha x + \ln|\beta \sin x + \gamma \cos x|\right) + C$$

From the equation:

$$\alpha = 1, \quad \beta = 1, \quad \gamma = 3$$

Finally:

$$\alpha + \frac{\gamma}{\beta} = 1 + \frac{3}{1} = 4$$

Ans. (3)

> Quick Tip
>
> For integrals involving trigonometric substitution, simplify using partial fractions, and carefully map back to trigonometric forms for final expressions.

Question 8.
A variable line L passes through the point $(3, 5)$ and intersects the positive coordinate axes at the points A and B. The minimum area of the triangle OAB, where O is the origin, is:

1. 30
2. 25
3. 40
4. 35

Correct Answer: (1)
Solution:

Step 1: Equation of the line L The line L passes through the point $(3, 5)$ and intersects the axes. Let the equation of the line L be:

$$\frac{x}{a} + \frac{y}{b} = 1,$$

where a and b are the intercepts on the x-axis and y-axis, respectively.
Since the line passes through $(3, 5)$, substitute $x = 3$ and $y = 5$:

$$\frac{3}{a} + \frac{5}{b} = 1. \quad (1)$$

Step 2: Area of triangle OAB The area of triangle OAB is given by:

$$\text{Area} = \frac{1}{2} \times a \times b.$$

From equation (1), express b in terms of a:

$$\frac{5}{b} = 1 - \frac{3}{a}.$$

$$b = \frac{5a}{a-3}. \quad (2)$$

Substitute $b = \frac{5a}{a-3}$ into the area formula:

$$\text{Area} = \frac{1}{2} \times a \times \frac{5a}{a-3}.$$

$$\text{Area} = \frac{5a^2}{2(a-3)}. \quad (3)$$

Step 3: Minimize the area Let $f(a) = \frac{5a^2}{2(a-3)}$. To find the minimum area, calculate $\frac{df}{da}$ and set it equal to zero:

$$f(a) = \frac{5a^2}{2(a-3)}.$$

Using the quotient rule:

$$\frac{df}{da} = \frac{(2(a-3)(10a)) - \left(5a^2(2)\right)}{4(a-3)^2}.$$

Simplify:

$$\frac{df}{da} = \frac{20a(a-3) - 10a^2}{4(a-3)^2}.$$

$$\frac{df}{da} = \frac{20a^2 - 60a - 10a^2}{4(a-3)^2}.$$

$$\frac{df}{da} = \frac{10a^2 - 60a}{4(a-3)^2}.$$

$$\frac{df}{da} = \frac{10a(a-6)}{4(a-3)^2}.$$

Set $\frac{df}{da} = 0$:

$$10a(a-6) = 0.$$

$$a = 0 \quad \text{or} \quad a = 6.$$

Since $a = 0$ is not valid (intercept cannot be zero), $a = 6$.
Step 4: Calculate b and the minimum area Substitute $a = 6$ into equation (2) to find b:

$$b = \frac{5(6)}{6-3} = \frac{30}{3} = 10.$$

The minimum area is:

$$\text{Area} = \frac{1}{2} \times 6 \times 10 = 30.$$

Final Answer: (1).

> Quick Tip
>
> To minimize the area of a triangle formed by a line and the axes, express the intercepts in terms of one variable and optimize using calculus.

Question 9.
Let

$$|\cos\theta \cos(60-\theta)\cos(60-\theta)| \leq \frac{1}{8}, \quad \theta \in [0, 2\pi].$$

Then, the sum of all $\theta \in [0, 2\pi]$, where $\cos 3\theta$ attains its maximum value, is:

1. 9π
2. 18π
3. 6π
4. 15π

Correct Answer: (3)
Solution:

Step 1: Simplify the inequality Using the trigonometric identity:

$$\cos\theta\cos(60^\circ-\theta)\cos(60^\circ+\theta)=\frac{1}{4}\cos 3\theta,$$

the inequality reduces to:

$$\left|\frac{1}{4}\cos 3\theta\right|\le\frac{1}{8}.$$

Simplify further:

$$|\cos 3\theta|\le\frac{1}{2}.$$

Step 2: Range of $\cos 3\theta$ The inequality becomes:

$$-\frac{1}{2}\le\cos 3\theta\le\frac{1}{2}.$$

The maximum value of $\cos 3\theta$ within this range is $\frac{1}{2}$. At this value:

$$\cos 3\theta=\frac{1}{2}.$$

Step 3: Solve for 3θ The general solution for $\cos 3\theta=\frac{1}{2}$ is:

$$3\theta=2n\pi\pm\frac{\pi}{3},\quad n\in Z.$$

Divide through by 3 to solve for θ:

$$\theta=\frac{2n\pi}{3}\pm\frac{\pi}{9}.$$

Step 4: Possible values of θ in $[0,2\pi]$ For $\theta\in[0,2\pi]$, substitute $n=0,1,2,\ldots$ until all possible values of θ are found.
- For $n=0$:

$$\theta=\pm\frac{\pi}{9}.$$

Since $\theta\ge 0$, take $\theta=\frac{\pi}{9}$.
- For $n=1$:

$$\theta=\frac{2\pi}{3}\pm\frac{\pi}{9}.$$

This gives $\theta=\frac{5\pi}{9}$ and $\frac{7\pi}{9}$.
- For $n=2$:

$$\theta=\frac{4\pi}{3}\pm\frac{\pi}{9}.$$

This gives $\theta=\frac{11\pi}{9}$ and $\frac{13\pi}{9}$.
- For $n=3$:

$$\theta=2\pi\pm\frac{\pi}{9}.$$

This gives $\theta=\frac{17\pi}{9}$ (as $\theta=\frac{19\pi}{9}>2\pi$ is invalid).
Thus, the possible values of θ are:

$$\theta=\frac{\pi}{9},\frac{5\pi}{9},\frac{7\pi}{9},\frac{11\pi}{9},\frac{13\pi}{9},\frac{17\pi}{9}.$$

Step 5: Sum of all θ The sum of these values is:

$$\text{Sum} = \frac{\pi}{9} + \frac{5\pi}{9} + \frac{7\pi}{9} + \frac{11\pi}{9} + \frac{13\pi}{9} + \frac{17\pi}{9}.$$

$$\text{Sum} = \frac{\pi(1+5+7+11+13+17)}{9} = \frac{\pi \cdot 54}{9} = 6\pi.$$

Final Answer: (3).

Quick Tip

When solving trigonometric inequalities, use identities to simplify and focus on periodicity for valid solutions within the given interval.

Question 10.
Let

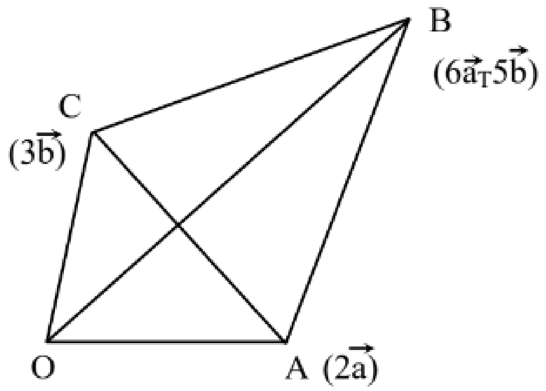
$$\overrightarrow{OA} = 2\vec{a}, \quad \overrightarrow{OB} = 6\vec{a} + 5\vec{b}, \quad \text{and} \quad \overrightarrow{OC} = 3\vec{b},$$

where O is the origin. If the area of the parallelogram with adjacent sides $\overrightarrow{OA}$ and $\overrightarrow{OC}$ is 15 sq. units, then the area (in sq. units) of the quadrilateral $OABC$ is equal to:

1. 38
2. 40
3. 32
4. 35

Correct Answer: (4)
Solution:

Given the quadrilateral $OABC$, we calculate its area step by step.



Area of the parallelogram:
The area of the parallelogram formed by sides $\overrightarrow{OA}$ and $\overrightarrow{OC}$ is given by:

$$\left|\overrightarrow{OA} \times \overrightarrow{OC}\right| = \left|2\,\vec{a} \times 3\,\vec{b}\right| = 15$$

Simplify:

$$6\left|\vec{a} \times \vec{b}\right| = 15$$

$$\left|\vec{a} \times \vec{b}\right| = \frac{5}{2} \quad \ldots(1)$$

Area of the quadrilateral $OABC$:
The area of the quadrilateral is:

$$\text{Area} = \frac{1}{2}\left|\vec{d_1} \times \vec{d_2}\right|$$

Here:

$$\vec{d_1} = \overrightarrow{AC}, \quad \vec{d_2} = \overrightarrow{OB}$$

Substituting:

$$\text{Area} = \frac{1}{2}\left|\overrightarrow{AC} \times \overrightarrow{OB}\right| = \frac{1}{2}\left|(3\vec{b} - 2\vec{a}) \times (6\vec{a} + 5\vec{b})\right|$$

Expanding the cross product:

$$= \frac{1}{2}\left|18\vec{b} \times \vec{a} - 10\vec{a} \times \vec{b}\right|$$

Using $\left|\vec{a} \times \vec{b}\right| = \frac{5}{2}$ from (1):

$$= \left(14\left|\vec{a} \times \vec{b}\right|\right)$$

$$= 14 \times \frac{5}{2} = 35$$

Final Answer: The area of the quadrilateral $OABC$ is 35.

Quick Tip

To calculate the area of a quadrilateral formed by vectors, find the cross product of diagonals or adjacent sides, and use the magnitude for the final calculation.

Question 11.
If the domain of the function

$$f(x) = \sin^{-1}\left(\frac{x-1}{2x+3}\right)$$

is $R - (\alpha, \beta)$, then $12\alpha\beta$ is equal to:

1. 36
2. 24
3. 40

4. 32

Correct Answer: (4)
Solution:

Step 1: Conditions for the domain of $f(x)$ The argument of $\sin^{-1}(x)$, $\frac{x-1}{2x+3}$, must satisfy two conditions: 1. $2x+3 \neq 0$ (denominator cannot be zero), so $x \neq -\frac{3}{2}$, 2. $\left|\frac{x-1}{2x+3}\right| \leq 1$.
Step 2: Solve $\left|\frac{x-1}{2x+3}\right| \leq 1$ Split the inequality into two cases: 1. For $\frac{x-1}{2x+3} \geq -1$:

$$x-1 \geq -(2x+3) \quad \Rightarrow \quad x-1 \geq -2x-3.$$

Simplify:

$$3x \geq -2 \quad \Rightarrow \quad x \geq -\frac{2}{3}.$$

2. For $\frac{x-1}{2x+3} \leq 1$:

$$x-1 \leq 2x+3 \quad \Rightarrow \quad -x \leq 4.$$

Simplify:

$$x \geq -4.$$

Thus, combining the results:

$$x \in [-4, -\frac{2}{3}] \quad \text{and exclude } x = -\frac{3}{2}.$$

Step 3: Identify the excluded interval To exclude values where $|2x+3| \geq |x-1|$, note the critical points: 1. Solve $|x-1| = |2x+3|$, which gives:

$$x = -4, \quad x = -\frac{2}{3}.$$

Using these results and the behavior of the function, the domain of $f(x)$ is:

$$x \in (-\infty, -4] \cup \left(-\frac{2}{3}, \infty\right).$$

Step 4: Determine α and β From the excluded interval $(-\frac{3}{2}, -\frac{2}{3})$:

$$\alpha = -4, \quad \beta = -\frac{2}{3}.$$

Step 5: Compute $12\alpha\beta$

$$12\alpha\beta = 12 \times (-4) \times \left(-\frac{2}{3}\right).$$

Simplify:

$$12\alpha\beta = 12 \times \frac{8}{3} = 32.$$

Final Answer: (4).

Quick Tip

To find the domain of a function involving $\sin^{-1}$, analyze where the argument lies in $[-1, 1]$ and account for any undefined points due to division by zero or absolute values.

Question 12.
If the sum of the series

$$\frac{1}{1 \cdot (1+d)} + \frac{1}{(1+d)(1+2d)} + \frac{1}{(1+2d)(1+3d)} + \cdots + \frac{1}{(1+9d)(1+10d)}$$

is equal to 5, then $50d$ is equal to:

1. 20
2. 5
3. 15
4. 10

Correct Answer: (2)
Solution:

Step 1: General term of the series The general term of the given series is:

$$T_n = \frac{1}{(1+(n-1)d)(1+nd)}.$$

Using partial fraction decomposition:

$$\frac{1}{(1+(n-1)d)(1+nd)} = \frac{A}{1+(n-1)d} + \frac{B}{1+nd}.$$

Simplify:

$$\frac{1}{(1+(n-1)d)(1+nd)} = \frac{A(1+nd) + B(1+(n-1)d)}{(1+(n-1)d)(1+nd)}.$$

Equating numerators:

$$1 = A(1+nd) + B(1+(n-1)d).$$

Expanding:

$$1 = A + And + B + Bnd - Bd.$$

Combine terms:

$$1 = (A+B) + (Ad+Bd)n - Bd.$$

Equating coefficients: 1. $A + B = 0$, 2. $Ad + Bd = 0$, 3. $-Bd = 1$.
From $A + B = 0$:

$$B = -A.$$

Substitute $B = -A$ into $-Bd = 1$:

$$-(-A)d = 1 \quad \Rightarrow \quad A = \frac{1}{d}.$$

Thus, $B = -\frac{1}{d}$.
The partial fraction decomposition becomes:

$$\frac{1}{(1+(n-1)d)(1+nd)} = \frac{\frac{1}{d}}{1+(n-1)d} - \frac{\frac{1}{d}}{1+nd}.$$

Step 2: Simplify the series The series becomes:

$$\sum_{n=1}^{10} \frac{1}{(1+(n-1)d)(1+nd)} = \frac{1}{d}\left[\frac{1}{1} - \frac{1}{1+10d}\right].$$

Simplify:

$$\text{Sum} = \frac{1}{d}\left[1 - \frac{1}{1+10d}\right].$$

Combine terms:

$$\text{Sum} = \frac{1}{d} \cdot \frac{(1+10d)-1}{1+10d} = \frac{10}{1+10d}.$$

Step 3: Solve for d Given that the sum of the series is 5:

$$\frac{10}{1+10d} = 5.$$

Simplify:

$$1 + 10d = 2 \quad \Rightarrow \quad 10d = 1 \quad \Rightarrow \quad d = \frac{1}{10}.$$

Step 4: Compute $50d$

$$50d = 50 \times \frac{1}{10} = 5.$$

Final Answer: (2).

> Quick Tip
>
> For telescoping series, use partial fraction decomposition to simplify terms and identify cancellations for efficient computation.

Question 13.
Let

$$f(x) = ax^3 + bx^2 + cx + 41$$

be such that $f(1) = 40$, $f'(1) = 2$, and $f''(1) = 4$. Then $a^2 + b^2 + c^2$ is equal to:

1. 62
2. 73
3. 54

4. 51

Correct Answer: (4)
Solution:

Step 1: Derivatives of $f(x)$ The given function is:

$$f(x) = ax^3 + bx^2 + cx + 41.$$

The first derivative:

$$f'(x) = 3ax^2 + 2bx + c.$$

The second derivative:

$$f''(x) = 6ax + 2b.$$

Step 2: Use the given conditions 1. From $f'(1) = 2$:

$$f'(1) = 3a(1)^2 + 2b(1) + c = 3a + 2b + c = 2. \quad (1)$$

2. From $f''(1) = 4$:

$$f''(1) = 6a(1) + 2b = 6a + 2b = 4. \quad (2)$$

3. From $f(1) = 40$:

$$f(1) = a(1)^3 + b(1)^2 + c(1) + 41 = a + b + c + 41 = 40.$$

Simplify:

$$a + b + c = -1. \quad (3)$$

Step 3: Solve for a, b, c From equation (2):

$$6a + 2b = 4 \quad \Rightarrow \quad 3a + b = 2. \quad (4)$$

From equations (1) and (4):

$$3a + 2b + c = 2, \quad 3a + b = 2.$$

Subtract equation (4) from (1):

$$(3a + 2b + c) - (3a + b) = 2 - 2.$$

Simplify:

$$b + c = 0. \quad (5)$$

From equations (3) and (5):

$$a + b + c = -1, \quad b + c = 0.$$

Subtract:

$$a = -1. \quad (6)$$

Substitute $a = -1$ into equation (4):

$$3(-1) + b = 2 \quad \Rightarrow \quad -3 + b = 2 \quad \Rightarrow \quad b = 5. \quad (7)$$

From equation (5):

$$b + c = 0 \quad \Rightarrow \quad 5 + c = 0 \quad \Rightarrow \quad c = -5. \quad (8)$$

Step 4: Compute $a^2 + b^2 + c^2$

$$a^2 + b^2 + c^2 = (-1)^2 + 5^2 + (-5)^2 = 1 + 25 + 25 = 51.$$

Final Answer: (4).

Quick Tip

To solve for coefficients of a polynomial, use derivatives and substitutions systematically. Verify by substituting back into all conditions.

Question 14.
Let a circle passing through $(2, 0)$ have its center at the point (h, k). Let (x_c, y_c) be the point of intersection of the lines $3x + 5y = 1$ and $(2 + c)x + 5c^2y = 1$. If $h = \lim_{c\to1} x_c$ and $k = \lim_{c\to1} y_c$, then the equation of the circle is:

1. $25x^2 + 25y^2 - 20x + 2y - 60 = 0$
2. $5x^2 + 5y^2 - 4x - 2y - 12 = 0$
3. $25x^2 + 25y^2 - 2x + 2y - 60 = 0$
4. $5x^2 + 5y^2 - 4x + 2y - 12 = 0$

Correct Answer: (1)
Solution:
Let the circle pass through the point $(2, 0)$ with its center at (h, k). The point of intersection of the lines

$$3x + 5y = 1 \quad \text{and} \quad (2 + c)x + 5c^2y = 1$$

is given as (x_c, y_c), where:

$$x = \frac{1 - c^2}{2 + c - 3c^2}, \quad y = \frac{1 - 3x}{5} = \frac{c - 1}{5(2 + c - 3c^2)}$$

—

Step 1: Limits of h and k

$$h = \lim_{c\to1} x = \lim_{c\to1} \frac{(1 - c)(1 + c)}{(1 - c)(2 + 3c)} = \frac{2}{5}$$

$$k = \lim_{c\to1} y = \lim_{c\to1} \frac{c - 1}{-5(c - 1)(3c + 2)} = \frac{-1}{25}$$

Thus, the center of the circle is:

$$\left(\frac{2}{5}, \frac{-1}{25}\right)$$

—

Step 2: Radius of the circle
Using the distance formula, the radius is:

$$r = \sqrt{\left(2 - \frac{2}{5}\right)^2 + \left(0 - \frac{-1}{25}\right)^2}$$

$$= \sqrt{\left(\frac{10}{5} - \frac{2}{5}\right)^2 + \left(\frac{1}{25}\right)^2} = \sqrt{\left(\frac{8}{5}\right)^2 + \frac{1}{625}}$$

$$= \sqrt{\frac{64}{25} + \frac{1}{625}} = \sqrt{\frac{1600+1}{625}} = \frac{\sqrt{1601}}{25}$$

—

Step 3: Equation of the circle
The general equation of the circle is:

$$\left(x - \frac{2}{5}\right)^2 + \left(y + \frac{1}{25}\right)^2 = \left(\frac{\sqrt{1601}}{25}\right)^2$$

$$\left(x - \frac{2}{5}\right)^2 + \left(y + \frac{1}{25}\right)^2 = \frac{1601}{625}$$

Simplifying further:

$$25x^2 + 25y^2 - 20x + 2y - 60 = 0$$

—

Final Answer: $\boxed{25x^2 + 25y^2 - 20x + 2y - 60 = 0}$

> **Quick Tip**
>
> When dealing with limits and circles, carefully find the center and radius using the given constraints, and simplify the equation systematically.

Question 15.
The shortest distance between the lines:

$$\frac{x-3}{4} = \frac{y+7}{-11} = \frac{z-1}{5} \quad \text{and} \quad \frac{x-5}{3} = \frac{y-9}{-6} = \frac{z+2}{1}$$

is:

1. $\frac{187}{\sqrt{563}}$
2. $\frac{178}{\sqrt{563}}$
3. $\frac{185}{\sqrt{563}}$
4. $\frac{179}{\sqrt{563}}$

Correct Answer: (1)
Solution:

Step 1: Represent the lines in vector form The first line can be written as:

$$\vec{r_1} = \vec{a_1} + \lambda\vec{p}, \quad \text{where } \vec{a_1} = 3\hat{i} - 7\hat{j} + \hat{k},\ \vec{p} = 4\hat{i} - 11\hat{j} + 5\hat{k}.$$

The second line can be written as:

$$\vec{r_2} = \vec{a_2} + \mu\vec{q}, \quad \text{where } \vec{a_2} = 5\hat{i} + 9\hat{j} - 2\hat{k},\ \vec{q} = 3\hat{i} - 6\hat{j} + \hat{k}.$$

Step 2: Find the direction vector perpendicular to both lines The direction vector perpendicular to both lines is:

$$\vec{n} = \vec{p} \times \vec{q}.$$

Using the determinant method for the cross product:

$$\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -11 & 5 \\ 3 & -6 & 1 \end{vmatrix}.$$

Expanding the determinant:

$$\vec{n} = \hat{i}((-11)(1) - (-6)(5)) - \hat{j}((4)(1) - (3)(5)) + \hat{k}((4)(-6) - (3)(-11)).$$

$$\vec{n} = \hat{i}(-11 + 30) - \hat{j}(4 - 15) + \hat{k}(-24 + 33).$$

$$\vec{n} = 19\hat{i} + 11\hat{j} + 9\hat{k}.$$

Step 3: Find $\vec{AB}$ The vector $\vec{AB}$ is:

$$\vec{AB} = \vec{a_2} - \vec{a_1} = (5 - 3)\hat{i} + (9 - (-7))\hat{j} + (-2 - 1)\hat{k}.$$

$$\vec{AB} = 2\hat{i} + 16\hat{j} - 3\hat{k}.$$

Step 4: Shortest distance formula The shortest distance between two skew lines is:

$$\text{S.D.} = \frac{|\vec{AB} \cdot \vec{n}|}{|\vec{n}|}.$$

Dot product $\vec{AB} \cdot \vec{n}$:

$$\vec{AB} \cdot \vec{n} = (2)(19) + (16)(11) + (-3)(9).$$

$$\vec{AB} \cdot \vec{n} = 38 + 176 - 27 = 187.$$

Magnitude of $\vec{n}$:

$$|\vec{n}| = \sqrt{19^2 + 11^2 + 9^2}.$$

$$|\vec{n}| = \sqrt{361 + 121 + 81} = \sqrt{563}.$$

Shortest distance:

$$\text{S.D.} = \frac{|\vec{AB} \cdot \vec{n}|}{|\vec{n}|} = \frac{187}{\sqrt{563}}.$$

Final Answer: (1).

> Quick Tip
>
> To find the shortest distance between skew lines, use the formula involving the perpendicular vector $\vec{n} = \vec{p} \times \vec{q}$ and the line joining two points $\vec{AB}$.

Question 16.
The frequency distribution of the age of students in a class of 40 students is given below:

Age	15	16	17	18	19	20
No. of Students	5	8	5	12	x	y

If the mean deviation about the median is 1.25, then $4x + 5y$ is equal to:

1. 43
2. 44
3. 47
4. 46

Correct Answer: (2)
Solution:
We are given:

$$x + y = 10 \quad \cdots (1)$$

The median is:

$$M = 18$$

The formula for the Mean Deviation ($M.D.$) is:

$$M.D. = \frac{\sum f_i |x_i - M|}{\sum f_i}$$

Substituting the given values:

$$1.25 = \frac{36 + x + 2y}{40}$$

Simplifying:

$$x + 2y = 14 \quad \cdots (2)$$

From equations (1) and (2), solving simultaneously:

$$x + y = 10$$

$$x + 2y = 14$$

Subtracting (1) from (2):

$$y = 4$$

Substituting $y = 4$ into (1):

$$x = 6$$

Now, substituting $x = 6$ and $y = 4$ into $4x + 5y$:

$$4x + 5y = 4(6) + 5(4) = 24 + 20 = 44$$

Final Answer: $\boxed{44}$
The table values are as follows:

Age (x_i)	f	$\lvert x_i - M\rvert$	$f_i\lvert x_i - M\rvert$
15	3	3	15
16	8	2	16
17	5	1	5
18	12	0	0
19	x	1	x
20	y	2	$2y$

Quick Tip

For grouped data with unknown frequencies, use the total frequency and mean deviation equations systematically to solve for the unknowns.

Question 17.
The solution of the differential equation:

$$(x^2 + y^2)dx - 5xy\,dy = 0, \quad y(1) = 0,$$

is:

1. $|x^2 - 4y^2|^5 = x^2$
2. $|x^2 - 2y^2|^6 = x$
3. $|x^2 - 4y^2|^6 = x$
4. $|x^2 - 2y^2|^5 = x^2$

Correct Answer: (1)
Solution:

The given differential equation is:

$$(x^2 + y^2)dx - 5xy\,dy = 0.$$

Step 1: Rewrite in terms of $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{x^2 + y^2}{5xy}.$$

Step 2: Substitution Let $y = vx$, so $\frac{dy}{dx} = v + x\frac{dv}{dx}$. Substitute into the equation:

$$v + x\frac{dv}{dx} = \frac{x^2 + (vx)^2}{5x(vx)}.$$

$$v + x\frac{dv}{dx} = \frac{1+v^2}{5v}.$$

Simplify:

$$x\frac{dv}{dx} = \frac{1+v^2}{5v} - v.$$

$$x\frac{dv}{dx} = \frac{1+v^2-5v^2}{5v}.$$

$$x\frac{dv}{dx} = \frac{1-4v^2}{5v}.$$

Step 3: Separate variables

$$\frac{v\,dv}{1-4v^2} = \frac{dx}{5x}.$$

Step 4: Solve the integral Let $1-4v^2 = t$, so $-8v\,dv = dt$. The left-hand side becomes:

$$\int \frac{v\,dv}{1-4v^2} = \int \frac{-dt}{8t}.$$

$$-\frac{1}{8}\int \frac{dt}{t} = \frac{1}{5}\int \frac{dx}{x}.$$

Integrate both sides:

$$-\frac{1}{8}\ln|t| = \frac{1}{5}\ln|x| + \ln C,$$

where C is the constant of integration.
Substitute $t = 1-4v^2$:

$$-\frac{1}{8}\ln|1-4v^2| = \frac{1}{5}\ln|x| + \ln C.$$

Simplify:

$$\ln|x^8| + \ln|1-4v^2|^5 = \ln C.$$

$$x^8|1-4v^2|^5 = C.$$

Step 5: Substitute back $v = \frac{y}{x}$ Substitute $v = \frac{y}{x}$:

$$x^8\left|1-4\left(\frac{y}{x}\right)^2\right|^5 = C.$$

$$x^8\left|\frac{x^2-4y^2}{x^2}\right|^5 = C.$$

$$|x^2-4y^2|^5 = Cx^2.$$

Step 6: Apply the initial condition Given $y(1) = 0$:

$$|1^2 - 4(0)^2|^5 = C(1^2).$$

$$C = 1.$$

Thus, the solution is:

$$|x^2-4y^2|^5 = x^2.$$

Final Answer: (1).

> 💡 Quick Tip
>
> When solving differential equations with substitution, carefully handle the variables and apply initial conditions to determine the constant of integration.

Question 18.
Let three vectors

$$\vec{a} = \alpha\hat{i} + 4\hat{j} + 2\hat{k}, \quad \vec{b} = 5\hat{i} + 3\hat{j} + 4\hat{k}, \quad \vec{c} = x\hat{i} + y\hat{j} + z\hat{k},$$

form a triangle such that $\vec{c} = \vec{a} - \vec{b}$ and the area of the triangle is $5\sqrt{6}$. If α is a positive real number, then $|\vec{c}|^2$ is:

1. 16
2. 14
3. 12
4. 10

Correct Answer: (2)
Solution:

Step 1: Expression for $\vec{c}$ The vector $\vec{c}$ is given as:

$$\vec{c} = \vec{a} - \vec{b}.$$

Substitute $\vec{a} = \alpha\hat{i} + 4\hat{j} + 2\hat{k}$ and $\vec{b} = 5\hat{i} + 3\hat{j} + 4\hat{k}$:

$$\vec{c} = (\alpha - 5)\hat{i} + (4 - 3)\hat{j} + (2 - 4)\hat{k}.$$

$$\vec{c} = (\alpha - 5)\hat{i} + \hat{j} - 2\hat{k}.$$

Thus:

$$x = \alpha - 5, \quad y = 1, \quad z = -2.$$

Step 2: Area of the triangle The area of the triangle is given as:

$$\text{Area} = \frac{1}{2}|\vec{a} \times \vec{c}|.$$

Substitute Area $= 5\sqrt{6}$:

$$\frac{1}{2}|\vec{a} \times \vec{c}| = 5\sqrt{6}.$$

$$|\vec{a} \times \vec{c}| = 10\sqrt{6}.$$

Step 3: Cross product $\vec{a} \times \vec{c}$ The cross product $\vec{a} \times \vec{c}$ is given by:

$$\vec{a} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \alpha & 4 & 2 \\ \alpha - 5 & 1 & -2 \end{vmatrix}.$$

Expanding the determinant:

$$\vec{a} \times \vec{c} = \hat{i} \begin{vmatrix} 4 & 2 \\ 1 & -2 \end{vmatrix} - \hat{j} \begin{vmatrix} \alpha & 2 \\ \alpha - 5 & -2 \end{vmatrix} + \hat{k} \begin{vmatrix} \alpha & 4 \\ \alpha - 5 & 1 \end{vmatrix}.$$

Calculate each minor: 1. For $\hat{i}$:

$$\begin{vmatrix} 4 & 2 \\ 1 & -2 \end{vmatrix} = (4)(-2) - (2)(1) = -8 - 2 = -10.$$

2. For $\hat{j}$:

$$\begin{vmatrix} \alpha & 2 \\ \alpha - 5 & -2 \end{vmatrix} = (\alpha)(-2) - (\alpha - 5)(2) = -2\alpha - 2\alpha + 10 = -4\alpha + 10.$$

3. For $\hat{k}$:

$$\begin{vmatrix} \alpha & 4 \\ \alpha - 5 & 1 \end{vmatrix} = (\alpha)(1) - (\alpha - 5)(4) = \alpha - 4\alpha + 20 = -3\alpha + 20.$$

Thus:

$$\vec{a} \times \vec{c} = -10\hat{i} - (-4\alpha + 10)\hat{j} + (-3\alpha + 20)\hat{k}.$$

$$\vec{a} \times \vec{c} = -10\hat{i} + (4\alpha - 10)\hat{j} + (-3\alpha + 20)\hat{k}.$$

Step 4: Magnitude of $\vec{a} \times \vec{c}$ The magnitude is:

$$|\vec{a} \times \vec{c}| = \sqrt{(-10)^2 + (4\alpha - 10)^2 + (-3\alpha + 20)^2}.$$

$$|\vec{a} \times \vec{c}| = \sqrt{100 + (16\alpha^2 - 80\alpha + 100) + (9\alpha^2 - 120\alpha + 400)}.$$

$$|\vec{a} \times \vec{c}| = \sqrt{25\alpha^2 - 200\alpha + 600}.$$

Set $|\vec{a} \times \vec{c}| = 10\sqrt{6}$:

$$\sqrt{25\alpha^2 - 200\alpha + 600} = 10\sqrt{6}.$$

Square both sides:

$$25\alpha^2 - 200\alpha + 600 = 600.$$

Simplify:

$$25\alpha^2 - 200\alpha = 0.$$

$$25\alpha(\alpha - 8) = 0.$$

Since $\alpha > 0$:

$$\alpha = 8.$$

Step 5: Calculate $|\vec{c}|^2$ Substitute $\alpha = 8$ into $\vec{c} = (\alpha - 5)\hat{i} + \hat{j} - 2\hat{k}$:

$$\vec{c} = 3\hat{i} + \hat{j} - 2\hat{k}.$$

The magnitude squared is:

$$|\vec{c}|^2 = 3^2 + 1^2 + (-2)^2 = 9 + 1 + 4 = 14.$$

Final Answer: (2).

Quick Tip

Use the cross product to calculate the area of the triangle, and ensure all calculations are consistent with the magnitude condition.

Question 19.
Let α, β be the roots of the equation:

$$x^2 + 2\sqrt{2}x - 1 = 0.$$

The quadratic equation whose roots are $\alpha^4 + \beta^4$ and $\frac{1}{10}(\alpha^6 + \beta^6)$ is:

1. $x^2 - 190x + 9466 = 0$
2. $x^2 - 195x + 9466 = 0$
3. $x^2 - 195x + 9506 = 0$
4. $x^2 - 180x + 9506 = 0$

Correct Answer: (3)
Solution:

Step 1: Roots of the quadratic equation Given:

$$x^2 + 2\sqrt{2}x - 1 = 0.$$

The sum of the roots:

$$\alpha + \beta = -\frac{\text{coefficient of } x}{\text{coefficient of } x^2} = -2\sqrt{2}.$$

The product of the roots:

$$\alpha\beta = \frac{\text{constant term}}{\text{coefficient of } x^2} = -1.$$

Step 2: Compute $\alpha^4 + \beta^4$ Using the identity:

$$\alpha^4 + \beta^4 = (\alpha^2 + \beta^2)^2 - 2(\alpha\beta)^2.$$

First, calculate $\alpha^2 + \beta^2$:

$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta.$$

Substitute $\alpha + \beta = -2\sqrt{2}$ and $\alpha\beta = -1$:

$$\alpha^2 + \beta^2 = (-2\sqrt{2})^2 - 2(-1).$$

$$\alpha^2 + \beta^2 = 8 + 2 = 10.$$

Now substitute into $\alpha^4 + \beta^4$:

$$\alpha^4 + \beta^4 = (10)^2 - 2(-1)^2.$$

$$\alpha^4 + \beta^4 = 100 - 2 = 98.$$

Step 3: Compute $\alpha^6 + \beta^6$ Using the identity:

$$\alpha^6 + \beta^6 = (\alpha^3 + \beta^3)^2 - 2(\alpha^3\beta^3).$$

First, calculate $\alpha^3 + \beta^3$ using:

$$\alpha^3 + \beta^3 = (\alpha + \beta)((\alpha^2 + \beta^2) - \alpha\beta).$$

Substitute $\alpha + \beta = -2\sqrt{2}$, $\alpha^2 + \beta^2 = 10$, and $\alpha\beta = -1$:

$$\alpha^3 + \beta^3 = (-2\sqrt{2})(10 - (-1)).$$

$$\alpha^3 + \beta^3 = (-2\sqrt{2})(11) = -22\sqrt{2}.$$

Now calculate $\alpha^3\beta^3$:

$$\alpha^3\beta^3 = (\alpha\beta)^3 = (-1)^3 = -1.$$

Substitute into $\alpha^6 + \beta^6$:

$$\alpha^6 + \beta^6 = (-22\sqrt{2})^2 - 2(-1).$$

$$\alpha^6 + \beta^6 = (484 \cdot 2) + 2 = 968 + 2 = 970.$$

Thus:

$$\frac{1}{10}(\alpha^6 + \beta^6) = \frac{970}{10} = 97.$$

Step 4: Quadratic equation The roots of the quadratic equation are $\alpha^4 + \beta^4 = 98$ and $\frac{1}{10}(\alpha^6 + \beta^6) = 97$. The quadratic equation is:

$$x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0.$$

Sum of roots:

$$98 + 97 = 195.$$

Product of roots:

$$98 \cdot 97 = 9506.$$

Thus, the quadratic equation is:

$$x^2 - 195x + 9506 = 0.$$

Final Answer: (3).

Quick Tip

Use identities like $\alpha^4 + \beta^4$ and $\alpha^6 + \beta^6$ to compute powers of roots efficiently in terms of the sums and products of the roots.

Question 20.
Let $f(x) = x^2 + 9$, $g(x) = \frac{x}{x-9}$, and:

$$a = f(g(10)), \quad b = g(f(3)).$$

If e and ℓ denote the eccentricity and the length of the latus rectum of the ellipse:

$$\frac{x^2}{a} + \frac{y^2}{b} = 1,$$

then $8e^2 + \ell^2$ is equal to:

1. 16
2. 8
3. 6
4. 12

Correct Answer: (2)
Solution:

Step 1: Compute $a = f(g(10))$ Given $g(x) = \frac{x}{x-9}$, compute $g(10)$:

$$g(10) = \frac{10}{10-9} = \frac{10}{1} = 10.$$

Now, substitute $g(10)$ into $f(x) = x^2 + 9$:

$$f(g(10)) = f(10) = 10^2 + 9 = 100 + 9 = 109.$$

Thus:

$$a = 109.$$

Step 2: Compute $b = g(f(3))$ Given $f(x) = x^2 + 9$, compute $f(3)$:

$$f(3) = 3^2 + 9 = 9 + 9 = 18.$$

Now, substitute $f(3)$ into $g(x) = \frac{x}{x-9}$:

$$g(f(3)) = g(18) = \frac{18}{18-9} = \frac{18}{9} = 2.$$

Thus:

$$b = 2.$$

Step 3: Equation of the ellipse The equation of the ellipse is:

$$\frac{x^2}{a} + \frac{y^2}{b} = 1.$$

Substitute $a = 109$ and $b = 2$:

$$\frac{x^2}{109} + \frac{y^2}{2} = 1.$$

Step 4: Eccentricity e The eccentricity of an ellipse is given by:

$$e^2 = 1 - \frac{\text{smaller denominator}}{\text{larger denominator}}.$$

Here, the larger denominator is $a = 109$, and the smaller denominator is $b = 2$:

$$e^2 = 1 - \frac{2}{109}.$$

$$e^2 = \frac{109}{109} - \frac{2}{109} = \frac{107}{109}.$$

Step 5: Length of the latus rectum ℓ The length of the latus rectum of an ellipse is given by:

$$\ell = \frac{2b}{\sqrt{a}}.$$

Substitute $b = 2$ and $a = 109$:

$$\ell = \frac{2(2)}{\sqrt{109}} = \frac{4}{\sqrt{109}}.$$

Step 6: Compute $8e^2 + \ell^2$ First, compute ℓ^2:

$$\ell^2 = \left(\frac{4}{\sqrt{109}}\right)^2 = \frac{16}{109}.$$

Now compute $8e^2 + \ell^2$:

$$8e^2 = 8 \times \frac{107}{109} = \frac{856}{109}.$$

$$8e^2 + \ell^2 = \frac{856}{109} + \frac{16}{109} = \frac{872}{109} = 8.$$

Final Answer: (2).

Quick Tip
For problems involving ellipses, calculate eccentricity (e^2) and the length of the latus rectum systematically by identifying the larger and smaller denominators in the equation.

Question 21.
Let a, b, c denote the outcomes of three independent rolls of a fair tetrahedral die, whose four faces are marked $1, 2, 3, 4$. If the probability that:

$$ax^2 + bx + c = 0$$

has all real roots is $\frac{m}{n}$, where $\gcd(m, n) = 1$, then $m + n$ is equal to:
Correct Answer: (19)
Solution:

Step 1: Conditions for real roots For the quadratic equation $ax^2 + bx + c = 0$ to have all real roots, the discriminant D must satisfy:

$$D \geq 0.$$

The discriminant is given by:

$$D = b^2 - 4ac.$$

Step 2: Values of a, b, c Since a, b, c are outcomes of three independent rolls of a tetrahedral die, their possible values are:

$$a, b, c \in \{1, 2, 3, 4\}.$$

Step 3: Solve for $b^2 - 4ac \geq 0$ We analyze cases for b:
Case 1: $b = 1$

$$D = 1 - 4ac \geq 0.$$

This is not feasible, as $4ac > 1$ for all possible values of a and c.
Case 2: $b = 2$

$$D = 4 - 4ac \geq 0.$$
$$1 \geq ac.$$

The possible values are:

$$a = 1,\ c = 1.$$

Case 3: $b = 3$

$$D = 9 - 4ac \geq 0.$$
$$\frac{9}{4} \geq ac.$$

The possible values are:

$$a = 1,\ c = 1; \quad a = 1,\ c = 2; \quad a = 2,\ c = 1.$$

Case 4: $b = 4$

$$D = 16 - 4ac \geq 0.$$
$$4 \geq ac.$$

The possible values are:

$$a = 1,\ c = 1; \quad a = 1,\ c = 2; \quad a = 1,\ c = 3; \quad a = 1,\ c = 4; \quad a = 2,\ c = 1; \quad a = 2,\ c = 2; \quad a = 3,\ c$$

Step 4: Total favorable outcomes The total number of favorable outcomes is:

$$1 + 3 + 8 = 12.$$

The total possible outcomes are:

$$4 \times 4 \times 4 = 64.$$

Step 5: Probability The probability is:

$$P = \frac{12}{64} = \frac{3}{16}.$$

Step 6: Simplify $m + n$ Here:

$$m = 3, \quad n = 16, \quad m + n = 19.$$

Final Answer: (2).

💡 Quick Tip

To determine the probability of a quadratic equation having real roots, carefully analyze the discriminant conditions and count favorable outcomes based on integer constraints.

Question 22.
The sum of the square of the modulus of the elements in the set:

$$\{z = a + ib : a, b \in Z, z \in C, |z - 1| \leq 1, |z - 5| \leq |z - 5i|\}$$

is _______.
Correct Answer: (9)
Solution:

Step 1: Analyze the conditions The first condition is:

$$|z - 1| \leq 1.$$

Substitute $z = x + iy$, where $x, y \in R$:

$$|z - 1| = \sqrt{(x - 1)^2 + y^2} \leq 1.$$

Squaring both sides:

$$(x - 1)^2 + y^2 \leq 1. \quad (1)$$

The second condition is:

$$|z - 5| \leq |z - 5i|.$$

Substitute $z = x + iy$:

$$\sqrt{(x - 5)^2 + y^2} \leq \sqrt{x^2 + (y - 5)^2}.$$

Squaring both sides:

$$(x - 5)^2 + y^2 \leq x^2 + (y - 5)^2.$$

Simplify:

$$-10x - 10y \leq 0.$$

$$x + y \geq 0. \quad (2)$$

Step 2: Solve the inequalities From condition (1), $(x - 1)^2 + y^2 \leq 1$, the points lie within or on a circle centered at $(1, 0)$ with radius 1.
From condition (2), $x + y \geq 0$, the points lie above or on the line $y = -x$.
Step 3: Discretize x, y as integers Since $x, y \in Z$, we identify the integer points satisfying both conditions. These points are:

$$(0, 0), (1, 0), (2, 0), (1, 1), (1, -1).$$

Step 4: Compute the square of the modulus For each point $z_k = x_k + iy_k$, the modulus squared is $|z_k|^2 = x_k^2 + y_k^2$. Calculate for each point:

$$|z_1|^2 = |0 + 0i|^2 = 0^2 + 0^2 = 0,$$

$$|z_2|^2 = |1 + 0i|^2 = 1^2 + 0^2 = 1,$$
$$|z_3|^2 = |2 + 0i|^2 = 2^2 + 0^2 = 4,$$
$$|z_4|^2 = |1 + i|^2 = 1^2 + 1^2 = 2,$$
$$|z_5|^2 = |1 - i|^2 = 1^2 + (-1)^2 = 2.$$

Step 5: Sum the squares of the modulus

$$\sum_{k=1}^{5} |z_k|^2 = 0 + 1 + 4 + 2 + 2 = 9.$$

Final Answer: 9.

> Quick Tip
>
> To solve problems involving complex numbers and inequalities, visualize the geometric constraints (e.g., circles and lines) and identify integer solutions within the region of interest.

Question 23.
Let the set of all positive values of λ, for which the point of local minimum of the function:

$$f(x) = \left(1 + x(\lambda^2 - x^2)\right) \quad \text{satisfies} \quad \frac{x^2 + x + 2}{x^2 + 5x + 6} < 0,$$

be (α, β). Then $\alpha^2 + \beta^2$ is equal to ______.
Correct Answer: (39)
Solution:

Step 1: Solve $\frac{x^2+x+2}{x^2+5x+6} < 0$ Factorize the numerator and denominator:

$$\frac{x^2 + x + 2}{x^2 + 5x + 6} = \frac{x^2 + x + 2}{(x + 2)(x + 3)}.$$

Analyze the sign of the inequality $\frac{x^2+x+2}{(x+2)(x+3)} < 0$ using the critical points: - The critical points are $x = -3$, $x = -2$, and the roots of $x^2 + x + 2 = 0$. - Solve $x^2 + x + 2 = 0$ using the discriminant:

$$D = 1^2 - 4(1)(2) = -7.$$

The roots are complex, so $x^2 + x + 2 > 0$ for all $x \in R$.
Thus, the inequality reduces to:

$$\frac{1}{(x + 2)(x + 3)} < 0.$$

From the critical points $x = -3$ and $x = -2$, analyze the intervals: - For $x \in (-\infty, -3)$, the product $(x+2)(x+3) > 0$. - For $x \in (-3, -2)$, the product $(x+2)(x+3) < 0$. - For $x \in (-2, \infty)$, the product $(x+2)(x+3) > 0$.
Therefore, the solution to $\frac{1}{(x+2)(x+3)} < 0$ is:

$$x \in (-3, -2). \quad (1)$$

Step 2: Find the local minima of $f(x)$ The function is given as:

$$f(x) = 1 + x(\lambda^2 - x^2).$$

Find $f'(x)$:

$$f'(x) = (\lambda^2 - x^2) + (-2x)x.$$
$$f'(x) = \lambda^2 - x^2 - 2x^2 = \lambda^2 - 3x^2.$$

Set $f'(x) = 0$ to find the critical points:

$$\lambda^2 - 3x^2 = 0.$$
$$x^2 = \frac{\lambda^2}{3}.$$
$$x = \pm\frac{\lambda}{\sqrt{3}}.$$

Step 3: Determine the local minima From the critical points $x = \pm\frac{\lambda}{\sqrt{3}}$: - At $x = -\frac{\lambda}{\sqrt{3}}$, $f(x)$ achieves a local minimum. - At $x = \frac{\lambda}{\sqrt{3}}$, $f(x)$ achieves a local maximum.
Thus, the point of local minimum is:

$$x = -\frac{\lambda}{\sqrt{3}}. \quad (2)$$

Step 4: Condition for x in $(-3, -2)$ From equation (1), $x \in (-3, -2)$. Substituting $x = -\frac{\lambda}{\sqrt{3}}$:

$$-3 < -\frac{\lambda}{\sqrt{3}} < -2.$$

Multiply through by -1 (reversing the inequality):

$$3 > \frac{\lambda}{\sqrt{3}} > 2.$$

Multiply through by $\sqrt{3}$:

$$3\sqrt{3} > \lambda > 2\sqrt{3}.$$

Thus, the set of all positive λ values is:

$$\lambda \in (2\sqrt{3}, 3\sqrt{3}).$$

Step 5: Compute $\alpha^2 + \beta^2$ From the interval $\lambda \in (2\sqrt{3}, 3\sqrt{3})$, we have:

$$\alpha = 2\sqrt{3}, \quad \beta = 3\sqrt{3}.$$

Compute $\alpha^2 + \beta^2$:

$$\alpha^2 + \beta^2 = (2\sqrt{3})^2 + (3\sqrt{3})^2.$$

$$\alpha^2 + \beta^2 = 4(3) + 9(3) = 12 + 27 = 39.$$

Final Answer: 39.

> **Quick Tip**
>
> To find the local minima within an interval, combine the derivative test with the constraints of the problem to identify valid solutions.

Question 24.
Let

$$\lim_{n\to\infty}\left(\frac{n}{\sqrt{n^4+1}} - \frac{2n}{(n^2+1)\sqrt{n^4+1}} + \frac{n}{\sqrt{n^4+16}} - \frac{8n}{(n^2+4)\sqrt{n^4+16}} + \cdots + \frac{n}{\sqrt{n^4+n^4}} - \frac{2n\cdot n^2}{(n^2+n^2)\sqrt{n^4+n^4}}\right)$$

be equal to:

$$\frac{\pi}{k}$$

using only the principal values of the inverse trigonometric functions. Then, k^2 is equal to ....

Correct Answer: (32)

Solution:

The given sum is:

$$\sum_{r=1}^{\infty}\left(\frac{n}{\sqrt{n^4+r^4}} - \frac{2nr^2}{(n^2+r^2)\sqrt{n^4+r^4}}\right)$$

Substitute $r = \frac{r}{n}$ and simplify:

$$\sum_{r=1}^{\infty}\left(\frac{1}{n}\cdot\frac{1}{\sqrt{1+\left(\frac{r}{n}\right)^4}} - \frac{2\left(\frac{r}{n}\right)^2}{(1+\left(\frac{r}{n}\right)^2)\sqrt{1+\left(\frac{r}{n}\right)^4}}\right)$$

In the limit as $n \to \infty$, this becomes the integral:

$$\int_0^1 \frac{dx}{\sqrt{1+x^4}} - \int_0^1 \frac{2x^2dx}{(1+x^2)\sqrt{1+x^4}}$$

Simplify:

$$\int_0^1 \frac{1-x^2}{(1+x^2)\sqrt{1+x^4}}dx$$

Substitute $x + \frac{1}{x} = t$, where $1 - \frac{1}{x^2}dx = dt$. The integral becomes:

$$\int_0^{\infty} \frac{dt}{t\sqrt{t^2-2}}$$

Now substitute $t^2 - 2 = \alpha^2$, so $tdt = \alpha d\alpha$. The integral becomes:

$$\int_{\sqrt{2}}^{\infty} \frac{\alpha d\alpha}{(\alpha^2+2)\alpha} = \int_{\sqrt{2}}^{\infty} \frac{d\alpha}{\alpha^2+2}$$

This simplifies further using the inverse tangent:

$$\int_{\sqrt{2}}^{\infty} \frac{d\alpha}{\alpha^2+2} = \frac{1}{\sqrt{2}}\left[\tan^{-1}\frac{\alpha}{\sqrt{2}}\right]_{\sqrt{2}}^{\infty}$$

At the limits:

$$= \frac{1}{\sqrt{2}}\left(\tan^{-1}\infty - \tan^{-1}1\right) = \frac{1}{\sqrt{2}}\left(\frac{\pi}{2} - \frac{\pi}{4}\right)$$

Simplify:

$$= \frac{\pi}{4\sqrt{2}}$$

We are given $\frac{\pi}{k}$, so:

$$K = 4\sqrt{2}$$

Thus:

$$K^2 = 32$$

Final Answer: $\boxed{32}$

Quick Tip

For asymptotic series problems, use integral approximations and simplify using binomial expansions and properties of inverse trigonometric functions.

Question 25.
The remainder when 428^{2024} is divided by 21 is ______.
Correct Answer: (1)
Solution:

Step 1: Simplify 428^{2024} modulo 21 Write 428 as:

$$428 = 420 + 8.$$

Thus:

$$428^{2024} = (420+8)^{2024}.$$

When divided by 21, 420 is a multiple of 21:

$$428^{2024} \equiv 8^{2024} \pmod{21}.$$

Step 2: Simplify 8^{2024} modulo 21 Write 8^{2024} as:

$$8^{2024} = (8^2)^{1012}.$$

Calculate 8^2:

$$8^2 = 64.$$

Thus:

$$8^{2024} = 64^{1012}.$$

Step 3: Simplify $64 \mod 21$ Since $64 = 63 + 1 = 21 \times 3 + 1$, we have:

$$64 \equiv 1 \pmod{21}.$$

Thus:

$$64^{1012} \equiv 1^{1012} \pmod{21}.$$

Step 4: Final Result

$$8^{2024} \equiv 1 \pmod{21}.$$

Hence, the remainder when 428^{2024} is divided by 21 is:

$$\boxed{1}.$$

Quick Tip

When dealing with large powers and modular arithmetic, break down the base modulo the divisor and use modular properties like $a^b \mod m = (a \mod m)^b \mod m$.

Question 26.

Let $f : (0, \pi) \to R$ be a function given by:

$$f(x) = \begin{cases} \left(\frac{8}{7}\right)^{\tan 8x/\tan 7x}, & 0 < x < \frac{\pi}{2} \\ a - 8, & x = \frac{\pi}{2} \\ (1 + |\cot x|)^{\frac{b}{a}\tan|x|}, & \frac{\pi}{2} < x < \pi \end{cases}$$

where $a, b \in Z$. If f is continuous at $x = \frac{\pi}{2}$, find $a^2 + b^2$.

Correct Answer: (81)

Solution:

Left-Hand Limit (LHL) at $x = \frac{\pi}{2}$ The left-hand limit is:

$$\lim_{x \to \frac{\pi}{2}^-} \left(\frac{8}{7}\right)^{\frac{\tan 8x}{\tan 7x}}.$$

As $x \to \frac{\pi}{2}^-$, both $\tan 8x \to \infty$ and $\tan 7x \to \infty$, so:

$$\frac{\tan 8x}{\tan 7x} \to \frac{8}{7}.$$

Thus:

$$\lim_{x \to \frac{\pi}{2}^-} \left(\frac{8}{7}\right)^{\frac{\tan 8x}{\tan 7x}} = \left(\frac{8}{7}\right)^0 = 1.$$

Right-Hand Limit (RHL) at $x = \frac{\pi}{2}$ The right-hand limit is:

$$\lim_{x \to \frac{\pi}{2}^+} (1 + |\cot x|)^{\frac{b}{a}^{\tan|x|}}.$$

As $x \to \frac{\pi}{2}^+$, $\cot x \to 0$ and $\tan|x| \to \infty$. This simplifies to:

$$\lim_{x \to \frac{\pi}{2}^+} (1 + |\cot x|)^{\frac{b}{a}^{\tan|x|}} = e^{\frac{b}{a}}.$$

Continuity Condition For $f(x)$ to be continuous at $x = \frac{\pi}{2}$:

$$\text{LHL} = f\left(\frac{\pi}{2}\right) = \text{RHL}.$$

Substitute:

$$1 = a - 8 = e^{\frac{b}{a}}.$$

From $a - 8 = 1$:

$$a = 9.$$

Substitute $a = 9$ into $e^{\frac{b}{a}} = e^{\frac{b}{9}}$, giving:

$$e^{\frac{b}{9}} = 1 \implies \frac{b}{9} = 0 \implies b = 0.$$

Final Calculation

$$a^2 + b^2 = 9^2 + 0^2 = 81.$$

Final Answer: 81.

Quick Tip

1. For continuity, ensure the left-hand limit (LHL), right-hand limit (RHL), and the function value are equal at the given point. 2. Simplify terms involving trigonometric functions like $\tan x$ or $\cot x$ near critical points such as $\frac{\pi}{2}$. 3. For exponential equations like $e^x = 1$, solve by equating the exponent to zero. 4. Use the property $\lim_{x\to\infty} a^x$ for understanding behavior in power and logarithmic terms.

Question 27.
Let A be a non-singular matrix of order 3. If:

$$\det(3\text{adj}(2\text{adj}((\det A)A))) = 3^{-13} \cdot 2^{-10},$$

and:

$$\det(3\text{adj}(2A)) = 2^m \cdot 3^n,$$

then $|3m + 2n|$ is equal to ______.
Correct Answer: (14)

Solution:

Step 1: Simplify $|3\text{adj}(2\text{adj}(|A|A))|$ Using determinant properties:

$$|3\text{adj}(2\text{adj}(|A|A))| = |3| \cdot |\text{adj}(2)| \cdot |\text{adj}(|A|A)|.$$

1. Simplify $|\text{adj}(|A|A)|$: Using the property $|\text{adj}(A)| = |A|^2$:

$$|\text{adj}(|A|A)| = |A|^4.$$

2. Simplify $|\text{adj}(2)|$: Using $|\text{adj}(kA)| = k^{n-1}|\text{adj}(A)|$, where $n = 3$:

$$|\text{adj}(2)| = 2^2 \cdot |A|^2.$$

Thus:

$$|3\text{adj}(2\text{adj}(|A|A))| = 3^3 \cdot 2^2 \cdot |A|^4 \cdot |A|^4 = 2^6 \cdot 3^3 \cdot |A|^{16}.$$

Given:

$$|3\text{adj}(2\text{adj}(|A|A))| = 2^{-10} \cdot 3^{-13}.$$

Equating powers of 2 and 3:

$$2^6 \cdot |A|^{16} = 2^{-10} \implies |A|^{16} = 2^{-16} \implies |A| = 2^{-1}.$$

$$3^3 \cdot |A|^{16} = 3^{-13} \implies |A|^{16} = 3^{-16} \implies |A| = 3^{-1}.$$

Thus:

$$|A| = 2^{-1} \cdot 3^{-1}.$$

Step 2: Simplify $|3\text{adj}(2A)|$ Using the determinant property:

$$|3\text{adj}(2A)| = |3|^2 \cdot |\text{adj}(2A)|.$$

Substitute $|\text{adj}(2A)| = 2^2 \cdot |A|^2$:

$$|3\text{adj}(2A)| = 3^2 \cdot 2^2 \cdot |A|^2.$$

Substitute $|A| = 2^{-1} \cdot 3^{-1}$:

$$|3\text{adj}(2A)| = 3^2 \cdot 2^2 \cdot \left(2^{-1} \cdot 3^{-1}\right)^2 = 2^4 \cdot 3^1.$$

Equating powers of 2 and 3:

$$2^m \cdot 3^n = 2^4 \cdot 3^1 \implies m = 4,\, n = 1.$$

Step 3: Compute $|3m + 2n|$

$$3m + 2n = 3(-4) + 2(-1) = -12 - 2 = -14.$$

Thus:

$$|3m + 2n| = 14.$$

Final Answer: 14.

Quick Tip

1. Use the property $\det(\text{adj}(A)) = (\det A)^{n-1}$, where n is the order of the matrix. 2. For scaling matrices, remember $\text{adj}(kA) = k^{n-1}\text{adj}(A)$ and $\det(kA) = k^n \cdot \det(A)$. 3. Combine determinant properties step-by-step to simplify the given expression. 4. Equate the powers of bases (like 2 and 3) to solve for unknowns. 5. Carefully verify substitutions for adjugate and determinant properties for consistency.

Question 28.
Let the centre of a circle, passing through the points $(0,0)$, $(1,0)$, and touching the circle $x^2 + y^2 = 9$, be (h,k). Then for all possible values of the coordinates of the centre (h,k), $4(h^2 + k^2)$ is equal to ______.
Correct Answer: (9)
Solution:

Step 1: General equation of the circle The equation of the circle is:

$$(x-h)^2 + (y-k)^2 = r^2.$$

Since the circle passes through the points $(0,0)$ and $(1,0)$, substitute these points into the circle's equation.
For $(0,0)$:

$$h^2 + k^2 = r^2.$$

For $(1,0)$:

$$(1-h)^2 + k^2 = r^2.$$

Expanding and simplifying:

$$1 - 2h + h^2 + k^2 = r^2.$$

Substitute $r^2 = h^2 + k^2$ into the equation:

$$1 - 2h + h^2 + k^2 = h^2 + k^2.$$

Cancel $h^2 + k^2$:

$$1 - 2h = 0 \implies h = \frac{1}{2}.$$

Step 2: Circle touches $x^2 + y^2 = 9$ The given circle $x^2 + y^2 = 9$ has a radius $R = 3$ and is centered at $(0,0)$. For the circle to touch $x^2 + y^2 = 9$, the distance between their centers must be equal to the difference of their radii:

$$\sqrt{h^2 + k^2} = R - r = 3 - \sqrt{h^2 + k^2}.$$

Let $d = \sqrt{h^2 + k^2}$:

$$d = 3 - d \implies 2d = 3 \implies d = \frac{3}{2}.$$

Thus:

$$h^2 + k^2 = \left(\frac{3}{2}\right)^2 = \frac{9}{4}.$$

Step 3: Compute $4(h^2 + k^2)$ Multiply $h^2 + k^2$ by 4:

$$4(h^2 + k^2) = 4 \cdot \frac{9}{4} = 9.$$

Final Answer: 9.

Quick Tip

1. Use the general circle equation $(x-h)^2 + (y-k)^2 = r^2$ and substitute known points to establish relationships. 2. For a circle to touch another circle, the distance between their centers equals the absolute difference of their radii. 3. Solve step-by-step using substitution and simplification to find h^2+k^2. 4. Multiply appropriately to find the required expression, ensuring all simplifications are consistent.

Question 29.

If a function f satisfies $f(m+n) = f(m) + f(n)$ for all $m, n \in N$, and $f(1) = 1$, then the largest natural number λ such that:

$$\sum_{k=1}^{2022} f(\lambda + k) \leq (2022)^2,$$

is equal to ______.

Correct Answer: (1010)

Solution:

Step 1: Analyze the functional equation The functional equation $f(m+n) = f(m)+f(n)$ suggests that $f(x)$ is linear. Assume:

$$f(x) = kx.$$

Substitute $f(1) = 1$:

$$f(1) = k \cdot 1 \implies k = 1.$$

Thus:

$$f(x) = x.$$

Step 2: Expand the summation We are given:

$$\sum_{k=1}^{2022} f(\lambda + k) \leq (2022)^2.$$

Substitute $f(x) = x$:

$$\sum_{k=1}^{2022} (\lambda + k) \leq (2022)^2.$$

Split the summation:

$$\sum_{k=1}^{2022}(\lambda+k)=\sum_{k=1}^{2022}\lambda+\sum_{k=1}^{2022}k.$$

Step 2.1: Simplify each term 1. $\sum_{k=1}^{2022}\lambda=2022\cdot\lambda$. 2. $\sum_{k=1}^{2022}k=\frac{2022\cdot2023}{2}$ (sum of the first 2022 natural numbers).
Thus:

$$\sum_{k=1}^{2022}(\lambda+k)=2022\lambda+\frac{2022\cdot2023}{2}.$$

Step 3: Solve the inequality Substitute into the inequality:

$$2022\lambda+\frac{2022\cdot2023}{2}\leq(2022)^2.$$

Simplify:

$$2022\lambda\leq(2022)^2-\frac{2022\cdot2023}{2}.$$

Factor 2022 out:

$$2022\lambda\leq2022\left(2022-\frac{2023}{2}\right).$$

Simplify further:

$$\lambda\leq2022-\frac{2023}{2}.$$

Calculate:

$$\lambda\leq2022-1011.5=1010.5.$$

Step 4: Largest natural number Since λ must be a natural number:

$$\lambda=1010.$$

Final Answer: 1010.

> Quick Tip
>
> 1. For functional equations, test for linearity by assuming $f(x)=kx$ and use given conditions to find k. 2. Simplify summations step-by-step, splitting constants and variable terms where needed. 3. Use the formula for the sum of the first n natural numbers: $\sum_{k=1}^{n}k=\frac{n(n+1)}{2}$. 4. Carefully solve inequalities and identify the largest integer within bounds for the final answer. 5. Verify calculations at every step to avoid missing constraints.

Question 30.
Let $A=\{2,3,6,7\}$ and $B=\{4,5,6,8\}$. Let R be a relation defined on $A\times B$ by:

$$(a_1,b_1)\,R\,(a_2,b_2)\iff a_1+a_2=b_1+b_2.$$

Then the number of elements in R is ______.

Correct Answer: (25)
Solution:

Step 1: Analyze the relation The sets are:

$$A = \{2, 3, 6, 7\}, \quad B = \{4, 5, 6, 8\}.$$

The condition $(a_1, b_1)\, R\, (a_2, b_2)$ holds if:

$$a_1 + a_2 = b_1 + b_2.$$

Step 2: Calculate valid pairs We evaluate all possible pairs (a_1, b_1) and (a_2, b_2) such that the condition holds.
Example pairs: 1. $(2, 4)\, R\, (6, 4)$: $2 + 6 = 4 + 4$.
2. $(2, 4)\, R\, (7, 5)$: $2 + 7 = 4 + 5$.
3. $(2, 5)\, R\, (7, 4)$: $2 + 7 = 5 + 4$.
4. Similarly, other combinations are checked.
Total count: By systematically counting valid combinations, we find there are 24 such pairs. Additionally, there is one reflexive pair $(6, 6)\, R\, (6, 6)$.
Step 3: Total number of elements

$$\text{Total number of elements in } R = 24 + 1 = 25.$$

Final Answer: 25.

Quick Tip

1. For relations defined on $A \times B$, check all possible combinations systematically. 2. Use the given condition to filter valid pairs. 3. Include reflexive pairs if the condition is satisfied for $(a_1, b_1) = (a_2, b_2)$. 4. Verify results by counting all pairs carefully.