MA : MATHEMATICS

Duration : Three Hours *Maximum Marks* :100

Read the following instructions carefully.

1. This question paper contains **16** printed pages including pages for rough work. Please check all pages and report discrepancy, if any.
2. Write your registration number, your name and name of the examination centre at the specified locations on the right half of the **O**ptical **R**esponse **S**heet (**ORS**).
3. Using HB pencil, darken the appropriate bubble under each digit of your registration number and the letters corresponding to your paper code.
4. All questions in this paper are of objective type.
5. Questions must be answered on **O**ptical **R**esponse **S**heet (**ORS**) by darkening the appropriate bubble (marked A, B, C, D) using HB pencil against the question number on the left hand side of the ORS. **Each question has only one correct answer**. In case you wish to change an answer, erase the old answer completely. More than one answer bubbled against a question will be treated as an incorrect response.
6. There are a total of 60 questions carrying 100 marks. Questions 1 through 20 are 1-mark questions, questions 21 through 60 are 2-mark questions.
7. Questions 51 through 56 (3 pairs) are common data questions and question pairs (57, 58) and (59, 60) are linked answer questions. The answer to the second question of the above 2 pairs depends on the answer to the first question of the pair. If the first question in the linked pair is wrongly answered or is un-attempted, then the answer to the second question in the pair will not be evaluated.
8. Un-attempted questions will carry zero marks.
9. Wrong answers will carry NEGATIVE marks. For Q.1 to Q.20, ⅓ mark will be deducted for each wrong answer. For Q. 21 to Q. 56, ⅔ mark will be deducted for each wrong answer. The question pairs (Q.57, Q.58), and (Q.59, Q.60) are questions with linked answers. There will be negative marks only for wrong answer to the first question of the linked answer question pair i.e. for Q.57 and Q.59, ⅔ mark will be deducted for each wrong answer. There is no negative marking for Q.58 and Q.60.
10. Calculator (without data connectivity) is allowed in the examination hall.
11. Charts, graph sheets or tables are NOT allowed in the examination hall.
12. Rough work can be done on the question paper itself. Additionally, blank pages are given at the end of the question paper for rough work.

Notations and Symbols used

$X \setminus Y$	:	$\{x \in X : x \notin Y\}$
$\mathbb{Z}$	:	The set of all integers
$\mathbb{Q}$	:	The set of all rational numbers
$\mathbb{R}$	:	The set of all real numbers
$\mathbb{C}$	:	The set of complex numbers
$\mathbb{R}^n$	:	$\{(x_1,\ldots, x_n) : x_i \in \mathbb{R}$ for $1 \le i \le n\}$
ℓ^1	:	The vector space of all sequences $\{x_n\}$ in $\mathbb{C}$ such that $\sum \lvert x_n \rvert < \infty$
c_{00}	:	The vector space of all sequences $\{x_n\}$ in $\mathbb{C}$ such that $x_n \neq 0$ for at most finitely many values of n
$\lVert \cdot \rVert_p$	:	The p-norm for $1 \le p < \infty$
A^T	:	The transpose of the matrix A
$U(a,b)$	:	Uniform distribution on the interval (a,b)
$f[x_0,\ldots, x_k]$	:	k th divided difference of f at $x_0,\ldots, x_k$
$\binom{n}{r}$	:	$\dfrac{n!}{r!\,(n-r)!}$
$E(X)$	:	Expectation of the random variable X

Q. 1 – Q. 20 carry one mark each.

Q.1 The dimension of the vector space $V = \{ A = (a_{ij})_{n\times n} : a_{ij} \in \mathbb{C}, a_{ij} = -a_{ji} \}$ over the field $\mathbb{R}$ is

(A) n^2 (B) $n^2 - 1$ (C) $n^2 - n$ (D) $\frac{n^2}{2}$

Q.2 The minimal polynomial associated with the matrix $\begin{bmatrix} 0 & 0 & 3 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}$ is

(A) $x^3 - x^2 - 2x - 3$ (B) $x^3 - x^2 + 2x - 3$
(C) $x^3 - x^2 - 3x - 3$ (D) $x^3 - x^2 + 3x - 3$

Q.3 For the function $f(z) = \sin\left(\frac{1}{\cos(1/z)}\right)$, the point $z = 0$ is

(A) a removable singularity (B) a pole
(C) an essential singularity (D) a non-isolated singularity

Q.4 Let $f(z) = \sum_{n=0}^{15} z^n$ for $z \in \mathbb{C}$. If $C : |z - i| = 2$ then $\oint_C \frac{f(z)\,dz}{(z-i)^{15}} =$

(A) $2\pi i(1+15i)$ (B) $2\pi i(1-15i)$ (C) $4\pi i(1+15i)$ (D) $2\pi i$

Q.5 For what values of α and β, the quadrature formula $\int_{-1}^{1} f(x)\,dx \approx \alpha f(-1) + f(\beta)$ is exact for all polynomials of degree ≤ 1?

(A) $\alpha = 1, \beta = 1$ (B) $\alpha = -1, \beta = 1$ (C) $\alpha = 1, \beta = -1$ (D) $\alpha = -1, \beta = -1$

Q.6 Let $f : [0,4] \to \mathbb{R}$ be a three times continuously differentiable function. Then the value of $f[1,2,3,4]$ is

(A) $\frac{f''(\xi)}{3}$ for some $\xi \in (0,4)$ (B) $\frac{f''(\xi)}{6}$ for some $\xi \in (0,4)$
(C) $\frac{f'''(\xi)}{3}$ for some $\xi \in (0,4)$ (D) $\frac{f'''(\xi)}{6}$ for some $\xi \in (0,4)$

Q.7 Which one of the following is TRUE ?

(A) Every linear programming problem has a feasible solution.
(B) If a linear programming problem has an optimal solution then it is unique.
(C) The union of two convex sets is necessarily convex.
(D) Extreme points of the disk $x^2+y^2 \le 1$ are the points on the circle $x^2+y^2=1$.

Q.8 The dual of the linear programming problem:

Minimize $\boldsymbol{c}^T \boldsymbol{x}$ subject to $A\boldsymbol{x} \ge \boldsymbol{b}$ and $\boldsymbol{x} \ge \boldsymbol{0}$ is

(A) Maximize $\boldsymbol{b}^T \boldsymbol{w}$ subject to $A^T \boldsymbol{w} \ge \boldsymbol{c}$ and $\boldsymbol{w} \ge \boldsymbol{0}$
(B) Maximize $\boldsymbol{b}^T \boldsymbol{w}$ subject to $A^T \boldsymbol{w} \le \boldsymbol{c}$ and $\boldsymbol{w} \ge \boldsymbol{0}$
(C) Maximize $\boldsymbol{b}^T \boldsymbol{w}$ subject to $A^T \boldsymbol{w} \le \boldsymbol{c}$ and $\boldsymbol{w}$ is unrestricted
(D) Maximize $\boldsymbol{b}^T \boldsymbol{w}$ subject to $A^T \boldsymbol{w} \ge \boldsymbol{c}$ and $\boldsymbol{w}$ is unrestricted

Q. 9 The resolvent kernel for the integral equation $u(x)=F(x)+\int_{\log 2}^{x} e^{(t-x)}\, u(t)\,dt$ is

(A) $\cos(x-t)$ (B) 1 (C) e^{t-x} (D) $e^{2(t-x)}$

Q.10 Consider the metrics $d_2(f,g)=\left(\int_a^b |f(t)-g(t)|^2 dt\right)^{1/2}$ and $d_\infty(f,g)=\sup_{t\in[a,b]} |f(t)-g(t)|$ on the space $X=C[a,b]$ of all real valued continuous functions on $[a,b]$. Then which of the following is TRUE ?

(A) Both (X,d_2) and (X,d_∞) are complete.
(B) (X,d_2) is complete but (X,d_∞) is NOT complete.
(C) (X,d_∞) is complete but (X,d_2) is NOT complete.
(D) Both (X,d_2) and (X,d_∞) are NOT complete.

Q.11 A function $f:\mathbb{R}\to\mathbb{R}$ need NOT be Lebesgue measurable if

(A) f is monotone
(B) $\{x\in\mathbb{R}: f(x)\ge\alpha\}$ is measurable for each $\alpha\in\mathbb{Q}$
(C) $\{x\in\mathbb{R}: f(x)=\alpha\}$ is measurable for each $\alpha\in\mathbb{R}$
(D) For each open set G in $\mathbb{R}$, $f^{-1}(G)$ is measurable

Q.12 Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal sequence in a Hilbert space H and let $x\ (\neq 0) \in H$. Then

(A) $\lim_{n\to\infty}\langle x, e_n\rangle$ does not exist

(B) $\lim_{n\to\infty}\langle x, e_n\rangle = \|x\|$

(C) $\lim_{n\to\infty}\langle x, e_n\rangle = 1$

(D) $\lim_{n\to\infty}\langle x, e_n\rangle = 0$

Q.13 The subspace $\mathbb{Q} \times [0, 1]$ of $\mathbb{R}^2$ (with the usual topology) is

(A) dense in $\mathbb{R}^2$
(B) connected
(C) separable
(D) compact

Q.14 $\mathbb{Z}_2[x]/\langle x^3 + x^2 + 1\rangle$ is

(A) a field having 8 elements
(B) a field having 9 elements
(C) an infinite field
(D) NOT a field

Q.15 The number of elements of a principal ideal domain can be

(A) 15 (B) 25 (C) 35 (D) 36

Q.16 Let F, G and H be pairwise independent events such that $P(F) = P(G) = P(H) = \frac{1}{3}$ and $(F \cap G \cap H) = \frac{1}{4}$. Then the probability that at least one event among F, G and H occurs is

(A) $\frac{11}{12}$ (B) $\frac{7}{12}$ (C) $\frac{5}{12}$ (D) $\frac{3}{4}$

Q.17 Let X be a random variable such that $E(X^2)=E(X)=1$. Then $E(X^{100})=$

(A) 0 (B) 1 (C) 2^{100} (D) $2^{100}+1$

Q.18 For which of the following distributions, the weak law of large numbers does NOT hold ?

(A) Normal (B) Gamma (C) Beta (D) Cauchy

Q.19 If $D\equiv\dfrac{d}{dx}$ then the value of $\dfrac{1}{(xD+1)}\left(x^{-1}\right)$ is

(A) $\log x$ (B) $\dfrac{\log x}{x}$ (C) $\dfrac{\log x}{x^2}$ (D) $\dfrac{\log x}{x^3}$

Q.20 The equation $(\alpha xy^3+y\cos x)\,dx+(x^2y^2+\beta\sin x)\,dy=0$ is exact for

(A) $\alpha=\dfrac{3}{2}, \beta=1$ (B) $\alpha=1, \beta=\dfrac{3}{2}$

(C) $\alpha=\dfrac{2}{3}, \beta=1$ (D) $\alpha=1, \beta=\dfrac{2}{3}$

Q. 21 to Q. 60 carry two marks each.

Q.21 If

$$A=\begin{pmatrix} 1 & 0 & 0 \\ i & \dfrac{-1+i\sqrt{3}}{2} & 0 \\ 0 & 1+2i & \dfrac{-1-i\sqrt{3}}{2} \end{pmatrix},$$

then the trace of A^{102} is

(A) 0 (B) 1 (C) 2 (D) 3

Q.22 Which of the following matrices is NOT diagonalizable ?

(A) $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ (B) $\begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix}$ (C) $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ (D) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

Q.23 Let V be the column space of the matrix $A=\begin{pmatrix}1 & -1\\ 1 & 2\\ 1 & -1\end{pmatrix}$. Then the orthogonal projection of $\begin{pmatrix}0\\1\\0\end{pmatrix}$ on V is

(A) $\begin{pmatrix}0\\1\\0\end{pmatrix}$ (B) $\begin{pmatrix}0\\0\\1\end{pmatrix}$ (C) $\begin{pmatrix}1\\1\\0\end{pmatrix}$ (D) $\begin{pmatrix}1\\0\\1\end{pmatrix}$

Q.24 Let $\sum_{n=-\infty}^{\infty} a_n(z+1)^n$ be the Laurent series expansion of $f(z)=\sin\left(\frac{z}{z+1}\right)$. Then $a_{-2}=$

(A) 1 (B) 0 (C) $\cos(1)$ (D) $\frac{-1}{2}\sin(1)$

Q.25 Let $u(x,y)$ be the real part of an entire function $f(z)=u(x,y)+iv(x,y)$ for $z=x+iy\in\mathbb{C}$. If C is the positively oriented boundary of a rectangular region R in $\mathbb{R}^2$, then $\oint_C \left[\frac{\partial u}{\partial y}dx-\frac{\partial u}{\partial x}dy\right]=$

(A) 1 (B) 0 (C) 2π (D) π

Q.26 Let $\phi:[0,1]\to\mathbb{R}$ be three times continuously differentiable. Suppose that the iterates defined by $x_{n+1}=\phi(x_n)$, $n\ge 0$ converge to the fixed point ξ of ϕ. If the order of convergence is three then

(A) $\phi'(\xi)=0$, $\phi''(\xi)=0$ (B) $\phi'(\xi)\ne 0$, $\phi''(\xi)=0$

(C) $\phi'(\xi)=0$, $\phi''(\xi)\ne 0$ (D) $\phi'(\xi)\ne 0$, $\phi''(\xi)\ne 0$

Q.27 Let $f:[0,2]\to\mathbb{R}$ be a twice continuously differentiable function. If $\int_0^2 f(x)\,dx\approx 2f(1)$, then the error in the approximation is

(A) $\frac{f'(\xi)}{12}$ for some $\xi\in(0,2)$ (B) $\frac{f'(\xi)}{2}$ for some $\xi\in(0,2)$

(C) $\frac{f''(\xi)}{3}$ for some $\xi\in(0,2)$ (D) $\frac{f''(\xi)}{6}$ for some $\xi\in(0,2)$

Q.28 For a fixed $t \in \mathbb{R}$, consider the linear programming problem:

$$\begin{aligned} &\text{Maximize} \quad z = 3x + 4y \\ &\text{subject to} \quad x + y \le 100 \\ &\qquad\qquad\quad x + 3y \le t \\ &\qquad\quad \text{and} \quad x \ge 0,\ y \ge 0 \end{aligned}$$

The maximum value of z is 400 for $t =$

(A) 50 (B) 100 (C) 200 (D) 300

Q.29 The minimum value of $z = 2x_1 - x_2 + x_3 - 5x_4 + 22x_5$ subject to

$$\begin{aligned} x_1 - 2x_4 + x_5 &= 6 \\ x_2 + x_4 - 4x_5 &= 3 \\ x_3 + 3x_4 + 2x_5 &= 10 \\ x_j \ge 0,\ j &= 1, 2 \ldots, 5 \end{aligned}$$

is

(A) 28 (B) 19 (C) 10 (D) 9

Q.30 Using the Hungarian method, the optimal value of the assignment problem whose cost matrix is given by

5	23	14	8
10	25	1	23
35	16	15	12
16	23	11	7

is

(A) 29 (B) 52 (C) 26 (D) 44

Q.31 Which of the following sequence $\{f_n\}_{n=1}^{\infty}$ of functions does NOT converge uniformly on $[0,1]$?

(A) $f_n(x) = \dfrac{e^{-x}}{n}$ (B) $f_n(x) = (1-x)^n$

(C) $f_n(x) = \dfrac{x^2 + nx}{n}$ (D) $f_n(x) = \dfrac{\sin(nx+n)}{n}$

Q.32 Let $E = \{(x, y) \in \mathbb{R}^2 : 0 < x < y\}$. Then $\iint_E y e^{-(x+y)}\, dx\, dy =$

(A) $\dfrac{1}{4}$ (B) $\dfrac{3}{2}$

(C) $\dfrac{4}{3}$ (D) $\dfrac{3}{4}$

Q.33 Let $f_n(x) = \frac{1}{n} \sum_{k=0}^{n} \sqrt{k(n-k)} \binom{n}{k} x^k (1-x)^{n-k}$ for $x \in [0,1]$, $n = 1, 2, \ldots$. If $\lim_{n\to\infty} f_n(x) = f(x)$ for $x \in [0,1]$, then the maximum value of $f(x)$ on $[0,1]$ is

(A) 1 (B) $\frac{1}{2}$ (C) $\frac{1}{3}$ (D) $\frac{1}{4}$

Q.34 Let $f : (c_{00}, \|\cdot\|_1) \to \mathbb{C}$ be a non-zero continuous linear functional. The number of Hahn-Banach extensions of f to $(\ell^1, \|\cdot\|_1)$ is

(A) one (B) two
(C) three (D) infinite

Q.35 If $I : (\ell^1, \|\cdot\|_2) \to (\ell^1, \|\cdot\|_1)$ is the identity map, then

(A) both I and I^{-1} are continuous
(B) I is continuous but I^{-1} is NOT continuous
(C) I^{-1} is continuous but I is NOT continuous
(D) neither I nor I^{-1} is continuous

Q.36 Consider the topology $\tau = \{G \subseteq \mathbb{R} : \mathbb{R} \setminus G \text{ is compact in } (\mathbb{R}, \tau_u)\} \cup \{\phi, \mathbb{R}\}$ on $\mathbb{R}$, where τ_u is the usual topology on $\mathbb{R}$ and ϕ is the empty set. Then $(\mathbb{R}, \tau)$ is

(A) a connected Hausdorff space
(B) connected but NOT Hausdorff
(C) Hausdorff but NOT connected
(D) neither connected nor Hausdorff

Q.37 Let

$$\tau_1 = \{G \subseteq \mathbb{R} : G \text{ is finite or } \mathbb{R} \setminus G \text{ is finite}\}$$

and

$$\tau_2 = \{G \subseteq \mathbb{R} : G \text{ is countable or } \mathbb{R} \setminus G \text{ is countable}\}.$$

Then

(A) neither τ_1 nor τ_2 is a topology on $\mathbb{R}$
(B) τ_1 is a topology on $\mathbb{R}$ but τ_2 is NOT a topology on $\mathbb{R}$
(C) τ_2 is a topology on $\mathbb{R}$ but τ_1 is NOT a topology on $\mathbb{R}$
(D) both τ_1 and τ_2 are topologies on $\mathbb{R}$

Q.38 Which one of the following ideals of the ring $\mathbb{Z}[i]$ of Gaussian integers is NOT maximal ?

(A) $\langle 1+i \rangle$ (B) $\langle 1-i \rangle$ (C) $\langle 2+i \rangle$ (D) $\langle 3+i \rangle$

Q.39 If $Z(G)$ denotes the centre of a group G, then the order of the quotient group $G/Z(G)$ cannot be

(A) 4 (B) 6 (C) 15 (D) 25

Q.40 Let $Aut(G)$ denote the group of automorphisms of a group G. Which one of the following is NOT a cyclic group ?

(A) $Aut\,(\mathbb{Z}_4)$ (B) $Aut\,(\mathbb{Z}_6)$ (C) $Aut\,(\mathbb{Z}_8)$ (D) $Aut\,(\mathbb{Z}_{10})$

Q.41 Let X be a non-negative integer valued random variable with $E(X^2)=3$ and $E(X)=1$. Then $\sum_{i=1}^{\infty} i\,P(X \geq i) =$

(A) 1 (B) 2 (C) 3 (D) 4

Q.42 Let X be a random variable with probability density function $f \in \{f_0, f_1\}$, where

$$f_0(x)=\begin{cases}2x, \text{if } 0< x<1\\ 0, \text{ otherwise}\end{cases} \quad \text{and} \quad f_1(x)=\begin{cases}3x^2, \text{ if } 0<x<1\\ 0, \text{ otherwise}\end{cases}$$

For testing the null hypothesis $H_0: f \equiv f_0$ against the alternative hypothesis $H_1: f \equiv f_1$ at level of significance $\alpha = 0.19$, the power of the most powerful test is

(A) 0.729 (B) 0.271 (C) 0.615 (D) 0.385

Q.43 Let X and Y be independent and identically distributed $U(0,1)$ random variables. Then $P\left(Y<\left(X-\frac{1}{2}\right)^2\right)=$

(A) $\frac{1}{12}$ (B) $\frac{1}{4}$ (C) $\frac{1}{3}$ (D) $\frac{2}{3}$

Q.44 Let X and Y be Banach spaces and let $T: X \to Y$ be a linear map. Consider the statements:

P: If $x_n \to x$ in X then $Tx_n \to Tx$ in Y.

Q: If $x_n \to x$ in X and $Tx_n \to y$ in Y then $Tx = y$.

Then

(A) P implies Q and Q implies P
(B) P implies Q but Q does not imply P
(C) Q implies P but P does not imply Q
(D) neither P implies Q nor Q implies P

Q.45 If $y(x)=x$ is a solution of the differential equation $y''-\left(\frac{2}{x^2}+\frac{1}{x}\right)(xy'-y)=0,\ 0<x<\infty$, then its general solution is

(A) $\left(\alpha+\beta e^{-2x}\right)x$ (B) $\left(\alpha+\beta e^{2x}\right)x$ (C) $\alpha x+\beta e^{x}$ (D) $\left(\alpha e^{x}+\beta\right)x$

Q.46 Let $P_n(x)$ be the Legendre polynomial of degree n such that $P_n(1)=1,\ n=1,2,\ldots$. If

$$\int_{-1}^{1}\left(\sum_{j=1}^{n}\sqrt{j(2j+1)}\,P_j(x)\right)^2 dx=20,$$

then $n=$

(A) 2 (B) 3 (C) 4 (D) 5

Q.47 The integral surface satisfying the equation $y\frac{\partial z}{\partial x}+x\frac{\partial z}{\partial y}=x^2+y^2$ and passing through the curve $x=1-t,\ y=1+t,\ z=1+t^2$ is

(A) $z=xy+\frac{1}{2}\left(x^2-y^2\right)^2$

(B) $z=xy+\frac{1}{4}\left(x^2-y^2\right)^2$

(C) $z=xy+\frac{1}{8}\left(x^2-y^2\right)^2$

(D) $z=xy+\frac{1}{16}\left(x^2-y^2\right)^2$

Q.48 For the diffusion problem $u_{xx}=u_t\ (0<x<\pi, t>0)$, $u(0,t)=0$, $u(\pi,t)=0$ and $u(x,0)=3\sin 2x$, the solution is given by

(A) $3e^{-t}\sin 2x$ (B) $3e^{-4t}\sin 2x$ (C) $3e^{-9t}\sin 2x$ (D) $3e^{-2t}\sin 2x$

Q.49 A simple pendulum, consisting of a bob of mass m connected with a string of length a, is oscillating in a vertical plane. If the string is making an angle θ with the vertical, then the expression for the Lagrangian is given as

(A) $ma^2\left(\dot\theta^2-\frac{2g}{a}\sin^2\left(\frac{\theta}{2}\right)\right)$

(B) $2mga\sin^2\left(\frac{\theta}{2}\right)$

(C) $ma^2\left(\frac{\dot\theta^2}{2}-\frac{2g}{a}\sin^2\left(\frac{\theta}{2}\right)\right)$

(D) $\frac{ma}{2}\left(\dot\theta^2-\frac{2g}{a}\cos\theta\right)$

Q.50 The extremal of the functional $\int_0^1\left(y+x^2+\frac{y'^2}{4}\right)dx$, $y(0)=0$, $y(1)=0$ is

(A) $4(x^2-x)$ (B) $3(x^2-x)$ (C) $2(x^2-x)$ (D) x^2-x

Common Data Questions

Common Data for Questions 51 and 52:

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by

$$T(x_1, x_2, x_3) = (x_1 + 3x_2 + 2x_3,\ 3x_1 + 4x_2 + x_3,\ 2x_1 + x_2 - x_3).$$

Q.51 The dimension of the range space of T^2 is

(A) 0 (B) 1 (C) 2 (D) 3

Q.52 The dimension of the null space of T^3 is

(A) 0 (B) 1 (C) 2 (D) 3

Common Data for Questions 53 and 54:

Let $y_1(x) = 1 + x$ and $y_2(x) = e^x$ be two solutions of $y''(x) + P(x)\,y'(x) + Q(x)\,y(x) = 0$.

Q.53 $P(x) =$

(A) $1+x$ (B) $-1-x$ (C) $\frac{1+x}{x}$ (D) $\frac{-1-x}{x}$

Q.54 The set of initial conditions for which the above differential equation has NO solution is

(A) $y(0)=2,\ y'(0)=1$ (B) $y(1)=0,\ y'(1)=1$

(C) $y(1)=1,\ y'(1)=0$ (D) $y(2)=1,\ y'(2)=2$

Common Data for Questions 55 and 56:

Let X and Y be random variables having the joint probability density function

$$f(x,y) = \begin{cases} \dfrac{1}{\sqrt{2\pi y}}\, e^{\frac{-1}{2y}(x-y)^2}, & \text{if } -\infty < x < \infty,\ 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

Q.55 The variance of the random variable X is

(A) $\frac{1}{12}$ (B) $\frac{1}{4}$ (C) $\frac{7}{12}$ (D) $\frac{5}{12}$

Q.56 The covariance between the random variables X and Y is

(A) $\frac{1}{3}$ (B) $\frac{1}{4}$ (C) $\frac{1}{6}$ (D) $\frac{1}{12}$

Linked Answer Questions

Statement for Linked Answer Questions 57 and 58 :

Consider the function $f(z) = \dfrac{e^{iz}}{z(z^2+1)}$.

Q.57 The residue of f at the isolated singular point in the upper half plane $\{z = x + iy \in \mathbb{C} : y > 0\}$ is

(A) $\dfrac{-1}{2e}$ (B) $\dfrac{-1}{e}$ (C) $\dfrac{e}{2}$ (D) 1

Q.58 The Cauchy Principal Value of the integral $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x \ dx}{x(x^2+1)}$ is

(A) $-2\pi(1+2e^{-1})$ (B) $\pi(1-e^{-1})$ (C) $2\pi(1+e)$ (D) $-\pi(1+e^{-1})$

Statement for Linked Answer Questions 59 and 60:

Let $f(x, y) = kxy - x^3y - xy^3$ for $(x, y) \in \mathbb{R}^2$, where k is a real constant. The directional derivative of f at the point $(1, 2)$ in the direction of the unit vector $u = \left(\dfrac{-1}{\sqrt{2}}, \dfrac{-1}{\sqrt{2}}\right)$ is $\dfrac{15}{\sqrt{2}}$.

Q.59 The value of k is

(A) 2 (B) 4 (C) 1 (D) –2

Q.60 The value of f at a local minimum in the rectangular region $R = \left\{(x, y) \in \mathbb{R}^2 : |x| < \dfrac{3}{2}, |y| < \dfrac{3}{2}\right\}$ is

(A) –2 (B) –3 (C) $\dfrac{-7}{8}$ (D) 0

END OF THE QUESTION PAPER