ETS® The People Who Make the Test!

Volume 1

Second Edition

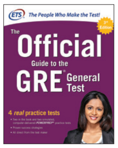
Official GRE® QUANTITATIVE REASONING Practice Questions

150 *real* test questions with explanations — straight from the maker of the GRE® General Test

Feeling ready for the *GRE*® General Test can help you do your best!

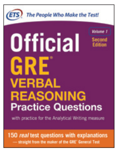
NEW! *POWERPREP PLUS*™ Online

Get the experience of taking the actual computer-delivered *GRE*® General Test and so much more! Two official practice tests, each containing a different set of never-before-published real test questions, simulate the computer-delivered test experience and provide scores within minutes for all three measures, explanations for correct answers and a score report that summarizes your performance.



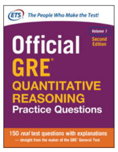
The Official Guide to the GRE® General Test, Third Edition

This guide features two actual, full-length practice tests, hundreds of authentic GRE test questions, explanations for many answers, test-taking strategies, sample essay responses with reader commentary and more. Two additional full-length tests are offered through the free *POWERPREP*™ Online. Available in print and as an eBook.



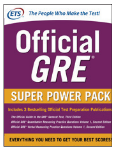
Official GRE® Verbal Reasoning Practice Questions Volume 1, Second Edition

For even more in-depth practice, this book provides an additional 150 real Verbal Reasoning questions with complete explanations and valuable hints and tips. This guide also includes new practice for the Analytical Writing measure. Available in print and as an eBook.



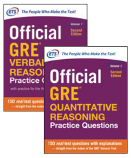
Official GRE® Quantitative Reasoning Practice Questions Volume 1, Second Edition

For even more in-depth practice, this book provides an additional 150 real Quantitative Reasoning questions with complete explanations and valuable hints and tips. This guide also includes a review of math topics you need to know for the test. Available in print and as an eBook.



Official *GRE*® Super Power Pack

Three of our most popular GRE General Test preparation products in one place! This bundle includes *The Official Guide to the GRE® General Test, Third Edition*, the *Official GRE® Quantitative Reasoning Practice Questions Volume 1, Second Edition* and the *Official GRE® Verbal Reasoning Practice Questions Volume 1, Second Edition*. Available in print and as an eBook.



Official *GRE*® Value Combo

This eBook bundle includes the *Official GRE® Quantitative Reasoning Practice Questions Volume 1, Second Edition* and the *Official GRE® Verbal Reasoning Practice Questions Volume 1, Second Edition* for a total of 300 never-before published questions.

ScoreItNow!™ Online Writing Practice

Use this online tool to sharpen your writing skills. Practice responding to GRE Analytical Writing tasks, receive immediate, confidential scores on your responses, review sample responses and general suggestions for improving your essay-writing skills and more.

Official GRE® Guide Mobile App

This app features authentic test questions with answers and explanations, test-taking tips and strategies plus more from *The Official Guide to the GRE® General Test*.

Download on the App Store

Learn more! **takethegre.com/prep**

37620

ETS GRE

Measuring the Power of Learning.®

Welcome to

Official GRE® Quantitative Reasoning Practice Questions Volume 1, Second Edition

The book you are holding offers 150 real GRE practice questions directly from the maker of the *GRE®* General Test. This book is specially created to give you in-depth practice and accurate test preparation for the Quantitative Reasoning measure.

Here's what you will find inside:

- **Authentic GRE Quantitative Reasoning test questions** arranged by content and question type—to help you build your test-taking skills. Plus, mixed practice sets.
- **Answers and explanations** for every question!
- **ETS's own test-taking strategies.** Learn valuable hints and tips that can help you get your best score.
- **GRE Math Review** covering math topics you need to know for the test.
- **Official information on the GRE Quantitative Reasoning measure.** Get the facts about the test content, structure, scoring, and more—straight from ETS.

About ETS

At ETS, we advance quality and equity in education for people worldwide by creating assessments based on rigorous research. ETS serves individuals, educational institutions, and government agencies by providing customized solutions for teacher certification, English language learning, and elementary, secondary, and post-secondary education, as well as conducting educational research, analysis, and policy studies. Founded as a nonprofit in 1947, ETS develops, administers, and scores more than 50 million tests annually—including the *TOEFL®* and *TOEIC®* tests, the *GRE®* tests, and *The Praxis Series®* assessments—in more than 180 countries at over 9,000 locations worldwide. For more information, visit **www.ets.org**.

Information wants to be free

IMPORTANT

ETS makes available free test preparation materials for individuals planning to take a GRE test. The *POWERPREP®* Online practice tests are available for individuals planning to take the computer-delivered GRE General Test, and the *Practice Book for the Paper-delivered GRE General Test*, Second edition, is available for individuals planning to take the paper-delivered test. The information about how to prepare for the Quantitative Reasoning measure of the GRE General Test, including test-taking strategies, question strategies, etc., that is included in the free test preparation is also included in this publication. This publication also provides you with 150 brand new practice questions with answers and explanations.

For more information about the GRE General Test, free and low-cost GRE test preparation materials, and other GRE products and services, please visit the GRE website at:

www.ets.org/gre

Inquiries concerning the practice test questions in this book should be sent to the GRE testing program at:

GRETestQuestionInquiries@ets.org

Volume 1

Second Edition

Official GRE® QUANTITATIVE REASONING Practice Questions

McGraw Hill Education

New York | Chicago | San Francisco | Athens | London | Madrid
Mexico City | Milan | New Delhi | Singapore | Sydney | Toronto

1 2 3 4 5 6 7 8 9 LOV 22 21 20 19 18 17

ISBN 978-1-259-86350-9
MHID 1-259-86350-6

e-ISBN 978-1-259-86351-6
e-MHID 1-259-86351-4

Interior Designer: Jane Tenenbaum

Contents

Information wants to be free

How to Use This Book

This book provides important information about the Quantitative Reasoning measure of the GRE General Test, including the knowledge and skills that are assessed and the types of questions that appear. The book will help you:

- Familiarize yourself with the test format and test question types
- Learn valuable test-taking strategies for each question type
- Review the math topics you need to know for the test
- Check your progress with Quantitative Reasoning practice questions

The following four-step program has been designed to help you make the best use of this book.

STEP 1 Learn About the GRE Quantitative Reasoning Measure

Chapter 1 of this book provides an overview of the GRE Quantitative Reasoning measure. Read this chapter to learn about the number of questions, time limits, and the test design features. You will also find valuable test-taking strategies from ETS and important information about how the measure is scored.

STEP 2 Study the Different GRE Quantitative Reasoning Question Types

Chapter 2 of this book describes the types of questions you will encounter in the Quantitative Reasoning measure. You will learn what the questions are designed to measure, and you will get tips for answering each question type. You will also see samples of each question type, with helpful explanations.

STEP 3 Practice Answering GRE Quantitative Reasoning Questions

Chapters 3, 4, 5, and 6 contain sets of authentic Quantitative Reasoning practice questions in the content areas of Arithmetic, Algebra, Geometry, and Data Analysis, respectively. Answer the questions in each set, and then read through the explanations to see which questions you found most challenging. Look for patterns. Did specific content areas give you trouble? You can refresh your math skills in these areas with the GRE Math Review in Appendix A. The GRE Math Review is a review of the math topics that are likely to appear on the Quantitative Reasoning measure. Each section of the GRE Math Review ends with exercises that will help you see how well you have mastered the material. Note that Appendix B provides information about the mathematical conventions that are used on the Quantitative Reasoning measure. Prior to answering the practice questions, it might be helpful to review these conventions.

STEP 4 Test Yourself With the Mixed Practice Sets

Once you have completed the practice sets for each content area in Chapters 3–6, it is time to practice with authentic GRE questions in the Mixed Practice Sets in Chapter 7. Each Mixed Practice Set includes the question types and content areas contained in an actual Quantitative Reasoning section of the GRE General Test.

1 Overview of the *GRE*® Quantitative Reasoning Measure

Your goal for this chapter

⇨ Review basic information on the structure of the *GRE*® Quantitative Reasoning measure, test-taking strategies, and scoring

Introduction to the *GRE*® General Test

The *GRE*® General Test—the most widely accepted graduate admissions test worldwide—measures verbal reasoning, quantitative reasoning, critical thinking, and analytical writing skills that are necessary for success in graduate and business school.

Prospective graduate and business school applicants from all around the world take the GRE General Test. Although applicants come from varying educational and cultural backgrounds, the GRE General Test provides a common measure for comparing candidates' qualifications. GRE scores are used by admissions committees and fellowship panels to supplement undergraduate records, recommendation letters, and other qualifications for graduate-level study.

The GRE General Test is available at test centers in more than 160 countries. In most regions of the world, the computer-delivered test is available on a continuous basis throughout the year. In areas of the world where computer-delivered testing is not available, the test is administered in a paper-delivered format up to three times a year.

Before taking the GRE General Test, it is important to become familiar with the content and structure of the test, and with each of the three measures—Verbal Reasoning, Quantitative Reasoning, and Analytical Writing. This book provides a close look at the GRE Quantitative Reasoning measure. Chapter 1 provides an overview of the structure and scoring of the Quantitative Reasoning measure. In Chapters 2 through 7, you will find information specific to the content of the Quantitative Reasoning measure. You can use the information in this publication to help you understand the type of material on which you will be tested. For the most up-to-date information about the GRE General Test, visit the GRE website at **www.ets.org/gre.**

The Quantitative Reasoning Measure of the Computer-delivered GRE General Test

Structure of the Quantitative Reasoning Measure

Measure	*Number of Questions*	*Allotted Time*
Quantitative Reasoning (Two sections)	20 questions per section	35 minutes per section

The Quantitative Reasoning sections may appear anytime in the test after section 1. The directions at the beginning of each Quantitative Reasoning section specify the total number of questions in the section and the time allowed for the section.

Test Design Features

The Quantitative Reasoning measure of the computer-delivered GRE General Test is section-level adaptive. This means the computer selects the second section of a measure based on your performance on the first section.

The advanced adaptive design also means you can freely move forward and backward throughout an entire section. Specific features include:

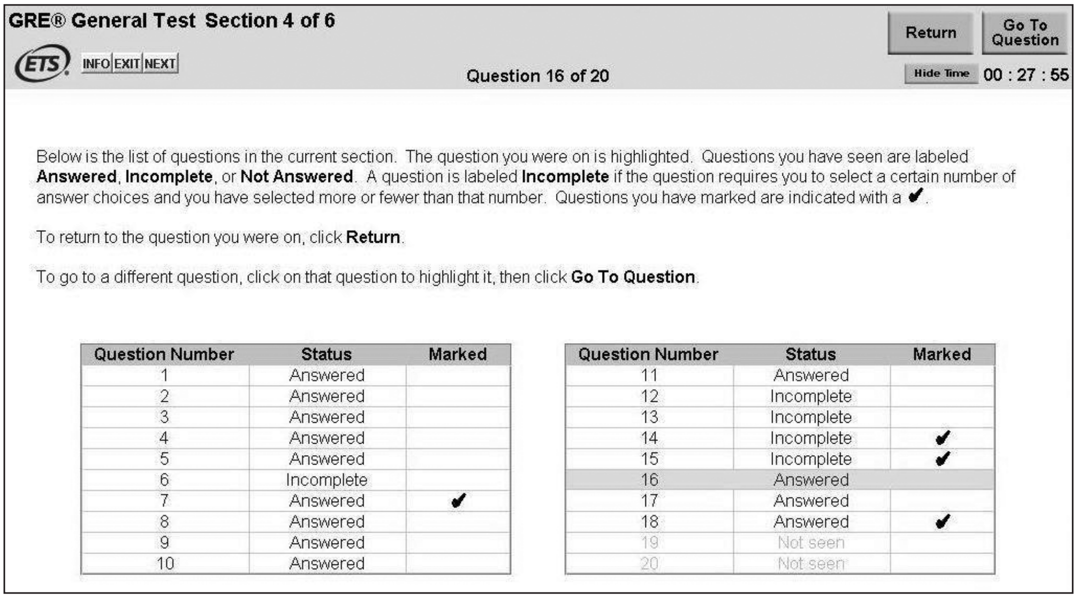
- Preview and review capabilities within a section
- “Mark” and “Review” features to tag questions, so you can skip them and return later if you have time remaining in the section
- The ability to change/edit answers within a section
- An on-screen calculator (More information about the calculator is given in Chapter 2.)

Test-taking Strategies

The questions in the Quantitative Reasoning measure are presented in a variety of formats. Some require you to select a single answer choice, others require you to select one or more answer choices, and yet others require you to enter a numeric answer. Make sure when answering a question that you understand what response is required. An on-screen calculator will be provided at the test center for use during the Quantitative Reasoning sections.

When taking the Quantitative Reasoning measure of the computer-delivered GRE General Test, you are free to skip questions that you might have difficulty answering within a section. The testing software has a “Mark” feature that enables you to mark questions you would like to revisit during the time provided to work on that section. The testing software also has a “Review” feature that lets you view a complete list of all the questions in the section on which you are working, that indicates whether you have answered each question, and that identifies the questions you have marked for review. Additionally, you can review questions you have already answered and change your answers, provided you still have time remaining to work on that section.

A sample review screen appears below. The review screen is intended to help you keep track of your progress on the test. Do not spend too much time on the review screen, as this will take away from the time allotted to read and answer the questions on the test.



Question Number	Status	Marked
1	Answered	
2	Answered	
3	Answered	
4	Answered	
5	Answered	
6	Incomplete	
7	Answered	✔
8	Answered	
9	Answered	
10	Answered	

Question Number	Status	Marked
11	Answered	
12	Incomplete	
13	Incomplete	
14	Incomplete	✔
15	Incomplete	✔
16	Answered	
17	Answered	
18	Answered	✔
19	Not seen	
20	Not seen	

Your Quantitative Reasoning score will be determined by the number of questions you answer correctly. Nothing is subtracted from a score if you answer a question incorrectly. Therefore, to maximize your scores on the Quantitative Reasoning measure, it is best to answer every question.

Work as rapidly as you can without being careless. Since no question carries greater weight than any other, do not waste time pondering individual questions you find extremely difficult or unfamiliar.

You may want to go through each of the Quantitative Reasoning sections rapidly first, stopping only to answer questions you can answer with certainty. Then go back and answer the questions that require greater thought, concluding with the difficult questions if you have time.

During the actual administration of the General Test, you may work only on one section at a time and only for the time allowed. Once you have completed a section, you may not go back to it.

Scratch Paper

You will receive a supply of scratch paper before you begin the test. You can replenish your supply of scratch paper as necessary throughout the test by asking the test administrator.

How the Quantitative Reasoning Measure Is Scored

The Quantitative Reasoning measure is section-level adaptive. This means the computer selects the second section of the measure based on your performance on the first section. Within each section, all questions contribute equally to the final score. First a raw score is computed. The raw score is the number of questions you answered correctly. The raw score is then converted to a scaled score through a process known as equating. The equating process accounts for minor variations in difficulty from test to test as well as the differences introduced by the section-level adaptation. Thus a given scaled score reflects the same level of performance regardless of which second section was selected and when the test was taken.

The Quantitative Reasoning Measure of the Paper-delivered GRE General Test

Structure of the Quantitative Reasoning Measure

Measure	*Number of Questions*	*Allotted Time*
Quantitative Reasoning (Two sections)	25 questions per section	40 minutes per section

The Quantitative Reasoning sections may appear in any order after section 2. The directions at the beginning of each section specify the total number of questions in the section and the time allowed for the section.

Test Design Features

- You are free, within any section, to skip questions and come back to them later or change the answer to a question.
- Answers are entered in the test book, rather than a separate answer sheet.
- You will be provided with an ETS calculator to use during the Quantitative Reasoning section; you may not use your own calculator.

Test-taking Strategies

The questions in the Quantitative Reasoning measure have a variety of formats. Some require you to select a single answer choice, others require you to select one or more answer choices, and yet others require you to enter a numeric answer. Make sure when answering a question that you understand what response is required. A calculator will be provided at the test center for use during the Quantitative Reasoning sections.

When taking a Quantitative Reasoning section, you are free, within that section, to skip questions that you might have difficulty answering and come back to them later during the time provided to work on that section. Also during that time you may change the answer to any question in that section by erasing it completely and filling in an alternative answer. Be careful not to leave any stray marks in the answer area, as they may be interpreted as incorrect responses. You can, however, safely make notes or perform calculations on other parts of the page. No additional scratch paper will be provided.

Your Quantitative Reasoning score will be determined by the number of questions you answer correctly. Nothing is subtracted from a score if you answer a question incorrectly. Therefore, to maximize your score on the Quantitative Reasoning measure, it is best to answer every question.

Work as rapidly as you can without being careless. Since no question carries greater weight than any other, do not waste time pondering individual questions you find extremely difficult or unfamiliar.

You may want to go through each of the Quantitative Reasoning sections rapidly first, stopping only to answer questions you can answer with certainty. Then go back and answer the questions that require greater thought, concluding with the difficult questions if you have time.

During the actual administration of the General Test, you may work only on the section the test center supervisor designates and only for the time allowed. You may *not* go back to an earlier section of the test after the supervisor announces, “Please stop work” for that section. The supervisor is authorized to dismiss you from the center for doing so.

All answers must be recorded in the test book.

How the Quantitative Reasoning Measure Is Scored

Scoring of the Quantitative Reasoning measure is essentially a two-step process. First a raw score is computed. The raw score is the number of questions answered correctly in the two sections for the measure. The raw score is then converted to a scaled score through a process known as equating. The equating process accounts for minor variations in difficulty among the different test editions. Thus a given scaled score reflects the same level of performance regardless of which edition of the test was taken.

Score Reporting

A Quantitative Reasoning score is reported on a 130–170 score scale, in 1-point increments. If you do not answer any questions at all for the measure, you will receive a No Score (NS) for that measure.

The *ScoreSelect®* Option

The *ScoreSelect®* option is available for both the GRE General Test and GRE Subject Tests and can be used by anyone with reportable scores from the last five years. This option lets you send institutions your best scores. For your four free score reports, you can send scores from your *Most Recent* test administration or scores from *All* test administrations in your reportable history. After test day, you can send scores from your *Most Recent, All,* or *Any* specific test administration(s) for a fee when ordering

Additional Score Reports. Just remember, scores for a test administration must be reported in their entirety. For more information, visit **www.ets.org/gre/scoreselect**.

Score Reporting Time Frames

Scores from computer-delivered GRE General Test administrations are reported approximately 10 to 15 days after the test date. Scores from paper-delivered administrations are reported about five weeks after the test date. If you are applying to a graduate or business school program, be sure to review the appropriate admissions deadlines and plan to take the test in time for your scores to reach the institution.

For more information on score reporting, visit the GRE website at **www.ets.org/gre/scores/get**.

2 Test Content

Your goals for this chapter

- ⇨ Learn general problem-solving steps and strategies
- ⇨ Learn the four types of *GRE*® Quantitative Reasoning questions and get tips for answering each question type
- ⇨ Study sample Quantitative Reasoning questions with solutions
- ⇨ Learn how to use the on-screen calculator

Overview of the Quantitative Reasoning Measure

The Quantitative Reasoning measure of the GRE General Test assesses your:

- basic mathematical skills
- understanding of elementary mathematical concepts
- ability to reason quantitatively and to model and solve problems with quantitative methods

Some of the Quantitative Reasoning questions are posed in real-life settings, while others are posed in purely mathematical settings. Many of the questions are "word problems," which must be translated and modeled mathematically. The skills, concepts, and abilities are assessed in the four content areas below.

Arithmetic topics include properties and types of integers, such as divisibility, factorization, prime numbers, remainders, and odd and even integers; arithmetic operations, exponents, and roots; and concepts such as estimation, percent, ratio, rate, absolute value, the number line, decimal representation, and sequences of numbers.

Algebra topics include operations with exponents; factoring and simplifying algebraic expressions; relations, functions, equations, and inequalities; solving linear and quadratic equations and inequalities; solving simultaneous equations and inequalities; setting up equations to solve word problems; and coordinate geometry, including graphs of functions, equations, and inequalities, intercepts, and slopes of lines.

Geometry topics include parallel and perpendicular lines, circles, triangles—including isosceles, equilateral, and 30°-60°-90° triangles—quadrilaterals, other polygons, congruent and similar figures, three-dimensional figures, area, perimeter, volume, the Pythagorean theorem, and angle measurement in degrees. The ability to construct proofs is not tested.

Data analysis topics include basic descriptive statistics, such as mean, median, mode, range, standard deviation, interquartile range, quartiles, and percentiles; interpretation of data in tables and graphs, such as line graphs, bar graphs, circle graphs, boxplots, scatterplots, and frequency distributions; elementary probability, such as probabilities of compound events and independent events; conditional probability; random variables and probability distributions, including normal distributions; and counting methods, such as combinations, permutations, and Venn diagrams. These topics are typically taught in high school algebra courses or introductory statistics courses. Inferential statistics is not tested.

The content in these areas includes high school mathematics and statistics at a level that is generally no higher than a second course in algebra; it does not include trigonometry, calculus, or other higher-level mathematics. The publication *Math Review for the GRE General Test,* which is available in Appendix A, provides detailed information about the content of the Quantitative Reasoning measure.

The mathematical symbols, terminology, and conventions used in the Quantitative Reasoning measure are those that are standard at the high school level. For example, the positive direction of a number line is to the right, distances are nonnegative, and prime numbers are greater than 1. Whenever nonstandard notation is used in a question, it is explicitly introduced in the question.

In addition to conventions, there are some important assumptions about numbers and figures that are listed in the Quantitative Reasoning section directions:

- All numbers used are real numbers.
- All figures are assumed to lie in a plane unless otherwise indicated.
- Geometric figures, such as lines, circles, triangles, and quadrilaterals, **are not necessarily** drawn to scale. That is, you should **not** assume that quantities such as lengths and angle measures are as they appear in a figure. You should assume, however, that lines shown as straight are actually straight, points on a line are in the order shown, and more generally, all geometric objects are in the relative positions shown. For questions with geometric figures, you should base your answers on geometric reasoning, not on estimating or comparing quantities by sight or by measurement.
- Coordinate systems, such as *xy*-planes and number lines, **are** drawn to scale; therefore, you can read, estimate, or compare quantities in such figures by sight or by measurement.
- Graphical data presentations, such as bar graphs, circle graphs, and line graphs, **are** drawn to scale; therefore, you can read, estimate, or compare data values by sight or by measurement.

More about conventions and assumptions appears in the publication *Mathematical Conventions for the GRE General Test,* which is available in Appendix B.

General Problem-solving Steps

Questions in the Quantitative Reasoning measure ask you to model and solve problems using quantitative, or mathematical, methods. Generally, there are three basic steps in solving a mathematics problem:

Step 1: Understand the problem
Step 2: Carry out a strategy for solving the problem
Step 3: Check your answer

Here is a description of the three steps, followed by a list of useful strategies for solving mathematics problems.

Step 1: Understand the Problem

The first step is to read the statement of the problem carefully to make sure you understand the information given and the problem you are being asked to solve.

Some information may describe certain quantities. Quantitative information may be given in words or mathematical expressions, or a combination of both. Also, in some problems you may need to read and understand quantitative information in data presentations, geometric figures, or coordinate systems. Other information may take the form of formulas, definitions, or conditions that must be satisfied by the quantities. For example, the conditions may be equations or inequalities, or may be words that can be translated into equations or inequalities.

In addition to understanding the information you are given, it is important to understand what you need to accomplish in order to solve the problem. For example, what unknown quantities must be found? In what form must they be expressed?

Step 2: Carry Out a Strategy for Solving the Problem

Solving a mathematics problem requires more than understanding a description of the problem, that is, more than understanding the quantities, the data, the conditions, the unknowns, and all other mathematical facts related to the problem. It requires determining *what* mathematical facts to use and *when* and *how* to use those facts to develop a solution to the problem. It requires a strategy.

Mathematics problems are solved by using a wide variety of strategies. Also, there may be different ways to solve a given problem. Therefore, you should develop a repertoire of problem-solving strategies, as well as a sense of which strategies are likely to work best in solving particular problems. Attempting to solve a problem without a strategy may lead to a lot of work without producing a correct solution.

After you determine a strategy, you must carry it out. If you get stuck, check your work to see if you made an error in your solution. It is important to have a flexible, open mind-set. If you check your solution and cannot find an error or if your solution strategy is simply not working, look for a different strategy.

Step 3: Check Your Answer

When you arrive at an answer, you should check that it is reasonable and computationally correct.

- Have you answered the question that was asked?
- Is your answer reasonable in the context of the question? Checking that an answer is reasonable can be as simple as recalling a basic mathematical fact and checking whether your answer is consistent with that fact. For example, the probability of an event must be between 0 and 1, inclusive, and the area of a geometric figure must be positive. In other cases, you can use estimation to check that your answer is reasonable. For example, if your solution involves adding three numbers, each of which is between 100 and 200, estimating the sum tells you that the sum must be between 300 and 600.
- Did you make a computational mistake in arriving at your answer? A key-entry error using the calculator? You can check for errors in each step in your solution. Or you may be able to check directly that your solution is correct. For example, if you solved the equation $7(3x-2)+4=95$ for x and got the answer $x=5$, you can check your answer by substituting $x=5$ into the equation to see that $7(3(5)-2)+4=95$.

Strategies

There are no set rules—applicable to all mathematics problems—to determine the best strategy. The ability to determine a strategy that will work grows as you solve more and more problems. What follows are brief descriptions of useful strategies, along with references to questions in this chapter that you can answer with the help of particular strategies. These strategies do not form a complete list, and, aside from grouping the first four strategies together, they are not presented in any particular order.

The first four strategies are translation strategies, where one representation of a mathematics problem is translated into another.

Strategy 1: Translate from Words to an Arithmetic or Algebraic Representation

Word problems are often solved by translating textual information into an arithmetic or algebraic representation. For example, an "odd integer" can be represented by the expression $2n+1$, where n is an integer; and the statement "the cost of a taxi trip is \$3.00, plus \$1.25 for each mile" can be represented by the equation $c=3+1.25m$. More generally, translation occurs when you understand a word problem in mathematical terms in order to model the problem mathematically.

- See question 4 on page 27 and question 5 on page 35.

Strategy 2: Translate from Words to a Figure or Diagram

To solve a problem in which a figure is described but not shown, draw your own figure. Draw the figure as accurately as possible, labeling as many parts as possible, including any unknowns.

Drawing figures can help in geometry problems as well as in other types of problems. For example, in probability and counting problems, drawing a diagram can sometimes make it easier to analyze the relevant data and to notice relationships and dependencies.

- See question 2 on page 25.

Strategy 3: Translate from an Algebraic to a Graphical Representation

Many algebra problems can be represented graphically in a coordinate system, whether the system is a number line if the problem involves one variable, or a coordinate plane if the problem involves two variables. Such graphs can clarify relationships that may be less obvious in algebraic representations.

- See question 3 on page 26.

Strategy 4: Translate from a Figure to an Arithmetic or Algebraic Representation

When a figure is given in a problem, it may be effective to express relationships among the various parts of the figure using arithmetic or algebra.

- See question 4 on page 18 and question 3 on page 34.

Strategy 5: Simplify an Arithmetic or Algebraic Representation

Arithmetic and algebraic representations include both expressions and equations. Your facility in simplifying a representation can often lead to a quick solution. Examples include converting from a percent to a decimal, converting from one measurement unit to another, combining like terms in an algebraic expression, and simplifying an equation until its solutions are evident.

- See question 6 on page 20 and question 4 on page 35.

Strategy 6: Add to a Geometric Figure

Sometimes you can add useful lines, points, or circles to a geometric figure to facilitate solving a problem. You can also add any given information—as well as any new information as you derive it—to the figure to help you see relationships within the figure more easily, for example, the length of a line segment or the measure of an angle.

- See question 3 on page 26.

Strategy 7: Find a Pattern

Patterns are found throughout mathematics. Identifying a pattern is often the first step in understanding a complex mathematical situation. Pattern recognition yields insight that may point in the direction of a complete solution to the problem or simply help you generate a hypothesis, which requires further exploration using another strategy. In a problem where you suspect there is a pattern but don't recognize it yet, working with particular instances can help you identify the pattern. Once a pattern is identified, it can be used to answer questions.

- See question 4 on page 31.

Strategy 8: Search for a Mathematical Relationship

More general than patterns, mathematical relationships exist throughout mathematics. Problems may involve quantities that are related algebraically, sets that are related logically, or figures that are related geometrically. Also, there may be relationships between information given textually, algebraically, graphically, etc. To express relationships between quantities, it is often helpful to introduce one or more variables to represent the quantities. Once a relationship is understood and expressed, it is often the key to solving a problem.

- See question 8 on page 22 and question 3 on page 30.

Strategy 9: Estimate

Sometimes it is not necessary to perform extensive calculations to solve a problem—it is sufficient to estimate the answer. The degree of accuracy needed depends on the particular question being asked. Care should be taken to determine how far off your estimate could possibly be from the actual answer to the question. Estimation can also be used to check whether the answer to a question is reasonable.

- See question 3 on page 17 and question 4 on page 27.

Strategy 10: Trial and Error

Version 1: Make a Reasonable Guess and Then Refine It

For some problems, the fastest way to a solution is to make a reasonable guess at the answer, check it, and then improve on your guess. This is especially useful if the number of possible answers is limited. In other problems, this approach may help you at least to understand better what is going on in the problem.

- See question 1 on page 29.

Version 2: Try More Than One Value of a Variable

To explore problems containing variables, it is useful to substitute values for the variables. It often helps to substitute more than one value for each variable. How many values to choose and what values are good choices depends on the problem. Also dependent on the problem is whether this approach, by itself, will yield a solution or whether the approach will simply help you generate a hypothesis that requires further exploration using another strategy.

- See question 2 on page 17 and question 5 on page 19.

Strategy 11: Divide into Cases

Some problems are quite complex. To solve such problems you may need to divide them into smaller, less complex problems, which are restricted cases of the original problem. When you divide a problem into cases, you should consider whether or not to include all possibilities. For example, if you want to prove that a certain statement is true for all integers, it may be best to show that it is true for all positive integers, then show it is true for all negative integers, and then show it is true for zero. In doing that, you will have shown that the statement is true for all integers, because each integer is either positive, negative, or zero.

- See question 1 on page 16 and question 2 on page 30.

Strategy 12: Adapt Solutions to Related Problems

When solving a new problem that seems similar to a problem that you know how to solve, you can try to solve the new problem by adapting the solution—both the strategies and the results—of the problem you know how to solve.

If the differences between the new problem and the problem you know how to solve are only surface features—for example, different numbers, different labels, or different categories—that is, features that are not fundamental to the structure of the problem, then solve the new problem using the same strategy as you used before.

If the differences between the new problem and the problem you know how to solve are more than just surface features, try to modify the solution to the problem you know how to solve to fit the conditions given in the new problem.

- See question 3 on page 30 and question 4 on page 31.

Strategy 13: Determine Whether a Conclusion Follows from the Information Given

In some problems, you are given information and a statement describing a possible conclusion, which may or may not follow from the information. You need to determine whether or not the conclusion is a logical consequence of the information given.

If you think that the conclusion follows from the information, try to show it. Using the information and any relevant mathematical relationships, try to reason step-by-step from the information to the conclusion. Another way to show that the conclusion follows from the information, is to show that in *all* cases in which the information is true, the conclusion is also true.

If you think that the conclusion does *not* follow from the information, try to show that instead. One way to show that a conclusion does not follow from the information is to produce a counterexample. A counterexample is a case where the given information is true but the conclusion is false. If you are unsuccessful in producing a counterexample, it does not necessarily mean that the conclusion does not follow from the information—it may mean that although a counterexample exists, you were not successful in finding it.

- See question 9 on page 23 and question 3 on page 38.

Strategy 14: Determine What Additional Information Is Sufficient to Solve a Problem

Some problems cannot be solved directly from the information given, and you need to determine what other information will help you answer the question. In that case, it is useful to list all the information given in the problem, along with the information that would be contained in a complete solution, and then evaluate what is missing. Sometimes the missing information can be derived from the information given, and sometimes it cannot.

- See question 3 on page 38.

Quantitative Reasoning Question Types

The Quantitative Reasoning measure has four types of questions:

- Quantitative Comparison questions
- Multiple-choice questions—Select One Answer Choice
- Multiple-choice questions—Select One or More Answer Choices
- Numeric Entry questions

Each question appears either independently as a discrete question or as part of a set of questions called a Data Interpretation set. All of the questions in a Data Interpretation set are based on the same data presented in tables, graphs, or other displays of data.

In the computer-delivered test, you are allowed to use a basic calculator—provided on-screen—on the Quantitative Reasoning measure. Information about using the calculator appears later in this chapter.

For the paper-delivered test, handheld calculators are provided at the test center for use during the test. Information about using the handheld calculator to help you answer questions appears in the free *Practice Book for the Paper-based GRE revised General Test,* which is available at **www.ets.org/gre/prepare.**

Quantitative Comparison Questions

Description

Questions of this type ask you to compare two quantities—Quantity A and Quantity B—and then determine which of the following statements describes the comparison.

- Quantity A is greater.
- Quantity B is greater.
- The two quantities are equal.
- The relationship cannot be determined from the information given.

Tips for Answering

- **Become familiar with the answer choices.** Quantitative Comparison questions always have the same answer choices, so get to know them, especially the last choice, "The relationship cannot be determined from the information given." Never select this last choice if it is clear that the values of the two quantities can be determined by computation. Also, if you determine that one quantity is greater than the other, make sure you carefully select the corresponding choice so as not to reverse the first two choices.
- **Avoid unnecessary computations.** Don't waste time performing needless computations in order to compare the two quantities. Simplify, transform, or estimate one or both of the given quantities only as much as is necessary to compare them.
- **Remember that geometric figures are not necessarily drawn to scale.** If any aspect of a given geometric figure is not fully determined, try to redraw the figure, keeping those aspects that are completely determined by the given information fixed but changing the aspects of the figure that are not determined. Examine the results. What variations are possible in the relative lengths of line segments or measures of angles?
- **Plug in numbers.** If one or both of the quantities are algebraic expressions, you can substitute easy numbers for the variables and compare the resulting quantities in your analysis. Consider all kinds of appropriate numbers before you give an answer: e.g., zero, positive and negative numbers, small and large numbers, fractions and decimals. If you see that Quantity A is greater than Quantity B in one case and Quantity B is greater than Quantity A in another case, choose "The relationship cannot be determined from the information given."
- **Simplify the comparison.** If both quantities are algebraic or arithmetic expressions and you cannot easily see a relationship between them, you can try to simplify the comparison. Try a step-by-step simplification that is similar to the steps involved when you solve the equation $5 = 4x + 3$ for x, or similar to the steps involved when you determine that the inequality $\frac{3y+2}{5} < y$ is equivalent to the simpler inequality $1 < y$. Begin by setting up a comparison involving the two quantities, as follows:

  Quantity A ? Quantity B

  where ? is a "placeholder" that could represent the relationship *greater than* (>), *less than* (<), or *equal to* (=) or could represent the fact that the relationship cannot be determined from the information given. Then try to simplify the comparison, step by step, until you can determine a relationship between simplified quantities. For example, you may conclude after the last step that ? represents equal to (=). Based on this conclusion, you may be able to compare Quantities A and B. To understand this strategy more fully, see sample questions 6 to 9.

Sample Questions

Compare Quantity A and Quantity B, using additional information centered above the two quantities if such information is given, and select one of the following four answer choices:

Ⓐ Quantity A is greater.
Ⓑ Quantity B is greater.
Ⓒ The two quantities are equal.
Ⓓ The relationship cannot be determined from the information given.

A symbol that appears more than once in a question has the same meaning throughout the question.

	Quantity A	Quantity B
1.	The least prime number greater than 24	The greatest prime number less than 28

Ⓐ Quantity A is greater.
Ⓑ Quantity B is greater.
Ⓒ The two quantities are equal.
Ⓓ The relationship cannot be determined from the information given.

Explanation

For the integers greater than 24, note that 25, 26, 27, and 28 are not prime numbers, but 29 is a prime number, as are 31 and many other greater integers. Thus, 29 is the least prime number greater than 24, and Quantity A is 29. For the integers less than 28, note that 27, 26, 25, and 24 are not prime numbers, but 23 is a prime number, as are 19 and several other lesser integers. Thus 23 is the greatest prime number less than 28, and Quantity B is 23.

The correct answer is Choice A, Quantity A is greater.

This explanation uses the following strategy.
Strategy 11: Divide into Cases

Lionel is younger than Maria.

	Quantity A	Quantity B
2.	Twice Lionel's age	Maria's age

Ⓐ Quantity A is greater.
Ⓑ Quantity B is greater.
Ⓒ The two quantities are equal.
Ⓓ The relationship cannot be determined from the information given.

Explanation

If Lionel's age is 6 years and Maria's age is 10 years, then Quantity A is greater, but if Lionel's age is 4 years and Maria's age is 10 years, then Quantity B is greater. Thus the relationship cannot be determined.

The correct answer is Choice D, the relationship cannot be determined from the information given.

This explanation uses the following strategies.
Strategy 10: Trial and Error
Strategy 13: Determine Whether a Conclusion Follows from the Information Given

---

	Quantity A	Quantity B
3.	54% of 360	150

Ⓐ Quantity A is greater.
Ⓑ Quantity B is greater.
Ⓒ The two quantities are equal.
Ⓓ The relationship cannot be determined from the information given.

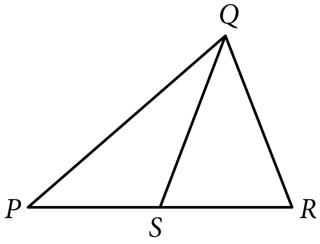
Explanation

Without doing the exact computation, you can see that 54 percent of 360 is greater than $\frac{1}{2}$ of 360, which is 180, and 180 is greater than Quantity B, 150.

Thus the correct answer is Choice A, Quantity A is greater.

This explanation uses the following strategy.
Strategy 9: Estimate

Figure 1



$PQ = PR$

	Quantity A	Quantity B
4.	PS	SR

Ⓐ Quantity A is greater.

Ⓑ Quantity B is greater.

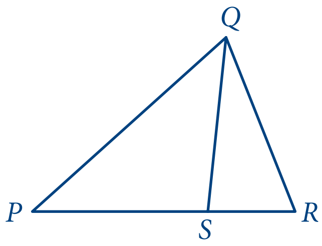
Ⓒ The two quantities are equal.

Ⓓ The relationship cannot be determined from the information given.

Explanation

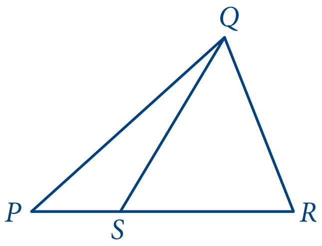
From Figure 1, you know that PQR is a triangle and that point S is between points P and R, so $PS < PR$ and $SR < PR$. You are also given that $PQ = PR$. However, this information is not sufficient to compare PS and SR. Furthermore, because the figure is not necessarily drawn to scale, you cannot determine the relative sizes of PS and SR visually from the figure, though they may appear to be equal. The position of S can vary along side PR anywhere between P and R. Following are two possible variations of Figure 1, each of which is drawn to be consistent with the information $PQ = PR$.

Figure 2



$PQ = PR$

Figure 3



$PQ = PR$

Note that Quantity A is greater in Figure 2 and Quantity B is greater in Figure 3.

Thus the correct answer is Choice D, the relationship cannot be determined from the information given.

This explanation uses the following strategies.

Strategy 4: Translate from a Figure to an Arithmetic or Algebraic Representation

Strategy 13: Determine Whether a Conclusion Follows from the Information Given

$$y = 2x^2 + 7x - 3$$

	Quantity A	Quantity B
5.	x	y

- (A) Quantity A is greater.
- (B) Quantity B is greater.
- (C) The two quantities are equal.
- (D) The relationship cannot be determined from the information given.

Explanation

If $x = 0$, then $y = 2(0^2) + 7(0) - 3 = -3$, so in this case, $x > y$; but if $x = 1$, then $y = 2(1^2) + 7(1) - 3 = 6$, so in that case, $y > x$.

Thus the correct answer is Choice D, the relationship cannot be determined from the information given.

This explanation uses the following strategies.
Strategy 10: Trial and Error
Strategy 13: Determine Whether a Conclusion Follows from the Information Given

Note that plugging numbers into expressions *may not* be conclusive. It *is* conclusive, however, if you get different results after plugging in different numbers: the conclusion is that the relationship cannot be determined from the information given. It is also conclusive if there are only a small number of possible numbers to plug in and all of them yield the same result, say, that Quantity B is greater.

Now suppose that there are an infinite number of possible numbers to plug in. If you plug many of them in and each time the result is, for example, that Quantity A is greater, you still cannot conclude that Quantity A is greater for every possible number that could be plugged in. Further analysis would be necessary and should focus on whether Quantity A is greater for all possible numbers or whether there are numbers for which Quantity A is not greater.

The following sample questions focus on simplifying the comparison.

$$y > 4$$

	Quantity A	Quantity B
6.	$\dfrac{3y+2}{5}$	y

Ⓐ Quantity A is greater.
Ⓑ Quantity B is greater.
Ⓒ The two quantities are equal.
Ⓓ The relationship cannot be determined from the information given.

Explanation

Set up the initial comparison:

$$\frac{3y+2}{5} \boxed{?}\, y$$

Then simplify:

Step 1: Multiply both sides by 5 to get

$$3y + 2 \boxed{?}\, 5y$$

Step 2: Subtract $3y$ from both sides to get

$$2 \boxed{?}\, 2y$$

Step 3: Divide both sides by 2 to get

$$1 \boxed{?}\, y$$

The comparison is now simplified as much as possible. In order to compare 1 and y, note that you are given the information $y > 4$ (above Quantities A and B). It follows from $y > 4$ that $y > 1$, or $1 < y$, so that in the comparison $1 \boxed{?}\, y$, the placeholder $\boxed{?}$ represents *less than* (<): $1 < y$.

However, the problem asks for a comparison between Quantity A and Quantity B, not a comparison between 1 and y. To go from the comparison between 1 and y to a comparison between Quantities A and B, start with the last comparison, $1 < y$, and carefully consider each simplification step in reverse order to determine what each comparison implies about the preceding comparison, all the way back to the comparison between Quantities A and B if possible. Since step 3 was "*divide* both sides by 2," *multiplying* both sides of the comparison $1 < y$ by 2 implies the preceding comparison $2 < 2y$, thus reversing step 3. Each simplification step can be reversed as follows:

- Reverse step 3: *multiply* both sides by 2.
- Reverse step 2: *add* $3y$ to both sides.
- Reverse step 1: *divide* both sides by 5.

When each step is reversed, the relationship remains *less than* (<), so Quantity A is less than Quantity B.

Thus the correct answer is Choice B, Quantity B is greater.

This explanation uses the following strategy.
Strategy 5: Simplify an Arithmetic or Algebraic Representation

While some simplification steps like subtracting 3 from both sides or dividing both sides by 10 are always reversible, it is important to note that some steps, like squaring both sides, may not be reversible.

Also, note that when you simplify an *inequality*, the steps of multiplying or dividing both sides by a negative number change the direction of the inequality; for example, if $x < y$, then $-x > -y$. So the relationship in the final, simplified inequality may be the *opposite* of the relationship between Quantities A and B. This is another reason to consider the impact of each step carefully.

	Quantity A	Quantity B
7.	$\dfrac{2^{30} - 2^{29}}{2}$	2^{28}

Ⓐ Quantity A is greater.
Ⓑ Quantity B is greater.
Ⓒ The two quantities are equal.
Ⓓ The relationship cannot be determined from the information given.

Explanation

Set up the initial comparison:

$$\frac{2^{30} - 2^{29}}{2} \ ? \ 2^{28}$$

Then simplify:

Step 1: Multiply both sides by 2 to get

$$2^{30} - 2^{29} \ ? \ 2^{29}$$

Step 2: Add 2^{29} to both sides to get

$$2^{30} \ ? \ 2^{29} + 2^{29}$$

Step 3: Simplify the right-hand side using the fact that $(2)(2^{29}) = 2^{30}$ to get

$$2^{30} \ ? \ 2^{30}$$

The resulting relationship is *equal to* (=). In reverse order, each simplification step implies *equal to* in the preceding comparison. So Quantities A and B are also equal.

Thus the correct answer is Choice C, the two quantities are equal.

This explanation uses the following strategy.
Strategy 5: Simplify an Arithmetic or Algebraic Representation

---

	Quantity A	Quantity B
8.	$x^2 + 1$	$2x - 1$

Ⓐ Quantity A is greater.
Ⓑ Quantity B is greater.
Ⓒ The two quantities are equal.
Ⓓ The relationship cannot be determined from the information given.

Explanation

Set up the initial comparison:

$$x^2 + 1 \;\boxed{?}\; 2x - 1$$

Then simplify by noting that the quadratic polynomial $x^2 - 2x + 1$ can be factored:

Step 1: Subtract $2x$ from both sides to get

$$x^2 - 2x + 1 \;\boxed{?}\; -1$$

Step 2: Factor the left-hand side to get

$$(x - 1)^2 \;\boxed{?}\; -1$$

The left-hand side of the comparison is the square of a number. Since the square of a number is always greater than or equal to 0, and 0 is greater than −1, the simplified comparison is the inequality $(x - 1)^2 > -1$ and the resulting relationship is *greater than* (>). In reverse order, each simplification step implies the inequality *greater than* (>) in the preceding comparison. Therefore Quantity A is greater than Quantity B.

The correct answer is Choice A, Quantity A is greater.

This explanation uses the following strategies.
Strategy 5: Simplify an Arithmetic or Algebraic Representation
Strategy 8: Search for a Mathematical Relationship

$$w > 1$$

	Quantity A	Quantity B
9.	$7w - 4$	$2w + 5$

Ⓐ Quantity A is greater.

Ⓑ Quantity B is greater.

Ⓒ The two quantities are equal.

Ⓓ The relationship cannot be determined from the information given.

Explanation

Set up the initial comparison:

$$7w - 4 \boxed{?} 2w + 5$$

Then simplify:

Step 1: Subtract $2w$ from both sides and add 4 to both sides to get

$$5w \boxed{?} 9$$

Step 2: Divide both sides by 5 to get

$$w \boxed{?} \frac{9}{5}$$

The comparison cannot be simplified any further. Although you are given that $w > 1$, you still don't know how w compares to $\frac{9}{5}$, or 1.8. For example, if $w = 1.5$, then $w < 1.8$, but if $w = 2$, then $w > 1.8$. In other words, the relationship between w and $\frac{9}{5}$ cannot be determined.

Note that each of these simplification steps is reversible, so in reverse order, each simplification step implies that the *relationship cannot be determined* in the preceding comparison. Thus the relationship between Quantities A and B cannot be determined.

The correct answer is Choice D, the relationship cannot be determined from the information given.

This explanation uses the following strategies.

Strategy 5: Simplify an Arithmetic or Algebraic Representation

Strategy 13: Determine Whether a Conclusion Follows from the Information Given

The strategy of simplifying the comparison works most efficiently when you note that a simplification step is reversible while actually taking the step. Here are some common steps that are always reversible:

- Adding any number or expression to both sides of a comparison
- Subtracting any number or expression from both sides
- Multiplying both sides by any nonzero number or expression
- Dividing both sides by any nonzero number or expression

Remember that if the relationship is an inequality, multiplying or dividing both sides by any *negative* number or expression will yield the opposite inequality. Be aware that some common operations like squaring both sides are generally not reversible and may require further analysis using other information given in the question in order to justify reversing such steps.

Multiple-choice Questions—Select One Answer Choice

Description

These questions are multiple-choice questions that ask you to select only one answer choice from a list of five choices.

Tips for Answering

- **Use the fact that the answer is there.** If your answer is not one of the five answer choices given, you should assume that your answer is incorrect and do the following:
  - Reread the question carefully—you may have missed an important detail or misinterpreted some information.
  - Check your computations—you may have made a mistake, such as mis-keying a number on the calculator.
  - Reevaluate your solution method—you may have a flaw in your reasoning.
- **Examine the answer choices.** In some questions you are asked explicitly which of the choices has a certain property. You may have to consider each choice separately, or you may be able to see a relationship between the choices that will help you find the answer more quickly. In other questions, it may be helpful to work backward from the choices, say, by substituting the choices in an equation or inequality to see which one works. However, be careful, as that method may take more time than using reasoning.
- **For questions that require approximations, scan the answer choices to see how close an approximation is needed.** In other questions, too, it may be helpful to scan the choices briefly before solving the problem to get a better sense of what the question is asking. If computations are involved in the solution, it may be necessary to carry out all computations exactly and round only your final answer in order to get the required degree of accuracy. In other questions, you may find that estimation is sufficient and will help you avoid spending time on long computations.

Sample Questions

Select a single answer choice.

1. If $5x + 32 = 4 - 2x$, what is the value of x ?

Ⓐ -4
Ⓑ -3
Ⓒ 4
Ⓓ 7
Ⓔ 12

Explanation

Solving the equation for x, you get $7x = -28$, and so $x = -4$.

The correct answer is Choice A, −4.

This explanation uses the following strategy.

Strategy 5: Simplify an Arithmetic or Algebraic Representation

---

2. Which of the following numbers is farthest from the number 1 on the number line?

Ⓐ -10
Ⓑ -5
Ⓒ 0
Ⓓ 5
Ⓔ 10

Explanation

Circling each of the answer choices in a sketch of the number line (Figure 4) shows that of the given numbers, −10 is the greatest distance from 1.

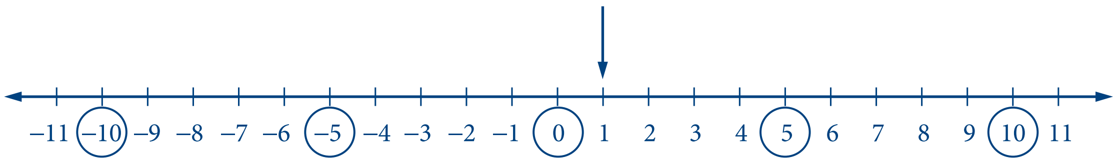


Figure 4

Another way to answer the question is to remember that the distance between two numbers on the number line is equal to the absolute value of the difference of the two numbers. For example, the distance between −10 and 1 is $|-10 - 1| = 11$, and the distance between 10 and 1 is $|10 - 1| = |9| = 9$.

The correct answer is Choice A, −10.

This explanation uses the following strategy.

Strategy 2: Translate from Words to a Figure or Diagram

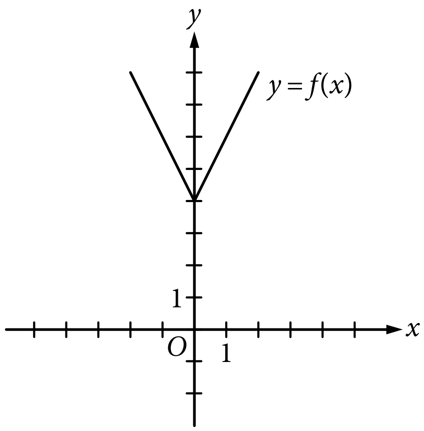


Figure 5

3. The figure above shows the graph of the function f defined by $f(x) = |2x| + 4$ for all numbers x. For which of the following functions g, defined for all numbers x, does the graph of g intersect the graph of f ?

   (A) $g(x) = x - 2$
   (B) $g(x) = x + 3$
   (C) $g(x) = 2x - 2$
   (D) $g(x) = 2x + 3$
   (E) $g(x) = 3x - 2$

Explanation

You can see that all five choices are linear functions whose graphs are lines with various slopes and y-intercepts. The graph of Choice A is a line with slope 1 and y-intercept -2, shown in Figure 6.

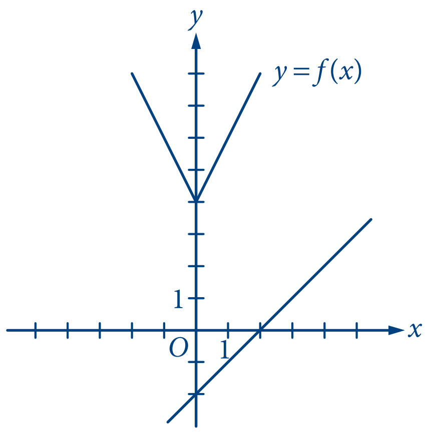


Figure 6

It is clear that this line will not intersect the graph of f to the left of the y-axis. To the right of the y-axis, the graph of f is a line with slope 2, which is greater than slope 1. Consequently, as the value of x increases, the value of y increases faster for f than for g, and therefore the graphs do not intersect to the right of the y-axis. Choice B is similarly ruled out. Note that if the y-intercept of either of the lines in Choices A and B were greater than or equal to 4 instead of less than 4, they would intersect the graph of f.

Choices C and D are lines with slope 2 and y-intercepts less than 4. Hence, they are parallel to the graph of f (to the right of the y-axis) and therefore will not intersect it. Any line with a slope greater than 2 and a y-intercept less than 4, like the line in Choice E, will intersect the graph of f (to the right of the y-axis).

The correct answer is Choice E, $g(x) = 3x - 2$.

This explanation uses the following strategies.
Strategy 3: Translate from an Algebraic to a Graphical Representation
Strategy 6: Add to a Geometric Figure
Strategy 8: Search for a Mathematical Relationship

---

4. A car got 33 miles per gallon using gasoline that cost $2.95 per gallon. Approximately what was the cost, in dollars, of the gasoline used in driving the car 350 miles?

Ⓐ $10
Ⓑ $20
Ⓒ $30
Ⓓ $40
Ⓔ $50

Explanation

Scanning the answer choices indicates that you can do at least some estimation and still answer confidently. The car used $\frac{350}{33}$ gallons of gasoline, so the cost was $\left(\frac{350}{33}\right)(2.95)$ dollars. You can estimate the product $\left(\frac{350}{33}\right)(2.95)$ by estimating $\frac{350}{33}$ a little low, 10, and estimating 2.95 a little high, 3, to get approximately $(10)(3) = 30$ dollars. You can also use the calculator to compute a more exact answer and then round the answer to the nearest 10 dollars, as suggested by the answer choices. The calculator yields the decimal 31.287. . . , which rounds to 30 dollars.

Thus the correct answer is Choice C, $30.

This explanation uses the following strategies.
Strategy 1: Translate from Words to an Arithmetic or Algebraic Representation
Strategy 9: Estimate

5. A certain jar contains 60 jelly beans—22 white, 18 green, 11 yellow, 5 red, and 4 purple. If a jelly bean is to be chosen at random, what is the probability that the jelly bean will be neither red nor purple?

   (A) 0.09
   (B) 0.15
   (C) 0.54
   (D) 0.85
   (E) 0.91

Explanation

Since there are 5 red and 4 purple jelly beans in the jar, there are 51 that are neither red nor purple, and the probability of selecting one of these is $\frac{51}{60}$. Since all of the answer choices are decimals, you must convert the fraction to its decimal equivalent, 0.85.

Thus the correct answer is Choice D, 0.85.

This explanation uses the following strategies.

Strategy 1: Translate from Words to an Arithmetic or Algebraic Representation
Strategy 5: Simplify an Arithmetic or Algebraic Representation

Multiple-choice Questions—Select One or More Answer Choices

Description

These questions are multiple-choice questions that ask you to select one or more answer choices from a list of choices. A question may or may not specify the number of choices to select. These questions are marked with square boxes beside the answer choices, not circles or ovals.

Tips for Answering

- **Note whether you are asked to indicate a specific number of answer choices or all choices that apply.** In the latter case, be sure to consider all of the choices, determine which ones are correct, and select all of those and only those choices. Note that there may be only one correct choice.
- **In some questions that involve conditions that limit the possible values of numerical answer choices, it may be efficient to determine the least and/or the greatest possible value.** Knowing the least and/or greatest possible value may enable you to quickly determine all of the choices that are correct.
- **Avoid lengthy calculations by recognizing and continuing numerical patterns.**

Sample Questions

Select one or more answer choices according to the specific question directions.

If the question does not specify how many answer choices to select, select all that apply.

- **The correct answer may be just one of the choices or as many as all of the choices, depending on the question.**
- **No credit is given unless you select all of the correct choices and no others.**

If the question specifies how many answer choices to select, select exactly that number of choices.

1. Which two of the following numbers have a product that is between -1 and 0 ?

   Indicate both of the numbers.

   A -20

   B -10

   C 2^{-4}

   D 3^{-2}

Explanation

For this question, you must select a pair of answer choices. The product of the pair must be negative, so the possible products are $(-20)(2^{-4})$, $(-20)(3^{-2})$, $(-10)(2^{-4})$, and $(-10)(3^{-2})$. The product must also be greater than -1. The first product is $\frac{-20}{2^4} = -\frac{20}{16} < -1$, the second product is $\frac{-20}{3^2} = -\frac{20}{9} < -1$, and the third product is $\frac{-10}{2^4} = -\frac{10}{16} > -1$, so you can stop there.

The correct answer consists of Choices B (-10) and C (2^{-4}).

This explanation uses the following strategies.

Strategy 1: Translate from Words to an Arithmetic or Algebraic Representation

Strategy 10: Trial and Error

2. Which of the following integers are multiples of both 2 and 3 ?

   Indicate all such integers.

   A 8
   B 9
   C 12
   D 18
   E 21
   F 36

Explanation

You can first identify the multiples of 2, which are 8, 12, 18, and 36, and then among the multiples of 2 identify the multiples of 3, which are 12, 18, and 36. Alternatively, if you realize that every number that is a multiple of 2 and 3 is also a multiple of 6, you can identify the choices that are multiples of 6.

The correct answer consists of Choices C (12), D (18), and F (36).

This explanation uses the following strategies.
Strategy 1: Translate from Words to an Arithmetic or Algebraic Representation
Strategy 11: Divide into Cases

---

3. Each employee of a certain company is in either Department X or Department Y, and there are more than twice as many employees in Department X as in Department Y. The average (arithmetic mean) salary is \$25,000 for the employees in Department X and \$35,000 for the employees in Department Y. Which of the following amounts could be the average salary for all of the employees of the company?

   Indicate all such amounts.

   A \$26,000
   B \$28,000
   C \$29,000
   D \$30,000
   E \$31,000
   F \$32,000
   G \$34,000

Explanation

One strategy for answering this kind of question is to find the least and/or greatest possible value. Clearly the average salary is between \$25,000 and \$35,000, and all of the answer choices are in this interval. Since you are told that there are more employees with the lower average salary, the average salary of all employees must be less than the average of \$25,000 and \$35,000, which is \$30,000. If there were exactly twice as many employees in Department X as in Department Y, then the average salary for all employees would be, to the nearest dollar, the following weighted mean,

$$\frac{(2)(25{,}000) + (1)(35{,}000)}{2 + 1} \approx 28{,}333 \text{ dollars}$$

where the weight for \$25,000 is 2 and the weight for \$35,000 is 1. Since there are *more* than twice as many employees in Department X as in Department Y, the actual average salary must be even closer to \$25,000 because the weight for \$25,000 is greater than 2. This means that \$28,333 is the greatest possible average. Among the choices given, the possible values of the average are therefore \$26,000 and \$28,000.

Thus the correct answer consists of Choices A (\$26,000) and B (\$28,000).

Intuitively, you might expect that any amount between \$25,000 and \$28,333 is a possible value of the average salary. To see that \$26,000 is possible, in the weighted mean above, use the respective weights 9 and 1 instead of 2 and 1. To see that \$28,000 is possible, use the respective weights 7 and 3.

This explanation uses the following strategies.
Strategy 1: Translate from Words to an Arithmetic or Algebraic Representation
Strategy 8: Search for a Mathematical Relationship
Strategy 12: Adapt Solutions to Related Problems

---

4. Which of the following could be the units digit of 57^n, where n is a positive integer?

   Indicate all such digits.

   [A] 0
   [B] 1
   [C] 2
   [D] 3
   [E] 4
   [F] 5
   [G] 6
   [H] 7
   [I] 8
   [J] 9

Explanation

The units digit of 57^n is the same as the units digit of 7^n for all positive integers n. To see why this is true for $n = 2$, compute 57^2 by hand and observe how its units digit results from the units digit of 7^2. Because this is true for every positive integer n, you need to consider only powers of 7. Beginning with $n = 1$ and proceeding consecutively, the units digits of 7, 7^2, 7^3, 7^4, and 7^5 are 7, 9, 3, 1, and 7, respectively. In this sequence, the first digit, 7, appears again, and the pattern of four digits, 7, 9, 3, 1, repeats without end. Hence, these four digits are the only possible units digits of 7^n and therefore of 57^n.

The correct answer consists of Choices B (1), D (3), H (7), and J (9).

This explanation uses the following strategies.
Strategy 7: Find a Pattern
Strategy 12: Adapt Solutions to Related Problems

Numeric Entry Questions

Description

Questions of this type ask you either to enter your answer as an integer or a decimal in a single answer box or to enter it as a fraction in two separate boxes—one for the numerator and one for the denominator. In the computer–delivered test, use the computer mouse and keyboard to enter your answer.

Tips for Answering

- **Make sure you answer the question that is asked.** Since there are no answer choices to guide you, read the question carefully and make sure you provide the type of answer required. Sometimes there will be labels before or after the answer box to indicate the appropriate type of answer. Pay special attention to units such as feet or miles, to orders of magnitude such as millions or billions, and to percents as compared with decimals.
- **If you are asked to round your answer, make sure you round to the required degree of accuracy.** For example, if an answer of 46.7 is to be rounded to the nearest integer, you need to enter the number 47. If your solution strategy involves intermediate computations, you should carry out all computations exactly and round only your final answer in order to get the required degree of accuracy. If no rounding instructions are given, enter the exact answer.
- **Examine your answer to see if it is reasonable with respect to the information given.** You may want to use estimation or another solution path to double-check your answer.

Sample Questions

Enter your answer as an integer or a decimal if there is a single answer box OR as a fraction if there are two separate answer boxes—one for the numerator and one for the denominator.

To enter an integer or a decimal, either type the number in the answer box using the keyboard or use the Transfer Display button on the calculator.

- **First, select the answer box—a cursor will appear in the box—and then type the number.**
- **For a negative sign, type a hyphen. For a decimal point, type a period.**
- **The Transfer Display button on the calculator will transfer the calculator display to the answer box.**
- **Equivalent forms of the correct answer, such as 2.5 and 2.50, are all correct.**
- **Enter the exact answer unless the question asks you to round your answer.**

To enter a fraction, type the numerator and the denominator in their respective answer boxes using the keyboard.

- **Select each answer box—a cursor will appear in the box—and then type an integer. A decimal point cannot be used in either box.**
- **For a negative sign, type a hyphen; to remove it, type the hyphen again.**
- **The Transfer Display button on the calculator cannot be used for a fraction.**
- **Fractions do not need to be reduced to lowest terms, though you may need to reduce your fraction to fit in the boxes.**

1. One pen costs \$0.25 and one marker costs \$0.35. At those prices, what is the total cost of 18 pens and 100 markers?

$ [ ]

Explanation

Multiplying \$0.25 by 18 yields \$4.50, which is the cost of the 18 pens; and multiplying \$0.35 by 100 yields \$35.00, which is the cost of the 100 markers. The total cost is therefore $\$4.50 + \$35.00 = \$39.50$. Equivalent decimals, such as \$39.5 or \$39.500, are considered correct.

Thus the correct answer is \$39.50 (or equivalent).

Note that the dollar symbol is in front of the answer box, so the symbol \$ does not need to be entered in the box. In fact, only numbers, a decimal point, and a negative sign can be entered in the answer box.

This explanation uses the following strategy.

Strategy 1: Translate from Words to an Arithmetic or Algebraic Representation

2. Rectangle R has length 30 and width 10, and square S has length 5. The perimeter of S is what fraction of the perimeter of R?

$$\frac{\boxed{}}{\boxed{}}$$

Explanation

The perimeter of R is $30 + 10 + 30 + 10 = 80$, and the perimeter of S is $(4)(5) = 20$. Therefore, the perimeter of S is $\frac{20}{80}$ of the perimeter of R. To enter the answer $\frac{20}{80}$, you should enter the numerator 20 in the top box and the denominator 80 in the bottom box. Because the fraction does not need to be reduced to lowest terms, any fraction that is equivalent to $\frac{20}{80}$ is also considered correct, as long as it fits in the boxes. For example, both of the fractions $\frac{2}{8}$ and $\frac{1}{4}$ are considered correct.

Thus the correct answer is $\frac{20}{80}$ (or any equivalent fraction).

This explanation uses the following strategy.
Strategy 1: Translate from Words to an Arithmetic or Algebraic Representation

---

RESULTS OF A USED-CAR AUCTION

	Small Cars	Large Cars
Number of cars offered	32	23
Number of cars sold	16	20
Projected sales total for cars offered (in thousands)	\$70	\$150
Actual sales total (in thousands)	\$41	\$120

Figure 7

3. For the large cars sold at an auction that is summarized in the table above, what was the average sale price per car?

\$ ☐

Explanation

From Figure 7, you see that the number of large cars sold was 20 and the sales total for large cars was \$120,000 (not \$120). Thus the average sale price per car was $\frac{\$120{,}000}{20} = \$6{,}000$.

The correct answer is \$6,000 (or equivalent).

(In numbers that are 1,000 or greater, you do not need to enter commas in the answer box.)

This explanation uses the following strategy.
Strategy 4: Translate from a Figure to an Arithmetic or Algebraic Representation

4. A merchant made a profit of $5 on the sale of a sweater that cost the merchant $15. What is the profit expressed as a percent of the merchant's cost?

   Give your answer to the nearest whole percent.

   [ ] %

Explanation

The percent profit is $\left(\frac{5}{15}\right)(100) = 33.333\ldots = 33.\overline{3}$ percent, which is 33%, to the nearest whole percent.

Thus the correct answer is 33% (or equivalent).

If you use the calculator and the Transfer Display button, the number that will be transferred to the answer box is 33.333333, which is incorrect since it is not given to the nearest whole percent. You will need to adjust the number in the answer box by deleting all of the digits to the right of the decimal point.

Also, since you are asked to give the answer as a percent, the decimal equivalent of 33 percent, which is 0.33, is incorrect. The percent symbol next to the answer box indicates that the form of the answer must be a percent. Entering 0.33 in the box would give the erroneous answer 0.33%.

This explanation uses the following strategies.

Strategy 1: Translate from Words to an Arithmetic or Algebraic Representation
Strategy 5: Simplify an Arithmetic or Algebraic Representation

---

5. Working alone at its constant rate, machine A produces k liters of a chemical in 10 minutes. Working alone at its constant rate, machine B produces k liters of the chemical in 15 minutes. How many minutes does it take machines A and B, working simultaneously at their respective constant rates, to produce k liters of the chemical?

   [ ] minutes

Explanation

Machine A produces $\frac{k}{10}$ liters per minute, and machine B produces $\frac{k}{15}$ liters per minute. So when the machines work simultaneously, the rate at which the chemical is produced is the sum of these two rates, which is $\frac{k}{10} + \frac{k}{15} = k\left(\frac{1}{10} + \frac{1}{15}\right) = k\left(\frac{25}{150}\right) = \frac{k}{6}$ liters per minute. To compute the time required to produce k liters at this rate, divide the amount k by the rate $\frac{k}{6}$ to get $\frac{k}{\frac{k}{6}} = 6$.

Therefore, the correct answer is 6 minutes (or equivalent).

One way to check that the answer of 6 minutes is reasonable is to observe that if the slower rate of machine B were the same as machine A's faster rate of k liters in 10 minutes, then the two machines, working simultaneously, would take half the time, or 5 minutes, to produce the k liters. So the answer has to be *greater than 5 minutes*. Similarly, if the faster rate of machine A were the same as machine B's slower rate of k liters in 15 minutes, then the two machines would take half the time, or 7.5 minutes, to produce the k liters. So the answer has to be *less than 7.5 minutes.* Thus the answer of 6 minutes is reasonable compared to the lower estimate of 5 minutes and the upper estimate of 7.5 minutes.

This explanation uses the following strategies.
Strategy 1: Translate from Words to an Arithmetic or Algebraic Representation
Strategy 5: Simplify an Arithmetic or Algebraic Representation
Strategy 8: Search for a Mathematical Relationship

Data Interpretation Sets

Description

Data Interpretation questions are grouped together and refer to the same table, graph, or other data presentation. These questions ask you to interpret or analyze the given data. The types of questions may be Multiple-choice (both types) or Numeric Entry.

Tips for Answering

- **Scan the data presentation briefly to see what it is about, but do not spend time studying all of the information in detail.** Focus on those aspects of the data that are necessary to answer the questions. Pay attention to the axes and scales of graphs; to the units of measurement or orders of magnitude (such as *billions*) that are given in the titles, labels, and legends; and to any notes that clarify the data.
- **When graphical data presentations, such as bar graphs and line graphs, are shown with scales, you should read, estimate, or compare quantities by sight or by measurement, according to the corresponding scales.** For example, you can use the relative sizes of bars or sectors to compare the quantities that they represent, but be aware of broken scales and of bars that do not start at 0.
- **The questions are to be answered only on the basis of the data presented, everyday facts (such as the number of days in a year), and your knowledge of mathematics.** Do not make use of specialized information you may recall from other sources about the particular context on which the questions are based unless the information can be derived from the data presented.

Sample Questions

Questions 1 to 3 are based on the following data.

ANNUAL PERCENT CHANGE IN DOLLAR AMOUNT OF SALES
AT FIVE RETAIL STORES FROM 2006 TO 2008

Store	Percent Change from 2006 to 2007	Percent Change from 2007 to 2008
P	10	−10
Q	−20	9
R	5	12
S	−7	−15
T	17	−8

Figure 8

1. If the dollar amount of sales at Store *P* was $800,000 for 2006, what was the dollar amount of sales at that store for 2008 ?

   (A) $727,200
   (B) $792,000
   (C) $800,000
   (D) $880,000
   (E) $968,000

Explanation

According to Figure 8, if the dollar amount of sales at Store *P* was $800,000 for 2006, then it was 10 percent greater for 2007, which is 110 percent of that amount, or $880,000. For 2008 the amount was 90 percent of $880,000, which is $792,000.

The correct answer is Choice B, $792,000.

Note that an increase of 10 percent for one year and a decrease of 10 percent for the following year does not result in the same dollar amount as the original dollar amount, because the base that is used in computing the percents is $800,000 for the first change but $880,000 for the second change.

This explanation uses the following strategies.

Strategy 4: Translate from a Figure to an Arithmetic or Algebraic Representation
Strategy 5: Simplify an Arithmetic or Algebraic Representation

2. At Store T, the dollar amount of sales for 2007 was what percent of the dollar amount of sales for 2008 ?

   Give your answer to the nearest 0.1 percent.

   $\boxed{}$ %

Explanation

If A is the dollar amount of sales at Store T for 2007, then 8 percent of A, or $0.08A$, is the amount of decrease from 2007 to 2008. Thus $A - 0.08A = 0.92A$ is the dollar amount for 2008. Therefore, the desired percent can be obtained by dividing A by $0.92A$, which equals $\frac{A}{0.92A} = \frac{1}{0.92} = 1.0869565\ldots.$ Expressed as a percent and rounded to the nearest 0.1 percent, this number is 108.7%.

Thus the correct answer is 108.7% (or equivalent).

This explanation uses the following strategies.

Strategy 4: Translate from a Figure to an Arithmetic or Algebraic Representation
Strategy 5: Simplify an Arithmetic or Algebraic Representation

---

3. Based on the information given, which of the following statements must be true?

   Indicate all such statements.

   A For 2008 the dollar amount of sales at Store R was greater than that at each of the other four stores.

   B The dollar amount of sales at Store S for 2008 was 22 percent less than that for 2006.

   C The dollar amount of sales at Store R for 2008 was more than 17 percent greater than that for 2006.

Explanation

For Choice A, since the only data given in Figure 8 are percent changes from year to year, there is no way to compare the actual dollar amount of sales at the stores for 2008 or for any other year. Even though Store R had the greatest percent increase from 2006 to 2008, its actual dollar amount of sales for 2008 may have been much smaller than that for any of the other four stores, and therefore Choice A is not necessarily true.

For Choice B, even though the sum of the two percent decreases would suggest a 22 percent decrease, the bases of the percents are different. If B is the dollar amount of sales at Store S for 2006, then the dollar amount for 2007 is 93 percent of B, or $0.93B$, and the dollar amount for 2008 is given by $(0.85)(0.93)B$, which is $0.7905B$. Note that this represents a percent decrease of $100 - 79.05$, or 20.95 percent, which is not equal to 22 percent, and so Choice B is not true.

For Choice C, if C is the dollar amount of sales at Store R for 2006, then the dollar amount for 2007 is given by $1.05C$ and the dollar amount for 2008 is given by $(1.12)(1.05)C$, which is $1.176C$. Note that this represents a 17.6 percent increase, which is greater than 17 percent, so Choice C must be true.

Therefore the correct answer consists of only Choice C (The dollar amount of sales at Store *R* for 2008 was more than 17 percent greater than that for 2006).

This explanation uses the following strategies.
Strategy 1: Translate from Words to an Arithmetic or Algebraic Representation
Strategy 4: Translate from a Figure to an Arithmetic or Algebraic Representation
Strategy 5: Simplify an Arithmetic or Algebraic Representation
Strategy 13: Determine Whether a Conclusion Follows from the Information Given
Strategy 14: Determine What Additional Information Is Sufficient to Solve a Problem

Using the Calculator

Sometimes the computations you need to do in order to answer a question in the Quantitative Reasoning measure are somewhat time-consuming, like long division, or involve square roots. For such computations, you can use the on-screen calculator provided in the computer-delivered test. The on-screen calculator is shown in Figure 9.

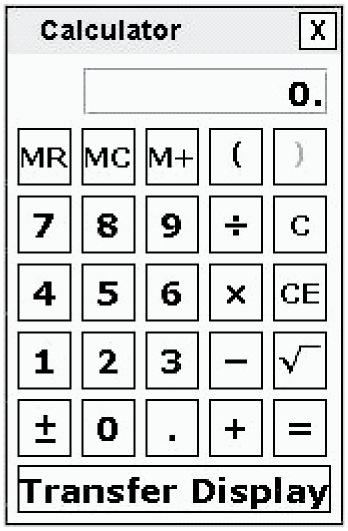


Figure 9

Although the calculator can shorten the time it takes to perform computations, keep in mind that the calculator provides results that supplement, but do not replace, your knowledge of mathematics. You must use your mathematical knowledge to determine whether the calculator's results are reasonable and how the results can be used to answer a question.

Here are some general guidelines for calculator use in the Quantitative Reasoning measure:

- Most of the questions don't require difficult computations, so don't use the calculator just because it's available.
- Use it for calculations that you know are tedious, such as long division, square roots, and addition, subtraction, or multiplication of numbers that have several digits.

- Avoid using it for simple computations that are quicker to do mentally, such as $10 - 490$, $(4)(70)$, $\frac{4{,}300}{10}$, $\sqrt{25}$, and 30^2.
- Avoid using it to introduce decimals if you are asked to give an answer as a fraction.
- Some questions can be answered more quickly by reasoning and estimating than by using the calculator.
- If you use the calculator, estimate the answer beforehand so that you can determine whether the calculator's answer is "in the ballpark." This may help you avoid key-entry errors.

The following guidelines are specific to the on-screen calculator in the computer-delivered test:

- When you use the computer mouse or the keyboard to operate the calculator, take care not to mis-key a number or operation.
- Note all of the calculator's buttons, including Transfer Display.
- The Transfer Display button can be used on Numeric Entry questions with a single answer box. This button will transfer the calculator display to the answer box. You should check that the transferred number has the correct form to answer the question. For example, if a question requires you to round your answer or convert your answer to a percent, make sure that you adjust the transferred number accordingly.
- Take note that the calculator respects *order of operations*, which is a mathematical convention that establishes which operations are performed before others in a mathematical expression that has more than one operation. The order is as follows: parentheses, exponentiation (including square roots), multiplications and divisions (from left to right), additions and subtractions (from left to right). With respect to order of operations, the value of the expression $1 + 2 \times 4$ is 9 because the expression is evaluated by first multiplying 2 and 4 and then by adding 1 to the result. This is how the on-screen calculator in the Quantitative Reasoning measure performs the operations. (Note that many basic calculators follow a different convention, whereby they perform multiple operations in the order that they are entered into the calculator. For such calculators, the result of entering $1 + 2 \times 4$ is 12. To get this result, the calculator adds 1 and 2, displays a result of 3, then multiplies 3 and 4, and displays a result of 12.)
- In addition to parentheses, the on-screen calculator has one memory location and three memory buttons that govern it: memory recall $\boxed{\text{MR}}$, memory clear $\boxed{\text{MC}}$, and memory sum $\boxed{\text{M+}}$. These buttons function as they normally do on most basic calculators.
- Some computations are not defined for real numbers: for example, division by zero or taking the square root of a negative number. If you enter $6\ \boxed{\div}\ 0\ \boxed{=}$, the word **Error** will be displayed. Similarly, if you enter $1\ \boxed{\pm}\ \boxed{\sqrt{\ }}$, then **Error** will be displayed. To clear the display, you must press the clear button $\boxed{\text{C}}$.
- The calculator displays up to eight digits. If a computation results in a number greater than 99,999,999, then **Error** will be displayed. For example, the calculation $10{,}000{,}000\ \boxed{\times}\ 10\ \boxed{=}$ results in **Error**. The clear button $\boxed{\text{C}}$ must be used to clear the display. If a computation results in a positive number less than 0.0000001, or 10^{-7}, then 0 will be displayed.

Below are some examples of computations using the calculator.

1. Compute $4+\dfrac{6.73}{2}$.

Explanation

Enter 4 $\boxed{+}$ 6.73 $\boxed{\div}$ 2 $\boxed{=}$ to get 7.365. Alternatively, enter 6.73 $\boxed{\div}$ 2 $\boxed{=}$ to get 3.365, and then enter $\boxed{+}$ 4 $\boxed{=}$ to get **7.365**.

---

2. Compute $-\dfrac{8.4+9.3}{70}$.

Explanation

Since division takes precedence over addition in the order of operations, you need to override that precedence in order to compute this fraction. Here are two ways to do that. You can use the parentheses for the addition in the numerator, entering $\boxed{(}$ 8.4 $\boxed{+}$ 9.3 $\boxed{)}$ $\boxed{\div}$ 70 $\boxed{=}$ $\boxed{\pm}$ to get –0.2528571. Or you can use the equals sign after 9.3, entering 8.4 $\boxed{+}$ 9.3 $\boxed{=}$ $\boxed{\div}$ 70 $\boxed{=}$ $\boxed{\pm}$ to get the same result. In the second way, note that pressing the first $\boxed{=}$ is essential, because without it, 8.4 $\boxed{+}$ 9.3 $\boxed{\div}$ 70 $\boxed{=}$ $\boxed{\pm}$ would erroneously compute $-\left(8.4+\dfrac{9.3}{70}\right)$ instead.

Incidentally, the exact value of the expression $-\dfrac{8.4+9.3}{70}$ is the repeating decimal $-0.25\overline{285714}$, where the digits 285714 repeat without end, but the calculator rounds the decimal to **–0.2528571.**

---

3. Find the length, to the nearest 0.01, of the hypotenuse of a right triangle with legs of length 21 and 54; that is, use the Pythagorean theorem and calculate $\sqrt{21^2+54^2}$.

Explanation

Enter 21 $\boxed{\times}$ 21 $\boxed{+}$ 54 $\boxed{\times}$ 54 $\boxed{=}$ $\boxed{\sqrt{\ }}$ to get 57.939624. Again, pressing the $\boxed{=}$ before the $\boxed{\sqrt{\ }}$ is essential because 21 $\boxed{\times}$ 21 $\boxed{+}$ 54 $\boxed{\times}$ 54 $\boxed{\sqrt{\ }}$ $\boxed{=}$ would erroneously compute $21^2 + 54\sqrt{54}$. This is because the square root would take precedence over the multiplication in the order of operations. Note that parentheses could be used, as in $\boxed{(}$ 21 $\boxed{\times}$ 21 $\boxed{)}$ $\boxed{+}$ $\boxed{(}$ 54 $\boxed{\times}$ 54 $\boxed{)}$ $\boxed{=}$ $\boxed{\sqrt{\ }}$, but they are not necessary because the multiplications already take precedence over the addition.

Incidentally, the exact answer is a nonterminating, nonrepeating decimal, or an irrational number, but the calculator rounds the decimal to 57.939624. Finally, note that the problem asks for the answer to the nearest 0.01, so the correct answer is **57.94**.

4. Compute $(-15)^3$.

Explanation

Enter 15 $\boxed{\pm}$ $\boxed{\times}$ 15 $\boxed{\pm}$ $\boxed{\times}$ 15 $\boxed{\pm}$ $\boxed{=}$ to get **−3,375.**

---

5. Convert 6 miles per hour to feet per second.

Explanation

The solution to this problem uses the conversion factors 1 mile = 5,280 feet and 1 hour = 3,600 seconds as follows:

$$\left(\frac{6\text{ miles}}{1\text{ hour}}\right)\left(\frac{5{,}280\text{ feet}}{1\text{ mile}}\right)\left(\frac{1\text{ hour}}{3{,}600\text{ seconds}}\right) = ?\ \frac{\text{feet}}{\text{second}}$$

Enter 6 $\boxed{\times}$ 5280 $\boxed{\div}$ 3600 $\boxed{=}$ to get 8.8. Alternatively, enter 6 $\boxed{\times}$ 5280 $\boxed{=}$ to get the result 31,680, and then enter $\boxed{\div}$ 3600 $\boxed{=}$ to get **8.8 feet per second.**

---

6. At a fund-raising event, 43 participants donated \$60 each, 21 participants donated \$80 each, and 16 participants donated \$100 each. What was the average (arithmetic mean) donation per participant, in dollars?

Explanation

The solution to this problem is to compute the weighted mean

$\dfrac{(43)(60)+(21)(80)+(16)(100)}{43+21+16}$. You can use the memory buttons and parentheses for this computation as follows:

Enter 43 $\boxed{\times}$ 60 $\boxed{=}$ $\boxed{M+}$ 21 $\boxed{\times}$ 80 $\boxed{=}$ $\boxed{M+}$ 16 $\boxed{\times}$ 100 $\boxed{=}$ $\boxed{M+}$ $\boxed{MR}$ $\boxed{\div}$ $\boxed{(}$ 43 $\boxed{+}$ 21 $\boxed{+}$ 16 $\boxed{)}$ $\boxed{=}$ to get 73.25, or **\$73.25 per participant.**

When the $\boxed{M+}$ button is first used, the number in the calculator display is stored in memory and an **M** appears to the left of the display to show that the memory function is in use. Each subsequent use of the $\boxed{M+}$ button adds the number in the current display to the number stored in memory and replaces the number stored in memory by the sum. When the $\boxed{MR}$ button is pressed in the computation above, the current value in memory, 5,860, is displayed. To clear the memory, use the $\boxed{MC}$ button, and the **M** next to the display disappears.

3 Arithmetic

Your goals for this chapter

- Practice answering *GRE*® questions in arithmetic
- Review answers and explanations, particularly for questions you answered incorrectly

This chapter contains GRE Quantitative Reasoning practice questions that involve arithmetic.

Arithmetic topics include properties and types of integers, such as divisibility, factorization, prime numbers, remainders, and odd and even integers; arithmetic operations, exponents, and roots; and concepts such as estimation, percent, ratio, rate, absolute value, the number line, decimal representation, and sequences of numbers.

The questions are arranged by question type: Quantitative Comparison questions, followed by both types of Multiple-choice questions, and then Numeric Entry questions.

Following the questions is an answer key for quick reference. Then, at the end of the chapter, you will find complete explanations for every question. Each explanation is presented with the corresponding question for easy reference.

Review the answers and explanations carefully, paying particular attention to explanations for questions that you answered incorrectly.

Before answering the practice questions, read the Quantitative Reasoning section directions that begin on the following page. Also, review the directions that precede each question type to make sure you understand how to answer the questions.

Quantitative Reasoning Section Directions

For each question, indicate the best answer, using the directions given.

Notes: All numbers used are real numbers.

All figures are assumed to lie in a plane unless otherwise indicated.

Geometric figures, such as lines, circles, triangles, and quadrilaterals, **are not necessarily** drawn to scale. That is, you should **not** assume that quantities such as lengths and angle measures are as they appear in a figure. You should assume, however, that lines shown as straight are actually straight, points on a line are in the order shown, and more generally, all geometric objects are in the relative positions shown. For questions with geometric figures, you should base your answers on geometric reasoning, not on estimating or comparing quantities by sight or by measurement.

Coordinate systems, such as *xy*-planes and number lines, **are** drawn to scale; therefore, you can read, estimate, or compare quantities in such figures by sight or by measurement.

Graphical data presentations, such as bar graphs, circle graphs, and line graphs, **are** drawn to scale; therefore, you can read, estimate, or compare data values by sight or by measurement.

Quantitative Comparison Questions

For Questions 1 to 7, compare Quantity A and Quantity B, using additional information centered above the two quantities if such information is given. Select one of the following four answer choices and fill in the corresponding oval to the right of the question.

Ⓐ Quantity A is greater.

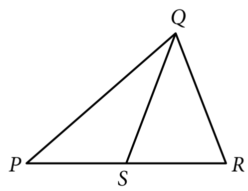
Ⓑ Quantity B is greater.

Ⓒ The two quantities are equal.

Ⓓ The relationship cannot be determined from the information given.

A symbol that appears more than once in a question has the same meaning throughout the question.

	Quantity A	Quantity B	Correct Answer
Example 1:	(2)(6)	2 + 6	Ⓐ (filled) Ⓑ Ⓒ Ⓓ



	Quantity A	Quantity B	Correct Answer
Example 2:	*PS*	*SR*	Ⓐ Ⓑ Ⓒ Ⓓ (filled)

(since equal lengths cannot be assumed, even though *PS* and *SR* appear equal)

D is the decimal form of the fraction $\frac{4}{11}$.

	Quantity A	Quantity B	
1.	The 25th digit to the right of the decimal point in D	4	Ⓐ Ⓑ Ⓒ Ⓓ

	Quantity A	Quantity B	
2.	$\sqrt[3]{270} - \sqrt[3]{10}$	$\sqrt[3]{80}$	Ⓐ Ⓑ Ⓒ Ⓓ

n is a positive integer, $x = 7n + 2$, and $y = 6n + 3$.

	Quantity A	Quantity B	
3.	The ones digit of $x + y$	5	Ⓐ Ⓑ Ⓒ Ⓓ

$r = 2$
$s = -7$

	Quantity A	Quantity B	
4.	$(r - s)^4$	$r^4 - s^4$	Ⓐ Ⓑ Ⓒ Ⓓ

---

n is an even negative integer.

	Quantity A	Quantity B	
5.	$\left(\frac{1}{3}\right)^n$	$(-3)^n$	Ⓐ Ⓑ Ⓒ Ⓓ

Today the price of a table was reduced by 20 percent from what it was yesterday, and the price of a lamp was reduced by 30 percent from what it was yesterday.

	Quantity A	Quantity B	
6.	The dollar amount of the reduction in the price of the table	The dollar amount of the reduction in the price of the lamp	Ⓐ Ⓑ Ⓒ Ⓓ

For 5 hours, a photocopier copied at a constant rate of 2 pages every 3 seconds.

	Quantity A	Quantity B	
7.	The number of pages the photocopier copied in the 5 hours	12,000	Ⓐ Ⓑ Ⓒ Ⓓ

Multiple-choice Questions—Select One Answer Choice

For Questions 8 to 13, select a single answer choice.

8. For each integer $n > 1$, let $A(n)$ denote the sum of the integers from 1 to n. For example, $A(100) = 1 + 2 + 3 + \cdots + 100 = 5{,}050$. What is the value of $A(200)$?

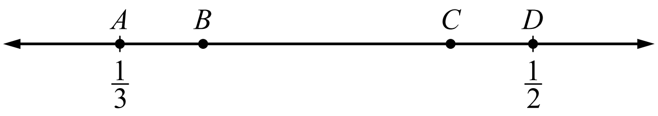
   Ⓐ 10,100
   Ⓑ 15,050
   Ⓒ 15,150
   Ⓓ 20,100
   Ⓔ 21,500

9. Which of the following integers CANNOT be expressed as the sum of two prime numbers?

   Ⓐ 8
   Ⓑ 9
   Ⓒ 10
   Ⓓ 11
   Ⓔ 12

10. When the positive integer n is divided by 45, the remainder is 18. Which of the following must be a divisor of n ?

   Ⓐ 11
   Ⓑ 9
   Ⓒ 7
   Ⓓ 6
   Ⓔ 4



11. Points A, B, C, and D are on the number line above, and $AB = CD = \frac{1}{3}(BC)$. What is the coordinate of C ?

Ⓐ $\frac{13}{30}$

Ⓑ $\frac{9}{20}$

Ⓒ $\frac{11}{24}$

Ⓓ $\frac{7}{15}$

Ⓔ $\frac{29}{60}$

12. Which of the following represents the total dollar amount that a customer would have to pay for an item that costs s dollars plus a sales tax of 8 percent, in terms of s ?

Ⓐ $\frac{s}{0.08}$

Ⓑ $\frac{s}{1.08}$

Ⓒ $\frac{s}{8}$

Ⓓ $0.08s$

Ⓔ $1.08s$

13. Marie earned \$0.75 for every mile she walked in a charity walkathon. If she earned a total of \$18.00 at that rate, how many miles did she walk?

Ⓐ 13.5

Ⓑ 17.5

Ⓒ 21

Ⓓ 22.5

Ⓔ 24

Multiple-choice Questions—Select One or More Answer Choices

For Questions 14 to 16, select all the answer choices that apply.

14. Which of the following operations carried out on both the numerator and the denominator of a fraction will always produce an equivalent fraction?

    Indicate all such operations.

    A Adding 2
    B Multiplying by 5
    C Dividing by 100

15. If $|z| \leq 1$, which of the following statements must be true?

    Indicate all such statements.

    A $z^2 \leq 1$
    B $z^2 \leq z$
    C $z^3 \leq z$

16. In a certain medical group, Dr. Schwartz schedules appointments to begin 30 minutes apart, Dr. Ramirez schedules appointments to begin 25 minutes apart, and Dr. Wu schedules appointments to begin 50 minutes apart. All three doctors schedule their first appointments to begin at 8:00 in the morning, which are followed by their successive appointments throughout the day without breaks. Other than at 8:00 in the morning, at what times before 1:30 in the afternoon do all three doctors schedule their appointments to begin at the same time?

    Indicate all such times.

    A 9:30 in the morning
    B 10:00 in the morning
    C 10:30 in the morning
    D 11:00 in the morning
    E 11:30 in the morning
    F 12:00 noon
    G 12:30 in the afternoon
    H 1:00 in the afternoon

Numeric Entry Questions

For Questions 17 to 19, enter your answer in the answer box(es) below the question.

- **Your answer may be an integer, a decimal, or a fraction, and it may be negative.**
- **If a question asks for a fraction, there will be two boxes—one for the numerator and one for the denominator. A decimal point cannot be used in a fraction.**
- **Equivalent forms of the correct answer, such as 2.5 and 2.50, are all correct. Fractions do not need to be reduced to lowest terms, though you may need to reduce your fraction to fit in the boxes.**
- **Enter the exact answer unless the question asks you to round your answer.**

17. The integers x and y are greater than 1. If $(4x)(7y) = 756,$ what is the value of $x + y$?

$x + y =$ [ ]

$$1, -3, 4, 1, -3, 4, 1, -3, 4, \ldots$$

18. In the sequence above, the first 3 terms repeat without end. What is the sum of the terms of the sequence from the 150th term to the 154th term?

[ ]

---

19. A manufacturing company has plants in three locations: Indonesia, Mexico, and Pakistan. The company has 6,000 employees, and each of the employees works at only one of the plants. If $\frac{3}{8}$ of the employees work at the plant in Indonesia and if twice as many employees work at the plant in Mexico as work at the plant in Pakistan, how many employees work at the plant in Mexico?

[ ] employees

ANSWER KEY

1. **Choice B**: Quantity B is greater.
2. **Choice C:** The two quantities are equal.
3. **Choice D**: The relationship cannot be determined from the information given.
4. **Choice A**: Quantity A is greater.
5. **Choice A**: Quantity A is greater.
6. **Choice D**: The relationship cannot be determined from the information given.
7. **Choice C**: The two quantities are equal.
8. **Choice D**: 20,100
9. **Choice D**: 11
10. **Choice B**: 9
11. **Choice D**: $\frac{7}{15}$
12. **Choice E**: 1.08s
13. **Choice E**: 24
14. **Choice B**: Multiplying by 5
    AND
    Choice C: Dividing by 100
15. **Choice A**: $z^2 \leq 1$
16. **Choice C**: 10:30 in the morning
    AND
    Choice H: 1:00 in the afternoon
17. **12**
18. **7**
19. **2,500**

Answers and Explanations

D is the decimal form of the fraction $\frac{4}{11}$.

	Quantity A	Quantity B	
1.	The 25th digit to the right of the decimal point in D	4	Ⓐ Ⓑ Ⓒ Ⓓ

Explanation

By dividing 4 by 11, you get the decimal form $D = 0.363636\ldots$, where the sequence of two digits "36" repeats without end. Continuing the repeating pattern, you see that the 1st digit, the 3rd digit, the 5th digit, and every subsequent odd-numbered digit to the right of the decimal point is 3. Therefore, Quantity A, the 25th digit to the right of the decimal point, is 3. Since Quantity A is 3 and Quantity B is 4, the correct answer is **Choice B**.

This explanation uses the following strategy.

Strategy 7: Find a Pattern

	Quantity A	Quantity B	
2.	$\sqrt[3]{270} - \sqrt[3]{10}$	$\sqrt[3]{80}$	Ⓐ Ⓑ Ⓒ Ⓓ

Explanation

You can simplify both quantities. Quantity A can be simplified as follows:

$$\sqrt[3]{270} - \sqrt[3]{10} = \left(\sqrt[3]{27}\right)\left(\sqrt[3]{10}\right) - \sqrt[3]{10} = 3\sqrt[3]{10} - \sqrt[3]{10} = 2\sqrt[3]{10}$$

Quantity B can be simplified as follows:

$$\sqrt[3]{80} = \left(\sqrt[3]{8}\right)\left(\sqrt[3]{10}\right) = 2\sqrt[3]{10}$$

Thus the correct answer is **Choice C**.

This explanation uses the following strategy.

Strategy 5: Simplify an Arithmetic or Algebraic Representation

n is a positive integer, $x = 7n + 2$, and $y = 6n + 3$.

	Quantity A	Quantity B	
3.	The ones digit of $x + y$	5	Ⓐ Ⓑ Ⓒ Ⓓ

Explanation

In the question, you are given that $x = 7n + 2$ and $y = 6n + 3$. Substituting the given expressions for x and y in Quantity A, you get $x + y = 13n + 5$.

Using trial and error, you can compare Quantity A, the ones digit of $13n + 5$, and Quantity B, 5, by plugging in a few values for the positive integer n.

If $n = 1$, then $x + y = 18$ and the ones digit is 8, which is greater than 5. So in this case Quantity A is greater than Quantity B.

If $n = 2$, then $x + y = 31$ and the ones digit is 1, which is less than 5. So in this case Quantity B is greater than Quantity A.

Since in one case Quantity A is greater than Quantity B, and in the other case Quantity B is greater than Quantity A, you can conclude that the correct answer is **Choice D**.

This explanation uses the following strategies.

Strategy 10: Trial and Error

Strategy 13: Determine Whether a Conclusion Follows from the Information Given

---

$$r = 2$$
$$s = -7$$

	Quantity A	Quantity B	
4.	$(r - s)^4$	$r^4 - s^4$	Ⓐ Ⓑ Ⓒ Ⓓ

Explanation

If you substitute the given numbers into the expressions in Quantities A and B, you can compare them.

Substituting in Quantity A gives $(r - s)^4 = (2 - (-7))^4 = 9^4$.

Substituting in Quantity B gives $r^4 - s^4 = 2^4 - (-7)^4 = 2^4 - 7^4$.

Without further calculation, you see that Quantity A is positive and Quantity B is negative. Thus the correct answer is **Choice A**.

This explanation uses the following strategy.

Strategy 5: Simplify an Arithmetic or Algebraic Representation

n is an even negative integer.

	Quantity A	Quantity B	
5.	$\left(\frac{1}{3}\right)^n$	$(-3)^n$	Ⓐ Ⓑ Ⓒ Ⓓ

Explanation

Try plugging the first few even negative integers into the expressions in Quantity A and Quantity B to see if a pattern emerges.

If $n = -2$, Quantity A is $\left(\frac{1}{3}\right)^{-2} = 3^2$ and Quantity B is $(-3)^{-2} = \frac{1}{(-3)^2} = \frac{1}{3^2}$.

If $n = -4$, Quantity A is $\left(\frac{1}{3}\right)^{-4} = 3^4$ and Quantity B is $(-3)^{-4} = \frac{1}{(-3)^4} = \frac{1}{3^4}$.

If $n = -6$, Quantity A is $\left(\frac{1}{3}\right)^{-6} = 3^6$ and Quantity B is $(-3)^{-6} = \frac{1}{(-3)^6} = \frac{1}{3^6}$.

From these three examples, it looks like Quantity A and Quantity B may always be reciprocals of each other, with Quantity A greater than 1 and Quantity B less than 1. You can see this as follows.

If n is an even negative integer, then n can be expressed as $-2k$ where k is a positive integer. Substituting $-2k$ for n in Quantity A and Quantity B, you get that Quantity A is $\left(\frac{1}{3}\right)^n = \left(\frac{1}{3}\right)^{-2k} = 3^{2k}$ and Quantity B is $(-3)^n = (-3)^{-2k} = \left(\frac{1}{3}\right)^{2k}$. Since for all positive integers k the value of 3^{2k} is greater than 1 and the value of $\left(\frac{1}{3}\right)^{2k}$ is less than 1, it follows that the correct answer is **Choice A**.

This explanation uses the following strategies.

Strategy 7: Find a Pattern

Strategy 8: Search for a Mathematical Relationship

Today the price of a table was reduced by 20 percent from what it was yesterday, and the price of a lamp was reduced by 30 percent from what it was yesterday.

	Quantity A	Quantity B	
6.	The dollar amount of the reduction in the price of the table	The dollar amount of the reduction in the price of the lamp	Ⓐ Ⓑ Ⓒ Ⓓ

Explanation

Quantity A is 20 percent of yesterday's price of the table. Since yesterday's price is not given, you cannot calculate this quantity. Similarly, you cannot calculate Quantity B. In the absence of further information with which to compare the two quantities, the correct answer is **Choice D**.

This explanation uses the following strategy.

Strategy 13: Determine Whether a Conclusion Follows from the Information Given

---

For 5 hours, a photocopier copied at a constant rate of 2 pages every 3 seconds.

	Quantity A	Quantity B	
7.	The number of pages the photocopier copied in the 5 hours	12,000	Ⓐ Ⓑ Ⓒ Ⓓ

Explanation

Translating the given information, you can calculate Quantity A, the number of pages the photocopier copied in the 5 hours. Copying at the rate of 2 pages every 3 seconds is the same as copying at a rate of 40 pages every 60 seconds, or 40 pages per minute. This rate is the same as copying 2,400 pages every hour, or 12,000 pages in 5 hours. Since Quantity A is equal to 12,000 and Quantity B is 12,000, the correct answer is **Choice C**.

This explanation uses the following strategy.

Strategy 1: Translate from Words to an Arithmetic or Algebraic Representation

8. For each integer $n > 1$, let $A(n)$ denote the sum of the integers from 1 to n.

   For example, $A(100) = 1 + 2 + 3 + \cdots + 100 = 5{,}050$. What is the value of $A(200)$?

   (A) 10,100
   (B) 15,050
   (C) 15,150
   (D) 20,100
   (E) 21,500

Explanation

In the question, you are given that $A(n)$ is equal to the sum of the integers from 1 to n, so $A(200) = 1 + 2 + 3 + \cdots + 100 + 101 + 102 + 103 + \cdots + 200$. In order to be able to use the given value of $A(100) = 5{,}050$, you can rewrite the sum as

$$\begin{aligned} A(200) &= A(100)+101+102+103+\cdots+200 \\ &= A(100)+(100+1)+(100+2)+(100+3)+\cdots+(100+100) \\ &= A(100)+(1+2+3+\cdots+100)+(100)(100) \\ &= A(100)+A(100)+(100)(100) \\ &= 5{,}050+5{,}050+10{,}000 \\ &= 20{,}100 \end{aligned}$$

Thus the correct answer is **Choice D**.

This explanation uses the following strategy.
Strategy 12: Adapt Solutions to Related Problems

---

9. Which of the following integers CANNOT be expressed as the sum of two prime numbers?

   (A) 8
   (B) 9
   (C) 10
   (D) 11
   (E) 12

Explanation

Trying to write each answer choice as a sum of two prime numbers by trial and error, you get:

Choice A: $8 = 3 + 5$
Choice B: $9 = 2 + 7$
Choice C: $10 = 3 + 7$
Choice D: $11 = 1 + 10 = 2 + 9 = 3 + 8 = 4 + 7 = 5 + 6$
Choice E: $12 = 5 + 7$

Of the answer choices given, only 11 cannot be expressed as the sum of two prime numbers. The correct answer is **Choice D**.

This explanation uses the following strategy.
Strategy 10: Trial and Error

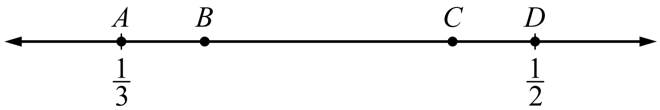
10. When the positive integer n is divided by 45, the remainder is 18. Which of the following must be a divisor of n ?

Ⓐ 11
Ⓑ 9
Ⓒ 7
Ⓓ 6
Ⓔ 4

Explanation

The given information tells you that n can be expressed in the form $n = 45k + 18$, where k can be any nonnegative integer. Consider how the divisors of 45 and 18 may be related to the divisors of n. Every common divisor of 45 and 18 is also a divisor of any sum of multiples of 45 and 18, like $45k + 18$. So any common divisor of 45 and 18 is also a divisor of n. Of the answer choices given, only 9 is a common divisor of 45 and 18. Thus the correct answer is **Choice B**.

This explanation uses the following strategies.
Strategy 1: Translate from Words to an Arithmetic or Algebraic Representation
Strategy 8: Search for a Mathematical Relationship



11. Points A, B, C, and D are on the number line above, and $AB = CD = \frac{1}{3}(BC)$. What is the coordinate of C ?

Ⓐ $\frac{13}{30}$
Ⓑ $\frac{9}{20}$
Ⓒ $\frac{11}{24}$
Ⓓ $\frac{7}{15}$
Ⓔ $\frac{29}{60}$

Explanation

From the figure you can see that since the coordinate of A is $\frac{1}{3}$, it follows that the coordinate of C is $\frac{1}{3} + AB + BC$. Since you are given that $AB = \frac{1}{3}(BC)$, the coordinate of C can be rewritten in terms of AB as follows:

$$\frac{1}{3} + AB + BC = \frac{1}{3} + AB + 3(AB) = \frac{1}{3} + 4(AB)$$

To find the coordinate of C, you need to know AB. From the figure, you know that $AD = AB + BC + CD = AB + 3(AB) + AB = 5(AB)$. On the other hand, since the coordinate of A is $\frac{1}{3}$ and the coordinate of D is $\frac{1}{2}$, it follows that $AD = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$. Therefore you can conclude that $\frac{1}{6} = 5(AB)$ and $AB = \frac{1}{30}$. Thus the coordinate of C is $\frac{1}{3} + 4(AB) = \frac{1}{3} + 4\left(\frac{1}{30}\right)$, or $\frac{7}{15}$. The correct answer is **Choice D**.

This explanation uses the following strategies.
Strategy 4: Translate from a Figure to an Arithmetic or Algebraic Representation
Strategy 8: Search for a Mathematical Relationship

---

12. Which of the following represents the total dollar amount that a customer would have to pay for an item that costs s dollars plus a sales tax of 8 percent, in terms of s ?

Ⓐ $\frac{s}{0.08}$

Ⓑ $\frac{s}{1.08}$

Ⓒ $\frac{s}{8}$

Ⓓ $0.08s$

Ⓔ $1.08s$

Explanation

The total dollar amount that the customer would have to pay is equal to the cost plus 8 percent of the cost. Translating to an algebraic expression, you get that the total amount is $s + 0.08s$, or $1.08s$. Thus the correct answer is **Choice E**.

This explanation uses the following strategy.
Strategy 1: Translate from Words to an Arithmetic or Algebraic Representation

13. Marie earned \$0.75 for every mile she walked in a charity walkathon. If she earned a total of \$18.00 at that rate, how many miles did she walk?

Ⓐ 13.5
Ⓑ 17.5
Ⓒ 21
Ⓓ 22.5
Ⓔ 24

Explanation

You can translate the given information into an algebraic equation. If Marie walks m miles, she earns $0.75m$ dollars. Since you know that she earned a total of \$18, you get $0.75m = 18$. Solving for m, you have $m = \frac{18}{0.75} = 24$. Thus the correct answer is **Choice E**.

This explanation uses the following strategy.
Strategy 1: Translate from Words to an Arithmetic or Algebraic Representation

---

14. Which of the following operations carried out on both the numerator and the denominator of a fraction will always produce an equivalent fraction?

Indicate all such operations.

[A] Adding 2
[B] Multiplying by 5
[C] Dividing by 100

Explanation

Multiplying both the numerator and the denominator of a fraction by the same nonzero number is equivalent to multiplying the fraction by 1, thus producing an equivalent fraction. The same is true for division. However, adding the same number to both the numerator and denominator does not usually produce an equivalent fraction. Here is an example:

$$\frac{1}{2} \neq \frac{1+2}{2+2} = \frac{3}{4}$$

Thus the correct answer consists of **Choices B and C**.

This explanation uses the following strategy.
Strategy 13: Determine Whether a Conclusion Follows from the Information Given

15. If $|z| \leq 1$, which of the following statements must be true?

Indicate all such statements.

$\boxed{A}$ $z^2 \leq 1$

$\boxed{B}$ $z^2 \leq z$

$\boxed{C}$ $z^3 \leq z$

Explanation

The condition stated in the question, $|z| \leq 1$, includes both positive and negative values of z. For example, both $\frac{1}{2}$ and $-\frac{1}{2}$ are possible values of z. Keep this in mind as you evaluate each of the inequalities in the answer choices to see whether the inequality must be true.

Choice A: $z^2 \leq 1$. First look at what happens for a positive and a negative value of z for which $|z| \leq 1$, say, $z = \frac{1}{2}$ and $z = -\frac{1}{2}$. If $z = \frac{1}{2}$, then $z^2 = \frac{1}{4}$. If $z = -\frac{1}{2}$, then $z^2 = \frac{1}{4}$. So in both these cases it is true that $z^2 \leq 1$.

Since the inequality $z^2 \leq 1$ is true for a positive and a negative value of z, try to prove that it is true for all values of z such that $|z| \leq 1$. Recall that if $0 \leq c \leq 1$, then $c^2 \leq 1$. Since $0 \leq |z| \leq 1$, letting $c = |z|$ yields $|z|^2 \leq 1$. Also, it is always true that $|z|^2 = z^2$, and so $z^2 \leq 1$.

Choice B: $z^2 \leq z$. As before, look at what happens when $z = \frac{1}{2}$ and when $z = -\frac{1}{2}$. If $z = \frac{1}{2}$, then $z^2 = \frac{1}{4}$. If $z = -\frac{1}{2}$, then $z^2 = \frac{1}{4}$. So when $z = \frac{1}{2}$, the inequality $z^2 \leq z$ is true, and when $z = -\frac{1}{2}$, the inequality $z^2 \leq z$ is false. Therefore, you can conclude that if $|z| \leq 1$, it is not necessarily true that $z^2 \leq z$.

Choice C: $z^3 \leq z$. As before, look at what happens when $z = \frac{1}{2}$ and when $z = -\frac{1}{2}$. If $z = \frac{1}{2}$, then $z^3 = \frac{1}{8}$. If $z = -\frac{1}{2}$, then $z^3 = -\frac{1}{8}$. So when $z = \frac{1}{2}$, the inequality $z^3 \leq z$ is true, and when $z = -\frac{1}{2}$, the inequality $z^3 \leq z$ is false. Therefore, you can conclude that if $|z| \leq 1$, it is not necessarily true that $z^3 \leq z$.

Thus when $|z| \leq 1$, Choice A, $z^2 \leq 1$, must be true, but the other two choices are not necessarily true. The correct answer consists of **Choice A**.

This explanation uses the following strategies.

Strategy 8: Search for a Mathematical Relationship

Strategy 10: Trial and Error

Strategy 13: Determine Whether a Conclusion Follows from the Information Given

16. In a certain medical group, Dr. Schwartz schedules appointments to begin 30 minutes apart, Dr. Ramirez schedules appointments to begin 25 minutes apart, and Dr. Wu schedules appointments to begin 50 minutes apart. All three doctors schedule their first appointments to begin at 8:00 in the morning, which are followed by their successive appointments throughout the day without breaks. Other than at 8:00 in the morning, at what times before 1:30 in the afternoon do all three doctors schedule their appointments to begin at the same time?

Indicate all such times.

- A 9:30 in the morning
- B 10:00 in the morning
- C 10:30 in the morning
- D 11:00 in the morning
- E 11:30 in the morning
- F 12:00 noon
- G 12:30 in the afternoon
- H 1:00 in the afternoon

Explanation

By examining the pattern of beginning times for the three types of appointments, you can see that the times will coincide when the number of minutes after 8:00 in the morning is a common multiple of 30, 25, and 50. The least common multiple of 30, 25, and 50 is 150, which represents 150 minutes, or 2.5 hours. So the times coincide every 2.5 hours after 8:00 in the morning, that is, at 10:30 in the morning, at 1:00 in the afternoon, and so on. The correct answer consists of **Choices C and H**.

This explanation uses the following strategy.
Strategy 7: Find a Pattern

17. The integers x and y are greater than 1. If $(4x)(7y) = 756,$ what is the value of $x + y$?

$$x + y = \boxed{}$$

Explanation

You can solve the given equation, $(4x)(7y) = 756,$ for the product xy:

$$(4x)(7y) = 756$$
$$28xy = 756$$
$$xy = 27$$

By trial and error, you find that 3 and 9 are the only two integers greater than 1 whose product is 27. So $x + y = 12,$ and the correct answer is **12**.

This explanation uses the following strategy.
Strategy 10: Trial and Error

$$1, -3, 4, 1, -3, 4, 1, -3, 4, \ldots$$

18. In the sequence above, the first 3 terms repeat without end. What is the sum of the terms of the sequence from the 150th term to the 154th term?

$$\boxed{}$$

Explanation

Examining the repeating pattern, you see that the 3rd term is 4, and every 3rd term after that, in other words, the 6th, 9th, 12th, 15th, and so on, is 4. Since 150 is a multiple of 3, the 150th term is 4. Therefore the 150th to the 154th terms are 4, 1, –3, 4, 1. The sum of these 5 terms is 7, so the correct answer is **7**.

This explanation uses the following strategy.
Strategy 7: Find a Pattern

19. A manufacturing company has plants in three locations: Indonesia, Mexico, and Pakistan. The company has 6,000 employees, and each of the employees works at only one of the plants. If $\frac{3}{8}$ of the employees work at the plant in Indonesia and if twice as many employees work at the plant in Mexico as work at the plant in Pakistan, how many employees work at the plant in Mexico?

[ ] employees

Explanation

Three-eighths of the company's 6,000 employees work in Indonesia, so the number of employees that do not work in Indonesia is $\frac{5}{8}(6{,}000),$ or 3,750. Of those employees, twice as many work in Mexico as work in Pakistan, so the number working in Mexico is $\frac{2}{3}(3{,}750),$ or 2,500. Thus the correct answer is **2,500** employees.

This explanation uses the following strategy.

Strategy 1: Translate from Words to an Arithmetic or Algebraic Representation

4 Algebra

Your goals for this chapter

- ⇨ Practice answering *GRE*® questions in algebra
- ⇨ Review answers and explanations, particularly for questions you answered incorrectly

This chapter contains GRE Quantitative Reasoning practice questions that involve algebra.

Algebra topics include operations with exponents; factoring and simplifying algebraic expressions; relations, functions, equations, and inequalities; solving linear and quadratic equations and inequalities; solving simultaneous equations and inequalities; setting up equations to solve word problems; and coordinate geometry, including graphs of functions, equations, and inequalities, intercepts, and slopes of lines.

The questions are arranged by question type: Quantitative Comparison questions, followed by both types of Multiple-choice questions, and then Numeric Entry questions.

Following the questions is an answer key for quick reference. Then, at the end of the chapter, you will find complete explanations for every question. Each explanation is presented with the corresponding question for easy reference.

Review the answers and explanations carefully, paying particular attention to explanations for questions that you answered incorrectly.

Before answering the practice questions, read the Quantitative Reasoning section directions that begin on the following page. Also, review the directions that precede each question type to make sure you understand how to answer the questions.

Quantitative Reasoning Section Directions

For each question, indicate the best answer, using the directions given.

Notes: All numbers used are real numbers.

All figures are assumed to lie in a plane unless otherwise indicated.

Geometric figures, such as lines, circles, triangles, and quadrilaterals, **are not necessarily** drawn to scale. That is, you should **not** assume that quantities such as lengths and angle measures are as they appear in a figure. You should assume, however, that lines shown as straight are actually straight, points on a line are in the order shown, and more generally, all geometric objects are in the relative positions shown. For questions with geometric figures, you should base your answers on geometric reasoning, not on estimating or comparing quantities by sight or by measurement.

Coordinate systems, such as *xy*-planes and number lines, **are** drawn to scale; therefore, you can read, estimate, or compare quantities in such figures by sight or by measurement.

Graphical data presentations, such as bar graphs, circle graphs, and line graphs, **are** drawn to scale; therefore, you can read, estimate, or compare data values by sight or by measurement.

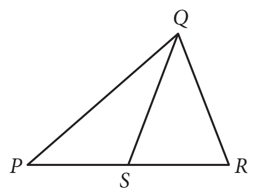
Quantitative Comparison Questions

For Questions 1 to 7, compare Quantity A and Quantity B, using additional information centered above the two quantities if such information is given. Select one of the following four answer choices and fill in the corresponding oval to the right of the question.

Ⓐ Quantity A is greater.
Ⓑ Quantity B is greater.
Ⓒ The two quantities are equal.
Ⓓ The relationship cannot be determined from the information given.

A symbol that appears more than once in a question has the same meaning throughout the question.

	Quantity A	Quantity B	Correct Answer
Example 1:	(2)(6)	2 + 6	Ⓐ (filled) Ⓑ Ⓒ Ⓓ



	Quantity A	Quantity B	Correct Answer
Example 2:	*PS*	*SR*	Ⓐ Ⓑ Ⓒ Ⓓ (filled) (since equal lengths cannot be assumed, even though *PS* and *SR* appear equal)

$$\frac{x(x-2)}{(x+3)(x-4)^2}=0$$

	Quantity A	Quantity B	
1.	x	-2	Ⓐ Ⓑ Ⓒ Ⓓ

$$x > 0$$

	Quantity A	Quantity B	
2.	$\frac{1}{x}$	$\frac{x+1}{x^2}$	Ⓐ Ⓑ Ⓒ Ⓓ

	Quantity A	Quantity B	
3.	$\lvert m + 25 \rvert$	$25 - m$	Ⓐ Ⓑ Ⓒ Ⓓ

During an experiment, the pressure of a fixed mass of gas increased from 40 pounds per square inch (psi) to 50 psi. Throughout the experiment, the pressure, P psi, and the volume, V cubic inches, of the gas varied in such a way that the value of the product PV was constant.

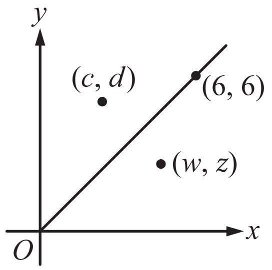
	Quantity A	Quantity B	
4.	The volume of the gas when the pressure was 40 psi	1.2 times the volume of the gas when the pressure was 50 psi	Ⓐ Ⓑ Ⓒ Ⓓ

$$x > 0$$

	Quantity A	Quantity B	
5.	x percent of $100x$	x^2	Ⓐ Ⓑ Ⓒ Ⓓ

$$(4x - 2y)(6x + 3y) = 18$$

	Quantity A	Quantity B	
6.	$4x^2 - y^2$	6	Ⓐ Ⓑ Ⓒ Ⓓ



	Quantity A	Quantity B	
7.	$w + d$	$c + z$	Ⓐ Ⓑ Ⓒ Ⓓ

Multiple-choice Questions—Select One Answer Choice

For Questions 8 to 13, select a single answer choice.

8. If $xy^2 = 12$ and $xy = 4$, then $x =$

Ⓐ 1

Ⓑ 2

Ⓒ $\sqrt{3}$

Ⓓ $\frac{2}{3}$

Ⓔ $\frac{4}{3}$

9. The total cost of 8 bagels at a bakery is x dollars. At this cost per bagel, which of the following represents the total cost, in dollars, of y bagels?

Ⓐ $\frac{8}{xy}$

Ⓑ $\frac{8x}{y}$

Ⓒ $\frac{8y}{x}$

Ⓓ $\frac{xy}{8}$

Ⓔ $\frac{x}{8y}$

10. Which of the following is equal to $\frac{2^{x-y}}{2^{x+y}}$ for all integers x and y ?

Ⓐ 4^{-x}

Ⓑ 4^{-y}

Ⓒ 2^{xy}

Ⓓ 4^{x}

Ⓔ 4^{y}

11. How many integers are in the solution set of the inequality $x^2 - 10 < 0$?

Ⓐ Two

Ⓑ Five

Ⓒ Six

Ⓓ Seven

Ⓔ Ten

12. A group of 5,000 investors responded to a survey asking whether they owned stocks and whether they owned bonds. Of the group, 20 percent responded that they owned only one of the two types of investments. If r is the number of investors in the group who owned stocks but not bonds, which of the following represents the number of investors in the group who owned bonds but not stocks, in terms of r ?

Ⓐ $5{,}000 - r$

Ⓑ $1{,}000 - r$

Ⓒ $r - 1{,}000$

Ⓓ $1{,}000r$

Ⓔ $(0.2)(5{,}000 - r)$

13. If $\dfrac{r+s}{4+5} = \dfrac{r}{4} + \dfrac{s}{5}$, which of the following statements must be true?

Ⓐ $r = s$

Ⓑ $5r = 4s$

Ⓒ $5r = -4s$

Ⓓ $25r = 16s$

Ⓔ $25r = -16s$

Multiple-choice Questions—Select One or More Answer Choices

For Questions 14 to 15, select all the answer choices that apply.

14. In the xy-plane, triangular region R is bounded by the lines $x = 0$, $y = 0$, and $4x + 3y = 60$. Which of the following points lie inside region R ?

    Indicate all such points.

    A (2, 18)
    B (5, 12)
    C (10, 7)
    D (12, 3)
    E (15, 2)

15. At the beginning of a trip, the tank of Diana's car was filled with gasoline to half of its capacity. During the trip, Diana used 30 percent of the gasoline in the tank. At the end of the trip, Diana added 8 gallons of gasoline to the tank. The capacity of the tank of Diana's car was x gallons. Which of the following expressions represent the number of gallons of gasoline in the tank after Diana added gasoline to the tank at the end of the trip?

    Indicate all such expressions.

    A $\frac{x}{2} - \frac{3x}{20} + 8$

    B $\frac{7x}{20} + 8$

    C $\frac{3x}{20} + 8$

    D $\frac{x}{2} + \frac{3x}{20} - 8$

    E $\frac{7x}{20} - 8$

Numeric Entry Questions

For Questions 16 to 17, enter your answer in the answer box(es) below the question.

- **Your answer may be an integer, a decimal, or a fraction, and it may be negative.**
- **If a question asks for a fraction, there will be two boxes—one for the numerator and one for the denominator. A decimal point cannot be used in a fraction.**
- **Equivalent forms of the correct answer, such as 2.5 and 2.50, are all correct. Fractions do not need to be reduced to lowest terms, though you may need to reduce your fraction to fit in the boxes.**
- **Enter the exact answer unless the question asks you to round your answer.**

16. Machine A, working alone at its constant rate, produces x pounds of peanut butter in 12 minutes. Machine B, working alone at its constant rate, produces x pounds of peanut butter in 18 minutes. How many minutes will it take machines A and B, working simultaneously at their respective constant rates, to produce x pounds of peanut butter?

$\boxed{}$ minutes

17. The function f has the property that $f(x) = f(x+1)$ for all numbers x. If $f(4) = 17$, what is the value of $f(8)$?

$\boxed{}$

ANSWER KEY

1. **Choice A**: Quantity A is greater.
2. **Choice B**: Quantity B is greater.
3. **Choice D**: The relationship cannot be determined from the information given.
4. **Choice A**: Quantity A is greater.
5. **Choice C**: The two quantities are equal.
6. **Choice B**: Quantity B is greater.
7. **Choice A**: Quantity A is greater.
8. **Choice E**: $\frac{4}{3}$
9. **Choice D**: $\frac{xy}{8}$
10. **Choice B**: 4^{-y}
11. **Choice D**: Seven
12. **Choice B**: $1{,}000 - r$
13. **Choice E**: $25r = -16s$
14. **Choice B**: (5, 12)

    AND

    Choice D: (12, 3)
15. **Choice A**: $\frac{x}{2} - \frac{3x}{20} + 8$

    AND

    Choice B: $\frac{7x}{20} + 8$
16. **7.2**
17. **17**

Answers and Explanations

$$\frac{x(x-2)}{(x+3)(x-4)^2}=0$$

	Quantity A	Quantity B	
1.	x	-2	Ⓐ Ⓑ Ⓒ Ⓓ

Explanation

To compare x with -2, you should first solve the equation $\frac{x(x-2)}{(x+3)(x-4)^2}=0$ for x.

To solve the equation, recall that a fraction $\frac{a}{b}$ is equal to 0 only if $a=0$ and $b \neq 0$

So the fraction $\frac{x(x-2)}{(x+3)(x-4)^2}$ is equal to 0 only if $x(x-2)$ is equal to 0 and $(x+3)(x-4)^2$ is not equal to 0. Note that the only values of x for which $x(x-2)=0$ are $x=0$ and $x=2$, and for these two values, $(x+3)(x-4)^2$ is not equal to 0. Therefore the only values of x for which the fraction $\frac{x(x-2)}{(x+3)(x-4)^2}$ is equal to 0 are $x=0$ and $x=2$. Since both 0 and 2 are greater than Quantity B, -2, the correct answer is **Choice A**.

This explanation uses the following strategy.
Strategy 8: Search for a Mathematical Relationship

$$x>0$$

	Quantity A	Quantity B	
2.	$\frac{1}{x}$	$\frac{x+1}{x^2}$	Ⓐ Ⓑ Ⓒ Ⓓ

Explanation

Note that Quantity B, $\frac{x+1}{x^2}$, can be expressed as $\frac{x}{x^2}+\frac{1}{x^2}$, which can be simplified to $\frac{1}{x}+\frac{1}{x^2}$. Note that Quantity A is $\frac{1}{x}$, and for all nonzero values of x, $\frac{1}{x^2}>0$. It follows that $\frac{1}{x}+\frac{1}{x^2}>\frac{1}{x}$; that is, Quantity B is greater than Quantity A. Thus the correct answer is **Choice B**.

This explanation uses the following strategy.
Strategy 8: Search for a Mathematical Relationship

	Quantity A	Quantity B	
3.	$\lvert m+25 \rvert$	$25-m$	Ⓐ Ⓑ Ⓒ Ⓓ

Explanation

To compare $|m+25|$ with $25-m$, first note that because the absolute value of any quantity is greater than or equal to 0, it must be true that $|m+25| \geq 0$ for all values of m. But you know that $25-m$ is less than 0 when m is greater than 25. So Quantity A is greater than Quantity B when $m > 25$.

On the other hand, both $|m+25|$ and $25-m$ are equal to 25 when $m=0$. So in one case Quantity A is greater than Quantity B, and in the other case the two quantities are equal. Thus the correct answer is **Choice D**.

This explanation uses the following strategies.

Strategy 11: Divide into Cases
Strategy 13: Determine Whether a Conclusion Follows from the Information Given

---

During an experiment, the pressure of a fixed mass of gas increased from 40 pounds per square inch (psi) to 50 psi. Throughout the experiment, the pressure, P psi, and the volume, V cubic inches, of the gas varied in such a way that the value of the product PV was constant.

	Quantity A	Quantity B	
4.	The volume of the gas when the pressure was 40 psi	1.2 times the volume of the gas when the pressure was 50 psi	Ⓐ Ⓑ Ⓒ Ⓓ

Explanation

You are given that the relationship between the pressure and volume of the gas throughout the experiment was $PV = C$, where C is a positive constant.

Quantity A is the volume of the gas when the pressure was 40 psi. At this pressure, the equation $PV = C$ becomes $40V = C$. Solving this equation for V, you get $V = \frac{C}{40} = 0.025C$; that is, the volume of the gas was $0.025C$ cubic inches.

Quantity B is 1.2 times the volume of the gas when the pressure was 50 psi. At this pressure, the equation $PV = C$ becomes $50V = C$, or $V = \frac{C}{50} = 0.02C$. Therefore $(1.2)V = (1.2)(0.02C) = 0.024C$; that is, Quantity B is $0.024C$ cubic inches. Thus the correct answer is **Choice A**.

This explanation uses the following strategies.

Strategy 1: Translate from Words to an Arithmetic or Algebraic Representation
Strategy 8: Search for a Mathematical Relationship

$$x > 0$$

	Quantity A	Quantity B	
5.	x percent of $100x$	x^2	Ⓐ Ⓑ Ⓒ Ⓓ

Explanation

Note that Quantity A can be expressed algebraically as $\left(\frac{x}{100}\right)(100x)$. Simplifying this algebraic expression gives x^2, which is Quantity B. The correct answer is therefore **Choice C**.

This explanation uses the following strategy.
Strategy 1: Translate from Words to an Arithmetic or Algebraic Representation

---

$$(4x - 2y)(6x + 3y) = 18$$

	Quantity A	Quantity B	
6.	$4x^2 - y^2$	6	Ⓐ Ⓑ Ⓒ Ⓓ

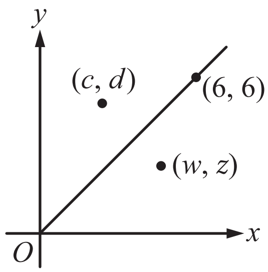
Explanation

The given equation $(4x - 2y)(6x + 3y) = 18$ can be simplified as follows.

Step 1: Note that $(4x - 2y) = 2(2x - y)$ and $(6x + 3y) = 3(2x + y)$, so the given equation can be rewritten as $(2)(2x - y)(3)(2x + y) = 18$.
Step 2: Dividing both sides of the rewritten equation by 6 gives $(2x - y)(2x + y) = 3$.
Step 3: Multiplying out the left side of the equation in Step 2 gives $4x^2 - y^2 = 3$.

Since Quantity A is $4x^2 - y^2$, it follows that Quantity A is equal to 3. Since Quantity B is 6, the correct answer is **Choice B**.

This explanation uses the following strategy.
Strategy 8: Search for a Mathematical Relationship



	Quantity A	Quantity B	
7.	$w + d$	$c + z$	Ⓐ Ⓑ Ⓒ Ⓓ

Explanation

The line in the xy-plane passes through the origin and the point with coordinates (6, 6), so the equation of the line is $y = x$. Since the point with coordinates (c, d) is above the line, it follows that $d > c$, and since the point with coordinates (w, z) is below the line, it follows that $w > z$. From the fact that $d > c$ and $w > z$, you get $w + d > c + z$; that is, Quantity A is greater than Quantity B. Thus the correct answer is **Choice A**.

This explanation uses the following strategy.
Strategy 8: Search for a Mathematical Relationship

---

8. If $xy^2 = 12$ and $xy = 4$, then $x =$

Ⓐ 1
Ⓑ 2
Ⓒ $\sqrt{3}$
Ⓓ $\frac{2}{3}$
Ⓔ $\frac{4}{3}$

Explanation

From the given equations $xy^2 = 12$ and $xy = 4$, it follows that $12 = xy^2 = (xy)y = 4y$, and so $y = 3$. Substituting $y = 3$ in the equation $xy = 4$ gives $3x = 4$, or $x = \frac{4}{3}$. Thus the correct answer is Choice E.

This explanation uses the following strategy.
Strategy 8: Search for a Mathematical Relationship

9. The total cost of 8 bagels at a bakery is x dollars. At this cost per bagel, which of the following represents the total cost, in dollars, of y bagels?

(A) $\frac{8}{xy}$

(B) $\frac{8x}{y}$

(C) $\frac{8y}{x}$

(D) $\frac{xy}{8}$

(E) $\frac{x}{8y}$

Explanation

The total cost of 8 bagels is x dollars, so one bagel costs $\frac{x}{8}$ dollars. Therefore the total cost of y bagels is $y\left(\frac{x}{8}\right)$, or $\frac{xy}{8}$ dollars. Thus the correct answer is **Choice D**.

This explanation uses the following strategy.

Strategy 1: Translate from Words to an Arithmetic or Algebraic Representation

---

10. Which of the following is equal to $\frac{2^{x-y}}{2^{x+y}}$ for all integers x and y ?

(A) 4^{-x}

(B) 4^{-y}

(C) 2^{xy}

(D) 4^{x}

(E) 4^{y}

Explanation

Simplifying the fraction $\frac{2^{x-y}}{2^{x+y}}$ yields $\frac{2^{x-y}}{2^{x+y}} = 2^{x-y-(x+y)} = 2^{x-y-x-y} = 2^{-2y}$. The only answer choice with 2 as the base is Choice C, 2^{xy}, which is clearly not the correct answer. Since all the other answer choices have 4 as the base, it is a good idea to rewrite 2^{-2y} as an expression with 4 as the base, as follows.

$$2^{-2y} = \frac{1}{2^{2y}} = \frac{1}{(2^2)^y} = \frac{1}{4^y} = 4^{-y}$$

Thus the correct answer is **Choice B**.

This explanation uses the following strategies.

Strategy 5: Simplify an Arithmetic or Algebraic Representation
Strategy 8: Search for a Mathematical Relationship

11. How many integers are in the solution set of the inequality $x^2 - 10 < 0$?

Ⓐ Two

Ⓑ Five

Ⓒ Six

Ⓓ Seven

Ⓔ Ten

Explanation

The inequality $x^2 - 10 < 0$ is equivalent to $x^2 < 10$. By inspection, the positive integers that satisfy this inequality are 1, 2, and 3. Note that 0 and the negative integers −1, −2, and −3 also satisfy the inequality, and there are no other integer solutions. So there are seven integers in the solution set: −3, −2, −1, 0, 1, 2, and 3. Thus the correct answer is **Choice D**.

This explanation uses the following strategy.
Strategy 8: Search for a Mathematical Relationship

---

12. A group of 5,000 investors responded to a survey asking whether they owned stocks and whether they owned bonds. Of the group, 20 percent responded that they owned only one of the two types of investments. If r is the number of investors in the group who owned stocks but not bonds, which of the following represents the number of investors in the group who owned bonds but not stocks, in terms of r ?

Ⓐ $5{,}000 - r$

Ⓑ $1{,}000 - r$

Ⓒ $r - 1{,}000$

Ⓓ $1{,}000r$

Ⓔ $(0.2)(5{,}000 - r)$

Explanation

Twenty percent of the 5,000 investors that responded to the survey said they owned either stocks or bonds, but not both. So the number of investors in that group is (0.20)(5,000), or 1,000. Given that r members of that group owned stocks but not bonds, the number of investors in that group who owned bonds but not stocks is $1{,}000 - r$. Thus the correct answer is **Choice B**.

This explanation uses the following strategies.
Strategy 1: Translate from Words to an Arithmetic or Algebraic Representation
Strategy 8: Search for a Mathematical Relationship

13. If $\frac{r+s}{4+5}=\frac{r}{4}+\frac{s}{5}$, which of the following statements must be true?

Ⓐ $r=s$

Ⓑ $5r=4s$

Ⓒ $5r=-4s$

Ⓓ $25r=16s$

Ⓔ $25r=-16s$

Explanation

You can simplify each side of the equation $\frac{r+s}{4+5}=\frac{r}{4}+\frac{s}{5}$ as follows.

$$\frac{r+s}{4+5}=\frac{r+s}{9}$$

$$\frac{r}{4}+\frac{s}{5}=\frac{5r+4s}{20}$$

So the equation $\frac{r+s}{4+5}=\frac{r}{4}+\frac{s}{5}$ can be rewritten as $\frac{r+s}{9}=\frac{5r+4s}{20}$.

Cross multiplying in the rewritten equation gives $20(r+s)=9(5r+4s)$, which simplifies to $20r+20s=45r+36s$, or $-16s=25r$. Thus the correct answer is **Choice E**.

This explanation uses the following strategy.

Strategy 5: Simplify an Arithmetic or Algebraic Representation

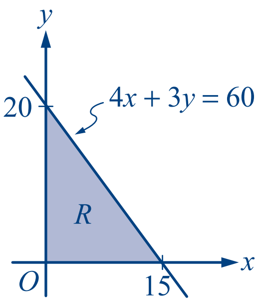
14. In the xy-plane, triangular region R is bounded by the lines $x = 0$, $y = 0$, and $4x + 3y = 60$. Which of the following points lie inside region R ?

Indicate all such points.

A (2, 18)
B (5, 12)
C (10, 7)
D (12, 3)
E (15, 2)

Explanation

Consider the three lines that bound triangular region R. The line $x = 0$ is the y-axis, and the line $y = 0$ is the x-axis. The line $4x + 3y = 60$ intersects the x-axis at (15, 0) and intersects the y-axis at (0, 20). The figure below shows region R.



From the figure, you can see that all points inside region R have positive coordinates and lie below the line $4x + 3y = 60$. Note that the equation $4x + 3y = 60$ can be rewritten in the form $y = 20 - \frac{4}{3}x$. In this form, you can see that points inside region R satisfy the inequality $y < 20 - \frac{4}{3}x$. Since all of the answer choices have positive coordinates, you need only to check whether the coordinates in each answer choice satisfy the inequality $y < 20 - \frac{4}{3}x$, or equivalently $4x + 3y < 60$.

Choice A, (2, 18): $4x + 3y = 4(2) + 3(18) = 62 > 60$. So Choice A is not in region R.

Choice B, (5, 12): $4x + 3y = 4(5) + 3(12) = 56 < 60$. So Choice B is in region R.

Choice C, (10, 7): $4x + 3y = 4(10) + 3(7) = 61 > 60$. So Choice C is in region R.

Choice D, (12, 3): $4x + 3y = 4(12) + 3(3) = 57 < 60$. So Choice D is in region R.

Choice E, (15, 2): $4x + 3y = 4(15) + 3(2) = 66 > 60$. So Choice E is not in region R.

Thus the correct answer consists of **Choices B and D**.

This explanation uses the following strategies.
Strategy 3: Translate from an Algebraic to a Graphical Representation
Strategy 8: Search for a Mathematical Relationship

15. At the beginning of a trip, the tank of Diana's car was filled with gasoline to half of its capacity. During the trip, Diana used 30 percent of the gasoline in the tank. At the end of the trip, Diana added 8 gallons of gasoline to the tank. The capacity of the tank of Diana's car was x gallons. Which of the following expressions represent the number of gallons of gasoline in the tank after Diana added gasoline to the tank at the end of the trip?

Indicate <u>all</u> such expressions.

- A $\frac{x}{2} - \frac{3x}{20} + 8$
- B $\frac{7x}{20} + 8$
- C $\frac{3x}{20} + 8$
- D $\frac{x}{2} + \frac{3x}{20} - 8$
- E $\frac{7x}{20} - 8$

Explanation

The capacity of the car's tank was x gallons of gasoline. Before the trip, the tank was half full and therefore contained $\frac{x}{2}$ gallons of gasoline. During the trip, 30 percent of the gasoline in the tank was used, so the number of gallons left was 70 percent of $\frac{x}{2}$, or $\left(\frac{7}{10}\right)\left(\frac{x}{2}\right) = \frac{7x}{20}$. After Diana added 8 gallons of gasoline to the tank, the total number of gallons in the tank was $\frac{7x}{20} + 8$. Thus one correct choice is Choice B, $\frac{7x}{20} + 8$.

However, the question asks you to find <u>all</u> of the answer choices that represent the number of gallons of gasoline in the tank at the end of the trip. So you need to determine whether any of the other choices are equivalent to $\frac{7x}{20} + 8$. Of the answer choices, only Choices A and C have the same constant term as Choice B: 8. So these are the only choices that need to be checked. Choice A, $\frac{x}{2} - \frac{3x}{20} + 8$, can be simplified as follows.

$$\frac{x}{2} - \frac{3x}{20} + 8 = \frac{10x}{20} - \frac{3x}{20} + 8 = \frac{7x}{20} + 8$$

So Choice A is equivalent to $\frac{7x}{20}+8$. Choice C, $\frac{3x}{20}+8$, is clearly not equivalent to $\frac{7x}{20}+8$. Thus the correct answer consists of **Choices A and B**.

This explanation uses the following strategies.
Strategy 1: Translate from Words to an Arithmetic or Algebraic Representation
Strategy 5: Simplify an Arithmetic or Algebraic Representation
Strategy 8: Search for a Mathematical Relationship

---

16. Machine A, working alone at its constant rate, produces x pounds of peanut butter in 12 minutes. Machine B, working alone at its constant rate, produces x pounds of peanut butter in 18 minutes. How many minutes will it take machines A and B, working simultaneously at their respective constant rates, to produce x pounds of peanut butter?

[ ] minutes

Explanation

Translating the given information into an algebraic expression, you see that machine A produces $\frac{x}{12}$ pounds of peanut butter in 1 minute, and machine B produces $\frac{x}{18}$ pounds of peanut butter in 1 minute. Therefore, working simultaneously, machine A and machine B produce $\frac{x}{12}+\frac{x}{18}$ pounds of peanut butter in 1 minute.

Letting t be the number of minutes it takes machines A and B, working simultaneously, to produce x pounds of peanut butter, you can set up the following equation.

$$\left(\frac{x}{12}+\frac{x}{18}\right)t = x$$

Solving for t, you get

$$\left(\frac{1}{12}+\frac{1}{18}\right)t = 1$$

$$\frac{5}{36}t = 1$$

$$t = \frac{36}{5} = 7.2$$

The correct answer is **7.2**.

This explanation uses the following strategies.
Strategy 1: Translate from Words to an Arithmetic or Algebraic Representation
Strategy 8: Search for a Mathematical Relationship

17. The function f has the property that $f(x) = f(x+1)$ for all numbers x. If $f(4) = 17$, what is the value of $f(8)$?

Explanation

The property that $f(x) = f(x+1)$ for all numbers x implies that $f(4) = f(5)$, $f(5) = f(6)$, $f(6) = f(7)$, and $f(7) = f(8)$. Therefore, since $f(4) = 17$, it follows that $f(8) = 17$. Thus the correct answer is **17**.

This explanation uses the following strategy.

Strategy 7: Find a Pattern

5 Geometry

Your goals for this chapter

- ⇨ Practice answering *GRE*® questions in geometry
- ⇨ Review answers and explanations, particularly for questions you answered incorrectly

This chapter contains GRE Quantitative Reasoning practice questions that involve geometry.

Geometry topics include parallel and perpendicular lines, circles, triangles—including isosceles, equilateral, and 30°-60°-90° triangles—quadrilaterals, other polygons, congruent and similar figures, three-dimensional figures, area, perimeter, volume, the Pythagorean theorem, and angle measurement in degrees. The ability to construct proofs is not tested.

The questions are arranged by question type: Quantitative Comparison questions, followed by both types of Multiple-choice questions, and then Numeric Entry questions.

Following the questions is an answer key for quick reference. Then, at the end of the chapter, you will find complete explanations for every question. Each explanation is presented with the corresponding question for easy reference.

Review the answers and explanations carefully, paying particular attention to explanations for questions that you answered incorrectly.

Before answering the practice questions, read the Quantitative Reasoning section directions that begin on the following page. Also, review the directions that precede each question type to make sure you understand how to answer the questions.

Quantitative Reasoning Section Directions

For each question, indicate the best answer, using the directions given.

Notes: All numbers used are real numbers.

All figures are assumed to lie in a plane unless otherwise indicated.

Geometric figures, such as lines, circles, triangles, and quadrilaterals, **are not necessarily** drawn to scale. That is, you should **not** assume that quantities such as lengths and angle measures are as they appear in a figure. You should assume, however, that lines shown as straight are actually straight, points on a line are in the order shown, and more generally, all geometric objects are in the relative positions shown. For questions with geometric figures, you should base your answers on geometric reasoning, not on estimating or comparing quantities by sight or by measurement.

Coordinate systems, such as *xy*-planes and number lines, **are** drawn to scale; therefore, you can read, estimate, or compare quantities in such figures by sight or by measurement.

Graphical data presentations, such as bar graphs, circle graphs, and line graphs, **are** drawn to scale; therefore, you can read, estimate, or compare data values by sight or by measurement.

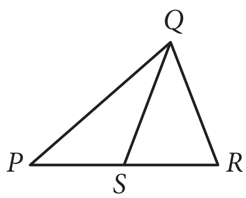
Quantitative Comparison Questions

For Questions 1 to 5, compare Quantity A and Quantity B, using additional information centered above the two quantities if such information is given. Select one of the following four answer choices and fill in the corresponding oval to the right of the question.

Ⓐ Quantity A is greater.
Ⓑ Quantity B is greater.
Ⓒ The two quantities are equal.
Ⓓ The relationship cannot be determined from the information given.

A symbol that appears more than once in a question has the same meaning throughout the question.

	Quantity A	Quantity B	Correct Answer
Example 1:	(2)(6)	2 + 6	● Ⓑ Ⓒ Ⓓ



	Quantity A	Quantity B	Correct Answer
Example 2:	*PS*	*SR*	Ⓐ Ⓑ Ⓒ ●

(since equal lengths cannot be assumed, even though *PS* and *SR* appear equal)

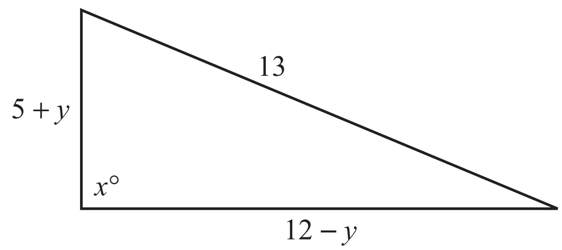
In the *xy*-plane, one of the vertices of square *S* is the point (2, 2). The diagonals of *S* intersect at the point (6, 6).

	Quantity A	Quantity B	
1.	The area of *S*	64	Ⓐ Ⓑ Ⓒ Ⓓ

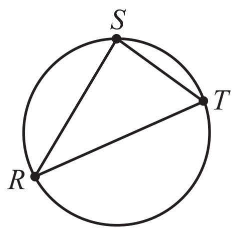
	Quantity A	Quantity B	
2.	The length of a side of a regular pentagon with a perimeter of 12.5	The length of a side of a regular hexagon with a perimeter of 15	A B C D

A line in the *xy*-plane contains the points (5, 4) and (2, −1).

	Quantity A	Quantity B	
3.	The slope of the line	0	A B C D



	Quantity A	Quantity B	
4.	x	90	A B C D



In the figure above, triangle *RST* is inscribed in a circle. The measure of angle *RST* is greater than 90°, and the area of the circle is 25π.

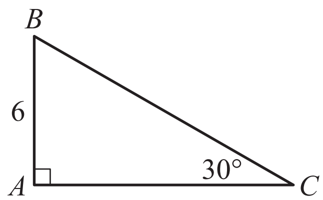
	Quantity A	Quantity B	
5.	The length of line segment *RT*	10	A B C D

Multiple-choice Questions—Select One Answer Choice

For Questions 6 to 10, select a single answer choice.

6. A construction company will produce identical metal supports in the shape of a right triangle with legs of length 3 feet and 4 feet. The three sides of each triangular support are to be constructed of metal stripping. If the company has a total of 6,000 feet of metal stripping and there is no waste of material in the construction of the supports, what is the greatest possible number of supports that the company can produce?

   (A) 428
   (B) 500
   (C) 545
   (D) 600
   (E) 1,000



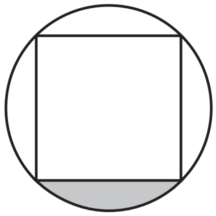
7. What is the area of triangle ABC shown above?

   (A) 18
   (B) 20
   (C) $12\sqrt{3}$
   (D) $18\sqrt{3}$
   (E) 36

---

8. The volume V of a right circular cylinder is $V = \pi r^2 h$, where r is the radius of the base and h is the height of the cylinder. If the volume of a right circular cylinder is 45π and its height is 5, what is the circumference of its base?

   (A) 3
   (B) 9
   (C) 3π
   (D) 6π
   (E) 9π



9. In the figure above, if the square inscribed in the circle has an area of 16, what is the area of the shaded region?

(A) $2\pi - 1$
(B) $2\pi - 4$
(C) $4\pi - 2$
(D) $4\pi - 4$
(E) $8\pi - 4$

10. The radius of circle A is r, and the radius of circle B is $\frac{3}{4}r$. What is the ratio of the area of circle A to the area of circle B?

(A) 1 to 4
(B) 3 to 4
(C) 4 to 3
(D) 9 to 16
(E) 16 to 9

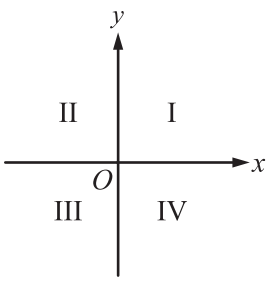
Multiple-choice Questions—Select One or More Answer Choices

For Questions 11 to 12, select all the answer choices that apply.

11. A flat, rectangular flower bed with an area of 2,400 square feet is bordered by a fence on three sides and by a walkway on the fourth side. If the entire length of the fence is 140 feet, which of the following could be the length, in feet, of one of the sides of the flower bed?

    Indicate all such lengths.

    [A] 20
    [B] 30
    [C] 40
    [D] 50
    [E] 60
    [F] 70
    [G] 80
    [H] 90



12. The quadrants of the xy-plane are shown in the figure above. In the xy-plane, line m (not shown) has a positive slope and a positive x-intercept. Line m intersects which of the quadrants?

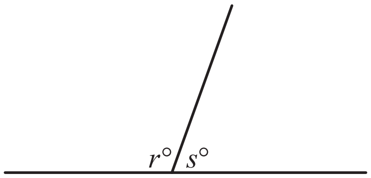
    Indicate all such quadrants.

    [A] Quadrant I
    [B] Quadrant II
    [C] Quadrant III
    [D] Quadrant IV

Numeric Entry Questions

For Question 13, enter your answer in the answer box(es) below the question.

- **Your answer may be an integer, a decimal, or a fraction, and it may be negative.**
- **If a question asks for a fraction, there will be two boxes—one for the numerator and one for the denominator. A decimal point cannot be used in a fraction.**
- **Equivalent forms of the correct answer, such as 2.5 and 2.50, are all correct. Fractions do not need to be reduced to lowest terms, though you may need to reduce your fraction to fit in the boxes.**
- **Enter the exact answer unless the question asks you to round your answer.**



13. In the figure above, if $\frac{r}{r+s}=\frac{5}{8}$, what is the value of r ?

$r =$ [ ]

ANSWER KEY

1. **Choice C**: The two quantities are equal.
2. **Choice C**: The two quantities are equal.
3. **Choice A**: Quantity A is greater.
4. **Choice D**: The relationship cannot be determined from the information given.
5. **Choice B**: Quantity B is greater.
6. **Choice B**: 500
7. **Choice D**: $18\sqrt{3}$
8. **Choice D**: 6π
9. **Choice B**: $2\pi - 4$
10. **Choice E**: 16 to 9
11. **Choice B**: 30
    AND
    Choice C: 40
    AND
    Choice E: 60
    AND
    Choice G: 80
12. **Choice A**: Quadrant I
    AND
    Choice C: Quadrant III
    AND
    Choice D: Quadrant IV
13. **112.5**

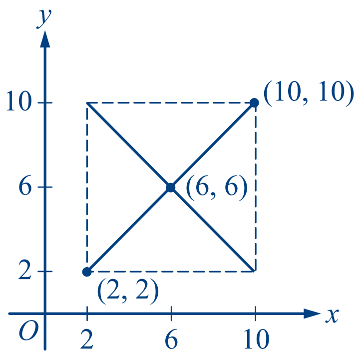
Answers and Explanations

In the xy-plane, one of the vertices of square S is the point (2, 2). The diagonals of S intersect at the point (6, 6).

	Quantity A	Quantity B	
1.	The area of S	64	Ⓐ Ⓑ Ⓒ Ⓓ

Explanation

Since the point (2, 2) is a vertex of square S and the point (6, 6) is the midpoint of the diagonals, it follows that the point (10, 10) is also a vertex of the square. Using this information you can sketch square S in the xy-plane, labeling the points (2, 2), (6, 6), and (10, 10) as shown in the figure below.



From the figure, you can see that S has sides of length 8. Therefore the area of S is $8^2 = 64$. Hence Quantity A is equal to Quantity B, and the correct answer is **Choice C**.

This explanation uses the following strategies.

Strategy 2: Translate from Words to a Figure or Diagram

Strategy 4: Translate from a Figure to an Arithmetic or Algebraic Representation

	Quantity A	Quantity B	
2.	The length of a side of a regular pentagon with a perimeter of 12.5	The length of a side of a regular hexagon with a perimeter of 15	Ⓐ Ⓑ Ⓒ Ⓓ

Explanation

A regular pentagon has 5 sides of equal length, so the length of a side of a regular pentagon is $\frac{1}{5}$ of its perimeter. Thus Quantity A is $\frac{12.5}{5}$, or 2.5. A regular hexagon has 6 sides of equal length, so the length of a side of a regular hexagon is $\frac{1}{6}$ of its perimeter. Thus Quantity B is $\frac{15}{6}$, or 2.5. So Quantity A and Quantity B are both equal to 2.5, and the correct answer is **Choice C**.

This explanation uses the following strategy.

Strategy 1: Translate from Words to an Arithmetic or Algebraic Representation

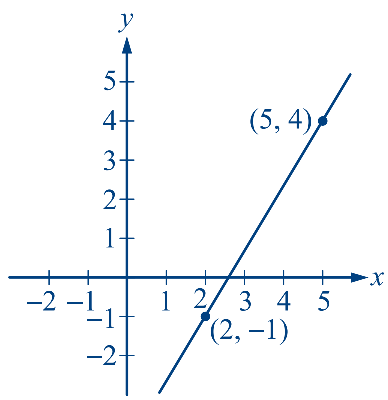
---

A line in the *xy*-plane contains the points (5, 4) and (2, −1).

	Quantity A	Quantity B	
3.	The slope of the line	0	Ⓐ Ⓑ Ⓒ Ⓓ

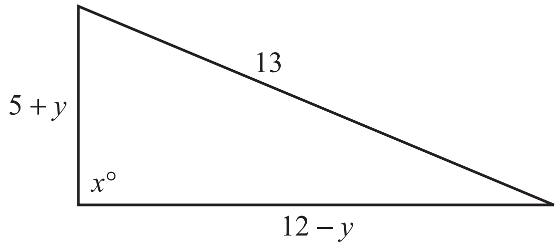
Explanation

You can begin by sketching the line in the *xy*-plane and labeling the points (5, 4) and (2, −1) on the line, as shown below.



From the figure, you can see that the line through the two points slants upward and to the right. So the slope of the line is greater than 0; that is, Quantity A is greater than Quantity B. The correct answer is **Choice A**. (Note that it is not necessary to calculate the slope of the line.)

This explanation uses the following strategy.
Strategy 2: Translate from Words to a Figure or Diagram



	Quantity A	Quantity B	
4.	x	90	Ⓐ Ⓑ Ⓒ Ⓓ

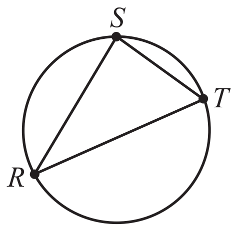
Explanation

The figure looks like a right triangle with legs of length $5 + y$ and $12 - y$ and hypotenuse of length 13. If $y = 0$, then the sides of the triangle have lengths 5, 12, and 13. This triangle is in fact a right triangle because $5^2 + 12^2 = 13^2$. So the angle labeled $x°$ is a right angle; that is, $x = 90$. In this case, Quantity A is equal to Quantity B.

Now consider another value of y, say $y = 1$, to see if the triangle is still a right triangle in this case. If $y = 1$, then the sides of the triangle have lengths 6, 11, and 13. This triangle is not a right triangle because $6^2 + 11^2 \neq 13^2$. So the angle labeled $x°$ is not a right angle; that is, $x \neq 90$. In this case, Quantity A is not equal to Quantity B.

Because Quantity A is equal to Quantity B in one case and Quantity A is not equal to Quantity B in another case, the correct answer is **Choice D**.

This explanation uses the following strategies.
Strategy 10: Trial and Error
Strategy 13: Determine Whether a Conclusion Follows from the Information Given



In the figure above, triangle RST is inscribed in a circle. The measure of angle RST is greater than 90°, and the area of the circle is 25π.

	Quantity A	Quantity B	
5.	The length of line segment RT	10	Ⓐ Ⓑ Ⓒ Ⓓ

Explanation

Since the area of the circle is 25π, it follows that the radius of the circle is 5 and the diameter is 10. Line segment RT is a diameter of the circle if and only if angle RST is a right angle. Since you are given that the measure of angle RST is greater than 90°, it follows that angle RST is not a right angle and that line segment RT is a chord but not a diameter. Therefore the length of line segment RT is less than 10, and the correct answer is **Choice B**.

This explanation uses the following strategy.
Strategy 8: Search for a Mathematical Relationship

---

6. A construction company will produce identical metal supports in the shape of a right triangle with legs of length 3 feet and 4 feet. The three sides of each triangular support are to be constructed of metal stripping. If the company has a total of 6,000 feet of metal stripping and there is no waste of material in the construction of the supports, what is the greatest possible number of supports that the company can produce?

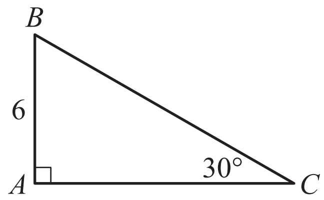
Ⓐ 428
Ⓑ 500
Ⓒ 545
Ⓓ 600
Ⓔ 1,000

Explanation

Since each support is in the shape of a right triangle with legs of length 3 feet and 4 feet, the length of the third side of the support is $\sqrt{3^2 + 4^2}$, or 5 feet. The total length of the stripping of each support is therefore $3 + 4 + 5$, or 12 feet. The company has 6,000 feet of metal stripping available. So, with no waste, the greatest possible number of supports that can be produced is $\frac{6,000}{12}$, or 500. The correct answer is **Choice B**.

This explanation uses the following strategy.
Strategy 1: Translate from Words to an Arithmetic or Algebraic Representation

---



7. What is the area of triangle *ABC* shown above?

   (A) 18
   (B) 20
   (C) $12\sqrt{3}$
   (D) $18\sqrt{3}$
   (E) 36

Explanation

The triangle is a 30°-60°-90° triangle, so the ratio of the lengths of the legs is 1 to $\sqrt{3}$. Since the length of the shorter leg, *AB*, is 6, it follows that the length of the longer leg, *AC*, is $6\sqrt{3}$. The area of the triangle is therefore $\frac{1}{2}(6)\left(6\sqrt{3}\right)$, or $18\sqrt{3}$. The correct answer is **Choice D**.

This explanation uses the following strategies.
Strategy 4: Translate from a Figure to an Arithmetic or Algebraic Representation
Strategy 8: Search for a Mathematical Relationship

8. The volume V of a right circular cylinder is $V = \pi r^2 h,$ where r is the radius of the base and h is the height of the cylinder. If the volume of a right circular cylinder is 45π and its height is 5, what is the circumference of its base?

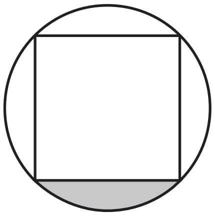
(A) 3
(B) 9
(C) 3π
(D) 6π
(E) 9π

Explanation

You are given that the volume of the right circular cylinder is 45π and the height is 5. It follows that $\pi r^2 h = 45\pi,$ or $r^2 h = 45.$ Since $h = 5$ and $r^2 h = 45,$ it follows that $r^2 = 9,$ or $r = 3.$ Therefore the circumference of the circular base is $2\pi r = 2\pi(3) = 6\pi,$ and the correct answer is **Choice D**.

This explanation uses the following strategy.
Strategy 1: Translate from Words to an Arithmetic or Algebraic Representation

---



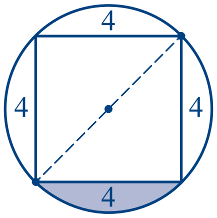
9. In the figure above, if the square inscribed in the circle has an area of 16, what is the area of the shaded region?

(A) $2\pi - 1$
(B) $2\pi - 4$
(C) $4\pi - 2$
(D) $4\pi - 4$
(E) $8\pi - 4$

Explanation

It is clear from the figure that the area of the shaded region is $\frac{1}{4}$ of the difference between the area of the circle and the area of the square. You are given that the area of the square is 16, so each side has length 4.

You can find the area of the circle if you know the radius of the circle. If you draw a diagonal of the square, as shown in the figure below, you can see that the diagonal is also a diameter of the circle.



Note that the diagonal divides the square into two isosceles right triangles with legs of length 4. By the Pythagorean theorem applied to one of the right triangles, the length of the diagonal is equal to $\sqrt{4^2+4^2}$, or $4\sqrt{2}$. Thus the radius of the circle is $r=\frac{4\sqrt{2}}{2}=2\sqrt{2}$, and the area of the circle is $\pi r^2=\pi\left(2\sqrt{2}\right)^2=8\pi$. Therefore the area of the shaded region is $\frac{8\pi-16}{4}$, or $2\pi-4$. The correct answer is **Choice B**.

This explanation uses the following strategies.
Strategy 4: Translate from a Figure to an Arithmetic or Algebraic Representation
Strategy 6: Add to a Geometric Figure
Strategy 8: Search for a Mathematical Relationship

---

10. The radius of circle A is r, and the radius of circle B is $\frac{3}{4}r$. What is the ratio of the area of circle A to the area of circle B ?

Ⓐ 1 to 4
Ⓑ 3 to 4
Ⓒ 4 to 3
Ⓓ 9 to 16
Ⓔ 16 to 9

Explanation

Circle A has radius r, so its area is πr^2. Circle B has radius $\frac{3r}{4}$, so its area is $\pi\left(\frac{3r}{4}\right)^2=\frac{9\pi r^2}{16}$. Therefore the ratio of the area of circle A to the area of circle B is πr^2 to $\frac{9\pi r^2}{16}$, which is the same as the ratio 1 to $\frac{9}{16}$, which is the same as the ratio 16 to 9.

The correct answer is **Choice E**.

This explanation uses the following strategy.
Strategy 1: Translate from Words to an Arithmetic or Algebraic Representation

11. A flat, rectangular flower bed with an area of 2,400 square feet is bordered by a fence on three sides and by a walkway on the fourth side. If the entire length of the fence is 140 feet, which of the following could be the length, in feet, of one of the sides of the flower bed?

Indicate all such lengths.

- A 20
- B 30
- C 40
- D 50
- E 60
- F 70
- G 80
- H 90

Explanation

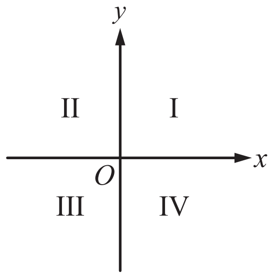
You know that the area of the rectangular flower bed is 2,400 square feet. So if the flower bed is a feet long and b feet wide, then $ab = 2{,}400$. If the side of the flower bed that is bordered by the walkway is one of the sides that are b feet long, then the total length of the three sides of the flower bed bordered by the fence is $2a + b$ feet. Since you are given that the total length of the fence is 140 feet, it follows that $2a + b = 140$. Since $ab = 2{,}400$, you can substitute $\frac{2{,}400}{a}$ for b in the equation $2a + b = 140$ to get the equation $2a + \frac{2{,}400}{a} = 140$. It follows that $2a^2 + 2{,}400 = 140a$, or $a^2 - 70a + 1{,}200 = 0$.

When you solve this equation for a (either by factoring or by using the quadratic formula), you get $a = 30$ or $a = 40$. If $a = 30$, then $b = \frac{2{,}400}{30} = 80$; if $a = 40$, then $b = \frac{2{,}400}{40} = 60$. So the possible lengths of the sides are 30, 40, 60, and 80. Thus the correct answer consists of **Choices B, C, E, and G**.

This explanation uses the following strategies.

Strategy 1: Translate from Words to an Arithmetic or Algebraic Representation

Strategy 8: Search for a Mathematical Relationship



12. The quadrants of the xy-plane are shown in the figure above. In the xy-plane, line m (not shown) has a positive slope and a positive x-intercept. Line m intersects which of the quadrants?

    Indicate all such quadrants.

    $\boxed{A}$ Quadrant I

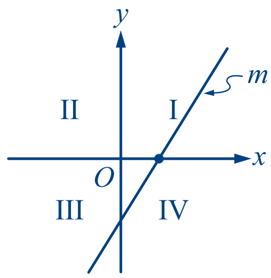
    $\boxed{B}$ Quadrant II

    $\boxed{C}$ Quadrant III

    $\boxed{D}$ Quadrant IV

Explanation

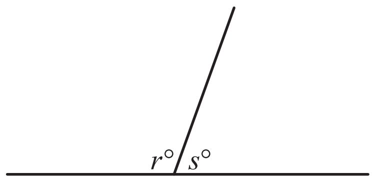
Since line m has a positive x-intercept, it must cross the x-axis to the right of the origin; and since the slope of line m is positive, the line must slant upward and to the right. Consequently, the line must have a negative y-intercept. The figure below shows a typical line that satisfies these conditions.



In the figure, the line intersects quadrants I, III, and IV. Thus the correct answer consists of **Choices A, C, and D**.

This explanation uses the following strategy.

Strategy 6: Add to a Geometric Figure



13. In the figure above, if $\frac{r}{r+s}=\frac{5}{8}$, what is the value of r ?

$r =$ ☐

Explanation

From the figure, note that $r° + s°$ must equal 180°. Therefore $\frac{r}{r+s}=\frac{r}{180}$. Since you are also given in the question that $\frac{r}{r+s}=\frac{5}{8}$, you can conclude that $\frac{r}{180}=\frac{5}{8}$. Thus $r=\frac{5(180)}{8}=112.5$, and the correct answer is **112.5**.

This explanation uses the following strategy.

Strategy 4: Translate from a Figure to an Arithmetic or Algebraic Representation

6 Data Analysis

Your goals for this chapter

- Practice answering *GRE*® questions in data analysis
- Review answers and explanations, particularly for questions you answered incorrectly

This chapter contains GRE Quantitative Reasoning practice questions that involve data analysis.

Data analysis topics include basic descriptive statistics, such as mean, median, mode, range, standard deviation, interquartile range, quartiles, and percentiles; interpretation of data in tables and graphs, such as line graphs, bar graphs, circle graphs, boxplots, scatterplots, and frequency distributions; elementary probability, such as probabilities of compound events and independent events; random variables and probability distributions, including normal distributions; and counting methods, such as combinations, permutations, and Venn diagrams. These topics are typically taught in high school algebra courses or introductory statistics courses. Inferential statistics is not tested.

The questions are arranged by question type: Quantitative Comparison questions, followed by both types of Multiple-choice questions, followed by Numeric Entry questions, and finally Data Interpretation sets.

Following the questions is an answer key for quick reference. Then, at the end of the chapter, you will find complete explanations for every question. Each explanation is presented with the corresponding question for easy reference.

Review the answers and explanations carefully, paying particular attention to explanations for questions that you answered incorrectly.

Before answering the practice questions, read the Quantitative Reasoning section directions that begin on the following page. Also, review the directions that precede each question type to make sure you understand how to answer the questions.

Quantitative Reasoning Section Directions

For each question, indicate the best answer, using the directions given.

Notes: All numbers used are real numbers.

All figures are assumed to lie in a plane unless otherwise indicated.

Geometric figures, such as lines, circles, triangles, and quadrilaterals, **are not necessarily** drawn to scale. That is, you should **not** assume that quantities such as lengths and angle measures are as they appear in a figure. You should assume, however, that lines shown as straight are actually straight, points on a line are in the order shown, and more generally, all geometric objects are in the relative positions shown. For questions with geometric figures, you should base your answers on geometric reasoning, not on estimating or comparing quantities by sight or by measurement.

Coordinate systems, such as *xy*-planes and number lines, **are** drawn to scale; therefore, you can read, estimate, or compare quantities in such figures by sight or by measurement.

Graphical data presentations, such as bar graphs, circle graphs, and line graphs, **are** drawn to scale; therefore, you can read, estimate, or compare data values by sight or by measurement.

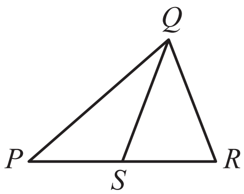
Quantitative Comparison Questions

For Questions 1 to 6, compare Quantity A and Quantity B, using additional information centered above the two quantities if such information is given. Select one of the following four answer choices and fill in the corresponding oval to the right of the question.

Ⓐ Quantity A is greater.
Ⓑ Quantity B is greater.
Ⓒ The two quantities are equal.
Ⓓ The relationship cannot be determined from the information given.

A symbol that appears more than once in a question has the same meaning throughout the question.

	Quantity A	Quantity B	Correct Answer
Example 1:	(2)(6)	2 + 6	Ⓐ (filled) Ⓑ Ⓒ Ⓓ



	Quantity A	Quantity B	Correct Answer
Example 2:	PS	SR	Ⓐ Ⓑ Ⓒ Ⓓ (filled) (since equal lengths cannot be assumed, even though PS and SR appear equal)

The average (arithmetic mean) of 4 donations to a charity was \$80. Two of the 4 donations were \$90 and \$60.

	Quantity A	Quantity B	
1.	The average of the other 2 donations	\$80	Ⓐ Ⓑ Ⓒ Ⓓ

AGE DISTRIBUTION OF
EMPLOYEES OF A BUSINESS

Age Interval	Number of Employees
15—24	17
25—24	24
35—44	26
45—54	21
55—64	18
Total	106

	Quantity A	Quantity B	
2.	The range of the ages of the 20 oldest employees of the business	11 years	Ⓐ Ⓑ Ⓒ Ⓓ

	Quantity A	Quantity B	
3.	The sum of the first 7 positive integers	7 times the median of the first 7 positive integers	Ⓐ Ⓑ Ⓒ Ⓓ

	Quantity A	Quantity B	
4.	The number of two-digit positive integers for which the units digit is not equal to the tens digit	80	Ⓐ Ⓑ Ⓒ Ⓓ

In a probability experiment, G and H are independent events. The probability that G will occur is r, and the probability that H will occur is s, where both r and s are greater than 0.

	Quantity A	Quantity B	
5.	The probability that either G will occur or H will occur, but not both	$r + s - rs$	Ⓐ Ⓑ Ⓒ Ⓓ

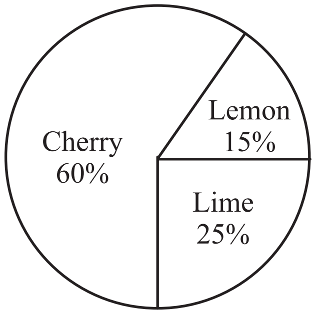
$$S = \{1, 4, 7, 10\}$$
$$T = \{2, 3, 5, 8, 13\}$$

x is a number in set S, and y is a number in set T.

	Quantity A	Quantity B	
6.	The number of different possible values of the product xy	20	Ⓐ Ⓑ Ⓒ Ⓓ

Multiple-choice Questions—Select One Answer Choice

For Questions 7 to 12, select a single answer choice.



7. The graph above shows the distribution of three different flavors of hard candies—cherry, lemon, and lime—in a candy jar. If all the lemon candies are removed and no other candies are added or removed, what fraction of the remaining candies in the jar will be lime candies?

Ⓐ $\frac{1}{7}$

Ⓑ $\frac{2}{9}$

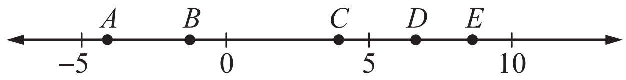
Ⓒ $\frac{1}{4}$

Ⓓ $\frac{5}{17}$

Ⓔ $\frac{5}{12}$

8. R is a list of 15 consecutive integers, and T is a list of 21 consecutive integers. The median of the integers in list R is equal to the least integer in list T. If the two lists are combined into one list of 36 integers, how many different integers are on the combined list?

Ⓐ 25
Ⓑ 27
Ⓒ 28
Ⓓ 32
Ⓔ 36



9. From the 5 points A, B, C, D, and E on the number line above, 3 different points are to be randomly selected. What is the probability that the coordinates of the 3 points selected will all be positive?

Ⓐ $\frac{1}{10}$

Ⓑ $\frac{1}{5}$

Ⓒ $\frac{3}{10}$

Ⓓ $\frac{2}{5}$

Ⓔ $\frac{3}{5}$

10. In a distribution of 850 different measurements, x centimeters is at the 73rd percentile. If there are 68 measurements in the distribution that are greater than y centimeters but less than x centimeters, then y is approximately at what percentile in the distribution?

Ⓐ 45th

Ⓑ 50th

Ⓒ 55th

Ⓓ 60th

Ⓔ 65th

11. Each of the following linear equations defines y as a function of x for all integers x from 1 to 100. For which of the following equations is the standard deviation of the y-values corresponding to all the x-values the greatest?

Ⓐ $y = \frac{x}{3}$

Ⓑ $y = \frac{x}{2} + 40$

Ⓒ $y = x$

Ⓓ $y = 2x + 50$

Ⓔ $y = 3x - 20$

12. For a certain distribution, the measurement 12.1 is 1.5 standard deviations below the mean, and the measurement 17.5 is 3.0 standard deviations above the mean. What is the mean of the distribution?

Ⓐ 13.8
Ⓑ 13.9
Ⓒ 14.0
Ⓓ 14.1
Ⓔ 14.2

Multiple-choice Questions—Select One or More Answer Choices

For Question 13, select all the answer choices that apply.

13. Set *A* has 50 members and set *B* has 53 members. At least 2 of the members in set *A* are not in set *B*. Which of the following could be the number of members in set *B* that are not in set *A* ?

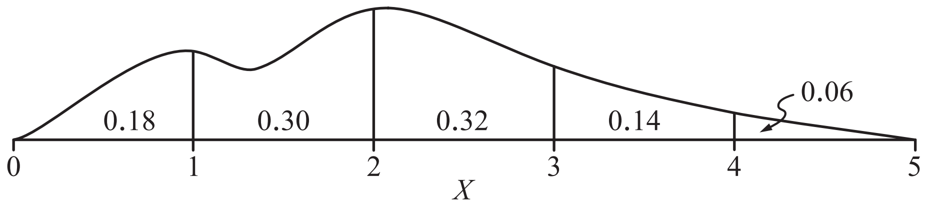
Indicate all such numbers.

[A] 3
[B] 5
[C] 13
[D] 25
[E] 50
[F] 53

Numeric Entry Questions

For Questions 14 to 15, enter your answer in the answer box(es) below the question.

- **Your answer may be an integer, a decimal, or a fraction, and it may be negative.**
- **If a question asks for a fraction, there will be two boxes—one for the numerator and one for the denominator. A decimal point cannot be used in a fraction.**
- **Equivalent forms of the correct answer, such as 2.5 and 2.50, are all correct. Fractions do not need to be reduced to lowest terms, though you may need to reduce your fraction to fit in the boxes.**
- **Enter the exact answer unless the question asks you to round your answer.**



14. The figure above shows the probability distribution of a continuous random variable X. For each of the five intervals shown, the figure gives the probability that the value of X is in that interval. What is the probability that $1 < X < 4$?

FIVE MOST POPULOUS CITIES IN THE UNITED STATES
APRIL 2000

City	Population (in thousands)
New York	8,008
Los Angeles	3,695
Chicago	2,896
Houston	1,954
Philadelphia	1,518

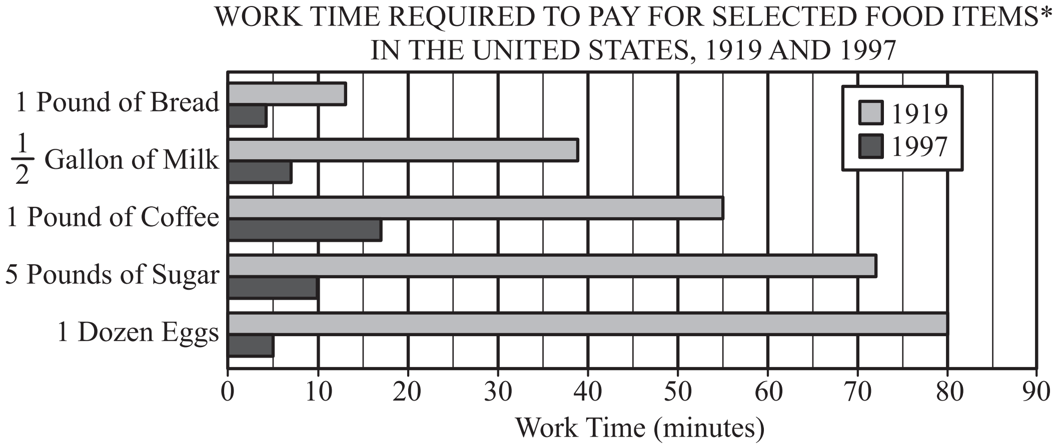
15. The populations of the five most populous cities in the United States in April 2000 are listed in the table above. The total population of the United States in April 2000 was 281,422,000. Based on the data shown, the population of the three most populous cities combined was what percent of the total population of the United States in April 2000 ?

Give your answer to the nearest whole percent.

%

Data Interpretation Sets

Questions 16 to 19 are based on the following data. For these questions, select a single answer choice.



*For each year, the work time, in hours, required to pay for a food item is the average price of the food item divided by the average hourly wage for rank-and-file manufacturing workers. The work time in the graph is given in minutes.

16. In 1997, at the rates shown in the graph, the work time required to pay for which of the following food items was greatest?

    Ⓐ 10 pounds of bread
    Ⓑ 5 gallons of milk
    Ⓒ 3 pounds of coffee
    Ⓓ 20 pounds of sugar
    Ⓔ 5 dozen eggs

17. If the average hourly wage of the rank-and-file manufacturing worker in 1919 was \$0.55, which of the following is closest to the average price of $\frac{1}{2}$ gallon of milk in 1919 ?

    Ⓐ \$0.80
    Ⓑ \$0.65
    Ⓒ \$0.50
    Ⓓ \$0.35
    Ⓔ \$0.20

18. At the rates shown in the graph, which of the following is closest to the number of hours of work time that was required to pay for 20 kilograms of sugar in 1919 ? (1 kilogram equals 2.2 pounds, rounded to the nearest 0.1 pound.)

   Ⓐ 11
   Ⓑ 14
   Ⓒ 20
   Ⓓ 31
   Ⓔ 53

19. Eight hours of work time paid for approximately how many more dozen eggs in 1997 than it did in 1919 ?

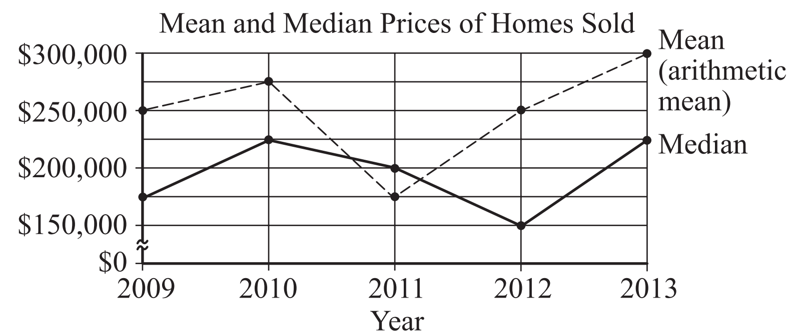
   Ⓐ 50
   Ⓑ 70
   Ⓒ 90
   Ⓓ 110
   Ⓔ 130

Questions 20 to 23 are based on the following data. For these questions, select a single answer choice unless otherwise directed.

HOMES SOLD IN COUNTY *T*, 2009–2013

Number of Homes Sold

Year	Number
2009	503
2010	351
2011	390
2012	410
2013	290



20. Which of the following is closest to the mean of the prices of the 700 homes sold in 2012 and 2013 combined?

   Ⓐ $265,000
   Ⓑ $270,000
   Ⓒ $275,000
   Ⓓ $280,000
   Ⓔ $285,000

21. By approximately what percent did the median price of homes sold in County *T* decrease from 2011 to 2012 ?

Ⓐ 10%
Ⓑ 15%
Ⓒ 25%
Ⓓ 33%
Ⓔ 50%

For Question 22, select all the answer choices that apply.

22. Based on the information given, which of the following statements about the sum of the prices of all the homes sold in a given year must be true?

Indicate <u>all</u> such statements.

[A] The sum of the prices for 2010 was greater than the sum for 2009.
[B] The sum of the prices for 2010 was greater than the sum for 2011.
[C] The sum of the prices for 2009 was greater than the sum for 2011.

23. County *T* collected a tax equal to 3 percent of the price of each home sold in the county in 2009. Approximately how much did County *T* collect in taxes from all homes sold in 2009 ?

Ⓐ $38,000
Ⓑ $260,000
Ⓒ $380,000
Ⓓ $2,600,000
Ⓔ $3,800,000

Questions 24 to 26 are based on the following data. For these questions, select a single answer choice.

PERSONAL INCOME AND
PUBLIC EDUCATION REVENUE
IN CONTRY X
(in constant 1998 dollars)

Year	Per Capita Income	Revenue per Student
1930	\$6,610	\$710
1940	\$6,960	\$950
1950	\$9,540	\$1,330
1960	\$12,780	\$2,020
1970	\$17,340	\$3,440
1980	\$20,150	\$4,400
1990	\$24,230	\$5,890

24. From 1930 to 1990, approximately what was the average increase per year in per capita income?

Ⓐ \$150
Ⓑ \$200
Ⓒ \$250
Ⓓ \$300
Ⓔ \$350

25. In 1950 the revenue per student was approximately what percent of the per capita income?

Ⓐ 8%
Ⓑ 11%
Ⓒ 14%
Ⓓ 17%
Ⓔ 20%

26. For how many of the seven years shown was the revenue per student less than $\frac{1}{5}$ of the per capita income for the year?

Ⓐ One
Ⓑ Two
Ⓒ Three
Ⓓ Four
Ⓔ Five

ANSWER KEY

1. **Choice A**: Quantity A is greater.
2. **Choice D**: The relationship cannot be determined from the information given.
3. **Choice C**: The two quantities are equal.
4. **Choice A**: Quantity A is greater.
5. **Choice B**: Quantity B is greater.
6. **Choice B**: Quantity B is greater.
7. **Choice D**: $\frac{5}{17}$
8. **Choice C**: 28
9. **Choice A**: $\frac{1}{10}$
10. **Choice E**: 65th
11. **Choice E**: $y = 3x - 20$
12. **Choice B**: 13.9
13. **Choice B**: 5
    AND
    Choice C: 13
    AND
    Choice D: 25
    AND
    Choice E: 50
    AND
    Choice F: 53
14. **0.76**
15. **5**
16. **Choice B**: 5 gallons of milk
17. **Choice D**: \$0.35
18. **Choice A**: 11
19. **Choice C**: 90
20. **Choice B**: \$270,000
21. **Choice C**: 25%
22. **Choice B**: The sum of the prices for 2010 was greater than the sum for 2011.
    AND
    Choice C: The sum of the prices for 2009 was greater than the sum for 2011.
23. **Choice E**: \$3,800,000
24. **Choice D**: \$300
25. **Choice C**: 14%
26. **Choice E**: Five

Answers and Explanations

The average (arithmetic mean) of 4 donations to a charity was \$80. Two of the 4 donations were \$90 and \$60.

	Quantity A	Quantity B	
1.	The average of the other 2 donations	\$80	Ⓐ Ⓑ Ⓒ Ⓓ

Explanation

Note that Quantity B, \$80, is the average of the 4 donations. The average of 2 of the 4 donations, \$90 and \$60, is \$75. Since \$75 is less than \$80, it follows that Quantity A, the average of the other 2 donations, is greater than Quantity B. Therefore the correct answer is **Choice A**.

This explanation uses the following strategy.
Strategy 8: Search for a Mathematical Relationship

AGE DISTRIBUTION OF
EMPLOYEES OF A BUSINESS

Age Interval	Number of Employees
15—24	17
25—24	24
35—44	26
45—54	21
55—64	18
Total	106

	Quantity A	Quantity B	
2.	The range of the ages of the 20 oldest employees of the business	11 years	Ⓐ Ⓑ Ⓒ Ⓓ

Explanation

Of the 20 oldest employees, 18 are in the 55–64 age-group, and 2 are in the 45–54 age-group. Therefore the youngest of the 20 employees is in the 45–54 age-group, and the oldest is in the 55–64 age-group. The youngest of the 20 employees could be 45 years old and the oldest could be 64 years old. In this case, the range of their ages would be 64–45, or 19 years.

On the other hand, the youngest could be 54 years old and the oldest could be 55 years old, so the range of their ages would be 1 year. Because there are cases where the range is greater than 11 years and cases where it is less than 11 years, the correct answer is **Choice D**.

This explanation uses the following strategies.
Strategy 11: Divide into Cases
Strategy 13: Determine Whether a Conclusion Follows from the Information Given

---

	Quantity A	Quantity B	
3.	The sum of the first 7 positive integers	7 times the median of the first 7 positive integers	Ⓐ Ⓑ Ⓒ Ⓓ

Explanation
Quantity A is $1 + 2 + 3 + 4 + 5 + 6 + 7,$ or 28. The median of the first 7 positive integers is the middle number when they are listed in order from least to greatest, which is 4. So Quantity B is $(7)(4),$ or 28. Thus the correct answer is **Choice C**.

This explanation uses the following strategy.
Strategy 8: Search for a Mathematical Relationship

	Quantity A	Quantity B	
4.	The number of two-digit positive integers for which the units digit is not equal to the tens digit	80	Ⓐ Ⓑ Ⓒ Ⓓ

Explanation
The two-digit positive integers are the integers from 10 to 99. There are 90 such integers. In 9 of these integers, namely, 11, 22, 33, . . . , 99, the units digit and tens digit are equal. Hence, Quantity A, the number of two-digit positive integers for which the units digit is not equal to the tens digit, is $90 - 9,$ or 81. Since Quantity B is 80, the correct answer is **Choice A**.

This explanation uses the following strategy.
Strategy 11: Divide into Cases

In a probability experiment, G and H are independent events. The probability that G will occur is r, and the probability that H will occur is s, where both r and s are greater than 0.

	Quantity A	Quantity B	
5.	The probability that either G will occur or H will occur, but not both	$r + s - rs$	Ⓐ Ⓑ Ⓒ Ⓓ

Explanation

By the rules of probability, you can conclude that the probability that event H will not occur is $1 - s$. Also, the fact that G and H are independent events implies that G and "not H" are independent events. Therefore the probability that G will occur and H will not occur is $r(1 - s)$. Similarly, the probability that H will occur and G will not occur is $s(1 - r)$. So Quantity A, the probability that either G will occur or H will occur, but not both, is $r(1 - s) + s(1 - r) = r + s - 2rs$, which is less than Quantity B, $r + s - rs$. Thus the correct answer is **Choice B**.

This explanation uses the following strategy.
Strategy 1: Translate from Words to an Arithmetic or Algebraic Representation

---

$$S = \{1, 4, 7, 10\}$$
$$T = \{2, 3, 5, 8, 13\}$$

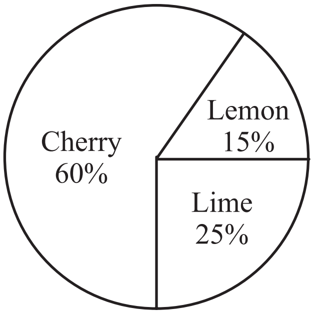
x is a number in set S, and y is a number in set T.

	Quantity A	Quantity B	
6.	The number of different possible values of the product xy	20	Ⓐ Ⓑ Ⓒ Ⓓ

Explanation

There are 4 numbers in S and 5 numbers in T, so the total number of possible products that can be formed using one number in each set is $(4)(5)$, or 20. However, some of these products have the same value; for example, $(1)(8) = (4)(2)$. Therefore, Quantity A, the number of different possible values of the product xy, is less than Quantity B, 20. Thus the correct answer is **Choice B**.

This explanation uses the following strategy.
Strategy 8: Search for a Mathematical Relationship



7. The graph above shows the distribution of three different flavors of hard candies—cherry, lemon, and lime—in a candy jar. If all the lemon candies are removed and no other candies are added or removed, what fraction of the remaining candies in the jar will be lime candies?

(A) $\frac{1}{7}$

(B) $\frac{2}{9}$

(C) $\frac{1}{4}$

(D) $\frac{5}{17}$

(E) $\frac{5}{12}$

Explanation

If the lemon candies are removed, then 85% of the original number of candies will remain. Of these, the fraction of lime candies will be $\frac{25}{85}$, or $\frac{5}{17}$. The correct answer is **Choice D**.

This explanation uses the following strategy.

Strategy 4: Translate from a Figure to an Arithmetic or Algebraic Representation

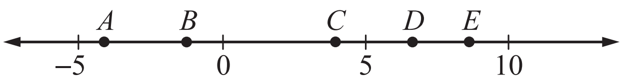
8. R is a list of 15 consecutive integers, and T is a list of 21 consecutive integers. The median of the integers in list R is equal to the least integer in list T. If the two lists are combined into one list of 36 integers, how many different integers are on the combined list?

   (A) 25
   (B) 27
   (C) 28
   (D) 32
   (E) 36

Explanation

The median of the numbers in list R is the middle number when the numbers are listed in order from least to greatest, that is, the 8th number. Since the median of the numbers in list R is equal to the least integer in list T, the 8 greatest integers in R are the 8 least integers in T, and the number of different integers in the combined list is $15 + 21 - 8$, or 28. The correct answer is **Choice C**.

This explanation uses the following strategy.
Strategy 8: Search for a Mathematical Relationship



9. From the 5 points A, B, C, D, and E on the number line above, 3 different points are to be randomly selected. What is the probability that the coordinates of the 3 points selected will all be positive?

   (A) $\frac{1}{10}$
   (B) $\frac{1}{5}$
   (C) $\frac{3}{10}$
   (D) $\frac{2}{5}$
   (E) $\frac{3}{5}$

Explanation

Of the 5 points, 3 have positive coordinates, points C, D, and E. The probability that the first point selected will have a positive coordinate is $\frac{3}{5}$. Since the second point selected must be different from the first point, there are 4 remaining points to select from, of which 2 are points with positive coordinates. Therefore, if the coordinate of the first point selected is positive, then the probability that the second point selected will have a positive coordinate is $\frac{2}{4}$.

Similarly, if the coordinates of the first 2 points selected are positive, then the probability that the third point selected will have a positive coordinate is $\frac{1}{3}$.

The probability that the coordinates of the 3 points selected will all be positive is the product of the three probabilities, $\left(\frac{3}{5}\right)\left(\frac{2}{4}\right)\left(\frac{1}{3}\right)$, or $\frac{1}{10}$. The correct answer is **Choice A**.

Alternatively, you can compute the probability as the following fraction.

$$\frac{\text{number of ways to select 3 points with positive coordinates}}{\text{number of ways to select 3 points from 5 points}}$$

Since there are only 3 points with positive coordinates, there is only 1 way to select them, so the numerator is 1. The denominator of the fraction is equal to the number of combinations of 5 objects taken 3 at a time, or "5 choose 3," which is $\frac{5!}{3!(5-3)!} = \frac{(5)(4)}{(2)(1)} = 10$. Therefore the probability is $\frac{1}{10}$, which is **Choice A**.

This explanation uses the following strategy.
Strategy 12: Adapt Solutions to Related Problems

---

10. In a distribution of 850 different measurements, x centimeters is at the 73rd percentile. If there are 68 measurements in the distribution that are greater than y centimeters but less than x centimeters, then y is approximately at what percentile in the distribution?

  Ⓐ 45th
  Ⓑ 50th
  Ⓒ 55th
  Ⓓ 60th
  Ⓔ 65th

Explanation

If x centimeters is at the 73rd percentile, then approximately 73% of the measurements in the distribution are less than or equal to x centimeters. The 68 measurements that are greater than y centimeters but less than x centimeters are $\left(\frac{68}{850}\right)(100\%)$, or 8%, of the distribution. Thus approximately $73\% - 8\%$, or 65%, of the measurements are less than or equal to y centimeters, that is, y is approximately at the 65th percentile in the distribution. The correct answer is **Choice E**.

This explanation uses the following strategy.
Strategy 8: Search for a Mathematical Relationship

11. Each of the following linear equations defines y as a function of x for all integers x from 1 to 100. For which of the following equations is the standard deviation of the y-values corresponding to all the x-values the greatest?

Ⓐ $y = \frac{x}{3}$

Ⓑ $y = \frac{x}{2} + 40$

Ⓒ $y = x$

Ⓓ $y = 2x + 50$

Ⓔ $y = 3x - 20$

Explanation

Recall that the standard deviation of the numbers in a data set is a measure of the spread of the numbers about the mean of the numbers. The standard deviation is directly related to the distances between the mean and each of the numbers when the mean and the numbers are considered on a number line. Note that each of the answer choices is an equation of the form $y = ax + b$, where a and b are constants. For every value of x in a data set, the corresponding value of y is $ax + b$, and if m is the mean of the values of x, then $am + b$ is the mean of the corresponding values of y.

In the question, the set of values of x consists of the integers from 1 to 100, and each answer choice gives a set of 100 values of y corresponding to the 100 values of x. For each value of x in the data set,

(1) the distance between x and the mean m is $|x - m|$, and
(2) the distance between the corresponding y-value, $ax + b$, and the mean, $am + b$, of the corresponding y-values is $|ax + b - am - b|$, which is equal to $|ax - am|$, or $|a||x - m|$.

Therefore the greater the absolute value of a in the equation $y = ax + b$, the greater the distance between each y-value and the mean of the y-values; hence, the greater the spread. Note that the value of b is irrelevant. Scanning the choices, you can see that the equation in which the absolute value of a is greatest is $y = 3x - 20$. Thus the correct answer is **Choice E**.

This explanation uses the following strategy.
Strategy 8: Search for a Mathematical Relationship

12. For a certain distribution, the measurement 12.1 is 1.5 standard deviations below the mean, and the measurement 17.5 is 3.0 standard deviations above the mean. What is the mean of the distribution?

Ⓐ 13.8
Ⓑ 13.9
Ⓒ 14.0
Ⓓ 14.1
Ⓔ 14.2

Explanation

If m represents the mean of the distribution and s represents the standard deviation, then the statement "the measurement 12.1 is 1.5 standard deviations below the mean" can be represented by the equation $12.1 = m - 1.5s$. Similarly, the statement "the measurement 17.5 is 3.0 standard deviations above the mean" can be represented by the equation $17.5 = m + 3.0s$.

One way to solve the two linear equations for m is to eliminate the s. To do this, you can multiply the equation $12.1 = m - 1.5s$ by 2 and then add the result to the equation $17.5 = m + 3.0s$ to get $41.7 = 3m$. Solving this equation for m gives the mean 13.9. Thus the correct answer is **Choice B**.

This explanation uses the following strategy.

Strategy 1: Translate from Words to an Arithmetic or Algebraic Representation

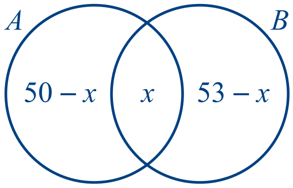
13. Set A has 50 members and set B has 53 members. At least 2 of the members in set A are not in set B. Which of the following could be the number of members in set B that are not in set A ?

Indicate all such numbers.

A 3
B 5
C 13
D 25
E 50
F 53

Explanation

Let x be the number of members in the intersection of set A and set B. Then the distribution of the members of A and B can be represented by the following Venn diagram.

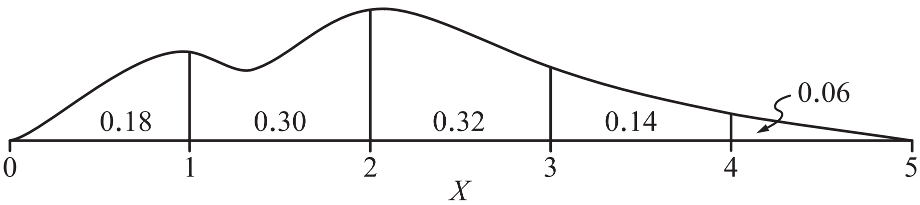


The question asks you to indicate which of the answer choices could be the number of members in set B that are not in set A. This is equivalent to determining which of the answer choices are possible values of $53 - x$.

You are given that the number of members in set A that are not in set B is at least 2, and clearly the number of members in set A that are not in set B is at most all 50 members of A; that is, $2 \le 50 - x \le 50$. Note that $53 - x$ is 3 more than $50 - x$. So by adding 3 to each part of $2 \le 50 - x \le 50$, you get the equivalent inequality $5 \le 53 - x \le 53$. Thus the number of members in set B that are not in set A can be any integer from 5 to 53. The correct answer consists of **Choices B, C, D, E, and F**.

This explanation uses the following strategies.

Strategy 1: Translate from Words to an Arithmetic or Algebraic Representation
Strategy 2: Translate from Words to a Figure or Diagram



14. The figure above shows the probability distribution of a continuous random variable X. For each of the five intervals shown, the figure gives the probability that the value of X is in that interval. What is the probability that $1 < X < 4$?

Explanation

In the distribution shown, the interval from 1 to 4 is divided into the three intervals—the interval from 1 to 2, the interval from 2 to 3, and the interval from 3 to 4. The probability that $1 < X < 4$ is the sum of the probability that $1 < X < 2$, the probability that $2 < X < 3$, and the probability that $3 < X < 4$, that is, $0.30 + 0.32 + 0.14 = 0.76$. The correct answer is **0.76**.

This explanation uses the following strategy.

Strategy 4: Translate from a Figure to an Arithmetic or Algebraic Representation

FIVE MOST POPULOUS CITIES IN THE UNITED STATES
APRIL 2000

City	Population (in thousands)
New York	8,008
Los Angeles	3,695
Chicago	2,896
Houston	1,954
Philadelphia	1,518

15. The populations of the five most populous cities in the United States in April 2000 are listed in the table above. The total population of the United States in April 2000 was 281,422,000. Based on the data shown, the population of the three most populous cities combined was what percent of the total population of the United States in April 2000 ?

Give your answer to the nearest whole percent.

[ ] %

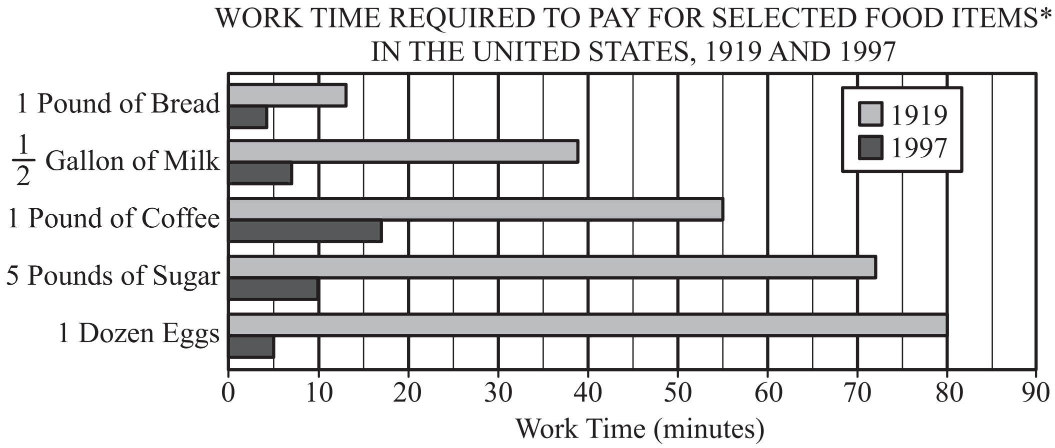
Explanation

From the data given, the three most populous cities were New York, Los Angeles, and Chicago, and the population of the three cities combined was $8{,}008{,}000 + 3{,}695{,}000 + 2{,}896{,}000,$ or 14,599,000. As a percent of the total population of the United States, this is $\left(\frac{14{,}599{,}000}{281{,}422{,}000}\right)(100\%) \approx 5.19\%,$ which, rounded to the nearest whole percent, is 5%.

The correct answer is **5**.

This explanation uses the following strategy.

Strategy 4: Translate from a Figure to an Arithmetic or Algebraic Representation



*For each year, the work time, in hours, required to pay for a food item is the average price of the food item divided by the average hourly wage for rank-and-file manufacturing workers. The work time in the graph is given in minutes.

16. In 1997, at the rates shown in the graph, the work time required to pay for which of the following food items was greatest?

(A) 10 pounds of bread
(B) 5 gallons of milk
(C) 3 pounds of coffee
(D) 20 pounds of sugar
(E) 5 dozen eggs

Explanation

Reading from the graph, you can compute the approximate work times for the quantities listed in the choices.

Choice A: The approximate work time required to pay for 1 pound of bread was 4 minutes, so the approximate work time to pay for 10 pounds of bread was $(10)(4)$, or 40 minutes.

Choice B: The approximate work time required to pay for $\frac{1}{2}$ gallon of milk was 7 minutes, so the approximate work time to pay for 5 gallons of milk was $(10)(7)$, or 70 minutes.

Choice C: The approximate work time required to pay for 1 pound of coffee was 17 minutes, so the approximate work time to pay for 3 pounds of coffee was $(3)(17)$, or 51 minutes.

Choice D: The approximate work time required to pay for 5 pounds of sugar was 10 minutes, so the approximate work time to pay for 20 pounds of sugar was $(4)(10)$, or 40 minutes.

Choice E: The approximate work time required to pay for 1 dozen eggs was 5 minutes, so the approximate work time to pay for 5 dozen eggs was $(5)(5)$, or 25 minutes.

Of these times, the greatest is 70 minutes for 5 gallons of milk. The correct answer is **Choice B**.

This explanation uses the following strategies.

Strategy 4: Translate from a Figure to an Arithmetic or Algebraic Representation
Strategy 5: Simplify an Arithmetic or Algebraic Representation
Strategy 9: Estimate

17. If the average hourly wage of the rank-and-file manufacturing worker in 1919 was \$0.55, which of the following is closest to the average price of $\frac{1}{2}$ gallon of milk in 1919 ?

(A) \$0.80
(B) \$0.65
(C) \$0.50
(D) \$0.35
(E) \$0.20

Explanation

From the graph, you see that in 1919 the work time required to pay for $\frac{1}{2}$ gallon of milk was approximately 38 minutes. Given an hourly wage of \$0.55, the wage for 38 minutes is $\left(\frac{38}{60}\right)(\$0.55)$, or about \$0.35. The correct answer is **Choice D**.

This explanation uses the following strategies.
Strategy 4: Translate from a Figure to an Arithmetic or Algebraic Representation
Strategy 5: Simplify an Arithmetic or Algebraic Representation
Strategy 9: Estimate

---

18. At the rates shown in the graph, which of the following is closest to the number of hours of work time that was required to pay for 20 kilograms of sugar in 1919 ? (1 kilogram equals 2.2 pounds, rounded to the nearest 0.1 pound.)

(A) 11
(B) 14
(C) 20
(D) 31
(E) 53

Explanation

If 1 kilogram equals 2.2 pounds, then 20 kilograms equals 44 pounds.

According to the graph, in 1919 the work time required to pay for 5 pounds of sugar was approximately 72 minutes, so the work time required to pay for 44 pounds of sugar was $\left(\frac{72}{5}\right)(44)$, or 633.6 minutes, which is approximately 10.6 hours. Of the given choices, the one closest to this number is 11 hours. The correct answer is **Choice A**.

This explanation uses the following strategies.
Strategy 4: Translate from a Figure to an Arithmetic or Algebraic Representation
Strategy 5: Simplify an Arithmetic or Algebraic Representation
Strategy 9: Estimate

19. Eight hours of work time paid for approximately how many more dozen eggs in 1997 than it did in 1919 ?

    Ⓐ 50
    Ⓑ 70
    Ⓒ 90
    Ⓓ 110
    Ⓔ 130

Explanation

Since the work times are given in minutes, first convert 8 hours to 480 minutes.

In 1919, the work time that paid for 1 dozen eggs was approximately 80 minutes, so 480 minutes paid for $\frac{480}{80}$, or 6 dozen eggs.

In 1997, the work time that paid for 1 dozen eggs was approximately 5 minutes, so 480 minutes paid for $\frac{480}{5}$, or 96 dozen eggs.

Thus 8 hours of work time paid for 90 dozen more eggs in 1997 than it paid for in 1919. The correct answer is **Choice C**.

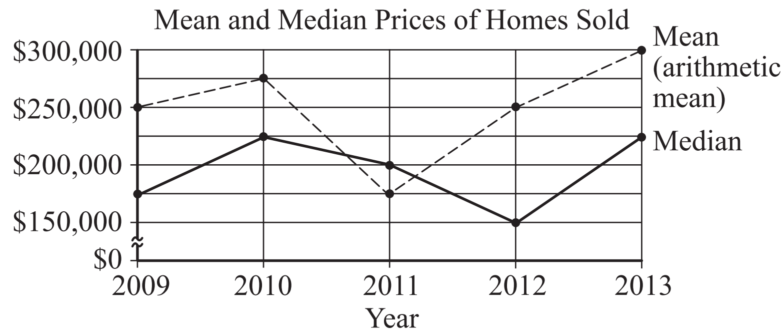
This explanation uses the following strategies.

Strategy 4: Translate from a Figure to an Arithmetic or Algebraic Representation
Strategy 5: Simplify an Arithmetic or Algebraic Representation
Strategy 9: Estimate

HOMES SOLD IN COUNTY *T*, 2009–2013

Number of Homes Sold

Year	Number
2009	503
2010	351
2011	390
2012	410
2013	290



20. Which of the following is closest to the mean of the prices of the 700 homes sold in 2012 and 2013 combined?

    (A) $265,000
    (B) $270,000
    (C) $275,000
    (D) $280,000
    (E) $285,000

Explanation

The number of homes sold is given in the table, and the mean of the prices is given in the line graph.

The mean price of the 700 homes sold in 2012 and 2013 is the weighted average of the mean price of the 410 homes sold in 2012, which is $250,000, and the mean price of the 290 homes sold in 2013, which is $300,000:

$$\frac{(410)(\$250{,}000) + (290)(\$300{,}000)}{700} \approx \$270{,}714$$

Of the choices given, the closest is $270,000. The correct answer is **Choice B**.

This explanation uses the following strategies.

Strategy 4: Translate from a Figure to an Arithmetic or Algebraic Representation
Strategy 9: Estimate

21. By approximately what percent did the median price of homes sold in County T decrease from 2011 to 2012 ?

Ⓐ 10%
Ⓑ 15%
Ⓒ 25%
Ⓓ 33%
Ⓔ 50%

Explanation

The median prices are given in the line graph. The median price decreased from \$200,000 in 2011 to \$150,000 in 2012, which is a decrease of \$50,000. As a percent of the 2011 price, this is $\left(\frac{50{,}000}{200{,}000}\right)(100\%)$, or 25%. The correct answer is **Choice C**.

This explanation uses the following strategy.

Strategy 4: Translate from a Figure to an Arithmetic or Algebraic Representation

---

22. Based on the information given, which of the following statements about the sum of the prices of all the homes sold in a given year must be true?

Indicate <u>all</u> such statements.

[A] The sum of the prices for 2010 was greater than the sum for 2009.
[B] The sum of the prices for 2010 was greater than the sum for 2011.
[C] The sum of the prices for 2009 was greater than the sum for 2011.

Explanation

For each year, the sum of the prices is equal to the number of homes sold times the mean price of the homes sold.

For 2010, the sum is equal to $(351)(\$275{,}000)$, or \$96,525,000.

For 2009, the sum is $(503)(\$250{,}000)$, or \$125,750,000, which is greater than the sum for 2010. So statement A is false.

For 2011, the sum is $(390)(\$175{,}000)$, or \$68,250,000, which is less than the sum for 2010. So statement B is true.

Since the sum for 2009 is greater than the sum for 2011, statement C is true.

The correct answer consists of **Choices B and C**.

This explanation uses the following strategy.

Strategy 4: Translate from a Figure to an Arithmetic or Algebraic Representation

23. County T collected a tax equal to 3 percent of the price of each home sold in the county in 2009. Approximately how much did County T collect in taxes from all homes sold in 2009 ?

Ⓐ $38,000
Ⓑ $260,000
Ⓒ $380,000
Ⓓ $2,600,000
Ⓔ $3,800,000

Explanation

The total price of all the homes sold in 2009 is equal to the number of homes sold times the mean price of the homes sold. The tax is 3% of this amount. Since the choices given are far apart, there is no need for accurate computations. Using estimation, you get a total price of about (500)($250,000), or $125 million. The tax of 3% is (0.03)($125 million), or approximately $3.75 million. Of the choices given, $3,800,000 is closest to this amount. The correct answer is **Choice E**.

This explanation uses the following strategies.
Strategy 4: Translate from a Figure to an Arithmetic or Algebraic Representation
Strategy 9: Estimate

PERSONAL INCOME AND
PUBLIC EDUCATION REVENUE
IN CONTRY X
(in constant 1998 dollars)

Year	Per Capita Income	Revenue per Student
1930	$6,610	$710
1940	$6,960	$950
1950	$9,540	$1,330
1960	$12,780	$2,020
1970	$17,340	$3,440
1980	$20,150	$4,400
1990	$24,230	$5,890

24. From 1930 to 1990, approximately what was the average increase per year in per capita income?

Ⓐ $150
Ⓑ $200
Ⓒ $250
Ⓓ $300
Ⓔ $350

Explanation

For the 60-year period from 1930 to 1990, the per capita income increased by $\$24{,}230 - \$6{,}610$, or $\$17{,}620$. The average annual increase is $\frac{\$17{,}620}{60}$, which is approximately $\frac{\$18{,}000}{60}$, or $300. (Since the choices are quite far apart, there is no need for an accurate calculation.) The correct answer is **Choice D**.

This explanation uses the following strategies.

Strategy 4: Translate from a Figure to an Arithmetic or Algebraic Representation
Strategy 9: Estimate

---

25. In 1950 the revenue per student was approximately what percent of the per capita income?

Ⓐ 8%
Ⓑ 11%
Ⓒ 14%
Ⓓ 17%
Ⓔ 20%

Explanation

The table shows that in 1950 the revenue per student was $1,330. As a percent of the per capita income of $9,540, this is $\left(\frac{1{,}330}{9{,}540}\right)(100\%)$, which is approximately 13.9%. Of the given choices, the closest is 14%. The correct answer is **Choice C**.

This explanation uses the following strategies.

Strategy 4: Translate from a Figure to an Arithmetic or Algebraic Representation
Strategy 9: Estimate

26. For how many of the seven years shown was the revenue per student less than $\frac{1}{5}$ of the per capita income for the year?

   Ⓐ One
   Ⓑ Two
   Ⓒ Three
   Ⓓ Four
   Ⓔ Five

Explanation

For most of the years, a rough estimate of $\frac{1}{5}$ of the per capita income is sufficient for the comparison to the revenue per student. For the years 1930, 1940, 1950, and 1960, you might estimate $\frac{1}{5}$ of the per capita incomes as \$1,300, \$1,400, \$2,000, and \$2,500, respectively, which are clearly greater than the corresponding revenues per student.

For 1980 and 1990, your estimates might be \$4,000 and \$5,000, which are less than the corresponding revenues per student.

For 1970, you do have to calculate $\frac{\$17{,}340}{5} = \$3{,}468,$ which is greater than \$3,440.

So the revenue per student was less than $\frac{1}{5}$ of the per capita income for the five years 1930, 1940, 1950, 1960, and 1970. The correct answer is **Choice E**.

This explanation uses the following strategies.

Strategy 4: Translate from a Figure to an Arithmetic or Algebraic Representation
Strategy 9: Estimate

7 Mixed Practice Sets

Your goals for this chapter

- ⇨ Practice answering *GRE*® Quantitative Reasoning questions in all four content areas
- ⇨ Review answers and explanations, particularly for questions you answered incorrectly

This chapter contains three sets of practice questions. Each set has 25 questions, with a mixture of content and question types.

In each set, the questions are arranged by question type: Quantitative Comparison questions, followed by both types of Multiple-choice questions, then by Numeric Entry questions, and ending with a Data Interpretation set.

After each set of questions, there is an answer key for quick reference, followed by explanations for every question. Each explanation is presented with the corresponding question for easy reference. Review the answers and explanations carefully, paying particular attention to explanations for questions that you answered incorrectly.

Before answering the practice questions, read the Quantitative Reasoning section directions on the next page. Also, review the directions that precede each question type to make sure you understand how to answer the questions.

Note that each set of 25 practice questions has about the same number of questions of each type as the individual Quantitative Reasoning sections in the paper-delivered GRE General Test, with 25 questions per 40-minute section. Therefore, to help you gauge the timed aspect of the Quantitative Reasoning measure, it may be useful to set aside a 40-minute block of time for each set of 25 questions.

If you are taking the computer-delivered GRE General Test, note that each Quantitative Reasoning section will contain 20 questions and you will have 35 minutes to answer them. If you can successfully complete each practice set in this chapter in 40 minutes, you should be able to answer 20 questions in 35 minutes. However, for a more realistic experience of taking the computer-delivered test under timed conditions, you should use the free *POWERPREP*® practice tests.

Quantitative Reasoning Section Directions

For each question, indicate the best answer, using the directions given.

Notes: All numbers used are real numbers.

All figures are assumed to lie in a plane unless otherwise indicated.

Geometric figures, such as lines, circles, triangles, and quadrilaterals, **are not necessarily** drawn to scale. That is, you should **not** assume that quantities such as lengths and angle measures are as they appear in a figure. You should assume, however, that lines shown as straight are actually straight, points on a line are in the order shown, and more generally, all geometric objects are in the relative positions shown. For questions with geometric figures, you should base your answers on geometric reasoning, not on estimating or comparing quantities by sight or by measurement.

Coordinate systems, such as *xy*-planes and number lines, **are** drawn to scale; therefore, you can read, estimate, or compare quantities in such figures by sight or by measurement.

Graphical data presentations, such as bar graphs, circle graphs, and line graphs, **are** drawn to scale; therefore, you can read, estimate, or compare data values by sight or by measurement.

PRACTICE SET 1

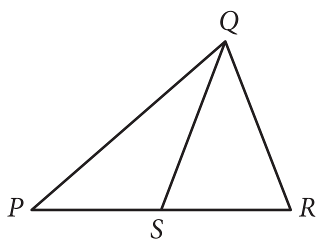
Quantitative Comparison Questions

For Questions 1 to 9, compare Quantity A and Quantity B, using additional information centered above the two quantities if such information is given. Select one of the following four answer choices and fill in the corresponding oval to the right of the question.

Ⓐ Quantity A is greater.
Ⓑ Quantity B is greater.
Ⓒ The two quantities are equal.
Ⓓ The relationship cannot be determined from the information given.

A symbol that appears more than once in a question has the same meaning throughout the question.

	Quantity A	Quantity B	Correct Answer
Example 1:	(2)(6)	2 + 6	Ⓐ Ⓑ Ⓒ Ⓓ



	Quantity A	Quantity B	Correct Answer
Example 2:	*PS*	*SR*	Ⓐ Ⓑ Ⓒ Ⓓ (since equal lengths cannot be assumed, even though *PS* and *SR* appear equal)

Points *R*, *S*, and *T* lie on a number line, where *S* is between *R* and *T*. The distance between *R* and *S* is 6, and the distance between *R* and *T* is 15.

	Quantity A	Quantity B	
1.	The distance between the midpoints of line segments *RS* and *ST*	The distance between *S* and *T*	Ⓐ Ⓑ Ⓒ Ⓓ

S is a set of 8 numbers, of which 4 are negative and 4 are positive.

	Quantity A	Quantity B	
2.	The average (arithmetic mean) of the numbers in S	The median of the numbers in S	Ⓐ Ⓑ Ⓒ Ⓓ

The length of each side of rectangle R is an integer, and the area of R is 36.

	Quantity A	Quantity B	
3.	The number of possible values of the perimeter of R	6	Ⓐ Ⓑ Ⓒ Ⓓ

$$x = (z - 1)^2$$
$$y = (z + 1)^2$$

	Quantity A	Quantity B	
4.	The average (arithmetic mean) of x and y	z^2	Ⓐ Ⓑ Ⓒ Ⓓ

---

x, y, and z are the lengths of the sides of a triangle.

	Quantity A	Quantity B	
5.	$x + y + z$	$2z$	Ⓐ Ⓑ Ⓒ Ⓓ

At a club meeting, there are 10 more club members than nonmembers. The number of club members at the meeting is c.

	Quantity A	Quantity B	
6.	The total number of people at the club meeting	$2c - 10$	Ⓐ Ⓑ Ⓒ Ⓓ

n is a positive integer that is greater than 3 and has d positive divisors.

	Quantity A	Quantity B	
7.	n	2^{d-1}	Ⓐ Ⓑ Ⓒ Ⓓ

$$m = 10^{32} + 2$$

When m is divided by 11, the remainder is r.

	Quantity A	Quantity B	
8.	r	3	Ⓐ Ⓑ Ⓒ Ⓓ

$xy = 8$ and $x = y - 2$.

	Quantity A	Quantity B	
9.	y	0	Ⓐ Ⓑ Ⓒ Ⓓ

Multiple-choice Questions—Select One Answer Choice

For Questions 10 to 17, select a single answer choice.

10. The area of circle W is 16π and the area of circle Z is 4π. What is the ratio of the circumference of W to the circumference of Z ?

Ⓐ 2 to 1
Ⓑ 4 to 1
Ⓒ 8 to 1
Ⓓ 16 to 1
Ⓔ 32 to 1

---

11. In the xy-plane, a quadrilateral has vertices at $(-1, 4)$, $(7, 4)$, $(7, -5)$, and $(-1, -5)$. What is the perimeter of the quadrilateral?

Ⓐ 17
Ⓑ 18
Ⓒ 19
Ⓓ 32
Ⓔ 34

DISTRIBUTION OF THE
HEIGHTS OF 80 STUDENTS

Height (centimeters)	Number of Students
140 -144	6
145 -149	26
150 -154	32
155 -159	12
160 -164	4
Total	80

12. The table above shows the frequency distribution of the heights of 80 students, where the heights are recorded to the nearest centimeter. What is the least possible range of the recorded heights of the 80 students?

Ⓐ 15
Ⓑ 16
Ⓒ 20
Ⓓ 24
Ⓔ 28

13. Which of the following functions f defined for all numbers x has the property that $f(-x) = -f(x)$ for all numbers x ?

Ⓐ $f(x) = \dfrac{x^3}{x^2+1}$

Ⓑ $f(x) = \dfrac{x^2-1}{x^2+1}$

Ⓒ $f(x) = x^2(x^2 - 1)$

Ⓓ $f(x) = x(x^3 - 1)$

Ⓔ $f(x) = x^2(x^3 - 1)$

---

14. If 10^x equals 0.1 percent of 10^y, where x and y are integers, which of the following must be true?

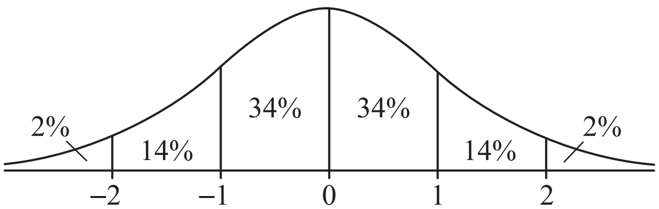
Ⓐ $y = x + 2$

Ⓑ $y = x + 3$

Ⓒ $x = y + 3$

Ⓓ $y = 1{,}000x$

Ⓔ $x = 1{,}000y$



15. The figure above shows the standard normal distribution, with mean 0 and standard deviation 1, including approximate percents of the distribution corresponding to the six regions shown.

The random variable Y is normally distributed with a mean of 470, and the value $Y = 340$ is at the 15th percentile of the distribution. Of the following, which is the best estimate of the standard deviation of the distribution?

Ⓐ 125

Ⓑ 135

Ⓒ 145

Ⓓ 155

Ⓔ 165

16. A car dealer received a shipment of cars, half of which were black, with the remainder consisting of equal numbers of blue, silver, and white cars. During the next month, 70 percent of the black cars, 80 percent of the blue cars, 30 percent of the silver cars, and 40 percent of the white cars were sold. What percent of the cars in the shipment were sold during that month?

    Ⓐ 36%
    Ⓑ 50%
    Ⓒ 55%
    Ⓓ 60%
    Ⓔ 72%

17. If an investment of P dollars is made today and the value of the investment doubles every 7 years, what will be the value of the investment, in dollars, 28 years from today?

    Ⓐ $8P^4$
    Ⓑ P^4
    Ⓒ $16P$
    Ⓓ $8P$
    Ⓔ $4P$

Multiple-choice Questions—Select One or More Answer Choices

For Questions 18 to 19, select all the answer choices that apply.

18. The distribution of the numbers of hours that students at a certain college studied for final exams has a mean of 12 hours and a standard deviation of 3 hours. Which of the following numbers of hours are within 2 standard deviations of the mean of the distribution?

    Indicate all such numbers.

    A 2
    B 5
    C 10
    D 14
    E 16
    F 20

19. In a certain sequence of numbers, each term after the first term is found by multiplying the preceding term by 2 and then subtracting 3 from the product. If the 4th term in the sequence is 19, which of the following numbers are in the sequence?

    Indicate all such numbers.

    A 5
    B 8
    C 11
    D 16
    E 22
    F 35

Numeric Entry Questions

For Questions 20 to 21, enter your answer in the answer box(es) below the question.

- **Your answer may be an integer, a decimal, or a fraction, and it may be negative.**
- **If a question asks for a fraction, there will be two boxes—one for the numerator and one for the denominator. A decimal point cannot be used in a fraction.**
- **Equivalent forms of the correct answer, such as 2.5 and 2.50, are all correct. Fractions do not need to be reduced to lowest terms, though you may need to reduce your fraction to fit in the boxes.**
- **Enter the exact answer unless the question asks you to round your answer.**

20. In a single line of people waiting to purchase tickets for a movie, there are currently 10 people behind Shandra. If 3 of the people who are currently in line ahead of Shandra purchase tickets and leave the line, and no one else leaves the line, there will be 8 people ahead of Shandra in line. How many people are in the line currently?

[ ] people

21. When the decimal point of a certain positive decimal number is moved six places to the right, the resulting number is 9 times the reciprocal of the original number. What is the original number?

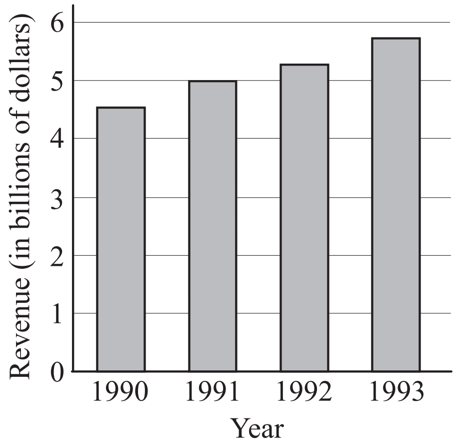
[ ]

Data Interpretation Set

Questions 22 to 25 are based on the following data. For these questions, select a single answer choice unless otherwise directed.

SELECTED DATA FOR GREETING CARD SALES

Annual Revenue from All Greeting Card Sales, 1990–1993



Number of Greeting Cards Sold for Ten Occasions in 1993

Occasion	Number of Cards
Christmas	2.4 billion
Valentine's Day	900 million
Easter	158 million
Mother's Day	155 million
Father's Day	102 million
Graduation	81 million
Thanksgiving	42 million
Halloween	32 million
St. Patrick's Day	18 million
Jewish New Year	12 million
Total	3.9 billion

Note: 1 billion = 1,000,000,000

22. In 1993 the number of Valentine's Day cards sold was approximately how many times the number of Thanksgiving cards sold?

Ⓐ 20
Ⓑ 30
Ⓒ 40
Ⓓ 50
Ⓔ 60

23. In 1993 a card company that sold 40 percent of the Mother's Day cards that year priced its cards for that occasion between \$1.00 and \$8.00 each. If the revenue from sales of the company's Mother's Day cards in 1993 was r million dollars, which of the following indicates all possible values of r ?

Ⓐ $155 < r < 1{,}240$
Ⓑ $93 < r < 496$
Ⓒ $93 < r < 326$
Ⓓ $62 < r < 744$
Ⓔ $62 < r < 496$

24. Approximately what was the percent increase in the annual revenue from all greeting card sales from 1990 to 1993 ?

Ⓐ 50%

Ⓑ 45%

Ⓒ 39%

Ⓓ 28%

Ⓔ 20%

For Question 25, select all the answer choices that apply.

25. In 1993 the average (arithmetic mean) price per card for all greeting cards sold was $1.25. For which of the following occasions was the number of cards sold in 1993 less than the total number of cards sold that year for occasions other than the ten occasions shown?

Indicate all such occasions.

[A] Christmas

[B] Valentine's Day

[C] Easter

[D] Mother's Day

[E] Father's Day

[F] Graduation

[G] Thanksgiving

[H] Halloween

ANSWER KEY

1. **Choice B:** Quantity B is greater.
2. **Choice D:** The relationship cannot be determined from the information given.
3. **Choice B:** Quantity B is greater.
4. **Choice A:** Quantity A is greater.
5. **Choice A:** Quantity A is greater.
6. **Choice C:** The two quantities are equal.
7. **Choice D:** The relationship cannot be determined from the information given.
8. **Choice C:** The two quantities are equal.
9. **Choice D:** The relationship cannot be determined from the information given.
10. **Choice A:** 2 to 1
11. **Choice E:** 34
12. **Choice B:** 16
13. **Choice A:** $f(x) = \dfrac{x^3}{x^2 + 1}$
14. **Choice B:** $y = x + 3$
15. **Choice A:** 125
16. **Choice D:** 60%
17. **Choice C:** $16P$
18. **Choice C:** 10
    AND
    Choice D: 14
    AND
    Choice E: 16
19. **Choice A**: 5
    AND
    Choice C: 11
    AND
    Choice F: 35
20. **22**
21. **0.003**
22. **Choice A**: 20
23. **Choice E**: 62 < r < 496
24. **Choice D**: 28%
25. **Choice C**: Easter
    AND
    Choice D: Mother's Day
    AND
    Choice E: Father's Day
    AND
    Choice F: Graduation
    AND
    Choice G: Thanksgiving
    AND
    Choice H: Halloween

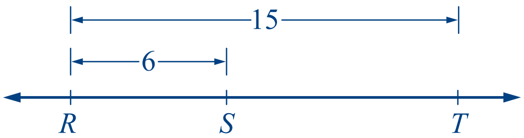
Answers and Explanations

Points R, S, and T lie on a number line, where S is between R and T. The distance between R and S is 6, and the distance between R and T is 15.

	Quantity A	Quantity B	
1.	The distance between the midpoints of line segments RS and ST	The distance between S and T	Ⓐ Ⓑ Ⓒ Ⓓ

Explanation

The figure below shows points R, S, and T on the number line.



From the figure, you can see that the distance between S and T is $15-6$, or 9, which is Quantity B. You can also see that Quantity A, the distance between the midpoints of line segments RS and ST, is equal to one-half of the distance between R and S plus one-half of the distance between S and T. So Quantity A is $\frac{6}{2}+\frac{9}{2}$, or 7.5. Since 7.5 is less than 9, the correct answer is **Choice B**.

This explanation uses the following strategy.

Strategy 2: Translate from Words to a Figure or Diagram

S is a set of 8 numbers, of which 4 are negative and 4 are positive.

	Quantity A	Quantity B	
2.	The average (arithmetic mean) of the numbers in S	The median of the numbers in S	Ⓐ Ⓑ Ⓒ Ⓓ

Explanation

In the question, the only information you are given about the 8 numbers in set S is that 4 are negative and 4 are positive. Sets with 4 negative and 4 positive numbers can vary greatly, so it is likely that the relationship between Quantity A, the average of the numbers in S, and Quantity B, the median of the numbers in S, cannot be determined from the information given. To explore this by trial and error, consider some different sets with 4 negative and 4 positive numbers. Here are some examples.

Example 1: $\{-4, -3, -2, -1, 1, 2, 3, 4\}$. In this case, the average of the numbers in the set is 0, and the median is also 0, so Quantity A is equal to Quantity B.

Example 2: $\{-100, -3, -2, -1, 1, 2, 3, 4\}$. In this case, the median of the numbers is 0, but the average of the numbers is less than 0, so Quantity B is greater than Quantity A.

Example 3: $\{-4, -3, -2, -1, 1, 2, 3, 100\}$. In this case, the median of the numbers is 0, but the average of the numbers is greater than 0, so Quantity A is greater than Quantity B.

From the three examples, you can see that the correct answer is **Choice D**.

This explanation uses the following strategies.
Strategy 10: Trial and Error
Strategy 13: Determine Whether a Conclusion Follows from the Information Given

---

The length of each side of rectangle R is an integer, and the area of R is 36.

	Quantity A	Quantity B	
3.	The number of possible values of the perimeter of R	6	Ⓐ Ⓑ Ⓒ Ⓓ

Explanation

Because the length of each side of rectangle R is an integer and each length is a factor of the area 36, there are 5 possible rectangles: a 1×36 rectangle, a 2×18 rectangle, a 3×12 rectangle, a 4×9 rectangle, and a 6×6 rectangle.

The perimeter of the 1×36 rectangle is $2(1 + 36)$, or 74.
The perimeter of the 2×18 rectangle is $2(2 + 18)$, or 40.
The perimeter of the 3×12 rectangle is $2(3 + 12)$, or 30.
The perimeter of the 4×9 rectangle is $2(4 + 9)$, or 26.
The perimeter of the 6×6 rectangle is $2(6 + 6)$, or 24.

Since each of the 5 possible rectangles has a different perimeter, Quantity A, the number of possible values of the perimeter, is 5. Since Quantity B is 6, the correct answer is **Choice B**.

This explanation uses the following strategy.
Strategy 11: Divide into Cases

$$x = (z - 1)^2$$
$$y = (z + 1)^2$$

	Quantity A	Quantity B	
4.	The average (arithmetic mean) of x and y	z^2	Ⓐ Ⓑ Ⓒ Ⓓ

Explanation

The average of x and y is $\frac{x+y}{2}$. Since you are given that $x = (z - 1)^2$ and $y = (z + 1)^2$, you can express Quantity A in terms of z as follows.

$$\frac{x+y}{2} = \frac{(z-1)^2 + (z+1)^2}{2}$$

This expression can be simplified as follows.

$$\frac{(z-1)^2 + (z+1)^2}{2} = \frac{z^2 - 2z + 1 + z^2 + 2z + 1}{2}$$

$$= \frac{2z^2 + 2}{2}$$

$$= z^2 + 1$$

In terms of z, Quantity A is $z^2 + 1$. Since $z^2 + 1$ is greater than z^2 for all values of z, the correct answer is **Choice A**.

This explanation uses the following strategies.

Strategy 1: Translate from Words to an Arithmetic or Algebraic Representation

Strategy 5: Simplify an Arithmetic or Algebraic Representation

x, y, and z are the lengths of the sides of a triangle.

	Quantity A	Quantity B	
5.	$x+y+z$	$2z$	Ⓐ Ⓑ Ⓒ Ⓓ

Explanation

In this question, you are comparing $x+y+z$ with $2z$. By subtracting z from both quantities, you can see that this is the same as comparing $x+y$ with z. Since x, y, and z are the lengths of the sides of a triangle, and in all triangles the length of each side must be less than the sum of the lengths of the other two sides, it follows that $z<x+y$. Thus the correct answer is **Choice A**.

This explanation uses the following strategies.

Strategy 1: Translate from Words to an Arithmetic or Algebraic Representation
Strategy 8: Search for a Mathematical Relationship

At a club meeting, there are 10 more club members than nonmembers. The number of club members at the meeting is c.

	Quantity A	Quantity B	
6.	The total number of people at the club meeting	$2c-10$	Ⓐ Ⓑ Ⓒ Ⓓ

Explanation

Since the number of club members is c and there are 10 more members than nonmembers, the number of nonmembers is $c-10$. Therefore Quantity A, the total number of people at the meeting, is $c+(c-10)$, or $2c-10$. Since Quantity B is also $2c-10$, the correct answer is **Choice C**.

This explanation uses the following strategy.

Strategy 1: Translate from Words to an Arithmetic or Algebraic Representation

n is a positive integer that is greater than 3 and has d positive divisors.

	Quantity A	Quantity B	
7.	n	2^{d-1}	Ⓐ Ⓑ Ⓒ Ⓓ

Explanation

Since there is no obvious relationship between the quantities n and 2^{d-1}, it is a good idea to try a few values of n to see what happens. Note that you are given that n is an integer greater than 3, so you can start comparing the quantities for the case $n = 4$ and proceed from there.

Case 1: $n = 4$. The integer 4 has three positive divisors, 1, 2, and 4. So in this case, $d = 3$. Therefore $2^{d-1} = 2^{3-1} = 4$, and the two quantities are equal.

Case 2: $n = 5$. The integer 5 has two positive divisors, 1 and 5. So in this case, $d = 2$. Therefore $2^{d-1} = 2^{2-1} = 2$, and Quantity A is greater than Quantity B.

In one case the two quantities are equal, and in the other case Quantity A is greater than Quantity B. Therefore the correct answer is **Choice D**.

This explanation uses the following strategies.

Strategy 10: Trial and Error

Strategy 13: Determine Whether a Conclusion Follows from the Information Given

$$m = 10^{32} + 2$$

When m is divided by 11, the remainder is r.

	Quantity A	Quantity B	
8.	r	3	Ⓐ Ⓑ Ⓒ Ⓓ

Explanation

Actually dividing $10^{32} + 2$ by 11 would be very time consuming, so it is worth trying to compare the quantities without actually doing the division.

A good approach would be to compute the remainders when $10^1 + 2$, $10^2 + 2$, $10^3 + 2$, $10^4 + 2$, etc., are divided by 11 to see if there is a pattern that can help you determine the remainder when $10^{32} + 2$ is divided by 11. The following table shows the first few cases.

n	Value of $10^n + 2$	Remainder When Divided by 11
1	$10^1 + 2 = 12 = 11 + 1$	1
2	$10^2 + 2 = 102 = 99 + 3 = 9(11) + 3$	3
3	$10^3 + 2 = 1{,}002 = 1{,}001 + 1 = 91(11) + 1$	1
4	$10^4 + 2 = 10{,}002 = 9{,}999 + 3 = 909(11) + 3$	3

Note that the remainder is 1 when 10 is raised to an odd power, and the remainder is 3 when 10 is raised to an even power. This pattern suggests that since 32 is even, the remainder when $10^{32} + 2$ is divided by 11 is 3.

To see that this is true, note that the integers 99 and 9,999 in the rows for $n = 2$ and $n = 4$, respectively, are multiples of 11. That is because they each consist of an even number of consecutive digits of 9. Also, these multiples of 11 are each 3 less than $10^2 + 2$ and $10^4 + 2$, respectively, so that is why the remainders are 3 when $10^2 + 2$ and $10^4 + 2$ are divided by 11. Similarly, for $n = 32$, the integer with 32 consecutive digits of 9 is a multiple of 11 because 32 is even. Also, that multiple of 11 is 3 less than $10^{32} + 2$, so the remainder is 3 when $10^{32} + 2$ is divided by 11. Thus the correct answer is **Choice C**.

An alternative approach is to rewrite the expression $10^{32}+2$ using the factoring technique $x^2-1=(x-1)(x+1)$ repeatedly, as follows.

$$\begin{aligned}10^{32}+2 &= (10^{32}-1)+3\\ &= (10^{16}-1)(10^{16}+1)+3\\ &= (10^{8}-1)(10^{8}+1)(10^{16}+1)+3\\ &= (10^{4}-1)(10^{4}+1)(10^{8}+1)(10^{16}+1)+3\\ &= (10^{2}-1)(10^{2}+1)(10^{4}+1)(10^{8}+1)(10^{16}+1)+3\\ &= (10+1)(10-1)(10^{2}+1)(10^{4}+1)(10^{8}+1)(10^{16}+1)+3\\ &= 11((10-1)(10^{2}+1)(10^{4}+1)(10^{8}+1)(10^{16}+1))+3\\ &= 11k+3\end{aligned}$$

where $k=(10-1)(10^2+1)(10^4+1)(10^8+1)(10^{16}+1)$ is an integer. Since $10^{32}+2$ is of the form $11k+3$, where k is an integer, it follows that when $10^{32}+2$ is divided by 11, the remainder is 3. The correct answer is **Choice C**.

This explanation uses the following strategies.
Strategy 5: Simplify an Arithmetic or Algebraic Representation
Strategy 7: Find a Pattern
Strategy 11: Divide into Cases

$$xy=8 \text{ and } x=y-2.$$

	Quantity A	Quantity B	
9.	y	0	Ⓐ Ⓑ Ⓒ Ⓓ

Explanation

In order to compare y and 0, you can try to determine the value of y from the two equations $xy=8$ and $x=y-2$. Substituting $y-2$ for x in the equation $xy=8$ gives $(y-2)y=8$, or $y^2-2y-8=0$. Factoring this quadratic equation yields $(y-4)(y+2)=0$. Therefore y can be either 4 or -2, so Quantity A can be greater than 0 or less than 0. Thus the correct answer is **Choice D**.

This explanation uses the following strategies.
Strategy 5: Simplify an Arithmetic or Algebraic Representation
Strategy 13: Determine Whether a Conclusion Follows from the Information Given

10. The area of circle W is 16π and the area of circle Z is 4π. What is the ratio of the circumference of W to the circumference of Z?

    (A) 2 to 1

    (B) 4 to 1

    (C) 8 to 1

    (D) 16 to 1

    (E) 32 to 1

Explanation

Recall that if a circle has radius r, then the area of the circle is πr^2 and the circumference is $2\pi r$. Since the area of circle W is 16π, it follows that $\pi r^2 = 16\pi$, so $r^2 = 16$ and $r = 4$. Therefore the circumference of circle W is $2\pi(4)$, or 8π. Similarly, since the area of circle Z is 4π, it follows that $\pi r^2 = 4\pi$, so $r^2 = 4$ and $r = 2$. Therefore the circumference of circle Z is $2\pi(2)$, or 4π. Thus the ratio of the circumference of W to the circumference of Z is 8π to 4π, or 2 to 1. The correct answer is **Choice A**.

This explanation uses the following strategies.

Strategy 1: Translate from Words to an Arithmetic or Algebraic Representation

Strategy 5: Simplify an Arithmetic or Algebraic Representation

11. In the xy-plane, a quadrilateral has vertices at $(-1, 4)$, $(7, 4)$, $(7, -5)$, and $(-1, -5)$. What is the perimeter of the quadrilateral?

Ⓐ 17

Ⓑ 18

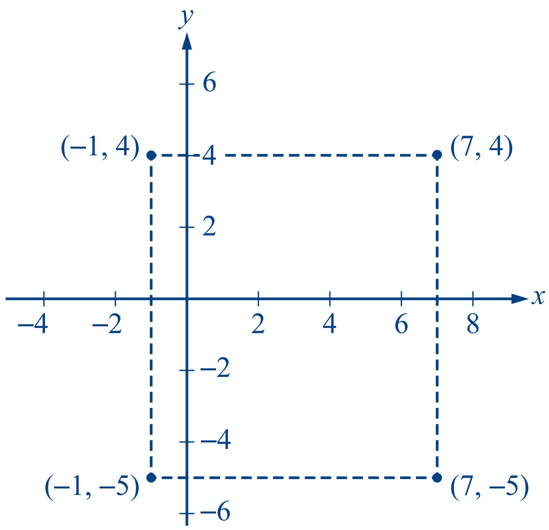
Ⓒ 19

Ⓓ 32

Ⓔ 34

Explanation

A sketch of the quadrilateral with vertices at $(-1, 4)$, $(7, 4)$, $(7, -5)$, and $(-1, -5)$ is shown in the xy-plane below.



From the figure you can see that two of the sides of the quadrilateral are horizontal and two are vertical. Therefore the quadrilateral is a rectangle. Since the points $(-1, -5)$ and $(7, -5)$ are endpoints of one of the horizontal sides, the length of each horizontal side is $7 - (-1)$, or 8. Since the points $(-1, 4)$ and $(-1, -5)$ are endpoints of one of the vertical sides, the length of each vertical side is $4 - (-5)$, or 9. Therefore the perimeter of the rectangle is $2(8 + 9)$, or 34. The correct answer is **Choice E**.

This explanation uses the following strategies.

Strategy 2: Translate from Words to a Figure or Diagram

Strategy 4: Translate from a Figure to an Arithmetic or Algebraic Representation

DISTRIBUTION OF THE
HEIGHTS OF 80 STUDENTS

Height (centimeters)	Number of Students
140 -144	6
145 -149	26
150 -154	32
155 -159	12
160 -164	4
Total	80

12. The table above shows the frequency distribution of the heights of 80 students, where the heights are recorded to the nearest centimeter. What is the least possible range of the recorded heights of the 80 students?

Ⓐ 15

Ⓑ 16

Ⓒ 20

Ⓓ 24

Ⓔ 28

Explanation

Recall that the range of the numbers in a group of data is the greatest number in the group minus the least number in the group. The table shows that the minimum recorded height of the 80 students can vary from 140 to 144 centimeters, and the maximum recorded height can vary from 160 to 164 centimeters. Thus the least possible range of the recorded heights is $160 - 144$, or 16 centimeters. The correct answer is **Choice B**.

This explanation uses the following strategy.

Strategy 4: Translate from a Figure to an Arithmetic or Algebraic Representation

13. Which of the following functions f defined for all numbers x has the property that $f(-x) = -f(x)$ for all numbers x ?

Ⓐ $f(x) = \dfrac{x^3}{x^2 + 1}$

Ⓑ $f(x) = \dfrac{x^2 - 1}{x^2 + 1}$

Ⓒ $f(x) = x^2(x^2 - 1)$

Ⓓ $f(x) = x(x^3 - 1)$

Ⓔ $f(x) = x^2(x^3 - 1)$

Explanation

To determine which of the functions among the five choices has the property that $f(-x) = -f(x)$ for all numbers x, you need to check each choice until you find one that has the property. In Choice A, $f(x) = \dfrac{x^3}{x^2 + 1}$:

$$f(-x) = \frac{(-x)^3}{(-x)^2 + 1} = \frac{-x^3}{x^2 + 1} \text{ and } -f(x) = -\left(\frac{x^3}{x^2 + 1}\right) = \frac{-x^3}{x^2 + 1}.$$

Therefore Choice A has the property $f(-x) = -f(x)$, and since only one of the five choices can be the correct answer, the correct answer is **Choice A**.

This explanation uses the following strategies.

Strategy 5: Simplify an Arithmetic or Algebraic Representation

Strategy 8: Search for a Mathematical Relationship

14. If 10^x equals 0.1 percent of 10^y, where x and y are integers, which of the following must be true?

Ⓐ $y = x + 2$

Ⓑ $y = x + 3$

Ⓒ $x = y + 2$

Ⓓ $y = 1{,}000x$

Ⓔ $x = 1{,}000y$

Explanation

The quantity 0.1 percent of m can be expressed as $\frac{0.1}{100}m$, which is equal to $\frac{1}{1{,}000}m$, or $\frac{1}{10^3}m$. Given that 10^x equals 0.1 percent of 10^y, it follows that

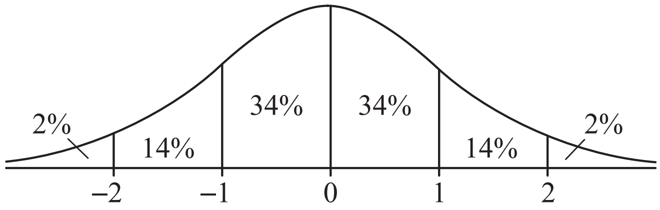
$$10^x = \left(\frac{1}{10^3}\right)(10^y) = \frac{10^y}{10^3} = 10^{y-3},$$

or $10^x = 10^{y-3}$. Therefore $x = y - 3$, or $y = x + 3$. The correct answer is **Choice B**.

This explanation uses the following strategies.

Strategy 1: Translate from Words to an Arithmetic or Algebraic Representation

Strategy 5: Simplify an Arithmetic or Algebraic Representation



15. The figure above shows the standard normal distribution, with mean 0 and standard deviation 1, including approximate percents of the distribution corresponding to the six regions shown.

    The random variable Y is normally distributed with a mean of 470, and the value $Y = 340$ is at the 15th percentile of the distribution. Of the following, which is the best estimate of the standard deviation of the distribution?

    (A) 125

    (B) 135

    (C) 145

    (D) 155

    (E) 165

Explanation

Since you know that the distribution of the random variable Y is normal with a mean of 470 and that the value 340 is at the 15th percentile of the distribution, you can estimate the standard deviation of the distribution of Y using the standard normal distribution. You can do this because the percent distributions of all normal distributions are the same in the following respect: The percentiles of every normal distribution are related to its standard deviation in exactly the same way as the percentiles of the standard normal distribution are related to its standard deviation. For example, approximately 14% of every normal distribution is between 1 and 2 standard deviations above the mean, just as the figure above illustrates for the standard normal distribution.

From the figure, approximately $2\% + 14\%$, or 16%, of the standard normal distribution is less than -1. Since $15\% < 16\%$, the 15th percentile of the distribution is at a value slightly below -1. For the standard normal distribution, the value -1 represents 1 standard deviation below the mean of 0. You can conclude that the 15th percentile of every normal distribution is at a value slightly below 1 standard deviation below the mean.

For the normal distribution of Y, the 15th percentile is 340, which is slightly below 1 standard deviation below the mean of 470. Consequently, the difference $470 - 340$, or 130, is a little greater than 1 standard deviation of Y; that is, the standard deviation of Y is a little less than 130. Of the answer choices given, the best estimate is 125, since it is close to, but a little less than, 130. The correct answer is **Choice A**.

This explanation uses the following strategies.

Strategy 4: Translate from a Figure to an Arithmetic or Algebraic Representation
Strategy 9: Estimate
Strategy 12: Adapt Solutions to Related Problems

16. A car dealer received a shipment of cars, half of which were black, with the remainder consisting of equal numbers of blue, silver, and white cars. During the next month, 70 percent of the black cars, 80 percent of the blue cars, 30 percent of the silver cars, and 40 percent of the white cars were sold. What percent of the cars in the shipment were sold during that month?

(A) 36%

(B) 50%

(C) 55%

(D) 60%

(E) 72%

Explanation

In the shipment, $\frac{1}{2}$ of the cars were black. Since the remainder of the cars consisted of equal numbers of blue, silver, and white cars, it follows that $\left(\frac{1}{3}\right)\left(\frac{1}{2}\right)$, or $\frac{1}{6}$, of the cars were blue, $\frac{1}{6}$ were silver, and $\frac{1}{6}$ were white. Based on the percents of the cars of each color that were sold during the next month, the percent of the cars in the shipment that were sold during that month was

$$\left(\frac{1}{2}\right)(70\%) + \left(\frac{1}{6}\right)(80\%) + \left(\frac{1}{6}\right)(30\%) + \left(\frac{1}{6}\right)(40\%) = 60\%$$

The correct answer is **Choice D**.

This explanation uses the following strategy.

Strategy 1: Translate from Words to an Arithmetic or Algebraic Representation

17. If an investment of P dollars is made today and the value of the investment doubles every 7 years, what will be the value of the investment, in dollars, 28 years from today?

Ⓐ $8P^4$

Ⓑ P^4

Ⓒ $16P$

Ⓓ $8P$

Ⓔ $4P$

Explanation

The investment of P dollars doubles every 7 years. Therefore 7 years from today, the value of the investment will be $2P$ dollars; 14 years from today, the value of the investment will be $4P$ dollars; 21 years from today, the value of the investment will be $8P$ dollars; and 28 years from today, the value of the investment will be $16P$ dollars. The correct answer is **Choice C**.

This explanation uses the following strategies.
Strategy 1: Translate from Words to an Arithmetic or Algebraic Representation
Strategy 7: Find a Pattern

---

18. The distribution of the numbers of hours that students at a certain college studied for final exams has a mean of 12 hours and a standard deviation of 3 hours. Which of the following numbers of hours are within 2 standard deviations of the mean of the distribution?

Indicate <u>all</u> such numbers.

[A] 2

[B] 5

[C] 10

[D] 14

[E] 16

[F] 20

Explanation

Given that the mean of the distribution is 12 hours and the standard deviation is 3 hours, the numbers of hours within 2 standard deviations of the mean are all numbers of hours between $12 - 2(3)$, or 6, and $12 + 2(3)$, or 18. Thus the correct answer consists of **Choices C, D, and E**.

This explanation uses the following strategy.
Strategy 1: Translate from Words to an Arithmetic or Algebraic Representation

19. In a certain sequence of numbers, each term after the first term is found by multiplying the preceding term by 2 and then subtracting 3 from the product. If the 4th term in the sequence is 19, which of the following numbers are in the sequence?

    Indicate all such numbers.

    $\boxed{A}$ 5

    $\boxed{B}$ 8

    $\boxed{C}$ 11

    $\boxed{D}$ 16

    $\boxed{E}$ 22

    $\boxed{F}$ 35

Explanation

Since the 4th term in the sequence is 19, it follows that the 5th term is $(19)(2) - 3,$ or 35. Proceeding backwards in the sequence from the 4th term to determine each preceding term, you would add 3 and then divide the result by 2. So the 3rd term is $\frac{19+3}{2},$ or 11; the 2nd term is $\frac{11+3}{2},$ or 7; and the 1st term is $\frac{7+3}{2},$ or 5. Hence the first 5 terms of the sequence are 5, 7, 11, 19, and 35, of which 5, 11, and 35 are among the answer choices.

Can you show that the other three answer choices, 8, 16, and 22, are not in the sequence? Note that 8, 16, and 22 are not among the first 5 terms of the sequence, and the 5th term of the sequence is 35. If you can show that each successive term in the sequence is greater than the term before it, you can conclude that 8, 16 and 22 are not terms in the sequence. If b is any term in the sequence, then the successive term is $2b - 3.$ Note that $b < 2b - 3$ is equivalent to $b > 3,$ so the successive term, $2b - 3,$ is greater than the term before it, b, if $b > 3.$ Since the first term of the sequence is 5, which is greater than 3, each successive term is greater than the term before it.

Thus the correct answer consists of **Choices A, C, and F**.

This explanation uses the following strategies.

Strategy 1: Translate from Words to an Arithmetic or Algebraic Representation

Strategy 7: Find a Pattern

20. In a single line of people waiting to purchase tickets for a movie, there are currently 10 people behind Shandra. If 3 of the people who are currently in line ahead of Shandra purchase tickets and leave the line, and no one else leaves the line, there will be 8 people ahead of Shandra in line. How many people are in the line currently?

[ ] people

Explanation

You are given that if 3 people currently ahead of Shandra leave the line and no one else leaves, there will be 8 people ahead of Shandra. This means that currently there are 11 people ahead of Shandra. In addition to the 11 people currently ahead of Shandra in line, Shandra herself is in line, and there are currently 10 people behind Shandra. Therefore the total number of people in line currently is $11 + 1 + 10$, or 22. The correct answer is **22**.

This explanation uses the following strategy.

Strategy 1: Translate from Words to an Arithmetic or Algebraic Representation

21. When the decimal point of a certain positive decimal number is moved six places to the right, the resulting number is 9 times the reciprocal of the original number. What is the original number?

Explanation

Moving the decimal point of a positive decimal number, n, six places to the right is equivalent to multiplying n by 10^6. In the question, you are given that the result of such a change is 9 times the reciprocal of the original number, or $9\left(\frac{1}{n}\right)$. Therefore $n(10^6) = 9\left(\frac{1}{n}\right)$. You can solve this equation for n as follows.

$$n(10^6) = 9\left(\frac{1}{n}\right)$$

$$n^2 = \frac{9}{10^6}$$

$$n = \sqrt{\frac{9}{10^6}}$$

$$n = \frac{3}{10^3}$$

$$n = 0.003$$

The correct answer is **0.003**.

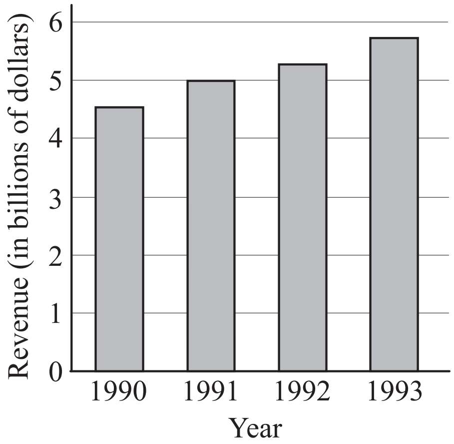
This explanation uses the following strategies.

Strategy 1: Translate from Words to an Arithmetic or Algebraic Representation

Strategy 5: Simplify an Arithmetic or Algebraic Representation

SELECTED DATA FOR GREETING CARD SALES

Annual Revenue from All Greeting Card Sales, 1990–1993



Number of Greeting Cards Sold for Ten Occasions in 1993

Occasion	Number of Cards
Christmas	2.4 billion
Valentine's Day	900 million
Easter	158 million
Mother's Day	155 million
Father's Day	102 million
Graduation	81 million
Thanksgiving	42 million
Halloween	32 million
St. Patrick's Day	18 million
Jewish New Year	12 million
Total	3.9 billion

Note: 1 billion = 1,000,000,000

22. In 1993 the number of Valentine's Day cards sold was approximately how many times the number of Thanksgiving cards sold?

(A) 20

(B) 30

(C) 40

(D) 50

(E) 60

Explanation

According to the table, the number of Valentine's Day cards sold in 1993 was 900 million, and the number of Thanksgiving cards sold was 42 million. Therefore the number of Valentine's Day cards sold was $\frac{900}{42}$, or approximately 21.4 times the number of Thanksgiving cards sold. Of the answer choices, the closest is 20. The correct answer is **Choice A**.

This explanation uses the following strategies.

Strategy 1: Translate from Words to an Arithmetic or Algebraic Representation
Strategy 4: Translate from a Figure to an Arithmetic or Algebraic Representation
Strategy 9: Estimate

23. In 1993 a card company that sold 40 percent of the Mother's Day cards that year priced its cards for that occasion between $1.00 and $8.00 each. If the revenue from sales of the company's Mother's Day cards in 1993 was r million dollars, which of the following indicates all possible values of r ?

Ⓐ $155 < r < 1{,}240$

Ⓑ $93 < r < 496$

Ⓒ $93 < r < 326$

Ⓓ $62 < r < 744$

Ⓔ $62 < r < 496$

Explanation

According to the table, 155 million Mother's Day cards were sold in 1993. The card company that sold 40 percent of the Mother's Day cards sold (0.4)(155) million, or 62 million cards. Since that company priced the cards between $1.00 and $8.00 each, the revenue, r million dollars, from selling the 62 million cards was between ($1.00)(62) million and ($8.00)(62) million, or between $62 million and $496 million; that is, $62 < r < 496.$ Thus the correct answer is **Choice E**.

This explanation uses the following strategies.

Strategy 1: Translate from Words to an Arithmetic or Algebraic Representation
Strategy 4: Translate from a Figure to an Arithmetic or Algebraic Representation
Strategy 8: Search for a Mathematical Relationship

---

24. Approximately what was the percent increase in the annual revenue from all greeting card sales from 1990 to 1993 ?

Ⓐ 50%

Ⓑ 45%

Ⓒ 39%

Ⓓ 28%

Ⓔ 20%

Explanation

According to the bar graph, the annual revenue from all greeting card sales in 1990 was approximately $4.5 billion, and the corresponding total in 1993 was approximately $5.75 billion. Therefore the percent increase from 1990 to 1993 was approximately $\left(\frac{5.75 - 4.5}{4.5}\right)(100\%)$, or approximately 28%. The correct answer is **Choice D**.

This explanation uses the following strategies.

Strategy 1: Translate from Words to an Arithmetic or Algebraic Representation
Strategy 4: Translate from a Figure to an Arithmetic or Algebraic Representation
Strategy 9: Estimate

25. In 1993 the average (arithmetic mean) price per card for all greeting cards sold was \$1.25. For which of the following occasions was the number of cards sold in 1993 less than the total number of cards sold that year for occasions other than the ten occasions shown?

Indicate all such occasions.

- A Christmas
- B Valentine's Day
- C Easter
- D Mother's Day
- E Father's Day
- F Graduation
- G Thanksgiving
- H Halloween

Explanation

According to the bar graph, the total annual revenue in 1993 was approximately \$5.75 billion. In the question, you are given that the average price per card for all greeting cards sold was \$1.25. Therefore the total number of cards sold for all occasions was $\frac{5.75}{1.25}$ billion, or 4.6 billion.

According to the table, the total number of cards sold in 1993 for the ten occasions shown was 3.9 billion. So the number of cards sold for occasions other than the ten occasions shown, in billions, was $4.6 - 3.9,$ or 0.7 billion. Note that 0.7 billion equals 700 million. From the table, you can see that less than 700 million cards were sold for each of six of the occasions in the answer choices: Easter, Mother's Day, Father's Day, Graduation, Thanksgiving, and Halloween. Thus the correct answer consists of **Choices C, D, E, F, G, and H**.

This explanation uses the following strategies.

Strategy 1: Translate from Words to an Arithmetic or Algebraic Representation
Strategy 4: Translate from a Figure to an Arithmetic or Algebraic Representation
Strategy 8: Search for a Mathematical Relationship
Strategy 9: Estimate

PRACTICE SET 2

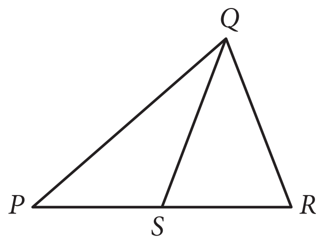
Quantitative Comparison Questions

For Questions 1 to 9, compare Quantity A and Quantity B, using additional information centered above the two quantities if such information is given. Select one of the following four answer choices and fill in the corresponding oval to the right of the question.

Ⓐ Quantity A is greater.
Ⓑ Quantity B is greater.
Ⓒ The two quantities are equal.
Ⓓ The relationship cannot be determined from the information given.

A symbol that appears more than once in a question has the same meaning throughout the question.

	Quantity A	Quantity B	Correct Answer
Example 1:	(2)(6)	2 + 6	Ⓐ Ⓑ Ⓒ Ⓓ



	Quantity A	Quantity B	Correct Answer
Example 2:	*PS*	*SR*	Ⓐ Ⓑ Ⓒ Ⓓ (since equal lengths cannot be assumed, even though *PS* and *SR* appear equal)

$x < 0$

	Quantity A	Quantity B	
1.	x^5	x^4	Ⓐ Ⓑ Ⓒ Ⓓ

	Quantity A	Quantity B	
2.	$(x+4)(y+3)$	$(x+3)(y+4)$	Ⓐ Ⓑ Ⓒ Ⓓ

$0.\overline{b}$ represents the decimal in which the digit b is repeated without end.

	Quantity A	Quantity B	
3.	$0.\overline{3} + 0.\overline{7}$	1.0	Ⓐ Ⓑ Ⓒ Ⓓ

A company plans to manufacture two types of hammers, type R and type S. The cost of manufacturing each hammer of type S is \$0.05 less than twice the cost of manufacturing each hammer of type R.

	Quantity A	Quantity B	
4.	The cost of manufacturing 1,000 hammers of type R and 1,000 hammers of type S	The cost of manufacturing 1,500 hammers of type S	Ⓐ Ⓑ Ⓒ Ⓓ

	Quantity A	Quantity B	
5.	$\dfrac{\left(\frac{1}{4}\right)^{-1} + \left(\frac{1}{4}\right)^{-2} + \left(\frac{1}{4}\right)^{-3}}{4}$	21	Ⓐ Ⓑ Ⓒ Ⓓ

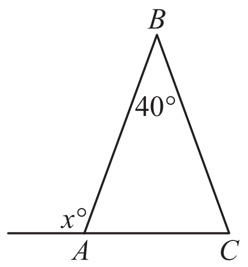
List X: 2, 5, s, t
List Y: 2, 5, t

The average (arithmetic mean) of the numbers in list X is equal to the average of the numbers in list Y.

	Quantity A	Quantity B	
6.	s	0	Ⓐ Ⓑ Ⓒ Ⓓ

The radius of circle A is 12 greater than the radius of circle B.

	Quantity A	Quantity B	
7.	The circumference of circle A minus the circumference of circle B	72	Ⓐ Ⓑ Ⓒ Ⓓ



$AB = BC$

	Quantity A	Quantity B	
8.	x	120	Ⓐ Ⓑ Ⓒ Ⓓ

$S = \{1, 2, 3, 4, 5, 6, 7, 8\}$

	Quantity A	Quantity B	
9.	The number of 4-member subsets of S	The number of 5-member subsets of S	Ⓐ Ⓑ Ⓒ Ⓓ

Multiple-choice Questions—Select One Answer Choice

For Questions 10 to 17, select a single answer choice.

10. In right triangle ABC, the ratio of the lengths of the two legs is 2 to 5. If the area of triangle ABC is 20, what is the length of the hypotenuse?

Ⓐ 7
Ⓑ 10
Ⓒ $4\sqrt{5}$
Ⓓ $\sqrt{29}$
Ⓔ $2\sqrt{29}$

11. According to surveys at a company, 20 percent of the employees owned cell phones in 1994, and 60 percent of the employees owned cell phones in 1998. From 1994 to 1998, what was the percent increase in the fraction of employees who owned cell phones?

Ⓐ 3%
Ⓑ 20%
Ⓒ 30%
Ⓓ 200%
Ⓔ 300%

12. If t is an integer and $8m = 16^t$, which of the following expresses m in terms of t?

Ⓐ 2^t
Ⓑ 2^{t-3}
Ⓒ $2^{3(t-3)}$
Ⓓ 2^{4t-3}
Ⓔ $2^{4(t-3)}$

13. Three pumps, P, R, and T, working simultaneously at their respective constant rates, can fill a tank in 5 hours. Pumps P and R, working simultaneously at their respective constant rates, can fill the tank in 7 hours. How many hours will it take pump T, working alone at its constant rate, to fill the tank?

Ⓐ 1.7
Ⓑ 10.0
Ⓒ 15.0
Ⓓ 17.5
Ⓔ 30.0

14. The perimeter of a flat rectangular lawn is 42 meters. The width of the lawn is 75 percent of its length. What is the area of the lawn, in square meters?

   (A) 40.5
   (B) 96
   (C) 108
   (D) 192
   (E) 432

15. The greatest of the 21 positive integers in a certain list is 16. The median of the 21 integers is 10. What is the least possible average (arithmetic mean) of the 21 integers?

   (A) 4
   (B) 5
   (C) 6
   (D) 7
   (E) 8

16. If x and y are integers and $x = \frac{(2)(3)(4)(5)(7)(11)(13)}{39y}$, which of the following could be the value of y ?

   (A) 15
   (B) 28
   (C) 38
   (D) 64
   (E) 143

17. Of the 40 specimens of bacteria in a dish, 3 specimens have a certain trait. If 5 specimens are to be selected from the dish at random and without replacement, which of the following represents the probability that only 1 of the 5 specimens selected will have the trait?

Ⓐ $\dfrac{\dbinom{5}{1}}{\dbinom{40}{3}}$

Ⓑ $\dfrac{\dbinom{5}{1}}{\dbinom{40}{5}}$

Ⓒ $\dfrac{\dbinom{40}{3}}{\dbinom{40}{5}}$

Ⓓ $\dfrac{\dbinom{3}{1}\dbinom{37}{4}}{\dbinom{40}{3}}$

Ⓔ $\dfrac{\dbinom{3}{1}\dbinom{37}{4}}{\dbinom{40}{5}}$

Multiple-choice Questions—Select One or More Answer Choices

For Questions 18 to 19, select all the answer choices that apply.

18. Two different positive integers x and y are selected from the odd integers that are less than 10. If $z = x + y$ and z is less than 10, which of the following integers could be the sum of x, y, and z ?

    Indicate <u>all</u> such integers.

    A 8

    B 9

    C 10

    D 12

    E 14

    F 15

    G 16

    H 18

19. For a certain probability experiment, the probability that event A will occur is $\frac{1}{2}$ and the probability that event B will occur is $\frac{1}{3}$. Which of the following values could be the probability that the event $A \cup B$ (that is, the event A or B, or both) will occur?

    Indicate <u>all</u> such values.

    A $\frac{1}{3}$

    B $\frac{1}{2}$

    C $\frac{3}{4}$

Numeric Entry Questions

For Questions 20 to 21, enter your answer in the answer box(es) below the question.

- **Your answer may be an integer, a decimal, or a fraction, and it may be negative.**
- **If a question asks for a fraction, there will be two boxes—one for the numerator and one for the denominator. A decimal point cannot be used in a fraction.**
- **Equivalent forms of the correct answer, such as 2.5 and 2.50, are all correct. Fractions do not need to be reduced to lowest terms, though you may need to reduce your fraction to fit in the boxes.**
- **Enter the exact answer unless the question asks you to round your answer.**

20. If a and b are the two solutions of the equation $x^2 - 5x + 4 = 0$, what is the value of $\left(\frac{1+a}{a}\right)\left(\frac{1+b}{b}\right)$?

    Give your answer as a fraction.

    $$\frac{\boxed{}}{\boxed{}}$$

21. From 2011 to 2012, Jack's annual salary increased by 10 percent and Arnie's annual salary decreased by 5 percent. If their annual salaries were equal in 2012, then Arnie's annual salary in 2011 was what percent greater than Jack's annual salary in 2011?

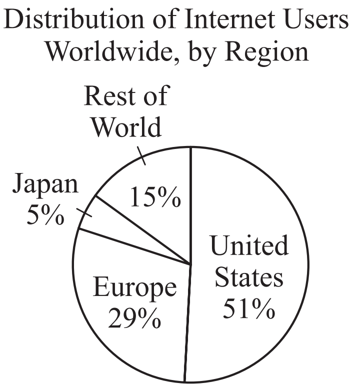
    Give your answer to the <u>nearest 0.1 percent</u>.

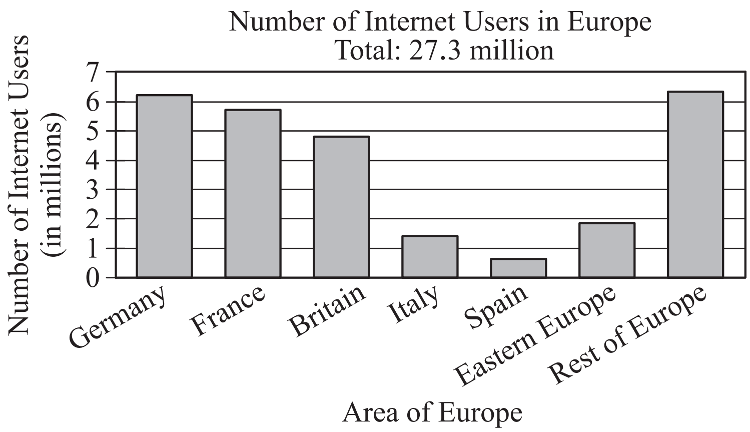
    $\boxed{}$ %

Data Interpretation Set

Questions 22 to 25 are based on the following data. For these questions, select a single answer choice unless otherwise directed.

INTERNET USE IN YEAR *X*





22. Which of the following is closest to the percent of Internet users in Europe who were in countries other than Germany, France, Britain, Italy, and Spain?

Ⓐ 30%

Ⓑ 34%

Ⓒ 38%

Ⓓ 42%

Ⓔ 46%

23. Approximately what was the range of the numbers of Internet users in the seven areas of Europe shown in the bar graph?

Ⓐ 6.5 million

Ⓑ 5.5 million

Ⓒ 3.5 million

Ⓓ 3.0 million

Ⓔ 2.5 million

24. The number of Internet users in the United States was approximately how many times the number of Internet users in Italy?

Ⓐ 5

Ⓑ 15

Ⓒ 20

Ⓓ 25

Ⓔ 35

For Question 25, select all the answer choices that apply.

25. Based on the information given, which of the following statements about Internet use in year *X* must be true?

Indicate all such statements.

- [A] The United States had more Internet users than all other countries in the world combined.
- [B] Spain had fewer Internet users than any country in Eastern Europe.
- [C] Germany and France combined had more than $\frac{1}{3}$ of the Internet users in Europe.

ANSWER KEY

1. **Choice B:** Quantity B is greater.
2. **Choice D:** The relationship cannot be determined from the information given.
3. **Choice A:** Quantity A is greater.
4. **Choice A:** Quantity A is greater.
5. **Choice C:** The two quantities are equal.
6. **Choice D:** The relationship cannot be determined from the information given.
7. **Choice A:** Quantity A is greater.
8. **Choice B:** Quantity B is greater.
9. **Choice A:** Quantity A is greater.
10. **Choice E:** $2\sqrt{29}$
11. **Choice D:** 200%
12. **Choice D:** 2^{4t-3}
13. **Choice D:** 17.5
14. **Choice C:** 108
15. **Choice C:** 6
16. **Choice B:** 28
17. **Choice E:** $\dfrac{\binom{3}{1}\binom{37}{4}}{\binom{40}{5}}$
18. **Choice A:** 8
    AND
    Choice D: 12
    AND
    Choice G: 16
19. **Choice B:** $\frac{1}{2}$
    AND
    Choice C: $\frac{3}{4}$
20. $\mathbf{\frac{5}{2}}$
21. **15.8**
22. **Choice A:** 30%
23. **Choice B:** 5.5 million
24. **Choice E:** 35
25. **Choice A:** The United States had more Internet users than all other countries in the world combined.
    AND
    Choice C: Germany and France combined had more than $\frac{1}{3}$ of the Internet users in Europe.

Answers and Explanations

$$x < 0$$

	Quantity A	Quantity B	
1.	x^5	x^4	Ⓐ Ⓑ Ⓒ Ⓓ

Explanation

If $x < 0$, then $x^5 < 0$ and $x^4 > 0$. Thus $x^5 < x^4$, and the correct answer is **Choice B**.

This explanation uses the following strategy.

Strategy 8: Search for a Mathematical Relationship

	Quantity A	Quantity B	
2.	$(x+4)(y+3)$	$(x+3)(y+4)$	Ⓐ Ⓑ Ⓒ Ⓓ

Explanation

To compare $(x+4)(y+3)$ and $(x+3)(y+4)$, try plugging in a few values for x and y.

Case 1: $x = 0$ and $y = 0$. In this case, Quantity A is equal to $(4)(3)$, or 12, and Quantity B is equal to $(3)(4)$, or 12. So Quantity A is equal to Quantity B.

Case 2: $x = 0$ and $y = 1$. In this case, Quantity A is equal to $(4)(4)$, or 16, and Quantity B is equal to $(3)(5)$, or 15. So Quantity A is greater than Quantity B.

In one case, Quantity A is equal to Quantity B, and in the other case, Quantity A is greater than Quantity B. Therefore the correct answer is **Choice D**.

This explanation uses the following strategies.

Strategy 10: Trial and Error

Strategy 13: Determine Whether a Conclusion Follows from the Information Given

---

$0.\overline{b}$ represents the decimal in which the digit b is repeated without end.

	Quantity A	Quantity B	
3.	$0.\overline{3} + 0.\overline{7}$	1.0	Ⓐ Ⓑ Ⓒ Ⓓ

Explanation

By the definition given in the question, $0.\overline{3}$ represents the decimal in which the digit 3 is repeated without end; that is, $0.\overline{3} = 0.333....$ It follows that $0.\overline{3}$ is greater than 0.3. Similarly, $0.\overline{7}$ is greater than 0.7. Therefore $0.\overline{3} + 0.\overline{7}$ is greater than 1; that is, Quantity A is greater than Quantity B. The correct answer is **Choice A**.

This explanation uses the following strategy.

Strategy 1: Translate from Words to an Arithmetic or Algebraic Representation

A company plans to manufacture two types of hammers, type R and type S. The cost of manufacturing each hammer of type S is \$0.05 less than twice the cost of manufacturing each hammer of type R.

	Quantity A	Quantity B	
4.	The cost of manufacturing 1,000 hammers of type R and 1,000 hammers of type S	The cost of manufacturing 1,500 hammers of type S	(A) (B) (C) (D)

Explanation

Note that Quantity A and Quantity B both include the cost of manufacturing 1,000 hammers of type S. If you remove that cost from both quantities, the problem is reduced to comparing the cost of manufacturing 1,000 hammers of type R with the cost of manufacturing 500 hammers of type S. Since the cost of manufacturing each hammer of type S is \$0.05 less than twice the cost of manufacturing each hammer of type R, it follows that the cost of manufacturing 1,000 hammers of type R is greater than the cost of manufacturing 500 hammers of type S. Thus the correct answer is **Choice A.**

This explanation uses the following strategies.

Strategy 8: Search for a Mathematical Relationship
Strategy 9: Estimate

	Quantity A	Quantity B	
5.	$\dfrac{\left(\frac{1}{4}\right)^{-1}+\left(\frac{1}{4}\right)^{-2}+\left(\frac{1}{4}\right)^{-3}}{4}$	21	(A) (B) (C) (D)

Explanation

Since $\left(\frac{1}{4}\right)^{-1}=4$, $\left(\frac{1}{4}\right)^{-2}=4^2$, and $\left(\frac{1}{4}\right)^{-3}=4^3$, you can simplify Quantity A as follows.

$$\frac{\left(\frac{1}{4}\right)^{-1}+\left(\frac{1}{4}\right)^{-2}+\left(\frac{1}{4}\right)^{-3}}{4}=\frac{4+4^2+4^3}{4}=1+4+4^2=21$$

Since Quantity B is also 21, the correct answer is **Choice C.**

This explanation uses the following strategy.

Strategy 5: Simplify an Arithmetic or Algebraic Representation

List X: 2, 5, s, t
List Y: 2, 5, t

The average (arithmetic mean) of the numbers in list X is equal to the average of the numbers in list Y.

	Quantity A	Quantity B	
6.	s	0	(A) (B) (C) (D)

Explanation

Since you are given that the average of the 4 numbers in list X is equal to the average of the 3 numbers in list Y, it follows that $\frac{2+5+s+t}{4}=\frac{2+5+t}{3}$. To make it easier to compare Quantity A and Quantity B, you can simplify this equation as follows.

$$\frac{2+5+s+t}{4}=\frac{2+5+t}{3}$$
$$\frac{7+s+t}{4}=\frac{7+t}{3}$$
$$3(7+s+t)=4(7+t)$$
$$21+3s+3t=28+4t$$
$$3s=7+t$$

From the equation $3s=7+t$, if $t=-7$, then $s=0$, but if $t=0$, then $s>0$. In one case, the quantities are equal, and in the other case, Quantity A is greater. Therefore the correct answer is **Choice D.**

This explanation uses the following strategies.
Strategy 1: Translate from Words to an Arithmetic or Algebraic Representation
Strategy 10: Trial and Error
Strategy 13: Determine Whether a Conclusion Follows from the Information Given

---

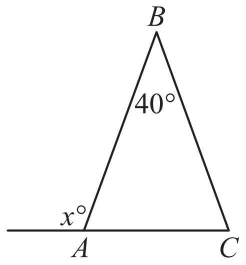
The radius of circle A is 12 greater than the radius of circle B.

	Quantity A	Quantity B	
7.	The circumference of circle A minus the circumference of circle B	72	(A) (B) (C) (D)

Explanation

Since the radius of circle A is 12 greater than the radius of circle B, if the radius of circle B is r, then the radius of circle A is $r+12$. The circumference of circle A minus the circumference of circle B is $2\pi(r+12)-2\pi r$, which simplifies to 24π. Since $\pi>3$, it follows that $24\pi>24(3)$; that is, $24\pi>72$. Since Quantity B is 72, the correct answer is **Choice A**.

This explanation uses the following strategies.
Strategy 8: Search for a Mathematical Relationship
Strategy 9: Estimate



$AB = BC$

	Quantity A	Quantity B	
8.	x	120	Ⓐ Ⓑ Ⓒ Ⓓ

Explanation

Note that there are four angles in the figure: the three interior angles of triangle ABC and the exterior angle at vertex A. Since the measure of the exterior angle at vertex A is x degrees, it follows that the measure of interior angle A is $(180 - x)°$. Also, since $AB = BC$, it follows that triangle ABC is isosceles and the measures of interior angles A and C are equal. Therefore the measure of interior angle C is also $(180 - x)°$.

Since the sum of the measures of the interior angles of a triangle is 180° and you are given that the measure of interior angle B is $40°$, it follows that

$$40 + (180 - x) + (180 - x) = 180.$$

Solving the equation for x gives $x = 110$. Since Quantity B is 120, the correct answer is **Choice B**.

This explanation uses the following strategies.

Strategy 4: Translate from a Figure to an Arithmetic or Algebraic Representation
Strategy 8: Search for a Mathematical Relationship

$$S = \{1, 2, 3, 4, 5, 6, 7, 8\}$$

	Quantity A	Quantity B	
9.	The number of 4-member subsets of S	The number of 5-member subsets of S	Ⓐ Ⓑ Ⓒ Ⓓ

Explanation

Recall that the number of r-member subsets of a set with n members is equal to $\frac{n!}{r!(n-r)!}$. So Quantity A is equal to $\frac{8!}{4!\,4!} = \frac{(8)(7)(6)(5)}{(4)(3)(2)(1)} = 70$. Similarly, Quantity B is equal to $\frac{8!}{5!\,3!} = \frac{(8)(7)(6)}{(3)(2)(1)} = 56$. Thus the correct answer is **Choice A**.

This explanation uses the following strategy.

Strategy 1: Translate from Words to an Arithmetic or Algebraic Representation

10. In right triangle ABC, the ratio of the lengths of the two legs is 2 to 5. If the area of triangle ABC is 20, what is the length of the hypotenuse?

Ⓐ 7
Ⓑ 10
Ⓒ $4\sqrt{5}$
Ⓓ $\sqrt{29}$
Ⓔ $2\sqrt{29}$

Explanation

The ratio of the lengths of the legs of right triangle ABC is 2 to 5, so you can represent the lengths as $2x$ and $5x$, respectively, where $x > 0$. Since the area of the triangle is 20, it follows that $\frac{(2x)(5x)}{2} = 20$. Solving this equation for x gives $x = 2$. So the lengths of the two legs are $2(2)$ and $2(5)$, or 4 and 10, respectively. Therefore, by the Pythagorean theorem, the length of the hypotenuse is $\sqrt{4^2 + 10^2}$. This square root can be simplified as follows.

$$\begin{aligned}\sqrt{4^2+10^2} &= \sqrt{116}\\ &= \sqrt{(4)(29)}\\ &= 2\sqrt{29}\end{aligned}$$

The correct answer is **Choice E**.

This explanation uses the following strategies.
Strategy 5: Simplify an Arithmetic or Algebraic Representation
Strategy 8: Search for a Mathematical Relationship

---

11. According to surveys at a company, 20 percent of the employees owned cell phones in 1994, and 60 percent of the employees owned cell phones in 1998.
From 1994 to 1998, what was the percent increase in the fraction of employees who owned cell phones?

Ⓐ 3%
Ⓑ 20%
Ⓒ 30%
Ⓓ 200%
Ⓔ 300%

Explanation

From 1994 to 1998, the percent of employees who owned cell phones increased from 20% to 60%. Thus the percent increase in the fraction of employees who owned cell phones was $\left(\frac{60\% - 20\%}{20\%}\right)(100\%)$, or 200%. The correct answer is **Choice D**.

This explanation uses the following strategy.
Strategy 1: Translate from Words to an Arithmetic or Algebraic Representation

12. If t is an integer and $8m = 16^t$, which of the following expresses m in terms of t ?

(A) 2^t

(B) 2^{t-3}

(C) $2^{3(t-3)}$

(D) 2^{4t-3}

(E) $2^{4(t-3)}$

Explanation

Note that all of the choices are expressions of the form 2 raised to a power. Expressing 8 and 16 as powers of 2, you can rewrite the given equation $8m = 16^t$ as $2^3m = (2^4)^t$, which is the same as $2^3m = 2^{4t}$. Solving for m and using the rules of exponents, you get $m = \frac{2^{4t}}{2^3} = 2^{4t-3}$. The correct answer is **Choice D**.

This explanation uses the following strategies.

Strategy 5: Simplify an Arithmetic or Algebraic Representation

Strategy 8: Search for a Mathematical Relationship

---

13. Three pumps, P, R, and T, working simultaneously at their respective constant rates, can fill a tank in 5 hours. Pumps P and R, working simultaneously at their respective constant rates, can fill the tank in 7 hours. How many hours will it take pump T, working alone at its constant rate, to fill the tank?

(A) 1.7

(B) 10.0

(C) 15.0

(D) 17.5

(E) 30.0

Explanation

Working simultaneously, pumps P and R fill $\frac{1}{7}$ of the tank in 1 hour. Working simultaneously, the three pumps fill $\frac{1}{5}$ of the tank in 1 hour. Therefore, working alone, pump T fills $\frac{1}{5} - \frac{1}{7}$, or $\frac{2}{35}$, of the tank in 1 hour. Thus, working alone, pump T takes $\frac{35}{2}$ hours, or 17.5 hours, to fill the tank. The correct answer is **Choice D**.

This explanation uses the following strategies.

Strategy 1: Translate from Words to an Arithmetic or Algebraic Representation

Strategy 8: Search for a Mathematical Relationship

14. The perimeter of a flat rectangular lawn is 42 meters. The width of the lawn is 75 percent of its length. What is the area of the lawn, in square meters?

Ⓐ 40.5
Ⓑ 96
Ⓒ 108
Ⓓ 192
Ⓔ 432

Explanation

Let s and t be the width and length, in meters, of the lawn, respectively. Then the perimeter is $2s + 2t$ meters, so that $2s + 2t = 42$. Also, the relationship between the width and the length can be translated as $s = 0.75t$. Substituting $0.75t$ for s in the equation for the perimeter yields $42 = 2(0.75t) + 2t = 3.5t$. So $t = \frac{42}{3.5} = 12$ and $s = (0.75)(12) = 9$. Since $st = 108$, the area of the lawn is 108 square meters. The correct answer is **Choice C**.

This explanation uses the following strategies.

Strategy 1: Translate from Words to an Arithmetic or Algebraic Representation
Strategy 8: Search for a Mathematical Relationship

---

15. The greatest of the 21 positive integers in a certain list is 16. The median of the 21 integers is 10. What is the least possible average (arithmetic mean) of the 21 integers?

Ⓐ 4
Ⓑ 5
Ⓒ 6
Ⓓ 7
Ⓔ 8

Explanation

You are given that the median of the 21 positive integers is 10 and the greatest of the 21 integers is 16. This means that when the 21 integers are listed in order from least to greatest,

- the 1st through 10th integers are between 1 and 10, inclusive;
- the 11th integer is 10;
- the 12th through 20th integers are between 10 and 16, inclusive; and
- the 21st integer is 16.

The least possible average of the 21 integers would be achieved by using the least possible value of each integer as described above in the reordered list:

- the 1st through 10th integers would each be 1;
- the 11th integer would be 10;
- the 12th through 20th integers would each be 10; and
- the 21st integer would be 16.

For the least possible integers, the sum is $(10)(1)+10+(9)(10)+16,$ or 126. Therefore the least possible average is $\frac{126}{21},$ or 6. The correct answer is **Choice C**.

This explanation uses the following strategies.
Strategy 1: Translate from Words to an Arithmetic or Algebraic Representation
Strategy 8: Search for a Mathematical Relationship

---

16. If x and y are integers and $x=\frac{(2)(3)(4)(5)(7)(11)(13)}{39y},$ which of the following could be the value of y ?

Ⓐ 15
Ⓑ 28
Ⓒ 38
Ⓓ 64
Ⓔ 143

Explanation

To simplify the equation $x=\frac{(2)(3)(4)(5)(7)(11)(13)}{39y},$ divide the numerator and denominator of the fraction by 39 to get $x=\frac{(2)(4)(5)(7)(11)}{y}.$ From the simplified equation, you can see that x is an integer if and only if y is a factor of $(2)(4)(5)(7)(11).$ To answer the question, you need to check each of the answer choices until you find the one that is a factor of $(2)(4)(5)(7)(11).$

Choice A: 15. Since $15=(3)(5)$ and 3 is not a factor of $(2)(4)(5)(7)(11),$ neither is 15.
Choice B: 28. Since $28=(4)(7)$ and both 4 and 7 are factors of $(2)(4)(5)(7)(11),$ so is 28.

You can check the other choices to confirm that none of them is a factor of $(2)(4)(5)(7)(11).$ The correct answer is **Choice B**.

This explanation uses the following strategies.
Strategy 5: Simplify an Arithmetic or Algebraic Representation
Strategy 8: Search for a Mathematical Relationship

17. Of the 40 specimens of bacteria in a dish, 3 specimens have a certain trait. If 5 specimens are to be selected from the dish at random and without replacement, which of the following represents the probability that only 1 of the 5 specimens selected will have the trait?

Ⓐ $\dfrac{\dbinom{5}{1}}{\dbinom{40}{3}}$

Ⓑ $\dfrac{\dbinom{5}{1}}{\dbinom{40}{5}}$

Ⓒ $\dfrac{\dbinom{40}{3}}{\dbinom{40}{5}}$

Ⓓ $\dfrac{\dbinom{3}{1}\dbinom{37}{4}}{\dbinom{40}{3}}$

Ⓔ $\dfrac{\dbinom{3}{1}\dbinom{37}{4}}{\dbinom{40}{5}}$

Explanation

In the context of this problem, $\dbinom{n}{r}$ represents the number of ways r specimens can be selected without replacement from n specimens.

The probability that only 1 of the 5 specimens selected from the 40 specimens will have the trait is equal to

$$\frac{\text{number of ways to select 5 specimens, only 1 of which has the trait}}{\text{number of ways to select 5 specimens}}.$$

The number of ways 5 specimens can be selected from the 40 specimens is $\dbinom{40}{5}$. To select 5 specimens, only 1 of which has the trait, you have to select 1 of the 3 specimens that have the trait and select 4 of the 37 specimens that do not have the trait. The number of such selections is the product $\binom{3}{1}\binom{37}{4}$. So the probability that only 1 of the 5 specimens selected will have the trait is represented by $\dfrac{\binom{3}{1}\binom{37}{4}}{\binom{40}{5}}$. The correct answer is **Choice E**.

This explanation uses the following strategies.
Strategy 1: Translate from Words to an Arithmetic or Algebraic Representation
Strategy 8: Search for a Mathematical Relationship

---

18. Two different positive integers x and y are selected from the odd integers that are less than 10. If $z = x + y$ and z is less than 10, which of the following integers could be the sum of x, y, and z?

    Indicate all such integers.

    [A] 8
    [B] 9
    [C] 10
    [D] 12
    [E] 14
    [F] 15
    [G] 16
    [H] 18

Explanation

The only pairs of positive odd integers x and y that are less than 10 and satisfy the condition $x + y < 10$ are the pair 1 and 3, the pair 1 and 5, the pair 1 and 7, and the pair 3 and 5. Since $z = x + y$, it follows that the sum of x, y, and z is equal to $2z$. The sum for each of the four possible pairs is found as follows.

- 1 and 3: $z = 4$, and the sum of x, y, and z is $2z$, or 8.
- 1 and 5: $z = 6$, and the sum of x, y, and z is 12.
- 1 and 7: $z = 8$, and the sum of x, y, and z is 16.
- 3 and 5: $z = 8$, and the sum of x, y, and z is 16.

Thus the only possible values of the sum of x, y, and z are 8, 12, and 16. The correct answer consists of **Choices A, D, and G**.

This explanation uses the following strategies.
Strategy 1: Translate from Words to an Arithmetic or Algebraic Representation
Strategy 8: Search for a Mathematical Relationship
Strategy 11: Divide into Cases

19. For a certain probability experiment, the probability that event A will occur is $\frac{1}{2}$ and the probability that event B will occur is $\frac{1}{3}$. Which of the following values could be the probability that the event $A \cup B$ (that is, the event A or B, or both) will occur?

Indicate all such values.

- A $\frac{1}{3}$
- B $\frac{1}{2}$
- C $\frac{3}{4}$

Explanation

Since you know that the probability of event A is $\frac{1}{2}$ and the probability of event B is $\frac{1}{3}$ but you are not given any information about the relationship between events A and B, you can compute only the minimum possible value and the maximum possible value of the probability of the event $A \cup B$.

The probability of $A \cup B$ is least if B is a subset of A; in that case, the probability of $A \cup B$ is just the probability of A, or $\frac{1}{2}$.

The probability of $A \cup B$ is greatest if A and B do not intersect at all; in that case, the probability of $A \cup B$ is the sum of the probabilities of A and B, or $\frac{1}{2} + \frac{3}{4} = \frac{5}{6}$.

With no further information about A and B, the probability that A or B, or both, will occur could be any number from $\frac{1}{2}$ to $\frac{5}{6}$. Of the answer choices given, only $\frac{1}{2}$ and $\frac{3}{4}$ are in this interval. The correct answer consists of **Choices B and C**.

This explanation uses the following strategies.

Strategy 8: Search for a Mathematical Relationship
Strategy 11: Divide into Cases

20. If a and b are the two solutions of the equation $x^2 - 5x + 4 = 0$, what is the value of $\left(\frac{1+a}{a}\right)\left(\frac{1+b}{b}\right)$?

Give your answer as a fraction.

$$\frac{\boxed{}}{\boxed{}}$$

Explanation

Factoring the quadratic equation $x^2 - 5x + 4 = 0$, you get $(x-1)(x-4) = 0$, so the two solutions are either $a = 1$ and $b = 4$ or $a = 4$ and $b = 1$. Note that a and b are interchangeable in the expression $\left(\frac{1+a}{a}\right)\left(\frac{1+b}{b}\right)$ so the value of the expression will be the same regardless of the choices of a and b. Thus

$$\left(\frac{1+a}{a}\right)\left(\frac{1+b}{b}\right) = \left(\frac{1+1}{1}\right)\left(\frac{1+4}{4}\right) = (2)\left(\frac{5}{4}\right) = \frac{5}{2},$$

and the correct answer is $\frac{5}{2}$.

This explanation uses the following strategies.

Strategy 5: Simplify an Arithmetic or Algebraic Representation
Strategy 8: Search for a Mathematical Relationship

---

21. From 2011 to 2012, Jack's annual salary increased by 10 percent and Arnie's annual salary decreased by 5 percent. If their annual salaries were equal in 2012, then Arnie's annual salary in 2011 was what percent greater than Jack's annual salary in 2011 ?

Give your answer to the nearest 0.1 percent.

$\boxed{}$ %

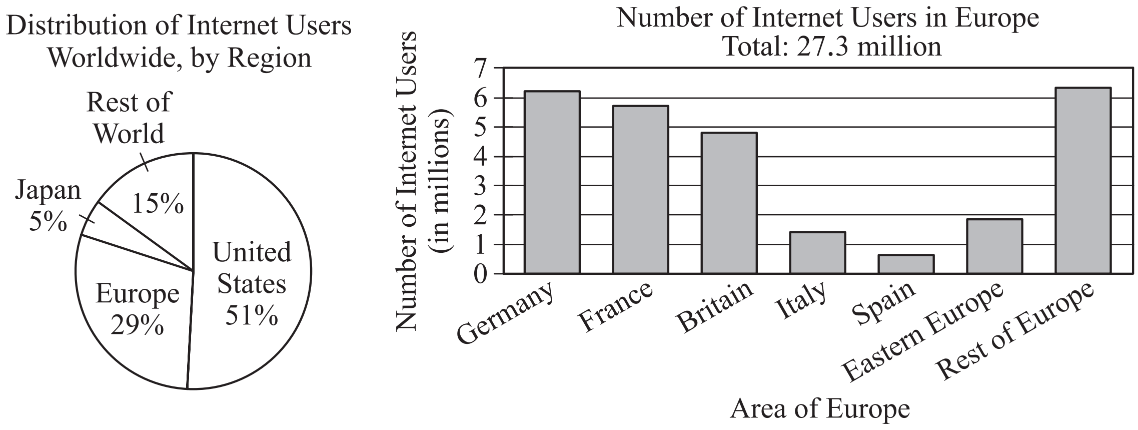
Explanation

Let k be Jack's annual salary in 2011, and let r be Arnie's annual salary in 2011. Then Jack's annual salary in 2012 was $1.1k$, and Arnie's was $0.95r$. Since their salaries in 2012 were equal to each other, you have $0.95r = 1.1k$. Solving the equation for r, you get $r = \frac{1.1}{0.95}k$. Since $\frac{1.1}{0.95} = 1.1578\ldots$, it follows that, rounded to the nearest 0.1%, Arnie's annual salary in 2011 was 15.8% greater than Jack's annual salary in 2011. The correct answer is **15.8**.

This explanation uses the following strategies.

Strategy 1: Translate from Words to an Arithmetic or Algebraic Representation
Strategy 5: Simplify an Arithmetic or Algebraic Representation

INTERNET USE IN YEAR X



22. Which of the following is closest to the percent of Internet users in Europe who were in countries other than Germany, France, Britain, Italy, and Spain?

Ⓐ 30%
Ⓑ 34%
Ⓒ 38%
Ⓓ 42%
Ⓔ 46%

Explanation

The distribution of Internet users in Europe is given in the bar graph. In the graph, the countries in Europe other than Germany, France, Britain, Italy, and Spain are grouped into two areas: Eastern Europe and Rest of Europe. According to the graph, there were approximately 1.9 million users in Eastern Europe and 6.3 million users in Rest of Europe. Thus the number of users in these two areas combined was approximately $1.9 + 6.3$, or 8.2 million.

From the title of the bar graph, the total number of Internet users in Europe was 27.3 million. Therefore the number of users in the two areas combined, expressed as a percent of all users in Europe, is about $\left(\frac{8.2}{27.3}\right)(100\%)$, or approximately 30%.

The answer choice that is closest is 30%. The correct answer is **Choice A**.

This explanation uses the following strategies.
Strategy 4: Translate from a Figure to an Arithmetic or Algebraic Representation
Strategy 5: Simplify an Arithmetic or Algebraic Representation
Strategy 9: Estimate

23. Approximately what was the range of the numbers of Internet users in the seven areas of Europe shown in the bar graph?

Ⓐ 6.5 million
Ⓑ 5.5 million
Ⓒ 3.5 million
Ⓓ 3.0 million
Ⓔ 2.5 million

Explanation

The range of the numbers of Internet users in the seven areas of Europe shown in the bar graph is equal to the greatest of the seven numbers minus the least of the seven numbers. Of the seven areas, Rest of Europe had the greatest number of users, approximately 6.3 million, and Spain had the least number of users, approximately 0.7 million. So the range was approximately $6.3 - 0.7$, or 5.6 million. The answer choice that is closest is 5.5 million. The correct answer is **Choice B**.

This explanation uses the following strategies.

Strategy 4: Translate from a Figure to an Arithmetic or Algebraic Representation
Strategy 9: Estimate

---

24. The number of Internet users in the United States was approximately how many times the number of Internet users in Italy?

Ⓐ 5
Ⓑ 15
Ⓒ 20
Ⓓ 25
Ⓔ 35

Explanation

According to the bar graph, the number of Internet users in Italy was approximately 1.4 million. The number of Internet users in the United States is not explicitly given, but you know from the circle graph that it was equal to 51% of the number of Internet users worldwide.

From the bar graph, the number of Internet users in Europe was 27.3 million, and from the circle graph this number was 29% of Internet users worldwide. It follows that the number of users worldwide was $\frac{27.3}{0.29}$ million, which is approximately 94 million. Thus the number of users in the United States was 51% of 94 million, or approximately 48 million.

Since the numbers of Internet users in the United States and in Italy were about 48 million and 1.4 million, respectively, it follows that the number in the United States was approximately $\frac{48}{1.4}$ times the number in Italy. Since $\frac{48}{1.4}$ is approximately 34, and the answer choice closest to 34 is 35, the correct answer is **Choice E**.

This explanation uses the following strategies.
Strategy 4: Translate from a Figure to an Arithmetic or Algebraic Representation
Strategy 8: Search for a Mathematical Relationship
Strategy 9: Estimate
Strategy 14: Determine What Additional Information Is Sufficient to Solve a Problem

---

25. Based on the information given, which of the following statements about Internet use in year X must be true?

    Indicate all such statements.

    A The United States had more Internet users than all other countries in the world combined.

    B Spain had fewer Internet users than any country in Eastern Europe.

    C Germany and France combined had more than $\frac{1}{3}$ of the Internet users in Europe.

Explanation

Each statement needs to be evaluated separately.

Statement A. According to the circle graph, 51% of Internet users worldwide were in the United States. Since 51% is greater than $\frac{1}{2}$, Statement A must be true.

Statement B. According to the bar graph, the number of Internet users in Eastern Europe, approximately 1.9 million, was greater than the number in Spain, approximately 0.7 million. However, since the bar graph does not give any information about the distribution of users in the individual countries in Eastern Europe, Statement B may or may not be true.

Statement C. According to the bar graph, Germany and France together had about 12 million Internet users. Since there were 27.3 million users in Europe and 12 is greater than $\frac{1}{3}$ of 27.3, Statement C must be true.

The correct answer consists of **Choices A and C**.

This explanation uses the following strategies.
Strategy 4: Translate from a Figure to an Arithmetic or Algebraic Representation
Strategy 13: Determine Whether a Conclusion Follows from the Information Given
Strategy 14: Determine What Additional Information Is Sufficient to Solve a Problem

PRACTICE SET 3

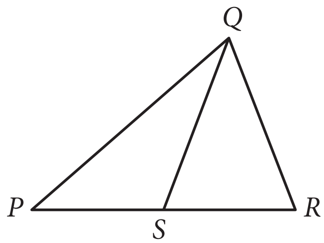
Quantitative Comparison Questions

For Questions 1 to 9, compare Quantity A and Quantity B, using additional information centered above the two quantities if such information is given. Select one of the following four answer choices and fill in the corresponding oval to the right of the question.

(A) Quantity A is greater.
(B) Quantity B is greater.
(C) The two quantities are equal.
(D) The relationship cannot be determined from the information given.

A symbol that appears more than once in a question has the same meaning throughout the question.

	Quantity A	Quantity B	Correct Answer
Example 1:	(2)(6)	2 + 6	(A) (B) (C) (D)



	Quantity A	Quantity B	Correct Answer
Example 2:	*PS*	*SR*	(A) (B) (C) (D) (since equal lengths cannot be assumed, even though *PS* and *SR* appear equal)

$$y < -6$$

	Quantity A	Quantity B	
1.	y	-5	(A) (B) (C) (D)

x is an integer and $23 < x < 27$.

	Quantity A	Quantity B	
2.	The median of the five integers 23, 24, 26, 27, and x	25	Ⓐ Ⓑ Ⓒ Ⓓ

---

r and t are consecutive integers and $p = r^2 + t$.

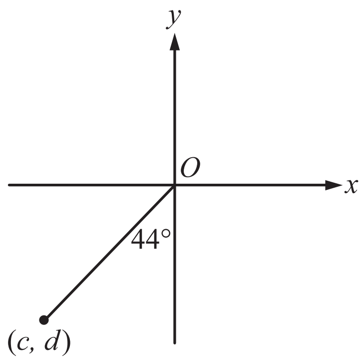
	Quantity A	Quantity B	
3.	$(-1)^p$	-1	Ⓐ Ⓑ Ⓒ Ⓓ

The function f is defined by $f\left(\frac{x+3}{2}\right) = 3x^2 - x + 5$ for all numbers x.

	Quantity A	Quantity B	
4	$f(4)$	75	Ⓐ Ⓑ Ⓒ Ⓓ

The sum of the annual salaries of the 21 teachers at School X is \$781,200. Twelve of the 21 teachers have an annual salary that is less than \$37,000.

	Quantity A	Quantity B	
5.	The average (arithmetic mean) of the annual salaries of the teachers at School X	The median of the annual salaries of the teachers at School X	Ⓐ Ⓑ Ⓒ Ⓓ



	Quantity A	Quantity B	
6.	c	d	Ⓐ Ⓑ Ⓒ Ⓓ

$N = 824^x$, where x is a positive integer.

	Quantity A	Quantity B	
7.	The number of possible values of the units digit of N	4	Ⓐ Ⓑ Ⓒ Ⓓ

$x > 0$ and $x \neq 1$.

	Quantity A	Quantity B	
8.	$(2x^{-4})(3x^2)$	$\frac{24x}{4x^2}$	Ⓐ Ⓑ Ⓒ Ⓓ

	Quantity A	Quantity B	
9.	The length of a leg of an isosceles right triangle with area R	The length of a side of a square with area R	Ⓐ Ⓑ Ⓒ Ⓓ

Multiple-choice Questions—Select One Answer Choice

For Questions 10 to 17, select a single answer choice.

10. The relationship between temperature C, in degrees Celsius, and temperature F, in degrees Fahrenheit, is given by the formula $F = \frac{9}{5}C + 32$. If a recipe calls for an oven temperature of 210 degrees Celsius, what is the oven temperature in degrees Fahrenheit?

(A) 320

(B) 350

(C) 410

(D) 420

(E) 500

11. Of the students in a school, 20 percent are in the science club and 30 percent are in the band. If 25 percent of the students in the school are in the band but are <u>not</u> in the science club, what percent of the students who are in the science club are <u>not</u> in the band?

(A) 5%

(B) 20%

(C) 25%

(D) 60%

(E) 75%

12. Each year, the members of a book club select novels and nonfiction books to discuss. The club meets 3 times to discuss each novel and 5 times to discuss each nonfiction book they select. The club met 52 times last year, and at each meeting the club discussed only one book. If the club discussed 12 books last year, how many of the books were novels?

(A) 2

(B) 4

(C) 5

(D) 7

(E) 14

13. If $-1 < x < y < 0$, which of the following shows the expressions xy, x^2y, and xy^2 listed in order from least to greatest?

Ⓐ $xy,\ x^2y,\ xy^2$

Ⓑ $xy,\ xy^2,\ x^2y$

Ⓒ $xy^2,\ xy,\ x^2y$

Ⓓ $xy^2,\ x^2y,\ xy$

Ⓔ $x^2y,\ xy^2,\ xy$

14. The 5 letters in the list G, H, I, J, K are to be rearranged so that G is the 3rd letter in the list and H is not next to G. How many such rearrangements are there?

Ⓐ 60

Ⓑ 36

Ⓒ 24

Ⓓ 12

Ⓔ 6

15. If j and k are even integers and $j < k$, which of the following equals the number of even integers that are greater than j and less than k?

Ⓐ $\dfrac{k-j-2}{2}$

Ⓑ $\dfrac{k-j-1}{2}$

Ⓒ $\dfrac{k-j}{2}$

Ⓓ $k-j$

Ⓔ $k-j-1$

16. Which of the following is closest to $\sqrt{2.3 \times 10^9}$?

Ⓐ 50,000

Ⓑ 150,000

Ⓒ 500,000

Ⓓ 1,500,000

Ⓔ 5,000,000

17. The interior dimensions of a rectangular tank are as follows: length 110 centimeters, width 90 centimeters, and height 270 centimeters. The tank rests on level ground. Based on the assumption that the volume of water increases by 10 percent when it freezes, which of the following is closest to the maximum height, in centimeters, to which the tank can be filled with water so that when the water freezes, the ice would not rise above the top of the tank?

Ⓐ 230

Ⓑ 235

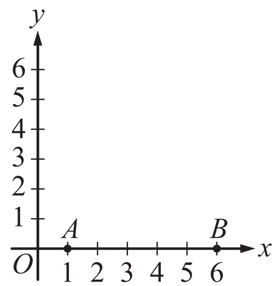
Ⓒ 240

Ⓓ 245

Ⓔ 250

Multiple-choice Questions—Select One or More Answer Choices

For Questions 18 to 19, select all the answer choices that apply.



18. Points A and B are shown in the xy-plane above. Point C (not shown) is above the x-axis so that the area of triangle ABC is 10. Which of the following could be the coordinates of C ?

    Indicate all such coordinates.

    A (0, 4)
    B (1, 3)
    C (2, 5)
    D (3, 4)
    E (4, 5)

19. In a factory, machine A operates on a cycle of 20 hours of work followed by 4 hours of rest, and machine B operates on a cycle of 40 hours of work followed by 8 hours of rest. Last week, the two machines began their respective cycles at 12 noon on Monday and continued until 12 noon on the following Saturday. On which days during that time period was there a time when both machines were at rest?

    Indicate all such days.

    A Monday
    B Tuesday
    C Wednesday
    D Thursday
    E Friday
    F Saturday

Numeric Entry Questions

For Questions 20 to 21, enter your answer in the answer box(es) below the question.

- **Your answer may be an integer, a decimal, or a fraction, and it may be negative.**
- **If a question asks for a fraction, there will be two boxes—one for the numerator and one for the denominator. A decimal point cannot be used in a fraction.**
- **Equivalent forms of the correct answer, such as 2.5 and 2.50, are all correct. Fractions do not need to be reduced to lowest terms, though you may need to reduce your fraction to fit in the boxes.**
- **Enter the exact answer unless the question asks you to round your answer.**

AVERAGE RATING OF PRODUCT *X*
GIVEN BY THREE GROUPS OF PEOPLE

Group	Number of People in Group	Average Rating
A	45	3.8
B	25	4.6
C	30	4.2

20. Each of the people in three groups gave a rating of Product *X* on a scale from 1 through 5. For each of the groups, the table above shows the number of people in the group and the average (arithmetic mean) of their ratings. What is the average of the ratings of the product given by the 100 people in the three groups combined?

    Give your answer to the nearest 0.1.

21. The first term in a certain sequence is 1, the 2nd term in the sequence is 2, and, for all integers $n \geq 3$, the *n*th term in the sequence is the average (arithmetic mean) of the first $n - 1$ terms in the sequence. What is the value of the 6th term in the sequence?

    Give your answer as a fraction.

Data Interpretation Set

Questions 22 to 25 are based on the following data. For these questions, select a single answer choice unless otherwise directed.

SIGHTINGS OF SELECTED BIRD SPECIES
IN PARK *H* IN 1999, BY SEASON

Species	Number of Sightings in Winter	Number of Sightings in Spring	Number of Sightings in Summer	Number of Sightings in Fall
Cardinal	30	18	11	20
Goldfinch	6	12	6	9
Junco	12	0	0	6
Nuthatch	8	2	0	4
Robin	6	12	28	18
Sparrow	20	19	23	22
Wren	0	18	30	12

22. In the winter, $\frac{2}{3}$ of the cardinal sightings, $\frac{1}{2}$ of the junco sightings, and $\frac{1}{4}$ of the sparrow sightings were in January. What fraction of the total number of sightings of these three bird species in the winter were in January?

   Ⓐ $\frac{1}{4}$

   Ⓑ $\frac{1}{3}$

   Ⓒ $\frac{1}{2}$

   Ⓓ $\frac{2}{3}$

   Ⓔ $\frac{3}{4}$

23. For which of the following bird species is the standard deviation of the numbers of sightings shown for the four seasons least?

   Ⓐ Cardinal

   Ⓑ Junco

   Ⓒ Robin

   Ⓓ Sparrow

   Ⓔ Wren

24. Which of the following is closest to the average (arithmetic mean) number of cardinal sightings for the 4 seasons?

    Ⓐ 12

    Ⓑ 14

    Ⓒ 16

    Ⓓ 18

    Ⓔ 20

For Question 25, use the directions for Numeric Entry questions.

25. By what percent did the number of wren sightings increase from spring to summer?

    Give your answer to the nearest whole percent.

    [ ] %

ANSWER KEY

1. **Choice B:** Quantity B is greater.
2. **Choice D:** The relationship cannot be determined from the information given.
3. **Choice C:** The two quantities are equal.
4. **Choice C:** The two quantities are equal.
5. **Choice A:** Quantity A is greater.
6. **Choice A:** Quantity A is greater.
7. **Choice B:** Quantity B is greater.
8. **Choice D:** The relationship cannot be determined from the information given.
9. **Choice A:** Quantity A is greater.
10. **Choice C:** 410
11. **Choice E:** 75%
12. **Choice B:** 4
13. **Choice E:** $x^2y, \ xy^2, \ xy$
14. **Choice D:** 12
15. **Choice A:** $\frac{k-j-2}{2}$
16. **Choice A:** 50,000
17. **Choice D:** 245
18. **Choice A:** (0, 4)
    AND
    Choice D: (3, 4)
19. **Choice C:** Wednesday
    AND
    Choice E: Friday
20. **4.1**
21. $\mathbf{\frac{3}{2}}$
22. **Choice C:** $\frac{1}{2}$
23. **Choice D:** Sparrow
24. **Choice E:** 20
25. **67**

Answers and Explanations

$$y < -6$$

	Quantity A	Quantity B	
1.	y	-5	(A) (B) (C) (D)

Explanation

You are given that y is a number that is less than -6. Since -6 is less than -5, it follows that y is less than -5. Thus the correct answer is **Choice B**.

This explanation uses the following strategy.
Strategy 8: Search for a Mathematical Relationship

---

x is an integer and $23 < x < 27$.

	Quantity A	Quantity B	
2.	The median of the five integers 23, 24, 26, 27, and x	25	(A) (B) (C) (D)

Explanation

You are given that x is an integer that is greater than 23 and less than 27. So x could be 24, 25, or 26. If $x = 24$, the median of the five integers 23, 24, 26, 27, and x is 24. Similarly, if $x = 25$, the median of the five integers is 25, and if $x = 26$, the median of the five integers is 26. So the median of the five integers is 24, 25, or 26. Thus the median of the five integers could be less than, equal to, or greater than 25. The correct answer is **Choice D**.

This explanation uses the following strategies.
Strategy 11: Divide into Cases
Strategy 13: Determine Whether a Conclusion Follows from the Information Given

r and t are consecutive integers and $p = r^2 + t$.

	Quantity A	Quantity B	
3.	$(-1)^p$	-1	Ⓐ Ⓑ Ⓒ Ⓓ

Explanation

Recall that

$$(-1)^p = \begin{cases} 1 \text{ if } p \text{ is an even integer} \\ -1 \text{ if } p \text{ is an odd integer} \end{cases}$$

Since $p = r^2 + t$, the value of $(-1)^p$ depends on whether $r^2 + t$ is odd or even.

If r is an odd integer, then r^2 is an odd integer and, since r and t are consecutive integers, t is an even integer. In this case, p is the sum of an odd integer and an even integer and is therefore an odd integer.

Similarly, if r is an even integer, then r^2 is an even integer and t is an odd integer. In this case, p is the sum of an even integer and an odd integer and is therefore an odd integer.

In both cases, p is an odd integer. It follows that $(-1)^p = -1$, and the correct answer is **Choice C**.

This explanation uses the following strategies.

Strategy 1: Translate from Words to an Arithmetic or Algebraic Representation

Strategy 11: Divide into Cases

The function f is defined by $f\left(\dfrac{x+3}{2}\right) = 3x^2 - x + 5$ for all numbers x.

	Quantity A	Quantity B	
4.	$f(4)$	75	Ⓐ Ⓑ Ⓒ Ⓓ

Explanation

The function f is defined by $f\left(\dfrac{x+3}{2}\right) = 3x^2 - x + 5$. To find the value of $f(4)$, you first need to find the value of x for which $\dfrac{x+3}{2} = 4$, and then you can plug that value of x into $f\left(\dfrac{x+3}{2}\right) = 3x^2 - x + 5$. Solving the equation $\dfrac{x+3}{2} = 4$ for x yields $x = 5$. Plugging $x = 5$ into $f\left(\dfrac{x+3}{2}\right) = 3x^2 - x + 5$, you get $f(4) = 3(5)^2 - 5 + 5$, or 75. Since Quantity B is 75, the correct answer is **Choice C**.

This explanation uses the following strategy.

Strategy 8: Search for a Mathematical Relationship

The sum of the annual salaries of the 21 teachers at School X is \$781,200. Twelve of the 21 teachers have an annual salary that is less than \$37,000.

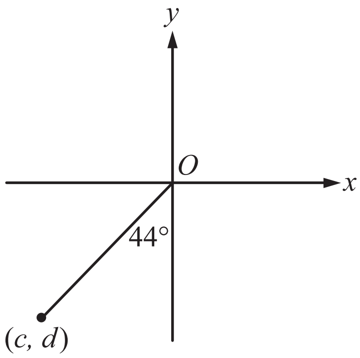
	Quantity A	Quantity B	
5.	The average (arithmetic mean) of the annual salaries of the teachers at School X	The median of the annual salaries of the teachers at School X	Ⓐ Ⓑ Ⓒ Ⓓ

Explanation

The average of the annual salaries of the 21 teachers is $\frac{\$781{,}200}{21}$, or \$37,200. By definition, the median of the 21 annual salaries is the 11th salary when the 21 salaries are listed in increasing order. Note that in the question you are given that 12 of the 21 salaries are less than \$37,000. Thus, when the 21 salaries are listed in increasing order, the first 12 salaries in the list are less than \$37,000. Thus the median of the salaries, which is the 11th salary in the list, is less than \$37,000, which is less than the average salary of \$37,200. The correct answer is **Choice A**.

This explanation uses the following strategy.

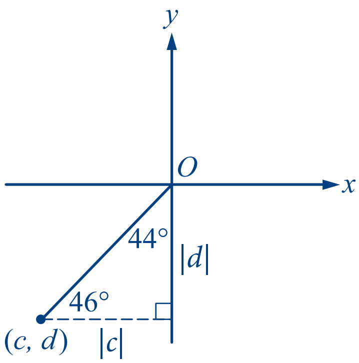
Strategy 8: Search for a Mathematical Relationship



	Quantity A	Quantity B	
6.	c	d	Ⓐ Ⓑ Ⓒ Ⓓ

Explanation

Note that if you draw a horizontal line segment from the point (c, d) to the y-axis, you form a 44°-46°-90° right triangle, with a horizontal leg of length $|c|$ and a vertical leg of length $|d|$, as shown in the figure below.



In the triangle, the horizontal leg is opposite the 44° angle and the vertical leg is opposite the 46° angle. Since the 44° angle is smaller than the 46° angle, the length of the horizontal leg is less than the length of the vertical leg; that is, $|c| < |d|$. Since c and d are both negative, it follows that $c > d$. The correct answer is **Choice A**.

This explanation uses the following strategies.

Strategy 4: Translate from a Figure to an Arithmetic or Algebraic Representation
Strategy 6: Add to a Geometric Figure
Strategy 8: Search for a Mathematical Relationship

$N = 824^x$, where x is a positive integer.

	Quantity A	Quantity B	
7.	The number of possible values of the units digit of N	4	Ⓐ Ⓑ Ⓒ Ⓓ

Explanation

Note that the units digit of a product of positive integers is equal to the units digit of the product of the units digits of those integers. In particular, since 824^x is a product of x integers, each of which is 824, it follows that the units digit of 824^x is equal to the units digit of 4^x. Also, the units digit of 4^x is equal to the units digit of the product (4)(units digit of 4^{x-1}).

The following table shows the units digit of 4^x for some values of x, beginning with $x = 2$.

x	(4)(units digit of 4^{x-1})	Units Digit of 4^x
2	$(4)(4) = 16$	6
3	$(4)(6) = 24$	4
4	$(4)(4) = 16$	6
5	$(4)(6) = 24$	4

From the table, you can see that the units digit of 4^x alternates, and will continue to alternate, between 4 and 6. Therefore, Quantity A, the number of possible values of the units digit of 824^x, is 2. Since Quantity B is 4, the correct answer is **Choice B**.

This explanation uses the following strategies.

Strategy 7: Find a Pattern

Strategy 12: Adapt Solutions to Related Problems

$x > 0$ and $x \neq 1$.

	Quantity A	Quantity B	
8.	$(2x^{-4})(3x^2)$	$\dfrac{24x}{4x^2}$	Ⓐ Ⓑ Ⓒ Ⓓ

Explanation

The two quantities can be simplified as follows.

Quantity A: $(2x^{-4})(3x^2) = \dfrac{6}{x^2}$

Quantity B: $\dfrac{24x}{4x^2} = \dfrac{6}{x}$

So comparing Quantity A with Quantity B is the same as comparing $\dfrac{6}{x^2}$ with $\dfrac{6}{x}$.

Since you are given that $x > 0$ and $x \neq 1$, and the quantities to be compared involve fractions and exponents, it is reasonable to consider two cases: $0 < x < 1$ and $x > 1$.

Case 1: $0 < x < 1$. If x is a number that satisfies $0 < x < 1$, then $x^2 < x$. Therefore $\frac{6}{x^2} > \frac{6}{x}$, and Quantity A is greater than Quantity B.

Case 2: $x > 1$. If x is a number that satisfies $x > 1$, then $x^2 > x$. Therefore $\frac{6}{x^2} < \frac{6}{x}$, and Quantity B is greater than Quantity A.

In one case Quantity A is greater, and in the other case Quantity B is greater. Thus the correct answer is **Choice D**.

This explanation uses the following strategies.
Strategy 5: Simplify an Arithmetic or Algebraic Representation
Strategy 8: Search for a Mathematical Relationship
Strategy 11: Divide into Cases
Strategy 13: Determine Whether a Conclusion Follows from the Information Given

---

	Quantity A	Quantity B	Ⓐ Ⓑ Ⓒ Ⓓ
9.	The length of a leg of an isosceles right triangle with area R	The length of a side of a square with area R	

Explanation

In an isosceles right triangle, both legs have the same length. The area of an isosceles right triangle with legs of length x is equal to $\frac{x^2}{2}$. The area of a square with sides of length s is s^2. Since you are given that an isosceles right triangle and a square have the same area R, it follows that $\frac{x^2}{2} = s^2$, and so $x = \sqrt{2}s$.

Since $\sqrt{2}$ is greater than 1, the length of a leg of the triangle is greater than the length of a side of the square. The correct answer is **Choice A**.

This explanation uses the following strategies.
Strategy 1: Translate from Words to an Arithmetic or Algebraic Representation
Strategy 5: Simplify an Arithmetic or Algebraic Representation
Strategy 8: Search for a Mathematical Relationship

10. The relationship between temperature C, in degrees Celsius, and temperature F, in degrees Fahrenheit, is given by the formula $F = \frac{9}{5}C + 32.$ If a recipe calls for an oven temperature of 210 degrees Celsius, what is the oven temperature in degrees Fahrenheit?

Ⓐ 320

Ⓑ 350

Ⓒ 410

Ⓓ 420

Ⓔ 500

Explanation

You are given the relationship $F = \frac{9}{5}C + 32.$ If the temperature is 210 degrees Celsius, then $C = 210$ and the temperature F, in degrees Fahrenheit, is $\frac{9}{5}(210) + 32,$ or 410. Thus the correct answer is **Choice C**.

This explanation uses the following strategy.
Strategy 5: Simplify an Arithmetic or Algebraic Representation

---

11. Of the students in a school, 20 percent are in the science club and 30 percent are in the band. If 25 percent of the students in the school are in the band but are <u>not</u> in the science club, what percent of the students who are in the science club are <u>not</u> in the band?

Ⓐ 5%

Ⓑ 20%

Ⓒ 25%

Ⓓ 60%

Ⓔ 75%

Explanation

You are given that 20% of the students are in the science club, 30% are in the band, and 25% are in the band but are not in the science club. You need to determine what percent of the students who are in the science club are not in the band. The information can be represented in a Venn diagram as follows.

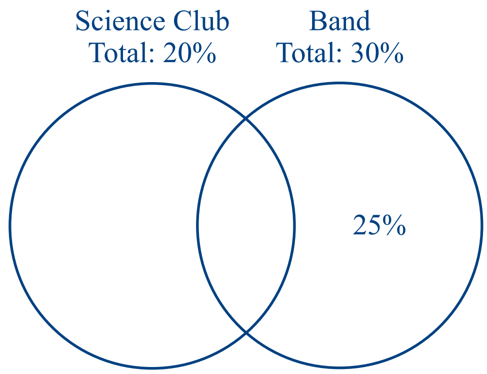


Figure I

From the Venn Diagram, you can see that the 30% who are in the band consists of 25% who are in the band but not in the science club, and 5% who are in both the band and the science club. Then you can see that the 20% who are in the science club consists of 5% who are in both the science club and the band, and 15% who are in the science club but not in the band, as shown in the revised Venn diagram below.

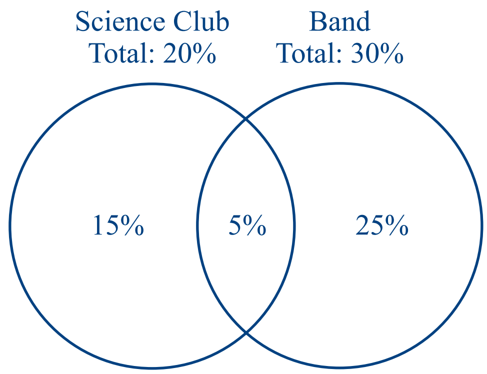


Figure II

Thus $\frac{15\%}{20\%}(100\%)$, or 75%, of the students in the science club are not in the band. The correct answer is **Choice E.**

This explanation uses the following strategies.
Strategy 1: Translate from Words to an Arithmetic or Algebraic Representation
Strategy 2: Translate from Words to a Figure or Diagram

12. Each year, the members of a book club select novels and nonfiction books to discuss. The club meets 3 times to discuss each novel and 5 times to discuss each nonfiction book they select. The club met 52 times last year, and at each meeting the club discussed only one book. If the club discussed 12 books last year, how many of the books were novels?

Ⓐ 2

Ⓑ 4

Ⓒ 5

Ⓓ 7

Ⓔ 14

Explanation

Let n represent the number of novels the club discussed last year. Since the club discussed a total of 12 books, the number of nonfiction books is represented by $12 - n$. This, together with the information that the club met 3 times to discuss each novel and 5 times to discuss each nonfiction book, tells you that the number of times the club met last year can be expressed as $3n + 5(12 - n)$. Since you know that the club met 52 times last year, it follows that $3n + 5(12 - n) = 52$. This equation can be solved for n as follows.

$$\begin{aligned} 3n + 5(12 - n) &= 52 \\ 3n + 60 - 5n &= 52 \\ 8 &= 2n \\ 4 &= n \end{aligned}$$

Thus the club discussed 4 novels. The correct answer is **Choice B**.

This explanation uses the following strategies.

Strategy 1: Translate from Words to an Arithmetic or Algebraic Representation
Strategy 5: Simplify an Arithmetic or Algebraic Representation
Strategy 8: Search for a Mathematical Relationship

---

13. If $-1 < x < y < 0$, which of the following shows the expressions xy, x^2y, and xy^2 listed in order from least to greatest?

Ⓐ $xy,\ x^2y,\ xy^2$

Ⓑ $xy,\ xy^2,\ x^2y$

Ⓒ $xy^2,\ xy,\ x^2y$

Ⓓ $xy^2,\ x^2y,\ xy$

Ⓔ $x^2y,\ xy^2,\ xy$

Explanation

You are given that $-1 < x < y < 0$. Since x and y are both negative numbers, it follows that xy is positive and both x^2y and xy^2 are negative. So xy is greater than both x^2y and xy^2. Now you need to determine which is greater, x^2y or xy^2. You can do this by multiplying the inequality $x < y$ by the positive number xy to get $x^2y < xy^2$. Thus $x^2y < xy^2 < xy$, and the correct answer is **Choice E**.

This explanation uses the following strategies.

Strategy 1: Translate from Words to an Arithmetic or Algebraic Representation
Strategy 8: Search for a Mathematical Relationship

---

14. The 5 letters in the list G, H, I, J, K are to be rearranged so that G is the 3rd letter in the list and H is not next to G. How many such rearrangements are there?

Ⓐ 60

Ⓑ 36

Ⓒ 24

Ⓓ 12

Ⓔ 6

Explanation

When the 5 letters are rearranged, G is to be listed in the 3rd position and H cannot be next to G, so there are only two possible positions for H: 1st and 5th.

Case 1: In the rearranged list, G is in the 3rd position, H is in the 1st position, and each of the remaining 3 letters can be in any of the remaining 3 positions. The number of ways these remaining 3 letters can be arranged is $3!$, or 6. Thus the total number of rearrangements in case 1 is 6.

Case 2: In the rearranged list, G is in the 3rd position, H is in the 5th position, and each of the remaining 3 letters can be in any of the remaining 3 positions. Thus, as in case 1, the total number of rearrangements in case 2 is 6.

Thus there are $6 + 6$, or 12, possible rearrangements, and the correct answer is **Choice D**.

This explanation uses the following strategies.

Strategy 1: Translate from Words to an Arithmetic or Algebraic Representation
Strategy 11: Divide into Cases

15. If j and k are even integers and $j < k$, which of the following equals the number of even integers that are greater than j and less than k ?

(A) $\dfrac{k-j-2}{2}$

(B) $\dfrac{k-j-1}{2}$

(C) $\dfrac{k-j}{2}$

(D) $k-j$

(E) $k-j-1$

Explanation

Since j and k are even integers, it follows that $k = j + 2n$ for some integer n. Consider the sequence of even integers from j to k.

$$j,\ j+(2)(1),\ j+(2)(2),\ j+(2)(3),\ldots,\ j+(2)(n-1),\ j+2n$$

Note that there are $n-1$ integers in the sequence between j and k, and these are the even integers greater than j and less than k. Therefore the answer is $n-1$, but the answer must be given in terms of j and k. Since $k = j + 2n$, you have $n = \dfrac{k-j}{2}$ and so

$$n-1 = \frac{k-j}{2} - 1 = \frac{k-j-2}{2}.$$

Thus the correct answer is **Choice A**.

This explanation uses the following strategies.
Strategy 5: Simplify an Arithmetic or Algebraic Representation
Strategy 8: Search for a Mathematical Relationship

16. Which of the following is closest to $\sqrt{2.3\times10^9}$?

Ⓐ 50,000

Ⓑ 150,000

Ⓒ 500,000

Ⓓ 1,500,000

Ⓔ 5,000,000

Explanation

The expression $\sqrt{2.3\times10^9}$ can be simplified as follows.

$$\begin{aligned}\sqrt{2.3\times10^9} &= \sqrt{2.3\times10\times10^8}\\ &= \sqrt{23\times10^8}\\ &= 10^4\sqrt{23}\end{aligned}$$

Since $\sqrt{23}$ is between 4 and 5, it follows that $10^4\sqrt{23}$ is between 40,000 and 50,000. Therefore, of the five answer choices listed, 50,000 is closest to $\sqrt{2.3\times10^9}$. The correct answer is **Choice A**.

This explanation uses the following strategies.
Strategy 5: Simplify an Arithmetic or Algebraic Representation
Strategy 9: Estimate

---

17. The interior dimensions of a rectangular tank are as follows: length 110 centimeters, width 90 centimeters, and height 270 centimeters. The tank rests on level ground. Based on the assumption that the volume of water increases by 10 percent when it freezes, which of the following is closest to the maximum height, in centimeters, to which the tank can be filled with water so that when the water freezes, the ice would not rise above the top of the tank?

Ⓐ 230

Ⓑ 235

Ⓒ 240

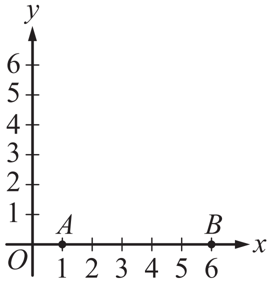
Ⓓ 245

Ⓔ 250

Explanation

Let x be the maximum height, in centimeters, to which the tank can be filled with water, so that when the water freezes, the ice will not rise above the top of the tank. Based on the assumption that the volume of water increases by 10 percent when it freezes and the fact that the length and width of the tank will not change, the maximum height of the ice is $1.1x$ centimeters. Since the height of the tank is 270 centimeters, it follows that $1.1x = 270$, or, to the nearest whole number, $x = 245$. The correct answer is **Choice D**.

This explanation uses the following strategy.
Strategy 1: Translate from Words to an Arithmetic or Algebraic Representation



18. Points A and B are shown in the xy-plane above. Point C (not shown) is above the x-axis so that the area of triangle ABC is 10. Which of the following could be the coordinates of C ?

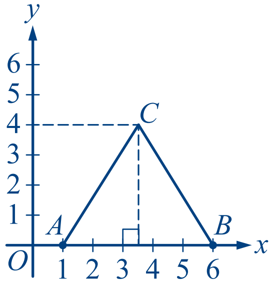
Indicate all such coordinates.

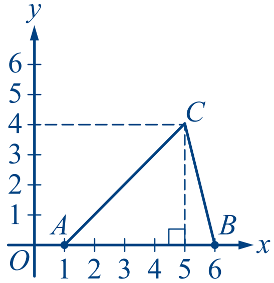
- A (0, 4)
- B (1, 3)
- C (2, 5)
- D (3, 4)
- E (4, 5)

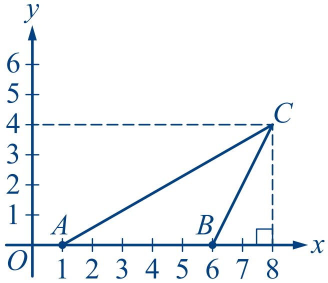
Explanation

From the figure, you can see that points A and B are on the x-axis. Since the x-coordinate of A is 1 and the x-coordinate of B is 6, the length of side AB is 5. Point C is not shown, but you know that C is above the x-axis and that the area of triangle ABC is 10. Using the formula for the area of a triangle, you can see that the height h from point C to the corresponding base AB satisfies the equation $10 = \frac{5h}{2}$. Therefore $h = 4$.

There are many possibilities for triangle ABC subject to these conditions. The following figures show three examples.







In the three examples, the dashed vertical line segments represent the height from point C to the base AB, and the coordinates of C are (3.5, 4), (5, 4), and (8, 4), respectively. Note that in all three cases, the y-coordinate of C is 4. In fact, for any triangle

ABC that satisfies the conditions in the question, point C is a point with a y-coordinate of 4. Of the answer choices, only (0, 4) and (3, 4) have a y-coordinate of 4. The correct answer consists of **Choices A and D**.

This explanation uses the following strategies.
Strategy 4: Translate from a Figure to an Arithmetic or Algebraic Representation
Strategy 6: Add to a Geometric Figure
Strategy 8: Search for a Mathematical Relationship

---

19. In a factory, machine A operates on a cycle of 20 hours of work followed by 4 hours of rest, and machine B operates on a cycle of 40 hours of work followed by 8 hours of rest. Last week, the two machines began their respective cycles at 12 noon on Monday and continued until 12 noon on the following Saturday. On which days during that time period was there a time when both machines were at rest?

Indicate <u>all</u> such days.

- [A] Monday
- [B] Tuesday
- [C] Wednesday
- [D] Thursday
- [E] Friday
- [F] Saturday

Explanation

Both machines began their respective cycles at 12 noon on Monday and continued until 12 noon on the following Saturday. Note that 1 day has 24 hours and 2 days have 48 hours.

Machine A operates on a cycle of 20 hours of work followed by 4 hours of rest, so it was at rest from 8 o'clock in the morning until 12 noon every day from Tuesday to Saturday.

Machine B operates on a cycle of 40 hours of work followed by 8 hours of rest, so it was at rest from 4 o'clock in the morning until 12 noon on Wednesday and Friday.

Thus both machines were at rest from 8 o'clock in the morning until 12 noon on Wednesday and Friday. The correct answer consists of **Choices C and E**.

This explanation uses the following strategies.
Strategy 1: Translate from Words to an Arithmetic or Algebraic Representation
Strategy 7: Find a Pattern

AVERAGE RATING OF PRODUCT *X*
GIVEN BY THREE GROUPS OF PEOPLE

Group	Number of People in Group	Average Rating
A	45	3.8
B	25	4.6
C	30	4.2

20. Each of the people in three groups gave a rating of Product *X* on a scale from 1 through 5. For each of the groups, the table above shows the number of people in the group and the average (arithmetic mean) of their ratings. What is the average of the ratings of the product given by the 100 people in the three groups combined?

    Give your answer to the nearest 0.1.

Explanation

For each of the three groups, the sum of the ratings given by the people in the group is equal to the number of people in the group times the average of their ratings. Therefore the sum of the ratings given by the 100 people in the three groups combined is $45(3.8) + 25(4.6) + 30(4.2)$, or 412, and the average of the 100 ratings is $\frac{412}{100}$, or 4.12. Therefore, to the nearest 0.1, the correct answer is **4.1**.

This explanation uses the following strategies.

Strategy 1: Translate from Words to an Arithmetic or Algebraic Representation
Strategy 4: Translate from a Figure to an Arithmetic or Algebraic Representation

21. The first term in a certain sequence is 1, the 2nd term in the sequence is 2, and, for all integers $n \geq 3$, the nth term in the sequence is the average (arithmetic mean) of the first $n - 1$ terms in the sequence. What is the value of the 6th term in the sequence?

Give your answer as a fraction.

$$\frac{\boxed{}}{\boxed{}}$$

Explanation

For all integers $n \geq 3$, the nth term in the sequence is the average of the first $n - 1$ terms in the sequence.

The 3rd term, which is the average of the first 2 terms, is $\frac{1+2}{2}$, or 1.5.

The 4th term, which is the average of the first 3 terms, is $\frac{1+2+1.5}{3}$, or 1.5.

Similarly, the 5th term is $\frac{1+2+1.5+1.5}{4} = \frac{6}{4}$, or 1.5, and the 6th term is $\frac{1+2+1.5+1.5+1.5}{5} = \frac{7.5}{5}$, or 1.5. Since you must give your answer as a fraction, the correct answer is $\frac{3}{2}$.

This explanation uses the following strategies.

Strategy 1: Translate from Words to an Arithmetic or Algebraic Representation
Strategy 7: Find a Pattern

SIGHTINGS OF SELECTED BIRD SPECIES
IN PARK *H* IN 1999, BY SEASON

Species	Number of Sightings in Winter	Number of Sightings in Spring	Number of Sightings in Summer	Number of Sightings in Fall
Cardinal	30	18	11	20
Goldfinch	6	12	6	9
Junco	12	0	0	6
Nuthatch	8	2	0	4
Robin	6	12	28	18
Sparrow	20	19	23	22
Wren	0	18	30	12

22. In the winter, $\frac{2}{3}$ of the cardinal sightings, $\frac{1}{2}$ of the junco sightings, and $\frac{1}{4}$ of the sparrow sightings were in January. What fraction of the total number of sightings of these three bird species in the winter were in January?

Ⓐ $\frac{1}{4}$

Ⓑ $\frac{1}{3}$

Ⓒ $\frac{1}{2}$

Ⓓ $\frac{2}{3}$

Ⓔ $\frac{3}{4}$

Explanation

The total number of sightings of cardinals, juncos, and sparrows in the winter was $30 + 12 + 20$, or 62. In the question, you are given that $\frac{2}{3}$ of the 30 cardinal sightings, $\frac{1}{2}$ of the 12 junco sightings, and $\frac{1}{4}$ of the 20 sparrow sightings were in January. Therefore the number of sightings of these three bird species in January was $\frac{2}{3}(30)+\frac{1}{2}(12)+\frac{1}{4}(20)$, or 31, which accounted for $\frac{31}{62}$, or $\frac{1}{2}$, of the total number of sightings of these three bird species in the winter. Thus the correct answer is **Choice C.**

This explanation uses the following strategies.

Strategy 1: Translate from Words to an Arithmetic or Algebraic Representation
Strategy 4: Translate from a Figure to an Arithmetic or Algebraic Representation

23. For which of the following bird species is the standard deviation of the numbers of sightings shown for the four seasons least?

Ⓐ Cardinal

Ⓑ Junco

Ⓒ Robin

Ⓓ Sparrow

Ⓔ Wren

Explanation

Recall that the standard deviation of the numbers in a list is a measure of the spread of the numbers about the mean of the numbers. The standard deviation is directly related to the differences $|x - m|$ between the mean m and each number x in the list. The smaller the differences, the smaller the standard deviation. Thus to answer the question, look at the values of $|x - m|$ for each of the five bird species in the answer choices and determine which species has the smallest values.

Choice A, Cardinal. For cardinals, the numbers of sightings are 30, 18, 11, and 20. The mean is approximately 20. Therefore the differences between the mean and the four numbers in the list are approximately 10, 2, 9, and 0.

Choice B, Junco. For juncos, the numbers of sightings are 12, 0, 0, and 6. The mean is approximately 5. Therefore the differences between the mean and the four numbers in the list are approximately 7, 5, 5, and 1.

Choice C, Robin. For robins, the numbers of sightings are 6, 12, 28, and 18. The mean is 16, and the differences between the mean and the four numbers in the list are 10, 4, 12, and 2.

Choice D, Sparrow. For sparrows, the numbers of sightings are 20, 19, 23, and 22. The mean is 21, and the differences between the mean and the four numbers in the list are 1, 2, 2, and 1.

Choice E, Wren. For wrens, the numbers of sightings are 0, 18, 30, and 12. The mean is 15, and the differences between the mean and the four numbers in the list are 15, 3, 15, and 3.

Consider how consistently small the differences are for the numbers of sightings of sparrows as compared to the differences for the other four species. From this you can judge that the numbers of sightings of sparrows has the least standard deviation. Thus the correct answer is **Choice D**.

This explanation uses the following strategies.

Strategy 4: Translate from a Figure to an Arithmetic or Algebraic Representation

Strategy 9: Estimate

24. Which of the following is closest to the average (arithmetic mean) number of cardinal sightings for the 4 seasons?

   Ⓐ 12

   Ⓑ 14

   Ⓒ 16

   Ⓓ 18

   Ⓔ 20

Explanation

The average (arithmetic mean) number of cardinal sightings for the 4 seasons is $\frac{30+18+11+20}{4}=\frac{79}{4}$, or 19.75. Of the answer choices given, 20 is closest to 19.75. Thus the correct answer is **Choice E**.

This explanation uses the following strategies.
Strategy 4: Translate from a Figure to an Arithmetic or Algebraic Representation
Strategy 9: Estimate

---

25. By what percent did the number of wren sightings increase from spring to summer?

   Give your answer to the nearest whole percent.

   [ ] %

Explanation

From spring to summer, the number of wren sightings increased from 18 to 30. Therefore the percent increase was $\left(\frac{30-18}{18}\right)(100\%)$, or $66.\overline{6}\%$. To the nearest whole percent, the percent increase was 67%. Thus the correct answer is **67**.

This explanation uses the following strategy.
Strategy 4: Translate from a Figure to an Arithmetic or Algebraic Representation

Appendix A GRE® Math Review

Your goals for this material

- ⇨ Review the math topics likely to appear on the *GRE*® General Test
- ⇨ Study examples with worked-out solutions
- ⇨ Test your skills with practice exercises

This Math Review will familiarize you with the mathematical skills and concepts that are important for solving problems and reasoning quantitatively on the Quantitative Reasoning measure of the GRE General Test. The skills and concepts are in the areas of Arithmetic, Algebra, Geometry, and Data Analysis. The material covered includes many definitions, properties, and examples, as well as a set of exercises (with answers) at the end of each part. Note, however, that this review is not intended to be all-inclusive—the test may include some concepts that are not explicitly presented in this review.

If any material in this review seems especially unfamiliar or is covered too briefly, you may also wish to consult appropriate mathematics texts for more information. Another resource is the Khan Academy® page on the GRE website at **www.ets.org/gre/khan,** where you will find links to free instructional videos about concepts in this review.

PART 1. ARITHMETIC

The review of arithmetic begins with integers, fractions, and decimals and progresses to the set of real numbers. The basic arithmetic operations of addition, subtraction, multiplication, and division are discussed, along with exponents and roots. The review of arithmetic ends with the concepts of ratio and percent.

1.1 Integers

The **integers** are the numbers 1, 2, 3, …, together with their negatives, −1, −2, −3, …, and 0. Thus, the set of integers is {…, −3, −2, −1, 0, 1, 2, 3, …}.

The positive integers are greater than 0, the negative integers are less than 0, and 0 is neither positive nor negative. When integers are added, subtracted, or multiplied, the result is always an integer; division of integers is addressed below. The many elementary number facts for these operations, such as $7 + 8 = 15$, $78 - 87 = -9$, $7 - (-18) = 25$, and $(7)(8) = 56$, should be familiar to you; they are not reviewed here. Here are three general facts regarding multiplication of integers.

Fact 1: The product of two positive integers is a positive integer.
Fact 2: The product of two negative integers is a positive integer.
Fact 3: The product of a positive integer and a negative integer is a negative integer.

When integers are multiplied, each of the multiplied integers is called a **factor** or **divisor** of the resulting product. For example, $(2)(3)(10) = 60$, so 2, 3, and 10 are factors of 60. The integers 4, 15, 5, and 12 are also factors of 60, since $(4)(15) = 60$ and $(5)(12) = 60$. The positive factors of 60 are 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, and 60. The negatives of these integers are also factors of 60, since, for example, $(-2)(-30) = 60$. There are no other factors of 60. We say that 60 is a **multiple** of each of its factors and that 60 is **divisible** by each of its divisors. Here are five more examples of factors and multiples.

Example 1.1.1: The positive factors of 100 are 1, 2, 4, 5, 10, 20, 25, 50, and 100.

Example 1.1.2: 25 is a multiple of only six integers: 1, 5, 25, and their negatives.

Example 1.1.3: The list of positive multiples of 25 has no end: 25, 50, 75, 100, … ; likewise, every nonzero integer has infinitely many multiples.

Example 1.1.4: 1 is a factor of every integer; 1 is not a multiple of any integer except 1 and −1.

Example 1.1.5: 0 is a multiple of every integer; 0 is not a factor of any integer except 0.

The **least common multiple** of two nonzero integers c and d is the least positive integer that is a multiple of both c and d. For example, the least common multiple of 30 and 75 is 150. This is because the positive multiples of 30 are 30, 60, 90, 120, 150, 180, 210, 240, 270, 300, 330, 390, 420, 450, … , and the positive multiples of 75 are 75, 150, 225, 300, 375, 450, … . Thus, the *common* positive multiples of 30 and 75 are 150, 300, 450, … , and the least of these is 150.

The **greatest common divisor** (or **greatest common factor**) of two nonzero integers c and d is the greatest positive integer that is a divisor of both c and d. For example, the greatest common divisor of 30 and 75 is 15. This is because the positive divisors of 30 are 1, 2, 3, 5, 6, 10, 15, and 30, and the positive divisors of 75 are 1, 3, 5, 15, 25, and 75. Thus, the *common* positive divisors of 30 and 75 are 1, 3, 5, and 15, and the greatest of these is 15.

When an integer c is divided by an integer d, where d is a divisor of c, the result is always a divisor of c. For example, when 60 is divided by 6 (one of its divisors), the result is 10, which is another divisor of 60. If d is *not* a divisor of c, then the result can be viewed in three different ways. The result can be viewed as a fraction or as a decimal, both of which are discussed later, or the result can be viewed as a **quotient** with a **remainder**, where both are integers. Each view is useful, depending on the context. Fractions and decimals are useful when the result must be viewed as a single number, while quotients with remainders are useful for describing the result in terms of integers only.

Regarding quotients with remainders, consider the integer c and the positive integer d, where d is *not* a divisor of c; for example, the integers 19 and 7. When 19 is divided by 7, the result is greater than 2, since $(2)(7) < 19$, but less than 3, since $19 < (3)(7)$. Because 19 is 5 more than (2)(7), we say that the result of 19 divided by 7 is the quotient 2 with remainder 5, or simply 2 remainder 5. In general, when an integer c is divided by a positive integer d, you first find the greatest multiple of d that is less than or equal to c. That multiple of d can be expressed as the product qd, where q is the quotient. Then the remainder is equal to c minus that multiple of d, or $r = c - qd$, where r is the remainder. The remainder is always greater than or equal to 0 and less than d.

Here are four examples that illustrate a few different cases of division resulting in a quotient and remainder.

Example 1.1.6: 100 divided by 45 is 2 remainder 10, since the greatest multiple of 45 that is less than or equal to 100 is (2)(45), or 90, which is 10 less than 100.

Example 1.1.7: 24 divided by 4 is 6 remainder 0, since the greatest multiple of 4 that is less than or equal to 24 is 24 itself, which is 0 less than 24. In general, the remainder is 0 if and only if c is divisible by d.

Example 1.1.8: 6 divided by 24 is 0 remainder 6, since the greatest multiple of 24 that is less than or equal to 6 is (0)(24), or 0, which is 6 less than 6.

Example 1.1.9: –32 divided by 3 is –11 remainder 1, since the greatest multiple of 3 that is less than or equal to –32 is (–11)(3), or –33, which is 1 less than –32.

Here are five more examples.

Example 1.1.10: 100 divided by 3 is 33 remainder 1, since $100 = (33)(3) + 1$.

Example 1.1.11: 100 divided by 25 is 4 remainder 0, since $100 = (4)(25) + 0$.

Example 1.1.12: 80 divided by 100 is 0 remainder 80, since $80 = (0)(100) + 80$.

Example 1.1.13: –13 divided by 5 is –3 remainder 2, since $-13 = (-3)(5) + 2$.

Example 1.1.14: –73 divided by 10 is –8 remainder 7, since $-73 = (-8)(10) + 7$.

If an integer is divisible by 2, it is called an **even integer**; otherwise, it is an **odd integer**. Note that when an odd integer is divided by 2, the remainder is always 1. The set of even integers is $\{\ldots, -6, -4, -2, 0, 2, 4, 6, \ldots\}$, and the set of odd integers is $\{\ldots, -5, -3, -1, 1, 3, 5, \ldots\}$. Here are six useful facts regarding the sum and product of even and odd integers.

Fact 1: The sum of two even integers is an even integer.
Fact 2: The sum of two odd integers is an even integer.
Fact 3: The sum of an even integer and an odd integer is an odd integer.
Fact 4: The product of two even integers is an even integer.
Fact 5: The product of two odd integers is an odd integer.
Fact 6: The product of an even integer and an odd integer is an even integer.

A **prime number** is an integer greater than 1 that has only two positive divisors: 1 and itself. The first ten prime numbers are 2, 3, 5, 7, 11, 13, 17, 19, 23, and 29. The integer 14 is not a prime number, since it has four positive divisors: 1, 2, 7, and 14. The integer 1 is not a prime number, and the integer 2 is the only prime number that is even.

Every integer greater than 1 either is a prime number or can be uniquely expressed as a product of factors that are prime numbers, or **prime divisors**. Such an expression is called a **prime factorization**. Here are six examples of prime factorizations.

Example 1.1.15: $12 = (2)(2)(3) = (2^2)(3)$
Example 1.1.16: $14 = (2)(7)$
Example 1.1.17: $81 = (3)(3)(3)(3) = 3^4$
Example 1.1.18: $338 = (2)(13)(13) = (2)(13^2)$
Example 1.1.19: $800 = (2)(2)(2)(2)(2)(5)(5) = (2^5)(5^2)$
Example 1.1.20: $1{,}155 = (3)(5)(7)(11)$

An integer greater than 1 that is not a prime number is called a **composite number**. The first ten composite numbers are 4, 6, 8, 9, 10, 12, 14, 15, 16, and 18.

1.2 Fractions

A **fraction** is a number of the form $\frac{c}{d}$, where c and d are integers and $d \neq 0$. The integer c is called the **numerator** of the fraction, and d is called the **denominator**. For example, $\frac{-7}{5}$ is a fraction in which -7 is the numerator and 5 is the denominator. Such numbers are also called **rational numbers**. Note that every integer n is a rational number, because n is equal to the fraction $\frac{n}{1}$.

If both the numerator c and the denominator d, where $d \neq 0$, are multiplied by the same nonzero integer, the resulting fraction will be equivalent to $\frac{c}{d}$.

Example 1.2.1: Multiplying the numerator and denominator of the fraction $\frac{-7}{5}$ by 4 gives

$$\frac{-7}{5} = \frac{(-7)(4)}{(5)(4)} = \frac{-28}{20}$$

Multiplying the numerator and denominator of the fraction $\frac{-7}{5}$ by -1 gives

$$\frac{-7}{5} = \frac{(-7)(-1)}{(5)(-1)} = \frac{7}{-5}$$

For all integers c and d, the fractions $\frac{-c}{d}$, $\frac{c}{-d}$, and $-\frac{c}{d}$ are equivalent.

Example 1.2.2: $\frac{-7}{5} = \frac{7}{-5} = -\frac{7}{5}$

If both the numerator and denominator of a fraction have a common factor, then the numerator and denominator can be factored and the fraction can be reduced to an equivalent fraction.

Example 1.2.3: $\frac{40}{72} = \frac{(8)(5)}{(8)(9)} = \frac{5}{9}$

Adding and Subtracting Fractions

To add two fractions with the same denominator, you add the numerators and keep the same denominator.

Example 1.2.4: $-\frac{8}{11}+\frac{5}{11}=\frac{-8+5}{11}=\frac{-3}{11}=-\frac{3}{11}$

To add two fractions with different denominators, first find a **common denominator**, which is a common multiple of the two denominators. Then convert both fractions to equivalent fractions with the same denominator. Finally, add the numerators and keep the common denominator.

Example 1.2.5: To add the two fractions $\frac{1}{3}$ and $-\frac{2}{5}$, first note that 15 is a common denominator of the fractions.

Then convert the fractions to equivalent fractions with denominator 15 as follows.

$$\frac{1}{3}=\frac{1(5)}{3(5)}=\frac{5}{15} \text{ and } -\frac{2}{5}=-\frac{2(3)}{5(3)}=-\frac{6}{15}$$

Therefore, the two fractions can be added as follows.

$$\frac{1}{3}+\frac{-2}{5}=\frac{5}{15}+\frac{-6}{15}=\frac{5+(-6)}{15}=-\frac{1}{15}$$

The same method applies to subtraction of fractions.

Multiplying and Dividing Fractions

To multiply two fractions, multiply the two numerators and multiply the two denominators. Here are two examples.

Example 1.2.6: $\left(\frac{10}{7}\right)\left(\frac{-1}{3}\right)=\frac{(10)(-1)}{(7)(3)}=\frac{-10}{21}=-\frac{10}{21}$

Example 1.2.7: $\left(\frac{8}{3}\right)\left(\frac{7}{3}\right)=\frac{56}{9}$

To divide one fraction by another, first **invert** the second fraction (that is, find its **reciprocal**), then multiply the first fraction by the inverted fraction. Here are two examples.

Example 1.2.8: $\frac{17}{8}\div\frac{3}{5}=\left(\frac{17}{8}\right)\left(\frac{5}{3}\right)=\frac{85}{24}$

Example 1.2.9: $\dfrac{\frac{3}{10}}{\frac{7}{13}}=\left(\frac{3}{10}\right)\left(\frac{13}{7}\right)=\frac{39}{70}$

Mixed Numbers

An expression such as $4\frac{3}{8}$ is called a **mixed number**. It consists of an integer part and a fraction part, where the fraction part has a value between 0 and 1; the mixed number $4\frac{3}{8}$ means $4+\frac{3}{8}$.

To convert a mixed number to a fraction, convert the integer part to an equivalent fraction with the same denominator as the fraction, and then add it to the fraction part.

Example 1.2.10: To convert the mixed number $4\frac{3}{8}$ to a fraction, first convert the integer 4 to a fraction with denominator 8, as follows.

$$4 = \frac{4}{1} = \frac{4(8)}{1(8)} = \frac{32}{8}$$

Then add $\frac{3}{8}$ to it to get

$$4\frac{3}{8} = \frac{32}{8} + \frac{3}{8} = \frac{35}{8}$$

Fractional Expressions

Numbers of the form $\frac{c}{d}$, where either c or d is not an integer and $d \neq 0$, are called fractional expressions. Fractional expressions can be manipulated just like fractions. Here are two examples.

Example 1.2.11: Add the numbers $\frac{\pi}{2}$ and $\frac{\pi}{3}$.

Solution: Note that 6 is a common denominator of both numbers.
The number $\frac{\pi}{2}$ is equivalent to the number $\frac{3\pi}{6}$, and the number $\frac{\pi}{3}$ is equivalent to the number $\frac{2\pi}{6}$.

Therefore

$$\frac{\pi}{2} + \frac{\pi}{3} = \frac{3\pi}{6} + \frac{2\pi}{6} = \frac{5\pi}{6}$$

Example 1.2.12: Simplify the number $\dfrac{\frac{1}{\sqrt{2}}}{\frac{3}{\sqrt{5}}}$.

Solution: Note that the numerator of the number is $\frac{1}{\sqrt{2}}$ and the denominator of the number is $\frac{3}{\sqrt{5}}$. Note also that the reciprocal of the denominator is $\frac{\sqrt{5}}{3}$.

Therefore,

$$\frac{\frac{1}{\sqrt{2}}}{\frac{3}{\sqrt{5}}} = \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{5}}{3}\right)$$

which can be simplified to $\frac{\sqrt{5}}{3\sqrt{2}}$.

Thus, the number $\dfrac{\frac{1}{\sqrt{2}}}{\frac{3}{\sqrt{5}}}$ simplifies to the number $\frac{\sqrt{5}}{3\sqrt{2}}$.

1.3 Exponents and Roots

Exponents

Exponents are used to denote the repeated multiplication of a number by itself; for example, $3^4 = (3)(3)(3)(3) = 81$ and $5^3 = (5)(5)(5) = 125$. In the expression 3^4, 3 is called the **base**, 4 is called the **exponent**, and we read the expression as "3 to the fourth power." Similarly, 5 to the third power is 125.

When the exponent is 2, we call the process **squaring**. Thus, 6 squared is 36; that is, $6^2 = (6)(6) = 36$. Similarly, 7 squared is 49; that is, $7^2 = (7)(7) = 49$.

When negative numbers are raised to powers, the result may be positive or negative; for example, $(-3)^2 = (-3)(-3) = 9$ and $(-3)^5 = (-3)(-3)(-3)(-3)(-3) = -243$. A negative number raised to an even power is always positive, and a negative number raised to an odd power is always negative. Note that $(-3)^2 = (-3)(-3) = 9$, but $-3^2 = -\big((3)(3)\big) = -9$. Exponents can also be negative or zero; such exponents are defined as follows.

The exponent zero: For all nonzero numbers a, $a^0 = 1$. The expression 0^0 is undefined.

Negative exponents: For all nonzero numbers a, $a^{-1} = \frac{1}{a}$, $a^{-2} = \frac{1}{a^2}$, $a^{-3} = \frac{1}{a^3}$, and so on.

Note that $(a)\left(a^{-1}\right) = (a)\left(\frac{1}{a}\right) = 1$.

Roots

A **square root** of a nonnegative number n is a number r such that $r^2 = n$. For example, 4 is a square root of 16 because $4^2 = 16$. Another square root of 16 is -4, since $(-4)^2 = 16$. All positive numbers have two square roots, one positive and one negative. The only square root of 0 is 0. The expression consisting of the square root symbol $\sqrt{}$ placed over a nonnegative number denotes the *nonnegative* square root (or the positive square root if the number is greater than 0) of that nonnegative number. Therefore, $\sqrt{100} = 10$, $-\sqrt{100} = -10$, and $\sqrt{0} = 0$. Square roots of negative numbers are not defined in the real number system.

Here are four important rules regarding operations with square roots, where $a > 0$ and $b > 0$.

Rule 1: $(\sqrt{a})^2 = a$

Example A: $(\sqrt{3})^2 = 3$

Example B: $(\sqrt{\pi})^2 = \pi$

Rule 2: $\sqrt{a^2} = a$

Example A: $\sqrt{4} = \sqrt{2^2} = 2$

Example B: $\sqrt{\pi^2} = \pi$

Rule 3: $\sqrt{a}\sqrt{b} = \sqrt{ab}$

Example A: $\sqrt{3}\sqrt{10} = \sqrt{30}$

Example B: $\sqrt{24} = \sqrt{4}\sqrt{6} = 2\sqrt{6}$

Rule 4: $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$

Example A: $\frac{\sqrt{5}}{\sqrt{15}} = \sqrt{\frac{5}{15}} = \sqrt{\frac{1}{3}}$

Example B: $\frac{\sqrt{18}}{\sqrt{2}} = \sqrt{\frac{18}{2}} = \sqrt{9} = 3$

A square root is a root of order 2. Higher order roots of a positive number n are defined similarly. For orders 3 and 4, the **cube root** of n, written as $\sqrt[3]{n}$, and **fourth root** of n, written as $\sqrt[4]{n}$, represent numbers such that when they are raised to the powers 3 and 4, respectively, the result is n. These roots obey rules similar to those above but with the exponent 2 replaced by 3 or 4 in the first two rules.

There are some notable differences between odd order roots and even order roots (in the real number system):

> For odd order roots, there is *exactly one* root for *every* number n, even when n is negative.
>
> For even order roots, there are *exactly two* roots for every *positive* number n and *no* roots for any *negative* number n.

For example, 8 has exactly one cube root, $\sqrt[3]{8} = 2$, but 8 has two fourth roots, $\sqrt[4]{8}$ and $-\sqrt[4]{8}$, whereas −8 has exactly one cube root, $\sqrt[3]{-8} = -2$, but −8 has no fourth root, since it is negative.

1.4 Decimals

The decimal number system is based on representing numbers using powers of 10. The place value of each digit corresponds to a power of 10. For example, the digits of the number 7,532.418 have the following place values.

Thousands		Hundreds	Tens	Ones or Units		Tenths	Hundredths	Thousandths
7	,	5	3	2	.	4	1	8

Arithmetic Figure 1

That is, the number 7,532.418 can be written as

$$7(1{,}000) + 5(100) + 3(10) + 2(1) + 4\left(\frac{1}{10}\right) + 1\left(\frac{1}{100}\right) + 8\left(\frac{1}{1{,}000}\right)$$

Alternatively, it can be written as

$$7\left(10^3\right) + 5\left(10^2\right) + 3\left(10^1\right) + 2\left(10^0\right) + 4\left(10^{-1}\right) + 1\left(10^{-2}\right) + 8\left(10^{-3}\right)$$

If there are a finite number of digits to the right of the decimal point, converting a decimal to an equivalent fraction with integers in the numerator and denominator is a straightforward process. Since each place value is a power of 10, every decimal can be converted to an integer divided by a power of 10. Here are three examples.

Example 1.4.1: $2.3 = 2 + \frac{3}{10} = \frac{20}{10} + \frac{3}{10} = \frac{23}{10}$

Example 1.4.2: $90.17 = 90 + \frac{17}{100} = \frac{9{,}000 + 17}{100} = \frac{9{,}017}{100}$

Example 1.4.3: $0.612 = \frac{612}{1{,}000}$

Conversely, every fraction with integers in the numerator and denominator can be converted to an equivalent decimal by dividing the numerator by the denominator using long division (which is not in this review). The decimal that results from the long division will either **terminate**, as in $\frac{1}{4} = 0.25$ and $\frac{52}{25} = 2.08$, or **repeat** without end, as in $\frac{1}{9} = 0.111\ldots$, $\frac{1}{22} = 0.0454545\ldots$, and $\frac{25}{12} = 2.08333\ldots$. One way to indicate the repeating part of a decimal that repeats without end is to use a bar over the digits that repeat. Here are four examples of fractions converted to decimals.

Example 1.4.4: $\frac{3}{8} = 0.375$

Example 1.4.5: $\frac{259}{40} = 6.475$

Example 1.4.6: $-\frac{1}{3} = -0.\overline{3}$

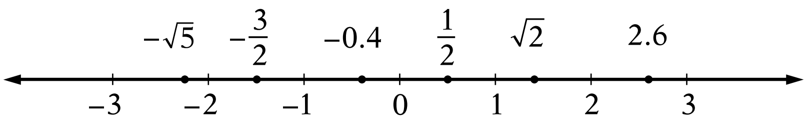
Example 1.4.7: $\frac{15}{14} = 1.0\overline{714285}$

Every fraction with integers in the numerator and denominator is equivalent to a decimal that either terminates or repeats. That is, every rational number can be expressed as a terminating or repeating decimal. The converse is also true; that is, every terminating or repeating decimal represents a rational number.

Not all decimals are terminating or repeating; for instance, the decimal that is equivalent to $\sqrt{2}$ is $1.41421356237\ldots$, and it can be shown that this decimal does not terminate or repeat. Another example is $0.020220222022220222220\ldots$, which has groups of consecutive 2s separated by a 0, where the number of 2s in each successive group increases by one. Since these two decimals do not terminate or repeat, they are not rational numbers. Such numbers are called **irrational numbers**.

1.5 Real Numbers

The set of **real numbers** consists of all rational numbers and all irrational numbers. The real numbers include all integers, fractions, and decimals. The set of real numbers can be represented by a number line called the **real number line**. Arithmetic Figure 2 below is a number line.



Arithmetic Figure 2

Every real number corresponds to a point on the number line, and every point on the number line corresponds to a real number. On the number line, all numbers to the left of 0 are negative and all numbers to the right of 0 are positive. As shown in Arithmetic Figure 2, the negative numbers $-0.4, -1, -\frac{3}{2}, -2, -\sqrt{5}$, and -3 are to the left of 0, and the positive numbers $\frac{1}{2}, 1, \sqrt{2}, 2, 2.6$, and 3 are to the right of 0. Only the number 0 is neither negative nor positive.

A real number x is **less than** a real number y if x is to the left of y on the number line, which is written as $x < y$. A real number y is **greater than** a real number x if y is to the right of x on the number line, which is written as $y > x$. For example, the number line in Arithmetic Figure 2 shows the following three relationships.

Relationship 1: $-\sqrt{5} < -2$

Relationship 2: $\frac{1}{2} > 0$

Relationship 3: $1 < \sqrt{2} < 2$

A real number x is **less than or equal to** a real number y if x is to the left of, *or corresponds to the same point as,* y on the number line, which is written as $x \le y$. A real number y is **greater than or equal to** a real number x if y is to the right of, *or corresponds to the same point as,* x on the number line, which is written as $y \ge x$.

To say that a real number x is between 2 and 3 on the number line means that $x > 2$ and $x < 3$, which can also be written as $2 < x < 3$. The set of all real numbers that are between 2 and 3 is called an **interval**, and $2 < x < 3$ is often used to represent that interval. Note that the endpoints of the interval, 2 and 3, are not included in the interval. Sometimes one or both of the endpoints are to be included in an interval. The following inequalities represent four types of intervals, depending on whether or not the endpoints are included.

Interval type 1: $2 < x < 3$
Interval type 2: $2 \le x < 3$
Interval type 3: $2 < x \le 3$
Interval type 4: $2 \le x \le 3$

There are also four types of intervals with only one endpoint, each of which consists of all real numbers to the right or to the left of the endpoint and include or do not include the endpoint. The following inequalities represent these types of intervals.

Interval type 1: $x < 4$
Interval type 2: $x \le 4$
Interval type 3: $x > 4$
Interval type 4: $x \ge 4$

The entire real number line is also considered to be an interval.

Absolute Value

The distance between a number x and 0 on the number line is called the **absolute value** of x, written as $|x|$. Therefore, $|3| = 3$ and $|-3| = 3$ because each of the numbers 3 and -3 is a distance of 3 from 0. Note that if x is positive, then $|x| = x$; if x is negative, then $|x| = -x$; and lastly, $|0| = 0$. It follows that the absolute value of any nonzero number is positive. Here are three examples.

Example 1.5.1: $\left|\sqrt{5}\right| = \sqrt{5}$

Example 1.5.2: $|-23| = -(-23) = 23$

Example 1.5.3: $|-10.2| = 10.2$

Properties of Real Numbers

Here are twelve general properties of real numbers that are used frequently. In each property, r, s, and t are real numbers.

Property 1: $r + s = s + r$ and $rs = sr$.

Example A: $8 + 2 = 2 + 8 = 10$

Example B: $(-3)(17) = (17)(-3) = -51$

Property 2: $(r + s) + t = r + (s + t)$ and $(rs)t = r(st)$.

Example A: $(7 + 3) + 8 = 7 + (3 + 8) = 18$

Example B: $\left(7\sqrt{2}\right)\sqrt{2} = 7\left(\sqrt{2}\sqrt{2}\right) = (7)(2) = 14$

Property 3: $r(s + t) = rs + rt$

Example: $5(3 + 16) = (5)(3) + (5)(16) = 95$

Property 4: $r + 0 = r$, $(r)(0) = 0$, and $(r)(1) = r$.

Property 5: If $rs = 0$, then either $r = 0$ or $s = 0$ or both.

Example: If $-2s = 0$, then $s = 0$.

Property 6: Division by 0 is undefined.

Example A: $5 \div 0$ is undefined.

Example B: $\frac{-7}{0}$ is undefined.

Example C: $\frac{0}{0}$ is undefined.

Property 7: If both r and s are positive, then both $r + s$ and rs are positive.

Property 8: If both r and s are negative, then $r + s$ is negative and rs is positive.

Property 9: If r is positive and s is negative, then rs is negative.

Property 10: $|r + s| \le |r| + |s|$. This is known as the **triangle inequality**.

Example: If $r = 5$ and $s = -2$, then $|5 + (-2)| = |5 - 2| = |3| = 3$ and $|5| + |-2| = 5 + 2 = 7$. Therefore, $|5 + (-2)| \le |5| + |-2|$.

Property 11: $|r||s| = |rs|$

Example: $|5||-2| = |(5)(-2)| = |-10| = 10$

Property 12: If $r > 1$, then $r^2 > r$. If $0 < s < 1$, then $s^2 < s$.

Example: $5^2 = 25 > 5$, but $\left(\frac{1}{5}\right)^2 = \frac{1}{25} < \frac{1}{5}$.

1.6 Ratio

The **ratio** of one quantity to another is a way to express their relative sizes, often in the form of a fraction, where the first quantity is the numerator and the second quantity is the denominator. Thus, if s and t are positive quantities, then the ratio of s to t can be written as the fraction $\frac{s}{t}$. The notation "s to t" and the notation "$s : t$" are also used to express this ratio. For example, if there are 2 apples and 3 oranges in a basket, we can say that the ratio of the number of apples to the number of oranges is $\frac{2}{3}$, or that it is 2 to 3, or that it is 2 : 3. Like fractions, ratios can be reduced to lowest terms.

For example, if there are 8 apples and 12 oranges in a basket, then the ratio of the number of apples to the number of oranges is still 2 to 3. Similarly, the ratio 9 to 12 is equivalent to the ratio 3 to 4.

If three or more positive quantities are being considered, say r, s, and t, then their relative sizes can also be expressed as a ratio with the notation "r to s to t". For example, if there are 5 apples, 30 pears, and 20 oranges in a basket, then the ratio of the number of apples to the number of pears to the number of oranges is 5 to 30 to 20. This ratio can be reduced to 1 to 6 to 4 by dividing each number by the greatest common divisor of 5, 30, and 20, which is 5.

A **proportion** is an equation relating two ratios; for example, $\frac{9}{12} = \frac{3}{4}$. To solve a problem involving ratios, you can often write a proportion and solve it by **cross multiplication**.

Example 1.6.1: To find a number x so that the ratio of x to 49 is the same as the ratio of 3 to 21, you can first write the following equation.

$$\frac{x}{49} = \frac{3}{21}$$

You can then cross multiply to get $21x = (3)(49)$, and finally you can solve for x to get $x = \frac{(3)(49)}{21} = 7$.

1.7 Percent

The term **percent** means *per hundred*, or *hundredths*. Percents are ratios that are often used to represent *parts of a whole*, where the whole is considered as having 100 parts. Percents can be converted to fraction or decimal equivalents. Here are three examples of percents.

Example 1.7.1: 1 percent means 1 part out of 100 parts. The fraction equivalent of 1 percent is $\frac{1}{100}$, and the decimal equivalent is 0.01.

Example 1.7.2: 32 percent means 32 parts out of 100 parts. The fraction equivalent of 32 percent is $\frac{32}{100}$, and the decimal equivalent is 0.32.

Example 1.7.3: 50 percent means 50 parts out of 100 parts. The fraction equivalent of 50 percent is $\frac{50}{100}$, and the decimal equivalent is 0.50.

Note that in the fraction equivalent, the *part* is the numerator of the fraction and the *whole* is the denominator. Percents are often written using the percent symbol, %, instead of the word "percent." Here are five examples of percents written using the % symbol, along with their fraction and decimal equivalents.

Example 1.7.4: $100\% = \frac{100}{100} = 1$

Example 1.7.5: $12\% = \frac{12}{100} = 0.12$

Example 1.7.6: $8\% = \frac{8}{100} = 0.08$

Example 1.7.7: $10\% = \frac{10}{100} = 0.1$

Example 1.7.8: $0.3\% = \frac{0.3}{100} = 0.003$

Be careful not to confuse 0.01 with 0.01%. The percent symbol matters. For example, $0.01 = 1\%$ but $0.01\% = \frac{0.01}{100} = 0.0001$.

To compute a *percent*, given the *part* and the *whole*, first divide the part by the whole to get the decimal equivalent, then multiply the result by 100. The percent is that number followed by the word "percent" or the % symbol.

Example 1.7.9: If the whole is 20 and the part is 13, you can find the percent as follows.

$$\frac{part}{whole} = \frac{13}{20} = 0.65 = 65\%$$

Example 1.7.10: What percent of 150 is 12.9 ?

Solution: Here, the whole is 150 and the part is 12.9, so

$$\frac{part}{whole} = \frac{12.9}{150} = 0.086 = 8.6\%$$

To find the *part* that is a certain *percent* of a *whole*, you can either multiply the *whole* by the decimal equivalent of the percent or set up a proportion to find the part.

Example 1.7.11: To find 30% of 350, you can multiply 350 by the decimal equivalent of 30%, or 0.3, as follows.

$$(350)(0.3) = 105$$

Alternatively, to use a proportion to find 30% of 350, you need to find the number of parts of 350 that yields the same ratio as 30 parts out of 100 parts. You want a number x that satisfies the proportion

$$\frac{part}{whole} = \frac{30}{100} \text{ or}$$

$$\frac{x}{350} = \frac{30}{100}$$

Solving for x yields $x = \frac{(30)(350)}{100} = 105$, so 30% of 350 is 105.

Given the *percent* and the *part*, you can calculate the *whole*. To do this, either you can use the decimal equivalent of the percent or you can set up a proportion and solve it.

Example 1.7.12: 15 is 60% of what number?

Solution: Use the decimal equivalent of 60%. Because 60% of some number z is 15, multiply z by the decimal equivalent of 60%, or 0.6.

$$0.6z = 15$$

Now solve for z by dividing both sides of the equation by 0.6 as follows.

$$z = \frac{15}{0.6} = 25$$

Using a proportion, look for a number z such that

$$\frac{part}{whole} = \frac{60}{100} \text{ or}$$

$$\frac{15}{z} = \frac{60}{100}$$

Hence, $60z = (15)(100)$, and therefore, $z = \frac{(15)(100)}{60} = \frac{1{,}500}{60} = 25$. That is, 15 is 60% of 25.

Percents Greater than 100%

Although the discussion about percent so far assumes a context of a *part* and a *whole*, it is not necessary that the part be less than the whole. In general, the whole is called the **base** of the percent. When the numerator of a percent is greater than the base, the percent is greater than 100%.

Example 1.7.13: 15 is 300% of 5, since

$$\frac{15}{5} = \frac{300}{100}$$

Example 1.7.14: 250% of 16 is 40, since

$$\left(\frac{250}{100}\right)(16) = (2.5)(16) = 40$$

Note that the decimal equivalent of 300% is 3.0 and the decimal equivalent of 250% is 2.5.

Percent Increase, Percent Decrease, and Percent Change

When a quantity changes from an initial positive amount to another positive amount (for example, an employee's salary that is raised), you can compute the amount of change as a percent of the initial amount. This is called **percent change**. If a quantity increases from 600 to 750, then the base of the increase is the initial amount, 600, and the amount of the increase is 750 – 600, or 150. The **percent increase** is found by dividing the amount of increase by the base, as follows.

$$\frac{\textit{amount of increase}}{\textit{base}} = \frac{750 - 600}{600} = \frac{150}{600} = \frac{25}{100} = 25\%$$

We say the percent increase is 25%. Sometimes this computation is written as follows.

$$\left(\frac{750 - 600}{600}\right)(100\%) = \left(\frac{150}{600}\right)(100\%) = 25\%$$

If a quantity doubles in size, then the percent increase is 100%. For example, if a quantity increases from 150 to 300, then the percent increase is calculated as follows.

$$\frac{\textit{amount of increase}}{\textit{base}} = \frac{300 - 150}{150} = \frac{150}{150} = 100\%$$

If a quantity decreases from 500 to 400, calculate the **percent decrease** as follows.

$$\frac{\textit{amount of decrease}}{\textit{base}} = \frac{500 - 400}{500} = \frac{100}{500} = \frac{20}{100} = 20\%$$

The quantity decreased by 20%.

When computing a percent *increase*, the base is the *smaller* number. When computing a percent *decrease*, the base is the *larger* number. In either case, the base is the initial number, before the change.

Example 1.7.15: An investment in a mutual fund increased by 12% in a single day. If the value of the investment before the increase was \$1,300, what was the value after the increase?

Solution: The percent increase is 12%. Therefore, the value of the increase is 12% of \$1,300, or, using the decimal equivalent, the increase is (0.12)(\$1,300) = \$156. Thus, the value of the investment after the change is

$$\$1{,}300 + \$156 = 1{,}456$$

Because the final result is the sum of the initial investment (100% of \$1,300) and the increase (12% of \$1,300), the final result is 100% + 12% = 112% of \$1,300. Thus, another way to get the final result is to multiply the value of the investment by the decimal equivalent of 112%, which is 1.12:

$$(\$1{,}300)(1.12) = \$1{,}456$$

A quantity may have several successive percent changes, where the base of each successive change is the result of the preceding percent change, as is the case in the following example.

Example 1.7.16: On September 1, 2013, the number of children enrolled in a certain preschool was 8% less than the number of children enrolled at the preschool on September 1, 2012. On September 1, 2014, the number of children enrolled in the preschool was 6% greater than the number of children enrolled in the preschool on September 1, 2013. By what percent did the number of students enrolled change from September 1, 2012, to September 1, 2014?

Solution: The *initial* base is the enrollment on September 1, 2012. The first percent change was the 8% decrease in the enrollment from September 1, 2012, to September 1, 2013. As a result of this decrease, the enrollment on September 1, 2013, was (100 – 8)%, or 92%, of the enrollment on September 1, 2012. The decimal equivalent of 92% is 0.92.

So, if n represents the number of children enrolled on September 1, 2012, then the number of children enrolled on September 1, 2013, is equal to $0.92n$.

The *new* base is the enrollment on September 1, 2013, which is $0.92n$. The second percent change was the 6% increase in enrollment from September 1, 2013, to September 1, 2014. As a result of this increase, the enrollment on September 1, 2014, was (100 + 6)%, or 106%, of the enrollment on September 1, 2013. The decimal equivalent of 106% is 1.06.

Thus, the number of children enrolled on September 1, 2014, was $(1.06)(0.92n)$, which is equal to $0.9752n$.

The percent equivalent of 0.9752 is 97.52%, which is 2.48% less than 100%. Thus, the percent change in the enrollment from September 1, 2012, to September 1, 2014, is a 2.48% decrease.

ARITHMETIC EXERCISES

Exercise 1. Evaluate the following.

(a) $15 - (6 - 4)(-2)$

(b) $(2 - 17) \div 5$

(c) $(60 \div 12) - (-7 + 4)$

(d) $3^4 - (-2)^3$

(e) $(-5)(-3) - 15$

(f) $(-2)^4(15 - 18)^4$

(g) $(20 \div 5)^2(-2 + 6)^3$

(h) $(-85)(0) - (-17)(3)$

Exercise 2. Evaluate the following.

(a) $\frac{1}{2} - \frac{1}{3} + \frac{1}{12}$

(b) $\left(\frac{3}{4} + \frac{1}{7}\right)\left(\frac{-2}{5}\right)$

(c) $\left(\frac{7}{8} - \frac{4}{5}\right)^2$

(d) $\left(\frac{3}{-8}\right) \div \left(\frac{27}{32}\right)$

Exercise 3. Which of the integers 312, 98, 112, and 144 are divisible by 8 ?

Exercise 4. (a) What is the prime factorization of 372 ?

(b) What are the positive divisors of 372 ?

Exercise 5. (a) What are the prime divisors of 100 ?

(b) What are the prime divisors of 144 ?

Exercise 6. Which of the integers 2, 9, 19, 29, 30, 37, 45, 49, 51, 83, 90, and 91 are prime numbers?

Exercise 7. What is the prime factorization of 585 ?

Exercise 8. Which of the following statements are true?

(a) $-5 < 3.1$

(b) $\sqrt{16} = 4$

(c) $7 \div 0 = 0$

(d) $0 < \left|-\frac{1}{7}\right|$

(e) $0.3 < \frac{1}{3}$

(f) $(-1)^{87} = -1$

(g) $\sqrt{(-3)^2} < 0$

(h) $\frac{21}{28} = \frac{3}{4}$

(i) $-\left|-23\right| = 23$

(j) $\frac{1}{2} > \frac{1}{17}$

(k) $(59^3)(59^2) = 59^6$

(l) $-\sqrt{25} < -4$

Exercise 9. Find the following.

(a) 40% of 15

(b) 150% of 48

(c) 0.6% of 800

(d) 15 is 30% of which number?

(e) 11 is what percent of 55?

Exercise 10. If a person's salary increases from \$200 per week to \$234 per week, what is the percent increase in the person's salary?

Exercise 11. If an athlete's weight decreases from 160 pounds to 152 pounds, what is the percent decrease in the athlete's weight?

Exercise 12. A particular stock is valued at \$40 per share. If the value increases by 20 percent and then decreases by 25 percent, what will be the value of the stock per share after the decrease?

Exercise 13. There are a total of 20 dogs and cats at a kennel. If the ratio of the number of dogs to the number of cats at the kennel is 3 to 2, how many cats are at the kennel?

Exercise 14. The integer c is even, and the integer d is odd. For each of the following integers, indicate whether the integer is even or odd.

(a) $c + 2d$

(b) $2c + d$

(c) cd

(d) c^d

(e) $(c + d)^2$

(f) $c^2 - d^2$

Exercise 15. When the positive integer n is divided by 3, the remainder is 2, and when n is divided by 5, the remainder is 1. What is the least possible value of n ?

ANSWERS TO ARITHMETIC EXERCISES

Exercise 1.
(a) 19
(b) –3
(c) 8
(d) 89
(e) 0
(f) 1,296
(g) 1,024
(h) 51

Exercise 2.
(a) $\frac{1}{4}$
(b) $-\frac{5}{14}$
(c) $\frac{9}{1{,}600}$
(d) $-\frac{4}{9}$

Exercise 3. 312, 112, and 144

Exercise 4. (a) $372 = (2^2)(3)(31)$

(b) The positive divisors of 372 are 1, 2, 3, 4, 6, 12, 31, 62, 93, 124, 186, and 372.

Exercise 5. (a) $100 = (2^2)(5^2)$, so the prime divisors are 2 and 5.

(b) $144 = (2^4)(3^2)$, so the prime divisors are 2 and 3.

Exercise 6. 2, 19, 29, 37, and 83

Exercise 7. $585 = (3^2)(5)(13)$

Exercise 8.
(a) True
(b) True
(c) False; division by 0 is undefined.
(d) True
(e) True
(f) True
(g) False; $\sqrt{(-3)^2} = \sqrt{9} = 3 > 0$
(h) True
(i) False; $-|-23| = -23$
(j) True
(k) False; $(59^3)(59^2) = 59^{3+2} = 59^5$
(l) True

Exercise 9.
(a) 6
(b) 72
(c) 4.8
(d) 50
(e) 20%

Exercise 10. 17%

Exercise 11. 5%

Exercise 12. $36 per share

Exercise 13. 8 cats

Exercise 14.
(a) $c + 2d$ is even.
(b) $2c + d$ is odd.
(c) cd is even.
(d) c^d is even.
(e) $(c + d)^2$ is odd.
(f) $c^2 - d^2$ is odd.

Exercise 15. 11

PART 2. ALGEBRA

The review of algebra begins with algebraic expressions, equations, inequalities, and functions and then progresses to several examples of applying them to solve real-life word problems. The review of algebra ends with coordinate geometry and graphs of functions as other important algebraic tools for solving problems.

2.1 Algebraic Expressions

A **variable** is a letter that represents a quantity whose value is unknown. The letters x and y are often used as variables, although any symbol can be used. An **algebraic expression** has one or more variables and can be written as a single **term** or as a sum of terms. Here are four examples of algebraic expressions.

Example 2.1.1: $2x$

Example 2.1.2: $y - \frac{1}{4}$

Example 2.1.3: $w^3z + 5z^2 - z^2 + 6$

Example 2.1.4: $\frac{8}{n+p}$

In the examples above, $2x$ is a single term, $y - \frac{1}{4}$ has two terms, $w^3z + 5z^2 - z^2 + 6$ has four terms, and $\frac{8}{n+p}$ has one term.

In the expression $w^3z + 5z^2 - z^2 + 6$, the terms $5z^2$ and $-z^2$ are called **like terms** because they have the same variables, and the corresponding variables have the same exponents. A term that has no variable is called a **constant** term. A number that is multiplied by variables is called the **coefficient** of a term.

A **polynomial** is the sum of a finite number of terms in which each term is either a constant term or a product of a coefficient and one or more variables with positive integer exponents. The **degree** of each term is the sum of the exponents of the variables in the term. A variable that is written without an exponent has degree 1. The degree of a constant term is 0. The **degree of a polynomial** is the greatest degree of its terms.

Polynomials of degrees 2 and 3 are known as quadratic and cubic polynomials, respectively.

Example 2.1.5: The expression $4x^6 + 7x^5 - 3x + 2$ is a polynomial in one variable, x. The polynomial has four terms.

The first term is $4x^6$. The coefficient of this term is 4, and its degree is 6.
The second term is $7x^5$. The coefficient of this term is 7, and its degree is 5.
The third term is $-3x$. The coefficient of this term is -3, and its degree is 1.
The fourth term is 2. This term is a constant, and its degree is 0.

Example 2.1.6: The expression $2x^2 - 7xy^3 - 5$ is a polynomial in two variables, x and y. The polynomial has three terms.

The first term is $2x^2$. The coefficient of this term is 2, and its degree is 2.

The second term is $-7xy^3$. The coefficient of this term is -7; and, since the degree of x is 1 and the degree of y^3 is 3, the degree of the term $-7xy^3$ is 4.
The third term is -5, which is a constant term. The degree of this term is 0.
In this example, the degrees of the three terms are 2, 4, and 0, so the degree of the polynomial is 4.

Example 2.1.7: The expression $4x^3 - 12x^2 - x + 36$ is a cubic polynomial in one variable.

Operations with Algebraic Expressions

The same rules that govern operations with numbers apply to operations with algebraic expressions.

In an algebraic expression, like terms can be combined by simply adding their coefficients, as the following three examples show.

Example 2.1.8: $2x + 5x = 7x$

Example 2.1.9: $w^3z + 5z^2 - z^2 + 6 = w^3z + 4z^2 + 6$

Example 2.1.10: $3xy + 2x - xy - 3x = 2xy - x$

A number or variable that is a factor of each term in an algebraic expression can be factored out, as the following three examples show.

Example 2.1.11: $4x + 12 = 4(x + 3)$

Example 2.1.12: $15y^2 - 9y = 3y\,(5y - 3)$

Example 2.1.13: For values of x where it is defined, the algebraic expression $\dfrac{7x^2 + 14x}{2x + 4}$ can be simplified as follows.

First factor the numerator and the denominator to get $\dfrac{7x(x+2)}{2(x+2)}$.

Since $x + 2$ occurs as a factor in both the numerator and the denominator of the expression, canceling it out will give an equivalent fraction for all values of x for which the expression is defined. Thus, for all values of x for which the expression is defined, the expression is equivalent to $\dfrac{7x}{2}$.

(A fraction is not defined when the denominator is equal to 0. The denominator of the original expression was $2(x + 2)$, which is equal to 0 when $x = -2$, so the original expression is defined for all $x \neq -2$.)

To multiply two algebraic expressions, each term of the first expression is multiplied by each term of the second expression and the results are added, as the following example shows.

Example 2.1.14: Multiply $(x + 2)(3x - 7)$ as follows.

First multiply each term of the expression $x + 2$ by each term of the expression $3x - 7$ to get the expression $x(3x) + x(-7) + 2(3x) + 2(-7)$.

Then simplify each term to get $3x^2 - 7x + 6x - 14$.
Finally, combine like terms to get $3x^2 - x - 14$.
So you can conclude that $(x + 2)(3x - 7) = 3x^2 - x - 14$.

A statement of equality between two algebraic expressions that is true for all possible values of the variables involved is called an **identity**. Here are seven examples of identities.

Identity 1: $ca + cb = c(a + b)$
Identity 2: $ca - cb = c(a - b)$
Identity 3: $(a + b)^2 = a^2 + 2ab + b^2$
Identity 4: $(a - b)^2 = a^2 - 2ab + b^2$
Identity 5: $a^2 - b^2 = (a + b)(a - b)$
Identity 6: $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
Identity 7: $(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

Identities can be used to modify and simplify algebraic expressions, as in the following example.

Example 2.1.15: Simplify the algebraic expression $\frac{x^2 - 9}{4x - 12}$.

Solution: Use the identity $a^2 - b^2 = (a + b)(a - b)$ to factor the numerator of the expression, and use the identity $ca - cb = c(a - b)$ to factor the denominator of the expression to get

$$\frac{x^2 - 9}{4x - 12} = \frac{(x + 3)(x - 3)}{4(x - 3)}$$

Now, since $x - 3$ occurs as a factor in both the numerator and the denominator, it can be canceled out when $x - 3 \neq 0$, that is, when $x \neq 3$ (since the fraction is not defined when the denominator is 0). Therefore, for all $x \neq 3$, the expression is equivalent to $\frac{x + 3}{4}$.

2.2 Rules of Exponents

In the algebraic expression x^a, where x is raised to the power a, x is called the **base** and a is called the **exponent**. For all integers a and b and all positive numbers x, except $x = 1$, the following property holds: If $x^a = x^b$, then $a = b$.

Example: If $2^{3c+1} = 2^{10}$, then $3c + 1 = 10$, and therefore, $c = 3$.

Here are seven basic rules of exponents. In each rule, the bases x and y are nonzero real numbers and the exponents a and b are integers, unless stated otherwise.

Rule 1: $x^{-a} = \frac{1}{x^a}$

Example A: $4^{-3} = \frac{1}{4^3} = \frac{1}{64}$

Example B: $x^{-10} = \frac{1}{x^{10}}$

Example C: $\frac{1}{2^{-a}} = 2^{a}$

Rule 2: $(x^a)(x^b) = x^{a+b}$

Example A: $(3^2)(3^4) = 3^{2+4} = 3^6 = 729$

Example B: $(y^3)(y^{-1}) = y^2$

Rule 3: $\frac{x^a}{x^b} = x^{a-b} = \frac{1}{x^{b-a}}$

Example A: $\frac{5^7}{5^4} = 5^{7-4} = 5^3 = 125$

Example B: $\frac{t^3}{t^8} = t^{-5} = \frac{1}{t^5}$

Rule 4: $x^0 = 1$

Example A: $7^0 = 1$

Example B: $(-3)^0 = 1$

Note that 0^0 is not defined.

Rule 5: $(x^a)(y^a) = (xy)^a$

Example A: $(2^3)(3^3) = 6^3 = 216$

Example B: $(10z)^3 = 10^3 z^3 = 1{,}000z^3$

Rule 6: $\left(\frac{x}{y}\right)^a = \frac{x^a}{y^a}$

Example A: $\left(\frac{3}{4}\right)^2 = \frac{3^2}{4^2} = \frac{9}{16}$

Example B: $\left(\frac{r}{4t}\right)^3 = \frac{r^3}{64t^3}$

Rule 7: $(x^a)^b = x^{ab}$

Example A: $(2^5)^2 = 2^{10} = 1{,}024$

Example B: $(3y^6)^2 = (3^2)(y^6)^2 = 9y^{12}$

The rules above are identities that are used to simplify expressions. Sometimes algebraic expressions look like they can be simplified in similar ways, but in fact they cannot. In order to avoid mistakes commonly made when dealing with exponents, keep the following six cases in mind.

Case 1: $(x^a)(y^b) \neq (xy)^{a+b}$

For example, $(2^4)(3^2) \neq (2 \times 3)^{4+2}$, since $(2^4)(3^2) = 144$ and $6^{4+2} = 6^6 = 46{,}656$.

Case 2: $(x^a)^b \neq x^a x^b$

Instead, $(x^a)^b = x^{ab}$ and $x^a x^b = x^{a+b}$; for example, $(4^2)^3 = 4^6$ and $4^2 4^3 = 4^5$.

Case 3: $(x+y)^a \neq x^a + y^a$.

In particular, note that $(x+y)^2 = x^2 + 2xy + y^2$; that is, the correct expansion contains the additional term $2xy$.

Case 4: $(-x)^2 \neq -x^2$

Instead, $(-x)^2 = x^2$. Note carefully where each negative sign appears.

Case 5: $\sqrt{x^2+y^2} \neq x+y$

Case 6: $\frac{a}{x+y} \neq \frac{a}{x} + \frac{a}{y}$

But it *is* true that $\frac{x+y}{a} = \frac{x}{a} + \frac{y}{a}$.

2.3 Solving Linear Equations

An **equation** is a statement of equality between two mathematical expressions. If an equation involves one or more variables, the values of the variables that make the equation true are called the **solutions** of the equation. To **solve an equation** means to find the values of the variables that make the equation true, that is, the values that **satisfy the equation**. Two equations that have the same solutions are called **equivalent equations**. For example, $x + 1 = 2$ and $2x + 2 = 4$ are equivalent equations; both are true when $x = 1$ and are false otherwise. The general method for solving an equation is to find successively simpler equivalent equations so that the simplest equivalent equation makes the solutions obvious.

The following three rules are important for producing equivalent equations.

Rule 1: When the same constant is added to or subtracted from both sides of an equation, the equality is preserved and the new equation is equivalent to the original equation.

Rule 2: When both sides of an equation are multiplied or divided by the same nonzero constant, the equality is preserved and the new equation is equivalent to the original equation.

Rule 3: When an expression that occurs in an equation is replaced by an equivalent expression, the equality is preserved and the new equation is equivalent to the original equation.

Example: Since the expression $2(x + 1)$ is equivalent to the expression $2x + 2$, when the expression $2(x + 1)$ occurs in an equation, it can be replaced by the expression $2x + 2$, and the new equation will be equivalent to the original equation.

A **linear equation** is an equation involving one or more variables in which each term in the equation is either a constant term or a variable multiplied by a coefficient. None of the variables are multiplied together or raised to a power greater than 1. For example, $2x + 1 = 7x$ and $10x - 9y - z = 3$ are linear equations, but $x + y^2 = 0$ and $xz = 3$ are not.

Linear Equations in One Variable

To solve a linear equation in one variable, find successively simpler equivalent equations by combining like terms and applying the rules for producing simpler equivalent equations until the solution is obvious.

Example 2.3.1: Solve the equation $11x - 4 - 8x = 2(x + 4) - 2x$ as follows.

Combine like terms on the left side to get $3x - 4 = 2(x + 4) - 2x$.

Replace $2(x + 4)$ by $2x + 8$ on the right side to get $3x - 4 = 2x + 8 - 2x$.

Combine like terms on the right side to get $3x - 4 = 8$.

Add 4 to both sides to get $3x = 12$.

Divide both sides by 3 to get $\frac{3x}{3} = \frac{12}{3}$.

Simplify to get $x = 4$.

You can always check your solution by substituting it into the original equation. If the resulting value of the right-hand side of the equation is equal to the resulting value of the left-hand side of the equation, your solution is correct.

Substituting the solution $x = 4$ into the left-hand side of the equation $11x - 4 - 8x = 2(x + 4) - 2x$ gives

$$11x - 4 - 8x = 11(4) - 4 - 8(4) = 44 - 4 - 32 = 8$$

Substituting the solution $x = 4$ into the right-hand side of the equation gives

$$2(x + 4) - 2x = 2(4 + 4) - 2(4) = 2(8) - 8 = 8$$

Since both substitutions give the same result, 8, the solution $x = 4$ is correct.

Note that it is possible for a linear equation to have no solutions. For example, the equation $2x + 3 = 2(7 + x)$ has no solution, since it is equivalent to the equation $3 = 14$, which is false. Also, it is possible that what looks to be a linear equation could turn out to be an identity when you try to solve it. For example, $3x - 6 = -3(2 - x)$ is true for all values of x, so it is an identity.

Linear Equations in Two Variables

A linear equation in two variables, x and y, can be written in the form

$$ax + by = c$$

where a, b, and c are real numbers and neither a nor b is equal to 0. For example, $3x + 2y = 8$ is a linear equation in two variables.

A solution of such an equation is an **ordered pair** of numbers (x, y) that makes the equation true when the values of x and y are substituted into the equation. For example, both the ordered pair $(2, 1)$ and the ordered pair $\left(-\frac{2}{3}, 5\right)$ are solutions of the equation $3x + 2y = 8$, but the ordered pair $(1, 2)$ is not a solution. Every linear equation in two variables has infinitely many solutions.

A set of equations in two or more variables is called a **system of equations**, and the equations in the system are called **simultaneous equations**. To solve a system of equations in two variables, x and y, means to find ordered pairs of numbers (x, y) that satisfy all of the equations in the system. Similarly, to solve a system of equations in three variables, x, y, and z, means to find ordered triples of numbers (x, y, z) that satisfy all of the equations in the system. Solutions of systems with more than three variables are defined in a similar way.

Generally, systems of linear equations in two variables consist of two linear equations, each of which contains one or both of the variables. Often, such systems have a unique solution; that is, there is only one ordered pair of numbers that satisfies both equations in the system. However, it is possible that the system will not have any solutions, or that it will have infinitely many solutions.

There are two basic methods for solving systems of linear equations, by **substitution** or by **elimination**. In the substitution method, one equation is manipulated to express one variable in terms of the other. Then the expression is substituted in the other equation.

Example 2.3.2: Use substitution to solve the following system of two equations.

$$\begin{aligned} 4x + 3y &= 13 \\ x + 2y &= 2 \end{aligned}$$

Solution:

Part 1: You can solve for y as follows.

Express x in the second equation in terms of y as $x = 2 - 2y$.

Substitute $2 - 2y$ for x in the first equation to get $4(2 - 2y) + 3y = 13$.

Replace $4(2 - 2y)$ by $8 - 8y$ on the left side to get $8 - 8y + 3y = 13$.

Combine like terms to get $8 - 5y = 13$.

Solving for y gives $y = -1$.

Part 2: Now, you can use the fact that $y = -1$ to solve for x as follows.

Substitute -1 for y in the second equation to get $x + 2(-1) = 2$.

Solving for x gives $x = 4$.

Thus, the solution of the system is $x = 4$ and $y = -1$, or $(x, y) = (4, -1)$.

In the elimination method, the object is to make the coefficients of one variable the same in both equations so that one variable can be eliminated either by adding the equations together or by subtracting one from the other.

Example 2.3.3: Use elimination to solve the following system of two equations.

$$\begin{aligned} 4x + 3y &= 13 \\ x + 2y &= 2 \end{aligned}$$

(Note that this is the same system of equations that was solved by substitution in Example 2.3.2.)

Solution: Multiplying both sides of the second equation by 4 yields $4(x + 2y) = 4(2)$, or $4x + 8y = 8$.

Now you have two equations with the same coefficient of x.

$$\begin{aligned} 4x + 3y &= 13 \\ 4x + 8y &= 8 \end{aligned}$$

If you subtract the equation $4x + 8y = 8$ from the equation $4x + 3y = 13$, the result is $-5y = 5$. Thus, $y = -1$, and substituting -1 for y in either of the original equations yields $x = 4$.

Again, the solution of the system is $x = 4$ and $y = -1$, or $(x, y) = (4, -1)$.

2.4 Solving Quadratic Equations

A **quadratic equation** in the variable x is an equation that can be written in the form

$$ax^2 + bx + c = 0$$

where a, b, and c are real numbers and $a \neq 0$. Quadratic equations have zero, one, or two real solutions.

The Quadratic Formula

One way to find solutions of a quadratic equation is to use the **quadratic formula**:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

where the notation ± is shorthand for indicating two solutions—one that uses the plus sign and the other that uses the minus sign.

Example 2.4.1: In the quadratic equation $2x^2 - x - 6 = 0$, we have $a = 2$, $b = -1$, and $c = -6$. Therefore, the quadratic formula yields

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(2)(-6)}}{2(2)}$$

When the expression under the square root sign is simplified, we get

$$x = \frac{-(-1) \pm \sqrt{49}}{2(2)}$$

which can be further simplified to

$$x = \frac{1 \pm \sqrt{49}}{4}$$

Finally, since $\sqrt{49} = 7$, we get that

$$x = \frac{1 \pm 7}{4}$$

Hence this quadratic equation has two real solutions: $x = \frac{1+7}{4} = 2$ and $x = \frac{1-7}{4} = -\frac{3}{2}$.

Example 2.4.2: In the quadratic equation $x^2 + 4x + 4 = 0$, we have $a = 1$, $b = 4$, and $c = 4$. Therefore, the quadratic formula yields

$$x = \frac{-4 \pm \sqrt{4^2 - 4(1)(4)}}{2(1)}$$

When the expression under the square root sign is simplified, we get

$$x = \frac{-4 \pm \sqrt{0}}{2(1)}$$

The expression under the square root sign is equal to 0 and $\sqrt{0} = 0$. Therefore, the expression can be simplified to

$$x = \frac{-4}{2(1)} = -2$$

Thus this quadratic equation has only one solution, $x = -2$.

Example 2.4.3: In the quadratic equation $x^2 + x + 5 = 0$, we have $a = 1$, $b = 1$, and $c = 5$. Therefore, the quadratic formula yields

$$x = \frac{-1 \pm \sqrt{(1)^2 - 4(1)(5)}}{2(1)}$$

The expression under the square root sign is equal to -19. Since square roots of negative numbers are not real numbers, x is not a real number, and there is no real solution to this quadratic equation.

Solving Quadratic Equations by Factoring

Some quadratic equations can be solved more quickly by factoring.

Example 2.4.4: The quadratic equation $2x^2 - x - 6 = 0$ in Example 2.4.1 can be factored as $(2x + 3)(x - 2) = 0$. When a product is equal to 0, at least one of the factors must be equal to 0, so either $2x + 3 = 0$ or $x - 2 = 0$.

If $2x + 3 = 0$, then $2x = -3$ and $x = -\frac{3}{2}$.

If $x - 2 = 0$, then $x = 2$.

Thus the solutions are $-\frac{3}{2}$ and 2.

Example 2.4.5: The quadratic equation $5x^2 + 3x - 2 = 0$ can be factored as $(5x - 2)(x + 1) = 0$.

Therefore, either $5x - 2 = 0$ or $x + 1 = 0$.

If $5x - 2 = 0$, then $x = \frac{2}{5}$.

If $x + 1 = 0$, then $x = -1$.

Thus the solutions are $\frac{2}{5}$ and -1.

2.5 Solving Linear Inequalities

A mathematical statement that uses one of the following four inequality signs is called an **inequality**.

$<$ less than
$>$ greater than
$\leq$ less than or equal to
$\geq$ greater than or equal to

Inequalities can involve variables and are similar to equations, except that the two sides are related by one of the inequality signs instead of the equality sign used in equations. For example, the inequality $4x + 1 \leq 7$ is a linear inequality in one variable, which states that $4x + 1$ is less than or equal to 7. To **solve an inequality** means to find the set of all values of the variable that make the inequality true. This set of values is also known as the **solution set** of an inequality. Two inequalities that have the same solution set are called **equivalent inequalities**.

The procedure used to solve a linear inequality is similar to that used to solve a linear equation, which is to simplify the inequality by isolating the variable on one side of the inequality, using the following two rules.

Rule 1: When the same constant is added to or subtracted from both sides of an inequality, the direction of the inequality is preserved and the new inequality is equivalent to the original.

Rule 2: When both sides of the inequality are multiplied or divided by the same nonzero constant, the direction of the inequality is *preserved if the constant is positive* but the direction is *reversed if the constant is negative*. In either case, the new inequality is equivalent to the original.

Example 2.5.1: The inequality $-3x + 5 \le 17$ can be solved as follows.

Subtract 5 from both sides to get $-3x \le 12$.

Divide both sides by -3 and reverse the direction of the inequality to get $\frac{-3x}{-3} \ge \frac{12}{-3}$.

That is, $x \ge -4$.

Therefore, the solution set of $-3x + 5 \le 17$ consists of all numbers greater than or equal to -4.

Example 2.5.2: The inequality $\frac{4x+9}{11} < 5$ can be solved as follows.

Multiply both sides by 11 to get $4x + 9 < 55$.

Subtract 9 from both sides to get $4x < 46$.

Divide both sides by 4 to get $x < \frac{46}{4}$.

That is, $x < 11.5$.

Therefore, the solution set of the inequality $\frac{4x+9}{11} < 5$ consists of all numbers less than 11.5.

2.6 Functions

An algebraic expression in one variable can be used to define a **function** of that variable. Functions are usually denoted by letters such as f, g, and h. For example, the algebraic expression $3x + 5$ can be used to define a function f by

$$f(x) = 3x + 5$$

where $f(x)$ is called the value of f at x and is obtained by substituting the value of x in the expression above. For example, if $x = 1$ is substituted in the expression above, the result is $f(1) = 3(1) + 5 = 8$.

It might be helpful to think of a function f as a machine that takes an input, which is a value of the variable x, and produces the corresponding output, $f(x)$. For any function, each input x gives exactly one output $f(x)$. However, more than one value of x can give the same output $f(x)$. For example, if g is the function defined by $g(x) = x^2 - 2x + 3$, then $g(0) = 3$ and $g(2) = 3$.

The **domain** of a function is the set of all permissible inputs, that is, all permissible values of the variable x. For the functions f and g defined above, the domain is the set of all real numbers. Sometimes the domain of the function is given explicitly and is restricted to a specific set of values of x. For example, we can define the function h by $h(x) = x^2 - 4$ for $-2 \le x \le 2$. Without an explicit restriction, the domain is assumed to be the set of all values of x for which $f(x)$ is a real number.

Example 2.6.1: Let f be the function defined by $f(x)=\frac{2x}{x-6}$. In this case, f is not defined at $x = 6$, because $\frac{12}{0}$ is not defined. Hence, the domain of f consists of all real numbers except for 6.

Example 2.6.2: Let g be the function defined by $g(x)=x^3+\sqrt{x+2}-10$. In this case, $g(x)$ is not a real number if $x < -2$. Hence, the domain of g consists of all real numbers x such that $x \geq -2$.

Example 2.6.3: Let h be the function defined by $h(x) = |x|$, which is the distance between x and 0 on the number line (see Arithmetic, Section 1.5). The domain of h is the set of all real numbers. Also, $h(x) = h(-x)$ for all real numbers x, which reflects the property that on the number line the distance between x and 0 is the same as the distance between $-x$ and 0.

2.7 Applications

Translating verbal descriptions into algebraic expressions is an essential initial step in solving word problems. Three examples of verbal descriptions and their translations are given below.

Example 2.7.1: If the square of the number x is multiplied by 3 and then 10 is added to that product, the result can be represented algebraically by $3x^2 + 10$.

Example 2.7.2: If John's present salary s is increased by 14 percent, then his new salary can be represented algebraically by $1.14s$.

Example 2.7.3: If y gallons of syrup are to be distributed among 5 people so that one particular person gets 1 gallon and the rest of the syrup is divided equally among the remaining 4, then the number of gallons of syrup that each of the 4 people will get can be represented algebraically by $\frac{y-1}{4}$.

The remainder of this section gives examples of various applications.

Average, Mixture, Rate, and Work Problems

Example 2.7.4: Ellen has received the following scores on 3 exams: 82, 74, and 90. What score will Ellen need to receive on the next exam so that the average (arithmetic mean) score for the 4 exams will be 85 ?

Solution: Let x represent the score on Ellen's next exam. This initial step of assigning a variable to the quantity that is sought is an important beginning to solving the problem. Then in terms of x, the average of the 4 exam scores is

$$\frac{82+74+90+x}{4}$$

which is supposed to equal 85. Now simplify the expression and set it equal to 85:

$$\frac{82+74+90+x}{4}=\frac{246+x}{4}=85$$

Solving the resulting linear equation for x, you get $246 + x = 340$, and so $x = 94$.

Therefore, Ellen will need to attain a score of 94 on the next exam.

Example 2.7.5: A mixture of 12 grams of vinegar and oil is 40 percent vinegar, where all of the measurements are by weight. How many grams of oil must be added to the mixture to produce a new mixture that is only 25 percent vinegar?

Solution: Let x represent the number of grams of oil to be added. Then the total number of grams of the new mixture will be $12 + x$, and the total number of grams of vinegar in the new mixture will be $(0.40)(12)$. Since the new mixture must be 25 percent vinegar,

$$\frac{(0.40)(12)}{12 + x} = 0.25$$

Therefore, $(0.40)(12) = (12 + x)(0.25)$.

Simplifying further gives $4.8 = 3 + 0.25x$, so $1.8 = 0.25x$, and $7.2 = x$.

Thus, 7.2 grams of oil must be added to produce a new mixture that is 25 percent vinegar.

Example 2.7.6: In a driving competition, Jeff and Dennis drove the same course at average speeds of 51 miles per hour and 54 miles per hour, respectively. If it took Jeff 40 minutes to drive the course, how long did it take Dennis?

Solution: Let x be the time, in minutes, that it took Dennis to drive the course. The distance d, in miles, is equal to the product of the rate r, in miles per hour, and the time t, in hours; that is,

$$d = rt$$

Note that since the rates are given in miles per *hour*, it is necessary to express the times in hours; for example, 40 minutes equals $\frac{40}{60}$ of an hour. Thus, the distance traveled by Jeff is the product of his speed and his time, $(51)\left(\frac{40}{60}\right)$ miles, and the distance traveled by Dennis is similarly represented by $(54)\left(\frac{x}{60}\right)$ miles.

Since the distances are equal, it follows that $(51)\left(\frac{40}{60}\right) = (54)\left(\frac{x}{60}\right)$.

From this equation it follows that $(51)(40) = 54x$ and $x = \frac{(51)(40)}{54} \approx 37.8$.

Thus, it took Dennis approximately 37.8 minutes to drive the course.

Example 2.7.7: A batch of computer parts consists of n identical parts, where n is a multiple of 60. Working alone at its constant rate, machine A takes 3 hours to produce a batch of computer parts. Working alone at its constant rate, machine B takes 2 hours to produce a batch of computer parts. How long will it take the two machines, working simultaneously at their respective constant rates, to produce a batch of computer parts?

Solution: Since machine A takes 3 hours to produce a batch, machine A can produce $\frac{1}{3}$ of the batch in 1 hour. Similarly, machine B can produce $\frac{1}{2}$ of the batch in 1 hour. If we let x represent the number of hours it takes both machines, working simultaneously, to produce the batch, then the two machines will produce $\frac{1}{x}$ of the batch in 1 hour. When the two machines work together, adding their individual production rates, $\frac{1}{3}$ and $\frac{1}{2}$, gives their combined production rate $\frac{1}{x}$. Therefore, it follows that $\frac{1}{3}+\frac{1}{2}=\frac{1}{x}$.

This equation is equivalent to $\frac{2}{6}+\frac{3}{6}=\frac{1}{x}$. So $\frac{5}{6}=\frac{1}{x}$ and $\frac{6}{5}=x$.

Thus, working together, the machines will take $\frac{6}{5}$ hours or 1 hour 12 minutes, to produce a batch of computer parts.

Example 2.7.8: At a fruit stand, apples can be purchased for $0.15 each and pears for $0.20 each. At these rates, a bag of apples and pears was purchased for $3.80. If the bag contained 21 pieces of fruit, how many of the pieces were pears?

Solution: If a represents the number of apples purchased and p represents the number of pears purchased, then the total cost of the fruit can be represented by the equation $0.15a + 0.20p = 3.80$, and the total number of pieces of fruit can be represented by the equation $a + p = 21$. Thus to answer the question, you need to solve the following system of equations.

$$\begin{aligned} 0.15a + 0.20p &= 3.80 \\ a + p &= 21 \end{aligned}$$

From the equation for the total number of fruit, $a = 21 - p$.

Substituting $21 - p$ for a in the equation for the total cost gives the equation

$$0.15(21 - p) + 0.20p = 3.80$$

So, $(0.15)(21) - 0.15p + 0.20p = 3.80$, which is equivalent to

$$3.15 - 0.15p + 0.20p = 3.80$$

Therefore $0.05p = 0.65$, and $p = 13$.

Thus, of the 21 pieces of fruit, 13 were pears.

Example 2.7.9: To produce a particular radio model, it costs a manufacturer $30 per radio, and it is assumed that if 500 radios are produced, all of them will be sold. What must be the selling price per radio to ensure that the profit (revenue from the sales minus the total production cost) on the 500 radios is greater than $8,200 ?

Solution: If the selling price per radio is y dollars, then the profit is $500(y - 30)$ dollars.

Therefore, $500(y - 30) > 8{,}200$.

Simplifying further gives $500y - 15{,}000 > 8{,}200$, which simplifies to $500y > 23{,}200$ and then to $y > 46.4$.

Thus, the selling price must be greater than $46.40 to ensure that the profit is greater than $8,200.

Interest

Some applications involve computing **interest** earned on an investment during a specified time period. The interest can be computed as simple interest or compound interest.

Simple interest is based only on the initial deposit, which serves as the amount on which interest is computed, called the **principal**, for the entire time period. If the amount P is invested at a *simple annual interest rate of r percent*, then the value V of the investment at the end of t years is given by the formula

$$V = P\left(1 + \frac{rt}{100}\right)$$

In the case of **compound interest**, interest is added to the principal at regular time intervals, such as annually, quarterly, and monthly. Each time interest is added to the principal, the interest is said to be compounded. After each compounding, interest is earned on the new principal, which is the sum of the preceding principal and the interest just added. If the amount P is invested at an *annual interest rate of r percent*, *compounded annually*, then the value V of the investment at the end of t years is given by the formula

$$V = P\left(1 + \frac{r}{100}\right)^t$$

If the amount P is invested at an *annual interest rate of r percent, compounded n times per year*, then the value V of the investment at the end of t years is given by the formula

$$V = P\left(1 + \frac{r}{100n}\right)^{nt}$$

Example 2.7.10: If \$10,000 is invested at a simple annual interest rate of 6 percent, what is the value of the investment after half a year?

Solution: According to the formula for simple interest, the value of the investment after $\frac{1}{2}$ year is

$$\$10{,}000\left(1 + 0.06\left(\frac{1}{2}\right)\right) = \$10{,}000(1.03) = \$10{,}300$$

Example 2.7.11: If an amount P is to be invested at an annual interest rate of 3.5 percent, compounded annually, what should be the value of P so that the value of the investment is \$1,000 at the end of 3 years? (Give your answer to the nearest dollar.)

Solution: According to the formula for 3.5 percent annual interest, compounded annually, the value of the investment after 3 years is

$$P(1 + 0.035)^3$$

and we set it to be equal to \$1,000 as follows.

$$P(1 + 0.035)^3 = \$1{,}000$$

To find the value of P, we divide both sides of the equation by $(1 + 0.035)^3$.

$$P = \frac{\$1{,}000}{(1 + 0.035)^3} \approx \$902$$

Thus, to the nearest dollar, \$902 should be invested.

Example 2.7.12: A college student expects to earn at least \$1,000 in interest on an initial investment of \$20,000. If the money is invested for one year at an annual interest rate of r percent, compounded quarterly, what is the least annual interest rate that would achieve the goal? (Give your answer to the nearest 0.1 percent.)

Solution: According to the formula for r percent annual interest, compounded quarterly, the value of the investment after 1 year is

$$\$20{,}000\left(1+\frac{r}{400}\right)^4$$

By setting this value greater than or equal to \$21,000 and solving for r, we get

$$\$20{,}000\left(1+\frac{r}{400}\right)^4 \geq \$21{,}000$$

which simplifies to

$$\left(1+\frac{r}{400}\right)^4 \geq 1.05$$

We can use the fact that taking the positive fourth root of each side of an inequality preserves the direction of the inequality. It is also true that taking the positive square root or any other positive root of each side of an inequality preserves the direction of the inequality. Using this fact, take the positive fourth root of both sides of

$$\left(1+\frac{r}{400}\right)^4 \geq 1.05$$

to get

$$1+\frac{r}{400} \geq \sqrt[4]{1.05}$$

which simplifies to

$$r \geq 400(\sqrt[4]{1.05}-1)$$

To compute the positive fourth root of 1.05, we can use the fact that for any number $x \geq 0$, $\sqrt[4]{x} = \sqrt{\sqrt{x}}$.

This allows us to compute the positive fourth root of 1.05 by taking the positive square root of 1.05 and then taking the positive square root of the result.

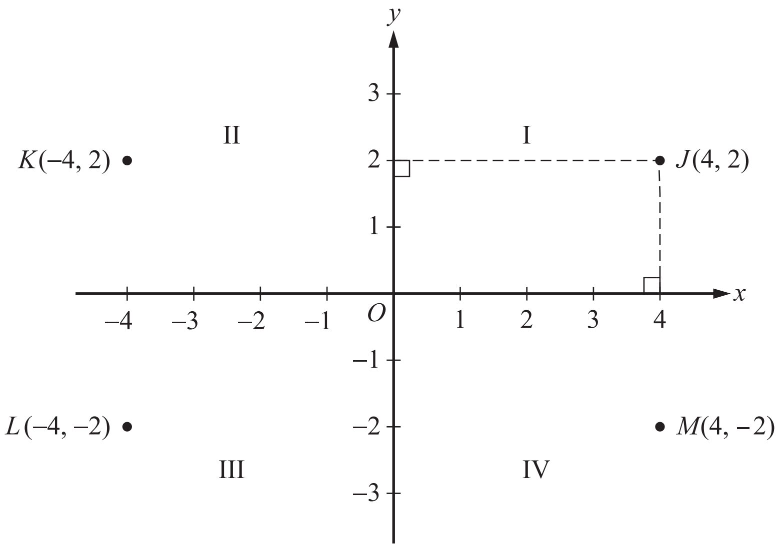
Therefore we can conclude that

$$400(\sqrt[4]{1.05}-1) = 400\left(\sqrt{\sqrt{1.05}}-1\right) \approx 4.9$$

Since $r \geq 400(\sqrt[4]{1.05}-1)$ and $400(\sqrt[4]{1.05}-1)$, rounded to the nearest 0.1, is 4.9, the least interest rate is approximately 4.9 percent per year, compounded quarterly.

2.8 Coordinate Geometry

Two real number lines that are perpendicular to each other and that intersect at their respective zero points define a **rectangular coordinate system**, often called the ***xy*-coordinate system** or ***xy*-plane**. The horizontal number line is called the ***x*-axis** and the vertical number line is called the ***y*-axis**. The point where the two axes intersect is called the **origin**, denoted by O. The positive half of the x-axis is to the right of the origin, and the positive half of the y-axis is above the origin. The two axes divide the plane into four regions called **quadrants.** The four quadrants are labeled I, II, III, and IV, as shown in Algebra Figure 1 below.



Algebra Figure 1

Each point J in the xy-plane can be identified with an ordered pair (x, y) of real numbers and is denoted by $J(x, y)$. The first number in the ordered pair is called the ***x*-coordinate**, and the second number is called the ***y*-coordinate**. A point with coordinates (x, y) is located $|x|$ units to the right of the y-axis if x is positive, or it is located $|x|$ units to the left of the y-axis if x is negative. Also, the point is located $|y|$ units above the x-axis if y is positive, or it is located $|y|$ units below the x-axis if y is negative. If $x = 0$, the point lies on the y-axis, and if $y = 0$, the point lies on the x-axis. The origin has coordinates $(0, 0)$. Unless otherwise noted, the units used on the x-axis and the y-axis are the same.

In Algebra Figure 1 above, the point $J(4, 2)$ is 4 units to the right of the y-axis and 2 units above the x-axis, the point $K(-4, 2)$ is 4 units to the left of the y-axis and 2 units above the x-axis, the point $L(-4, -2)$ is 4 units to the left of the y-axis and 2 units below the x-axis, and the point $M(4, -2)$ is 4 units to the right of the y-axis and 2 units below the x-axis.

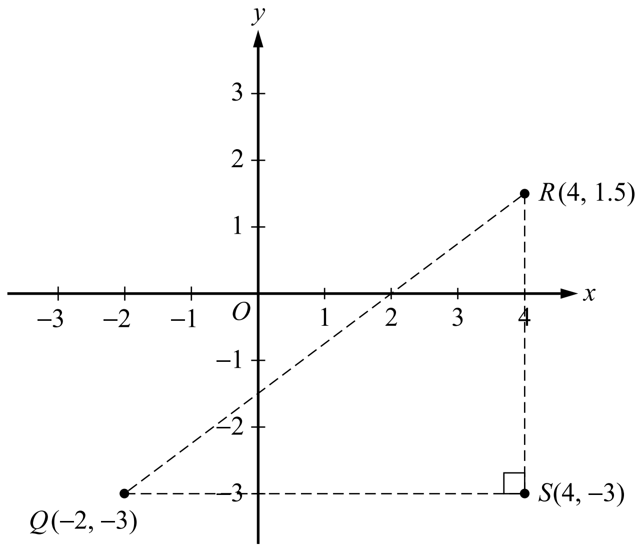
Note that the three points $K(-4, 2)$, $L(-4, -2)$, and $M(4, -2)$ have the same coordinates as J except for the signs. These points are geometrically related to J as follows.

- M is the **reflection of *J* about the *x*-axis**, or M and J are **symmetric about the *x*-axis**.
- K is the **reflection of *J* about the *y*-axis**, or K and J are **symmetric about the *y*-axis**.
- L is the **reflection of *J* about the origin**, or L and J are **symmetric about the origin**.

Calculating the Distance Between Two Points

The distance between two points in the xy-plane can be found by using the Pythagorean theorem. For example, the distance between the two points $Q(-2, -3)$ and $R(4, 1.5)$ in

Algebra Figure 2 below is the length of line segment QR. To find this length, construct a right triangle with hypotenuse QR by drawing a vertical line segment downward from R and a horizontal line segment rightward from Q until these two line segments intersect at the point $S(4, -3)$ forming a right angle, as shown in Algebra Figure 2. Then note that the horizontal side of the triangle has length $4 - (-2) = 6$ and the vertical side of the triangle has length $1.5 - (-3) = 4.5$.



Algebra Figure 2

Since line segment QR is the hypotenuse of the triangle, you can apply the Pythagorean theorem:

$$QR = \sqrt{6^2 + 4.5^2} = \sqrt{56.25} = 7.5$$

(For a discussion of right triangles and the Pythagorean theorem, see Geometry, Section 3.3.)

Graphing Linear Equations and Inequalities

Equations in two variables can be represented as graphs in the coordinate plane. In the xy-plane, the **graph of an equation** in the variables x and y is the set of all points whose ordered pairs (x, y) satisfy the equation.

The graph of a linear equation of the form $y = mx + b$ is a straight line in the xy-plane, where m is called the **slope** of the line and b is called the ***y*-intercept**.

The ***x*-intercepts** of a graph are the x-coordinates of the points at which the graph intersects the x-axis. Similarly, the ***y*-intercepts** of a graph are the y-coordinates of the points at which the graph intersects the y-axis. Sometimes the terms ***x*-intercept** and ***y*-intercept** refer to the actual intersection points.

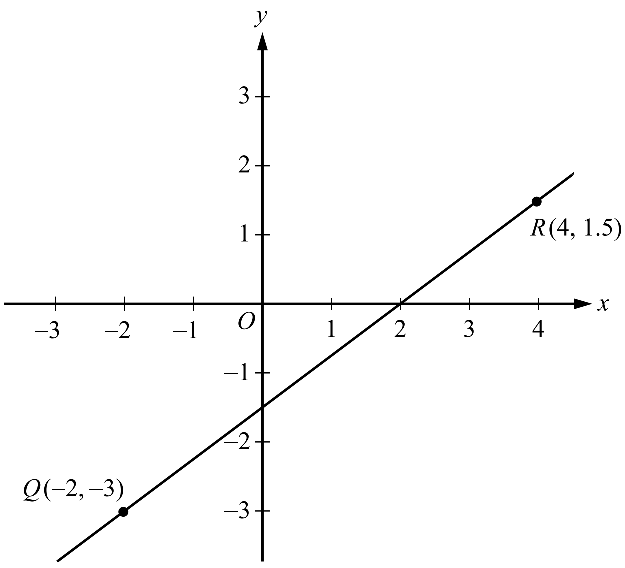
The slope of a line passing through two points $Q(x_1, y_1)$ and $R(x_2, y_2)$, where $x_1 \neq x_2$, is defined as

$$\frac{y_2 - y_1}{x_2 - x_1}$$

This ratio is often called "rise over run," where *rise* is the change in y when moving from Q to R and *run* is the change in x when moving from Q to R. A horizontal line has a slope of 0, since the rise is 0 for any two points on the line. Therefore, the equation of every horizontal line has the form $y = b$, where b is the y-intercept. The slope of a vertical line is not defined, since the run is 0. The equation of every vertical line has the form $x = a$, where a is the x-intercept.

Two lines are **parallel** if their slopes are equal. Two lines are **perpendicular** if their slopes are negative reciprocals of each other. For example, the line with equation $y = 2x + 5$ is perpendicular to the line with equation $y = -\frac{1}{2}x + 9$.

Example 2.8.1: Algebra Figure 3 below shows the graph of the line through the points $Q(-2, -3)$ and $R(4, 1.5)$.



Algebra Figure 3

In Algebra Figure 3 above, the slope of the line passing through the points

$Q(-2, -3)$ and $R(4, 1.5)$ is

$$\frac{1.5 - (-3)}{4 - (-2)} = \frac{4.5}{6} = 0.75$$

Line QR appears to intersect the y-axis close to the point $(0, -1.5)$ so the y-intercept of the line must be close to -1.5. To get the exact value of the y-intercept, substitute the coordinates of any point on the line into the equation $y = 0.75x + b$, and solve it for b.

For example, if you pick the point $Q(-2, -3)$ and substitute its coordinates into the equation, you get $-3 = (0.75)(-2) + b$.

Then adding $(0.75)(2)$ to both sides of the equation yields $b = -3 + (0.75)(2)$, or $b = -1.5$.

Therefore, the equation of line QR is $y = 0.75x - 1.5$.

You can see from the graph in Algebra Figure 3 that the x-intercept of line QR is 2, since QR passes through the point $(2, 0)$. More generally, you can find the x-intercept of a line by setting $y = 0$ in an equation of the line and solving it for x. So you can find the x-intercept of line QR by setting $y = 0$ in the equation $y = 0.75x - 1.5$ and solving it for x as follows.

Setting $y = 0$ in the equation $y = 0.75x - 1.5$ gives the equation $0 = 0.75x - 1.5$. Then adding 1.5 to both sides yields $1.5 = 0.75x$. Finally, dividing both sides by 0.75 yields $x = \frac{1.5}{0.75} = 2$.

Graphs of linear equations can be used to illustrate solutions of systems of linear equations and inequalities, as can be seen in Examples 2.8.2 and 2.8.3.

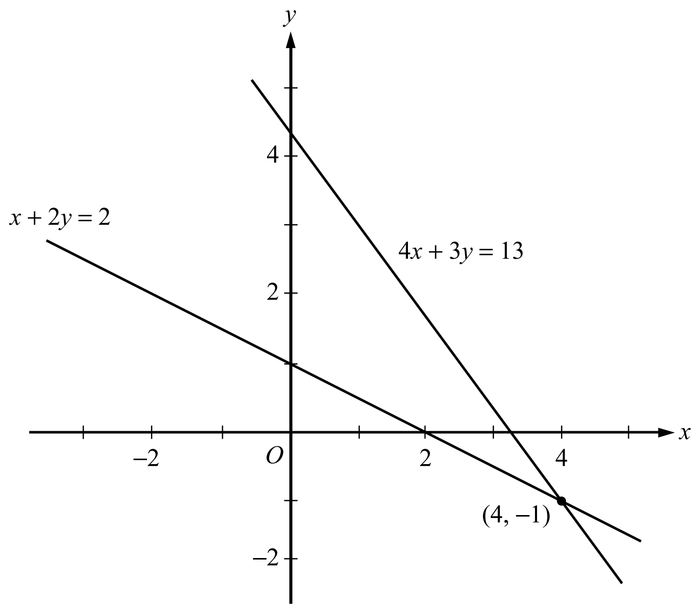
Example 2.8.2: Consider the following system of two linear equations in two variables.

$$4x + 3y = 13$$
$$x + 2y = 2$$

(Note that this system was solved by substitution and by elimination in Section 2.3.) Solving each equation for y in terms of x yields the following equivalent system of equations.

$$y = -\frac{4}{3}x + \frac{13}{3}$$
$$y = -\frac{1}{2}x + 1$$

Algebra Figure 4 below shows the graphs of the two equations in the xy-plane. The solution of the system of equations is the point at which the two graphs intersect, which is $(4, -1)$.



Algebra Figure 4

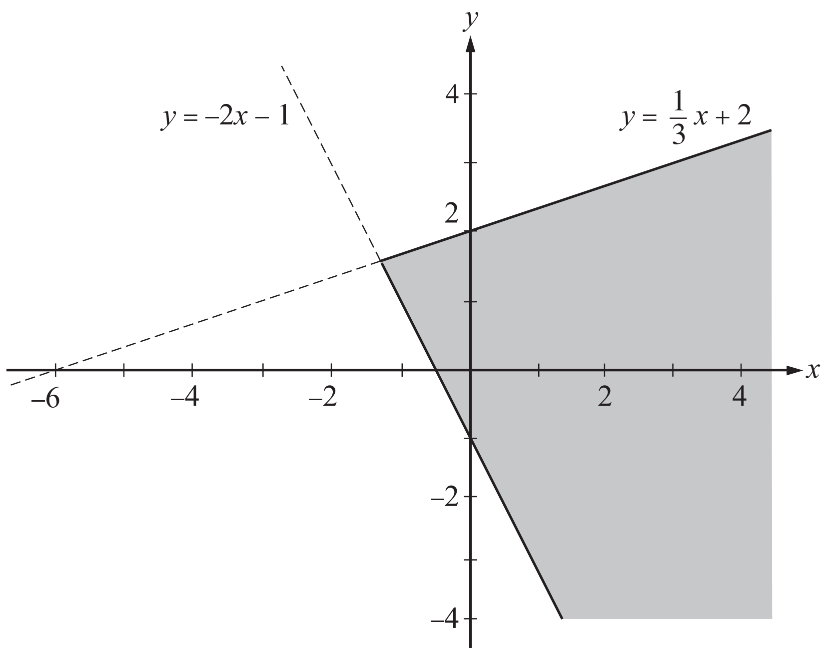
Example 2.8.3: Consider the following system of two linear inequalities.

$$x - 3y \geq -6$$
$$2x + y \geq -1$$

Solving each inequality for y in terms of x yields the following equivalent system of inequalities.

$$y \le \frac{1}{3}x + 2$$
$$y \ge -2x - 1$$

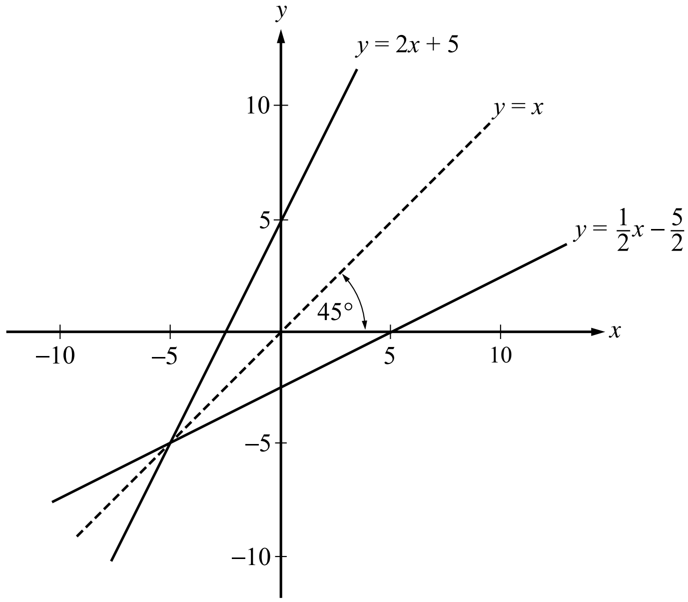
Each point (x, y) that satisfies the first inequality, $y \le \frac{1}{3}x + 2$, is either on the line $y = \frac{1}{3}x + 2$ or *below* the line because the y-coordinate is either equal to or *less than* $\frac{1}{3}x + 2$. Therefore, the graph of $y \le \frac{1}{3}x + 2$ consists of the line $y = \frac{1}{3}x + 2$ and the entire region below it. Similarly, the graph of $y \ge -2x - 1$ consists of the line $y = -2x - 1$ and the entire region *above* it. Thus, the solution set of the system of inequalities consists of all of the points that lie in the intersection of the two graphs described, which is represented by the shaded region shown in Algebra Figure 5 below, including the two half-lines that form the boundary of the shaded region.



Algebra Figure 5

Symmetry with respect to the x-axis, the y-axis, and the origin is mentioned earlier in this section. Another important symmetry is symmetry with respect to the line with equation $y = x$. The line $y = x$ passes through the origin, has a slope of 1, and makes a 45-degree angle with each axis. For any point with coordinates (a, b), the point with interchanged coordinates (b, a) is the reflection of (a, b) about the line $y = x$; that is, (a, b) and (b, a) are symmetric about the line $y = x$. It follows that interchanging x and y in the equation of any graph yields another graph that is the reflection of the original graph about the line $y = x$.

Example 2.8.4: Consider the line whose equation is $y = 2x + 5$. Interchanging x and y in the equation yields $x = 2y + 5$. Solving this equation for y yields $y = \frac{1}{2}x - \frac{5}{2}$. The line $y = 2x + 5$ and its reflection $y = \frac{1}{2}x - \frac{5}{2}$ are graphed in Algebra Figure 6 that follows.



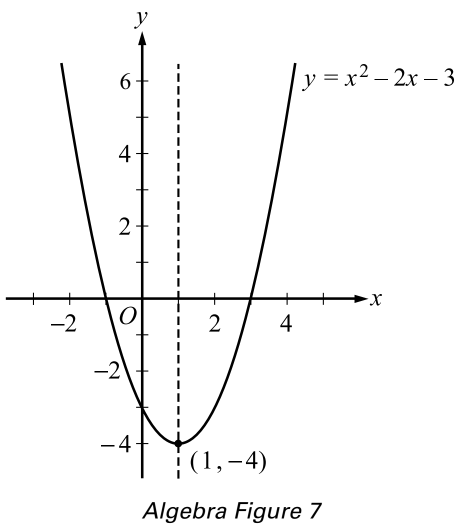
Algebra Figure 6

The line $y = x$ is a **line of symmetry** for the graphs of $y = 2x + 5$ and $y = \frac{1}{2}x - \frac{5}{2}$.

Graphing Quadratic Equations

The graph of a quadratic equation of the form $y = ax^2 + bx + c$, where a, b, and c are constants and $a \neq 0$, is a **parabola**. The x-intercepts of the parabola are the solutions of the equation $ax^2 + bx + c = 0$. If a is positive, the parabola opens upward and the **vertex** is its lowest point. If a is negative, the parabola opens downward and the vertex is its highest point. Every parabola that is the graph of a quadratic equation of the form $y = ax^2 + bx + c$ is symmetric with itself about the vertical line that passes through its vertex. In particular, the two x-intercepts are equidistant from this line of symmetry.

Example 2.8.5: The quadratic equation $y = x^2 - 2x - 3$ has the graph shown in Algebra Figure 7 below.



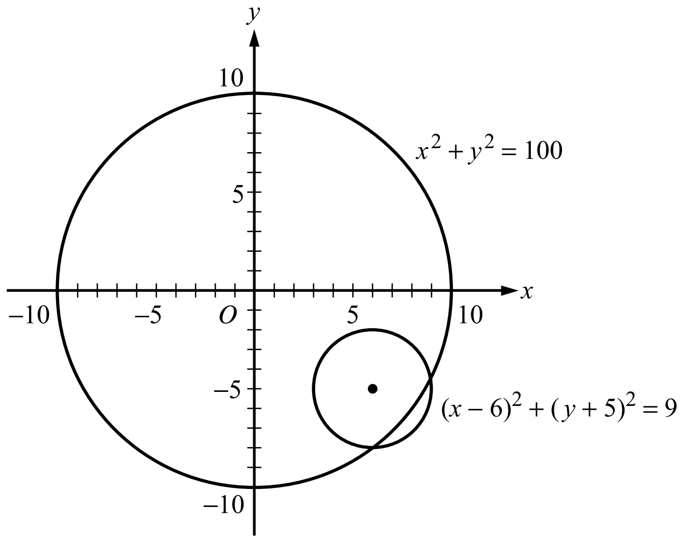
Algebra Figure 7

The graph indicates that the x-intercepts of the parabola are −1 and 3. The values of the x-intercepts can be confirmed by solving the quadratic equation $x^2 - 2x - 3 = 0$ to get $x = -1$ and $x = 3$. The point $(1, -4)$ is the vertex of the parabola, and the line $x = 1$ is its line of symmetry. The y-intercept is the y-coordinate of the point on the parabola at which $x = 0$, which is $y = 0^2 - 2(0) - 3 = -3$.

Graphing Circles

The graph of an equation of the form $(x - a)^2 + (y - b)^2 = r^2$ is a **circle** with its center at the point (a, b) and with radius $r > 0$.

Example 2.8.6: Algebra Figure 8 below shows the graph of two circles in the xy-plane. The larger of the two circles is centered at the origin and has radius 10, so its equation is $x^2 + y^2 = 100$. The smaller of the two circles has center $(6, -5)$ and radius 3, so its equation is $(x-6)^2 + (y+5)^2 = 9$.



Algebra Figure 8

2.9 Graphs of Functions

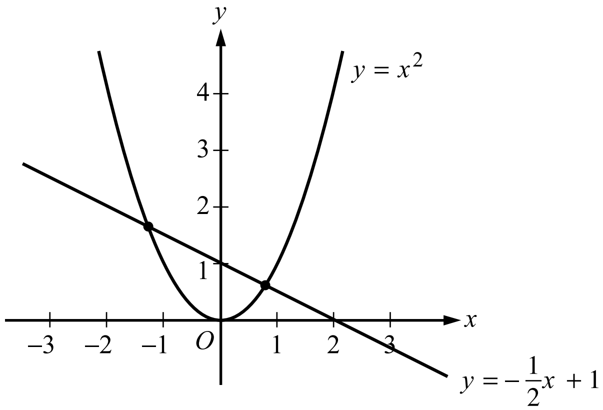
The coordinate plane can be used for graphing functions. To graph a function in the xy-plane, you represent each input x and its corresponding output $f(x)$ as a point (x, y), where $y = f(x)$. In other words, you use the x-axis for the input and the y-axis for the output.

Below are several examples of graphs of elementary functions.

Example 2.9.1: Consider the linear function defined by $f(x) = -\frac{1}{2}x + 1$. Its graph in the xy-plane is the line with the linear equation $y = -\frac{1}{2}x + 1$.

Example 2.9.2: Consider the quadratic function defined by $g(x) = x^2$. The graph of g is the parabola with the quadratic equation $y = x^2$.

The graph of both the linear equation $y = -\frac{1}{2}x + 1$ and the quadratic equation $y = x^2$ are shown in Algebra Figure 9 below.



Algebra Figure 9

Note that the graphs of f and g in Algebra Figure 9 above intersect at two points. These are the points at which $g(x) = f(x)$. We can find these points algebraically as follows.

Set $g(x) = f(x)$ and get $x^2 = -\frac{1}{2}x + 1$, which is equivalent to $x^2 + \frac{1}{2}x - 1 = 0$, or $2x^2 + x - 2 = 0$.

Then solve the equation $2x^2 + x - 2 = 0$ for x using the quadratic formula and get

$$x = \frac{-1 \pm \sqrt{1+16}}{4}$$

which represents the x-coordinates of the two solutions

$$x = \frac{-1+\sqrt{17}}{4} \approx 0.78 \quad \text{and} \quad x = \frac{-1-\sqrt{17}}{4} \approx -1.28.$$

With these input values, the corresponding y-coordinates can be found using either f or g:

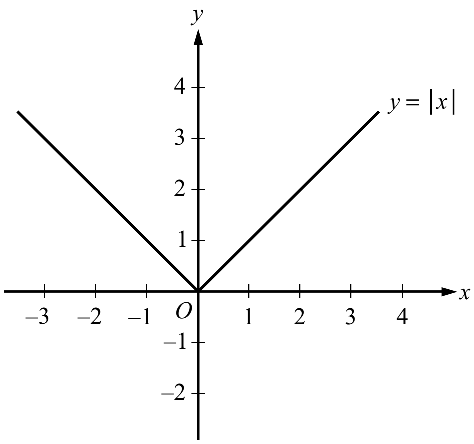
$$g\left(\frac{-1+\sqrt{17}}{4}\right) = \left(\frac{-1+\sqrt{17}}{4}\right)^2 \approx 0.61 \quad \text{and} \quad g\left(\frac{-1-\sqrt{17}}{4}\right) = \left(\frac{-1-\sqrt{17}}{4}\right)^2 \approx 1.64.$$

Thus, the two intersection points can be approximated by (0.78, 0.61) and (–1.28, 1.64).

Example 2.9.3: Consider the absolute value function defined by $h(x) = |x|$. By using the definition of absolute value (see Arithmetic, Section 1.5), h can be expressed as a **piecewise-defined** function:

$$h(x) = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

The graph of this function is V-shaped and consists of two linear pieces, $y = x$ and $y = -x$, joined at the origin, as shown in Algebra Figure 10 that follows.

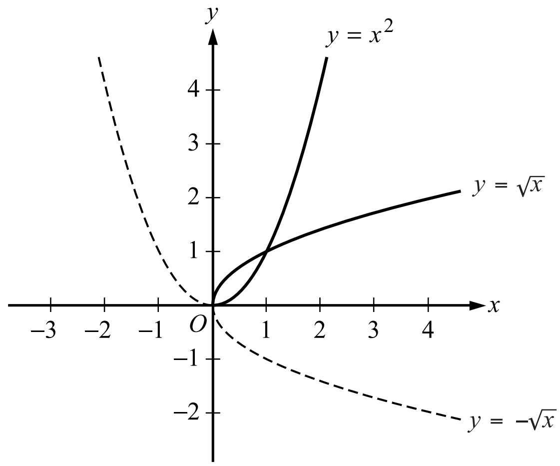


Algebra Figure 10

Example 2.9.4: Consider the positive square root function defined by $j(x) = \sqrt{x}$ for $x \geq 0$. The graph of this function is the upper half of a parabola lying on its side.

Also consider the negative square root function defined by $k(x) = -\sqrt{x}$ for $x \geq 0$. The graph of this function is the lower half of the parabola lying on its side.

The graphs of both of these functions, along with the graph of the parabola $y = x^2$, are shown in Algebra Figure 11 below.



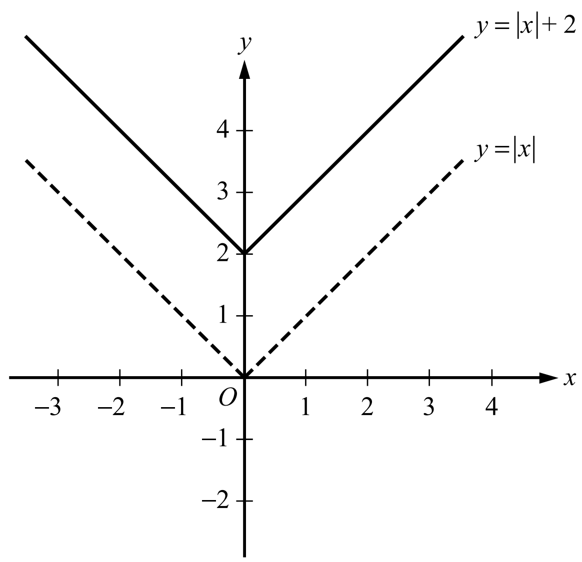
Algebra Figure 11

The graphs of $y = \sqrt{x}$ and $y = -\sqrt{x}$ are halves of a parabola because they are reflections of the right and left halves, respectively, of the parabola $y = x^2$ about the line $y = x$. This follows from squaring both sides of the two square root equations to get $y^2 = x$ and then interchanging x and y to get $y = x^2$.

Also note that $y = -\sqrt{x}$ is the reflection of $y = \sqrt{x}$ about the x-axis. In general, for any function h, the graph of $y = -h(x)$ is the **reflection** of the graph of $y = h(x)$ about the x-axis.

Example 2.9.5: Consider the function defined by $f(x) = |x| + 2$.

The graph of $f(x) = |x| + 2$ is the graph of $y = |x|$ shifted upward by 2 units, as shown in Algebra Figure 12 that follows.

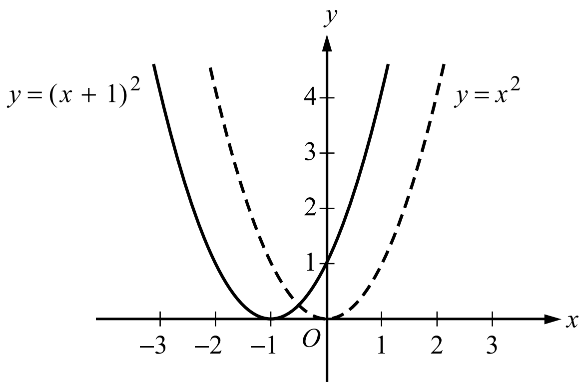


Algebra Figure 12

Similarly, the graph of the function $k(x) = |x| - 5$ is the graph of $y = |x|$ shifted downward by 5 units. (The graph of this function is not shown.)

Example 2.9.6: Consider the function defined by $g(x) = (x + 1)^2$.

The graph of $g(x) = (x + 1)^2$ is the graph of $y = x^2$ shifted to the left by 1 unit, as shown in Algebra Figure 13 below.



Algebra Figure 13

Similarly, the graph of the function $j(x) = (x - 4)^2$ is the graph of $y = x^2$ shifted to the right by 4 units. (The graph of this function is not shown.)

Note that in Example 2.9.5, the graph of the function $y = |x|$ was shifted upward and downward, and in Example 2.9.6, the graph of the function $y = x^2$ was shifted to the left and to the right. To double-check the direction of a shift, you can plot some corresponding values of the original function and the shifted function.

In general, for any function $h(x)$ and any positive number c, the following are true.

The graph of $h(x) + c$ is the graph of $h(x)$ **shifted upward** by c units.

The graph of $h(x) - c$ is the graph of $h(x)$ **shifted downward** by c units.

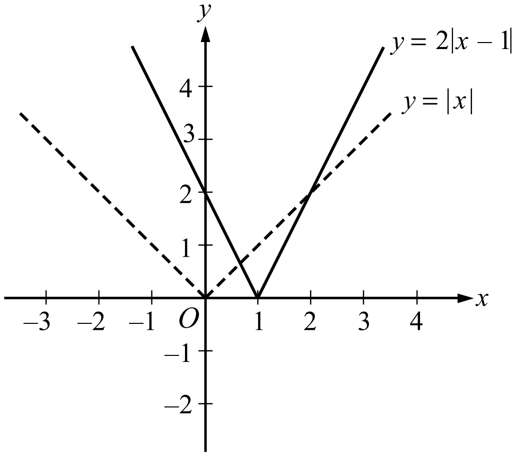
The graph of $h(x + c)$ is the graph of $h(x)$ **shifted to the left** by c units.

The graph of $h(x - c)$ is the graph of $h(x)$ **shifted to the right** by c units.

Example 2.9.7: Consider the functions defined by $f(x) = 2|x - 1|$ and $g(x) = -\dfrac{x^2}{4}$.

These functions are related to the absolute value function $|x|$ and the quadratic function x^2, respectively, in more complicated ways than in the preceding two examples.

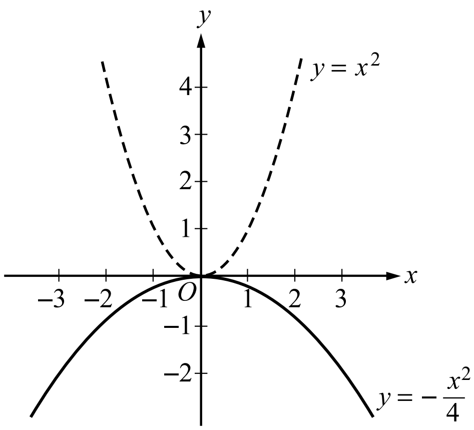
The graph of $f(x) = 2|x - 1|$ is the graph of $y = |x|$ shifted to the right by 1 unit and then stretched, or dilated, vertically away from the x-axis by a factor of 2, as shown in Algebra Figure 14 below.



Algebra Figure 14

Similarly, the graph of the function $h(x) = \dfrac{1}{2}|x - 1|$ is the graph of $y = |x|$ shifted to the right by 1 unit and then shrunk, or contracted, vertically toward the x-axis by a factor of $\dfrac{1}{2}$. (The graph of this function is not shown.)

The graph of $g(x) = -\dfrac{x^2}{4}$ is the graph of $y = x^2$ contracted vertically toward the x-axis by a factor of $\dfrac{1}{4}$ and then reflected in the x-axis, as shown in Algebra Figure 15 below.



Algebra Figure 15

In general, for any function $h(x)$ and any positive number c, the following are true.

The graph of $ch(x)$ is the graph of $h(x)$ **stretched vertically** by a factor of c if $c > 1$.

The graph of $ch(x)$ is the graph of $h(x)$ **shrunk vertically** by a factor of c if $0 < c < 1$.

ALGEBRA EXERCISES

Exercise 1. Find an algebraic expression to represent each of the following.

(a) The square of y is subtracted from 5, and the result is multiplied by 37.

(b) Three times x is squared, and the result is divided by 7.

(c) The product of $x + 4$ and y is added to 18.

Exercise 2. Simplify each of the following algebraic expressions.

(a) $3x^2 - 6 + x + 11 - x^2 + 5x$

(b) $3(5x - 1) - x + 4$

(c) $\dfrac{x^2 - 16}{x - 4}$, where $x \neq 4$

(d) $(2x + 5)(3x - 1)$

Exercise 3. (a) What is the value of $f(x) = 3x^2 - 7x + 23$, when $x = -2$?

(b) What is the value of $h(x) = x^3 - 2x^2 + x - 2$, when $x = 2$?

(c) What is the value of $k(x) = \dfrac{5}{3}x - 7$ when $x = 0$?

Exercise 4. If the function g is defined for all nonzero numbers y by $g(y) = \dfrac{y}{|y|}$, find the value of each of the following.

(a) $g(2)$

(b) $g(-2)$

(c) $g(2) - g(-2)$

Exercise 5. Use the rules of exponents to simplify the following.

(a) $(n^5)(n^{-3})$

(b) $(s^7)(t^7)$

(c) $\dfrac{r^{12}}{r^4}$

(d) $\left(\dfrac{2a}{b}\right)^5$

(e) $\left(w^5\right)^{-3}$

(f) $(5^0)(d^3)$

(g) $\dfrac{\left(x^{10}\right)\left(y^{-1}\right)}{\left(x^{-5}\right)\left(y^5\right)}$

(h) $\left(\dfrac{3x}{y}\right)^2 \div \left(\dfrac{1}{y}\right)^5$

Exercise 6. Solve each of the following equations for x.

(a) $5x - 7 = 28$

(b) $12 - 5x = x + 30$

(c) $5(x + 2) = 1 - 3x$

(d) $(x + 6)(2x - 1) = 0$

(e) $x^2 + 5x - 14 = 0$

(f) $x^2 - x - 1 = 0$

Exercise 7. Solve each of the following systems of equations for x and y.

(a) $x + y = 24$
$x - y = 18$

(b) $3x - y = -5$
$x + 2y = 3$

(c) $15x - 18 - 2y = -3x + y$
$10x + 7y + 20 = 4x + 2$

Exercise 8. Solve each of the following inequalities for x.

(a) $-3x > 7 + x$

(b) $25x + 16 \geq 10 - x$

(c) $16 + x > 8x - 12$

Exercise 9. For a given two-digit positive integer, the tens digit is 5 more than the units digit. The sum of the digits is 11. Find the integer.

Exercise 10. If the ratio of $2x$ to $5y$ is 3 to 4, what is the ratio of x to y ?

Exercise 11. Kathleen's weekly salary was increased by 8 percent to \$712.80. What was her weekly salary before the increase?

Exercise 12. A theater sells children's tickets for half the adult ticket price. If 5 adult tickets and 8 children's tickets cost a total of \$81, what is the cost of an adult ticket?

Exercise 13. Pat invested a total of \$3,000. Part of the money was invested in a money market account that paid 10 percent simple annual interest, and the remainder of the money was invested in a fund that paid 8 percent simple annual interest. If the total interest earned at the end of the first year from these investments was \$256, how much did Pat invest at 10 percent and how much at 8 percent?

Exercise 14. Two cars started from the same point and traveled on a straight course in opposite directions for 2 hours, at which time they were 208 miles apart. If one car traveled, on average, 8 miles per hour faster than the other car, what was the average speed of each car for the 2-hour trip?

Exercise 15. A group can charter a particular aircraft at a fixed total cost. If 36 people charter the aircraft rather than 40 people, then the cost per person is greater by \$12.

(a) What is the fixed total cost to charter the aircraft?

(b) What is the cost per person if 40 people charter the aircraft?

Exercise 16. An antiques dealer bought c antique chairs for a total of x dollars. The dealer sold each chair for y dollars.

(a) Write an algebraic expression for the profit, P, earned from buying and selling the chairs.

(b) Write an algebraic expression for the profit per chair.

Exercise 17. Algebra Figure 16 that follows shows right triangle PQR in the xy-plane. Find the following.

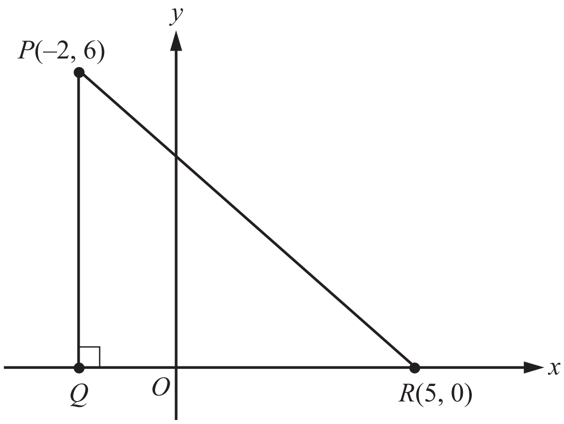
(a) The coordinates of point Q

(b) The lengths of line segment PQ, line segment QR, and line segment PR

(c) The perimeter of triangle PQR

(d) The area of triangle PQR

(e) The slope, y-intercept, and equation of the line passing through points P and R



Algebra Figure 16

Exercise 18. In the xy-plane, find the following.

(a) The slope and y-intercept of the line with equation $2y + x = 6$

(b) The equation of the line passing through the point (3, 2) with y-intercept 1

(c) The y-intercept of a line with slope 3 that passes through the point (–2, 1)

(d) The x-intercepts of the graphs in parts (a), (b), and (c)

Exercise 19. For the parabola $y = x^2 - 4x - 12$ in the xy-plane, find the following.

(a) The x-intercepts

(b) The y-intercept

(c) The coordinates of the vertex

Exercise 20. For the circle $(x - 1)^2 + (y + 1)^2 = 20$ in the xy-plane, find the following.

(a) The coordinates of the center

(b) The radius

(c) The area

Exercise 21. For each of the following functions, give the domain and a description of the graph $y = f(x)$ in the xy-plane, including its shape, and the x- and y- intercepts.

(a) $f(x) = -4$

(b) $f(x) = 100 - 900x$

(c) $f(x) = 5 - (x + 20)^2$

(d) $f(x) = \sqrt{x + 2}$

(e) $f(x) = x + |x|$

ANSWERS TO ALGEBRA EXERCISES

Exercise 1.
(a) $37(5 - y^2)$, or $185 - 37y^2$
(b) $\frac{(3x)^2}{7}$, or $\frac{9x^2}{7}$
(c) $18 + (x + 4)(y)$, or $18 + xy + 4y$

Exercise 2.
(a) $2x^2 + 6x + 5$
(b) $14x + 1$
(c) $x + 4$
(d) $6x^2 + 13x - 5$

Exercise 3.
(a) 49
(b) 0
(c) –7

Exercise 4.
(a) 1
(b) –1
(c) 2

Exercise 5.
(a) n^2
(b) $(st)^7$
(c) r^8
(d) $\frac{32a^5}{b^5}$
(e) $\frac{1}{w^{15}}$
(f) d^3
(g) $\frac{x^{15}}{y^6}$
(h) $9x^2y^3$

Exercise 6.
(a) 7
(b) –3
(c) $-\frac{9}{8}$
(d) The two solutions are –6 and $\frac{1}{2}$.
(e) The two solutions are –7 and 2.
(f) The two solutions are $\frac{1+\sqrt{5}}{2}$ and $\frac{1-\sqrt{5}}{2}$.

Exercise 7.
(a) $x = 21$ and $y = 3$
(b) $x = -1$ and $y = 2$
(c) $x = \frac{1}{2}$ and $y = -3$

Exercise 8.
(a) $x < -\frac{7}{4}$
(b) $x \geq -\frac{3}{13}$
(c) $x < 4$

Exercise 9. 83

Exercise 10. 15 to 8

Exercise 11. \$660

Exercise 12. \$9

Exercise 13. \$800 at 10% and \$2,200 at 8%

Exercise 14. 48 miles per hour and 56 miles per hour

Exercise 15. (a) \$4,320

(b) \$108

Exercise 16. (a) $P = cy - x$

(b) Profit per chair: $\frac{P}{c} = \frac{cy - x}{c} = y - \frac{x}{c}$

Exercise 17. (a) The coordinates of point Q are $(-2, 0)$.

(b) The length of PQ is 6, the length of QR is 7, and the length of PR is $\sqrt{85}$.

(c) $13 + \sqrt{85}$

(d) 21

(e) Slope: $-\frac{6}{7}$; y-intercept: $\frac{30}{7}$

equation of line: $y = -\frac{6}{7}x + \frac{30}{7}$, or $7y + 6x = 30$

Exercise 18. (a) Slope: $-\frac{1}{2}$; y-intercept: 3

(b) $y = \frac{x}{3} + 1$

(c) 7

(d) 6, -3, and $-\frac{7}{3}$

Exercise 19. (a) $x = -2$ and $x = 6$

(b) $y = -12$

(c) $(2, -16)$

Exercise 20. (a) $(1, -1)$

(b) $\sqrt{20}$

(c) 20π

Exercise 21. (a) Domain: the set of all real numbers. The graph is a horizontal line with y-intercept -4 and no x-intercept.

(b) Domain: the set of all real numbers. The graph is a line with slope -900, y-intercept 100, and x-intercept $\frac{1}{9}$.

(c) Domain: the set of all real numbers. The graph is a parabola opening downward with vertex at $(-20, 5)$, line of symmetry $x = -20$, y-intercept -395, and x-intercepts $-20 \pm \sqrt{5}$.

(d) Domain: the set of numbers greater than or equal to -2. The graph is the upper half of a parabola opening to the right with vertex at $(-2, 0)$, x-intercept -2, and y-intercept $\sqrt{2}$.

(e) Domain: the set of all real numbers. The graph is two half-lines joined at the origin: one half-line is the negative x-axis and the other is a line starting at the origin with slope 2. Every nonpositive number is an x-intercept, and the y-intercept is 0. The function is equal to the following piecewise-defined function

$$f(x) = \begin{cases} 2x, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

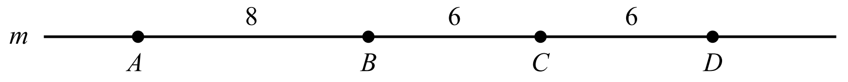
PART 3. GEOMETRY

The review of geometry begins with lines and angles and progresses to other plane figures, such as polygons, triangles, quadrilaterals, and circles. The review of geometry ends with some basic three-dimensional figures. Coordinate geometry is covered in the Algebra part.

3.1 Lines and Angles

A **line** is understood to be a straight line that extends in both directions without ending. Given any two points on a line, a **line segment** is the part of the line that contains the two points and all the points between them. The two points are called **endpoints**. Line segments that have equal lengths are called **congruent line segments**. The point that divides a line segment into two congruent line segments is called the **midpoint** of the line segment.

In Geometry Figure 1 below, A, B, C, and D are points on line m.

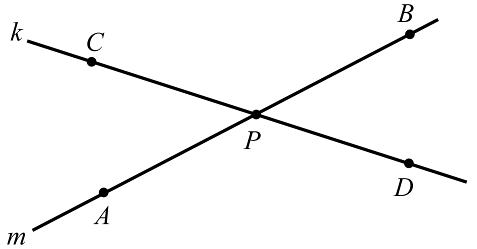


Geometry Figure 1

Line segment AB is the part of line m that consists of points A and B and all the points between A and B. According to Geometry Figure 1 above, the lengths of line segments AB, BC, and CD are 8, 6, and 6, respectively. Hence, line segments BC and CD are congruent. Since C is halfway between B and D, point C is the midpoint of line segment BD.

Sometimes the notation AB denotes line segment AB, and sometimes it denotes the **length** of line segment AB. The meaning of the notation can be determined from the context.

When two lines intersect at a point, they form four **angles**. Each angle has a **vertex** at the point of intersection of the two lines. For example, in Geometry Figure 2 below, lines k and m intersect at point P, forming the four angles APC, CPB, BPD, and DPA.

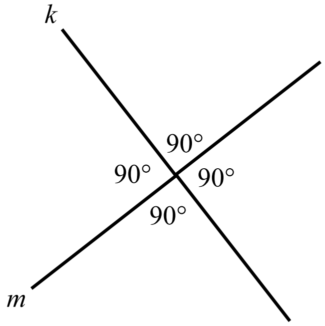


Geometry Figure 2

The first and third of the angles, that is, angles APC and BPD, are called **opposite angles**, also known as **vertical angles**. The second and fourth of the angles, that is, angles CPB and DPA, are also opposite angles. Opposite angles have equal measure, and angles that have equal measure are called **congruent angles**. Hence, opposite angles are congruent. The sum of the measures of the four angles is 360°.

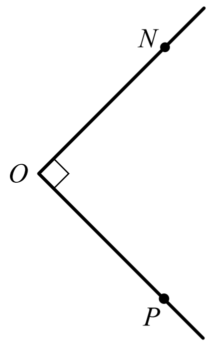
Sometimes the angle symbol $\angle$ is used instead of the word "angle." For example, angle APC can be written as $\angle APC$.

Two lines that intersect to form four congruent angles are called **perpendicular lines**. Each of the four angles has a measure of 90°. An angle with a measure of 90° is called a **right angle**. Geometry Figure 3 below shows two lines, k and m, that are perpendicular, denoted by $k \perp m$.



Geometry Figure 3

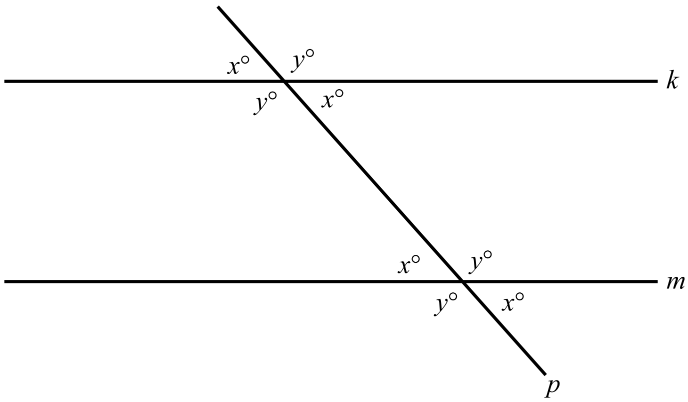
A common way to indicate that an angle is a right angle is to draw a small square at the vertex of the angle, as shown in Geometry Figure 4 below, where PON is a right angle.



Geometry Figure 4

An angle with measure less than 90° is called an **acute angle**, and an angle with measure between 90° and 180° is called an **obtuse angle**.

Two lines in the same plane that do not intersect are called **parallel lines**. Geometry Figure 5 below shows two lines, k and m, that are parallel, denoted by $k \parallel m$. The two lines are intersected by a third line, p, forming eight angles.

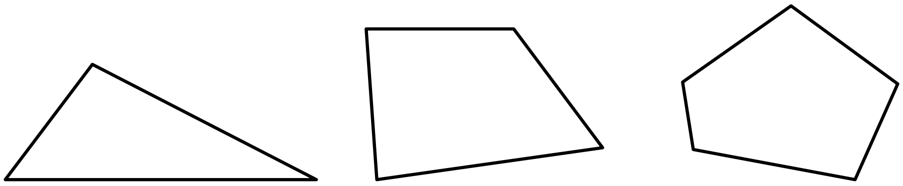


Geometry Figure 5

Note that four of the eight angles in Geometry Figure 5 have the measure $x°$, and the remaining four angles have the measure $y°$, where $x + y = 180$.

3.2 Polygons

A **polygon** is a closed figure formed by three or more line segments, all of which are in the same plane. The line segments are called the **sides** of the polygon. Each side is joined to two other sides at its endpoints, and the endpoints are called **vertices**. In this discussion, the term “polygon” means “convex polygon,” that is, a polygon in which the measure of each interior angle is less than 180°. Geometry Figure 6 below contains examples of a triangle, a quadrilateral, and a pentagon, all of which are convex.



Triangle (3 sides) Quadrilateral (4 sides) Pentagon (5 sides)

Geometry Figure 6

The simplest polygon is a **triangle**. Note that a **quadrilateral** can be divided into 2 triangles by drawing a diagonal; and a **pentagon** can be divided into 3 triangles by selecting one of the vertices and drawing 2 line segments connecting the selected vertex to the two nonadjacent vertices, as shown in Geometry Figure 7 below.

Geometry Figure 7

If a polygon has n sides, it can be divided into $n - 2$ triangles. Since the sum of the measures of the interior angles of a triangle is 180°, it follows that the sum of the measures of the interior angles of an n-sided polygon is $(n - 2)(180°)$. For example, since a quadrilateral has 4 sides, the sum of the measures of the interior angles of a quadrilateral is $(4 - 2)(180°) = 360°$, and since a **hexagon** has 6 sides, the sum of the measures of the interior angles of a hexagon is $(6 - 2)(180°) = 720°$.

A polygon in which all sides are congruent and all interior angles are congruent is called a **regular polygon**. For example, since an **octagon** has 8 sides, the sum of the measures of the interior angles of an octagon is $(8 - 2)(180°) = 1{,}080°$. Therefore, in a **regular octagon** the measure of each angle is $\frac{1{,}080°}{8} = 135°$.

The **perimeter** of a polygon is the sum of the lengths of its sides. The **area** of a polygon refers to the area of the region enclosed by the polygon.

In the next two sections, we will look at some basic properties of triangles and quadrilaterals.

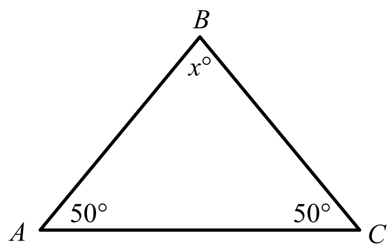
3.3 Triangles

Every triangle has three sides and three interior angles. The measures of the interior angles add up to 180°. The length of each side must be less than the sum of the lengths of the other two sides. For example, the sides of a triangle could not have the lengths 4, 7, and 12 because 12 is greater than 4 + 7.

The following are 3 types of special triangles.

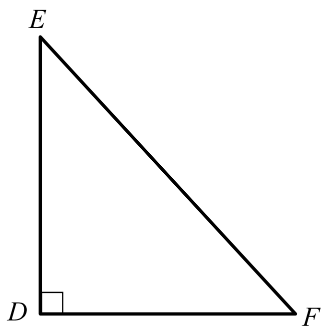
Type 1: A triangle with three congruent sides is called an **equilateral triangle**. The measures of the three interior angles of such a triangle are equal, and each measure is 60°.

Type 2: A triangle with at least two congruent sides is called an **isosceles triangle**. If a triangle has two congruent sides, then the angles opposite the two congruent sides are congruent. The converse is also true. For example, in triangle ABC in Geometry Figure 8 below, the measure of angle A is 50°, the measure of angle C is 50°, and the measure of angle B is $x°$. Since both angle A and angle C measure 50°, it follows that the length of AB is equal to the length of BC. Also, since the sum of the three angles of a triangle is 180°, it follows that $50 + 50 + x = 180$, and the measure of angle B is 80°.



Geometry Figure 8

Type 3: A triangle with an interior right angle is called a **right triangle**. The side opposite the right angle is called the **hypotenuse**; the other two sides are called **legs**. For example, in right triangle DEF in Geometry Figure 9 below, side EF is the side opposite right angle D; therefore EF is the hypotenuse and DE and DF are legs.



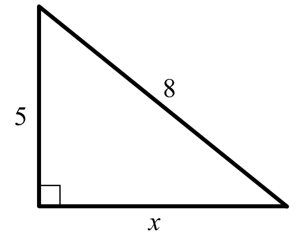
Geometry Figure 9

The Pythagorean Theorem

The **Pythagorean theorem** states that in a right triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the legs. Thus, for triangle *DEF* in Geometry Figure 9,

$$(EF)^2 = (DE)^2 + (DF)^2$$

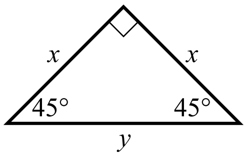
This relationship can be used to find the length of one side of a right triangle if the lengths of the other two sides are known. For example, consider a right triangle with hypotenuse of length 8, a leg of length 5, and another leg of unknown length x, as shown in Geometry Figure 10 below.



Geometry Figure 10

By the Pythagorean theorem, $8^2 = 5^2 + x^2$. Therefore, $64 = 25 + x^2$ and $39 = x^2$. Since $x^2 = 39$ and x must be positive, it follows that $x = \sqrt{39}$, or approximately 6.2.

The Pythagorean theorem can be used to determine the ratios of the lengths of the sides of two special right triangles. One special right triangle is an isosceles right triangle, as shown in Geometry Figure 11 below.



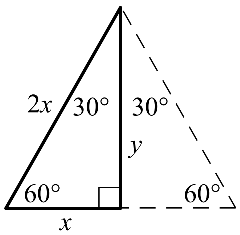
Geometry Figure 11

In Geometry Figure 11, the hypotenuse of the right triangle is of length y, both legs are of length x, and the angles opposite the legs are both 45-degree angles.

Applying the Pythagorean theorem to the isosceles right triangle in Geometry Figure 11 above shows that the lengths of its sides are in the ratio 1 to 1 to $\sqrt{2}$, as follows.

By the Pythagorean theorem, $y^2 = x^2 + x^2$. Therefore $y^2 = 2x^2$ and $y = \sqrt{2}x$. So the lengths of the sides are in the ratio x to x to $\sqrt{2}x$, which is the same as the ratio 1 to 1 to $\sqrt{2}$.

The other special right triangle is a 30°-60°-90° right triangle, which is half of an equilateral triangle, as shown in Geometry Figure 12 below.



Geometry Figure 12

Note that the length of the horizontal side, x, is one-half the length of the hypotenuse, $2x$. Applying the Pythagorean theorem to the 30°-60°-90° right triangle shows that the lengths of its sides are in the ratio 1 to $\sqrt{3}$ to 2, as follows.

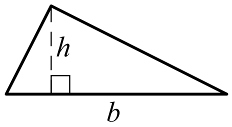
By the Pythagorean theorem, $x^2 + y^2 = (2x)^2$, which simplifies to $x^2 + y^2 = 4x^2$. Subtracting x^2 from both sides gives $y^2 = 4x^2 - x^2$ or $y^2 = 3x^2$. Therefore, $y = \sqrt{3}x$. Hence, the ratio of the lengths of the three sides of a 30°-60°-90° right triangle is x to $\sqrt{3}x$ to $2x$, which is the same as the ratio 1 to $\sqrt{3}$ to 2.

The Area of a Triangle

The **area** A of a triangle is given by the formula

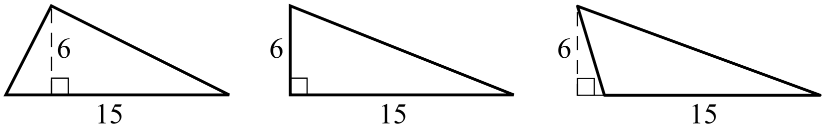
$$A = \frac{bh}{2}$$

where b is the length of a base, and h is the length of the corresponding height. Geometry Figure 13 below shows a triangle: the length of the horizontal base of the triangle is denoted by b and the length of the corresponding vertical height is denoted by h.



Geometry Figure 13

Any side of a triangle can be used as a base; the height that corresponds to the base is the perpendicular line segment from the opposite vertex to the base (or an extension of the base). Depending on the context, the term "base" can also refer to the *length* of a side of the triangle, and the term "height" can refer to the *length* of the perpendicular line segment from the opposite vertex to that side. The examples in Geometry Figure 14 below show three different configurations of a base and the corresponding height.



Geometry Figure 14

In all three triangles in Geometry Figure 14 above, the area is $\frac{(15)(6)}{2}$, or 45.

Congruent Triangles and Similar Triangles

Two triangles that have the same shape and size are called **congruent triangles**. More precisely, two triangles are congruent if their vertices can be matched up so that the corresponding angles and the corresponding sides are congruent.

By convention, the statement "triangles PQR and STU are congruent" does not just tell you that the two triangles are congruent, it also tells you what the corresponding parts of the two triangles are. In particular, because the letters in the name of the first triangle are given in the order PQR, and the letters in the name of the second triangle are given in the order STU, the statement tells you that angle P is congruent to angle S, angle Q is congruent to angle T, and angle R is congruent to angle U. It also tells you that sides PQ, QR, and PR in triangle PQR are congruent to sides ST, TU, and SU in triangle STU, respectively.

The following three propositions can be used to determine whether two triangles are congruent by comparing only some of their sides and angles.

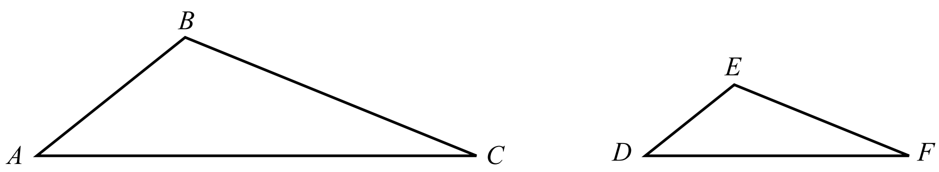
Proposition 1: If the three sides of one triangle are congruent to the three sides of another triangle, then the triangles are congruent. This proposition is called Side-Side-Side, or SSS, congruence.

Proposition 2: If two sides and the included angle of one triangle are congruent to two sides and the included angle of another triangle, then the triangles are congruent. This proposition is called Side-Angle-Side, or SAS, congruence.

Proposition 3: If two angles and the included side of one triangle are congruent to two angles and the included side of another triangle, then the triangles are congruent. This proposition is called Angle-Side-Angle, or ASA, congruence. Note that if two angles of one triangle are congruent to two angles of another triangle, then the remaining angles are also congruent to each other, since the sum of the angle measures in any triangle is 180 degrees. Therefore, a similar proposition, called Angle-Angle-Side, or AAS, congruence, follows from ASA congruence.

Two triangles that have the same shape but not necessarily the same size are called **similar triangles**. More precisely, two triangles are similar if their vertices can be matched up so that the corresponding angles are congruent or, equivalently, the lengths of the corresponding sides have the same ratio, called the **scale factor of similarity**. For example, all 30°-60°-90° right triangles are similar triangles, though they may differ in size.

Geometry Figure 15 below shows two similar triangles, triangle ABC and triangle DEF.



Geometry Figure 15

As with the convention for congruent triangles, the letters in similar triangles ABC and DEF indicate their corresponding parts.

Since triangles ABC and DEF are similar, we have $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$. By cross multiplication, we can obtain other proportions, such as $\frac{AB}{BC} = \frac{DE}{EF}$.

3.4 Quadrilaterals

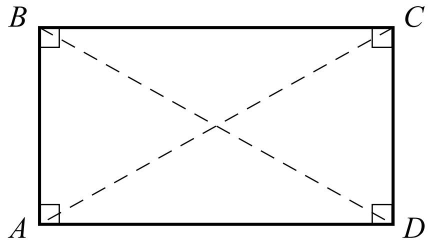
Every quadrilateral has four sides and four interior angles. The measures of the interior angles add up to 360°.

Special Types of Quadrilaterals

The following are four special types of quadrilaterals.

Type 1: A quadrilateral with four right angles is called a **rectangle**. Opposite sides of a rectangle are parallel and congruent, and the two diagonals are also congruent.

Geometry Figure 16 below shows rectangle *ABCD*. In rectangle *ABCD*, opposite sides *AD* and *BC* are parallel and congruent, opposite sides *AB* and *DC* are parallel and congruent, and diagonal *AC* is congruent to diagonal *BD*.



Geometry Figure 16

Type 2: A rectangle with four congruent sides is called a **square**.

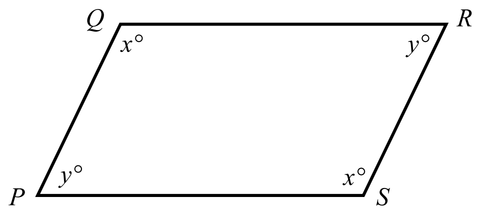
Type 3: A quadrilateral in which both pairs of opposite sides are parallel is called a **parallelogram**. In a parallelogram, opposite sides are congruent and opposite angles are congruent.

Note that all rectangles are parallelograms.

Geometry Figure 17 below shows parallelogram *PQRS*. In parallelogram *PQRS*:

Opposite sides *PQ* and *SR* are parallel and congruent.
Opposite sides *QR* and *PS* are parallel and congruent.
Opposite angles *Q* and *S* are congruent.
Opposite angles *P* and *R* are congruent.

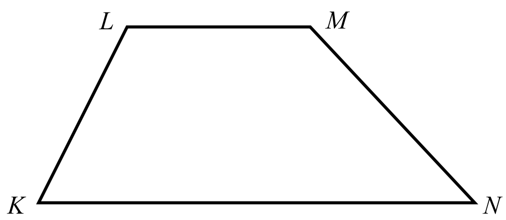
In the figure, angles *Q* and *S* are both labeled $x°$, and angles *P* and *R* are both labeled $y°$.



Geometry Figure 17

Type 4: A quadrilateral in which at least one pair of opposite sides is parallel is called a **trapezoid**. Two opposite, parallel sides of the trapezoid are called **bases** of the trapezoid.

Geometry Figure 18 below shows trapezoid *KLMN*. In trapezoid *KLMN*, horizontal side *KN* is parallel to horizontal side *LM*. Sides *KN* and *LM* are the bases of the trapezoid.



Geometry Figure 18

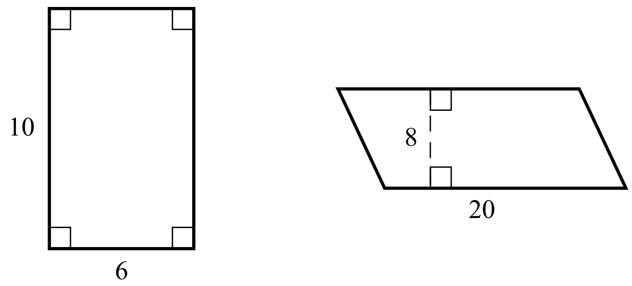
The Areas of the Special Types of Quadrilaterals

For all parallelograms, including rectangles and squares, the **area** A is given by the formula

$$A = bh$$

where b is the length of a base and h is the length of the corresponding height.

Any side of a parallelogram can be used as a base. The height corresponding to the base is the perpendicular line segment from any point on the side opposite the base to the base (or an extension of that base). Depending on the context, the term "base" can also refer to the *length* of a side of the parallelogram, and the term "height" can refer to the *length* of the perpendicular line segment from that side to the opposite side. Examples of finding the areas of a rectangle and a parallelogram are shown in Geometry Figure 19 below.



$A = (6)(10) = 60$ $\qquad$ $A = (20)(8) = 160$

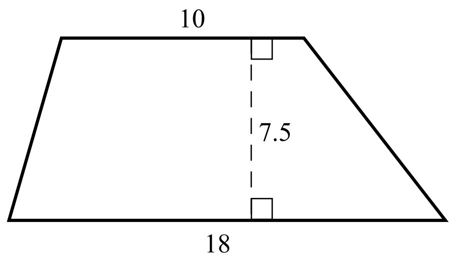
Geometry Figure 19

The **area** A of a trapezoid is given by the formula

$$A = \frac{1}{2}(b_1 + b_2)(h)$$

where b_1 and b_2 are the lengths of the bases of the trapezoid, and h is the corresponding height. For example, for the trapezoid in Geometry Figure 20 below with bases of length 10 and 18 and a height of 7.5, the area is

$$\frac{1}{2}(10 + 18)(7.5) = 105$$

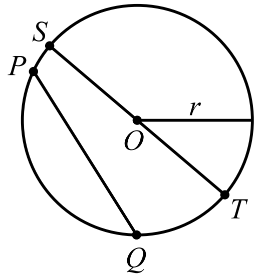


Geometry Figure 20

3.5 Circles

Given a point O in a plane and a positive number r, the set of points in the plane that are a distance of r units from O is called a **circle**. The point O is called the **center** of the circle and the distance r is called the **radius** of the circle. The **diameter** of the circle is twice the radius. Two circles with equal radii are called **congruent circles**.

Any line segment joining two points on the circle is called a **chord**. The terms "radius" and "diameter" can also refer to line segments: A **radius** is any line segment joining a point on the circle and the center of the circle, and a **diameter** is a chord that passes through the center of the circle. In Geometry Figure 21 below, O is the center of the circle, r is length of a radius, PQ is a chord, and ST is a diameter, as well as a chord.



Geometry Figure 21

The distance around a circle is called the **circumference** of the circle, which is analogous to the perimeter of a polygon. The ratio of the circumference C to the diameter d is the same for all circles and is denoted by the Greek letter π; that is,

$$\frac{C}{d} = \pi$$

The value of π is approximately 3.14 and can also be approximated by the fraction $\frac{22}{7}$.

If r is the radius of a circle, then $\frac{C}{d} = \frac{C}{2r} = \pi$, and so the circumference is related to the radius by the equation

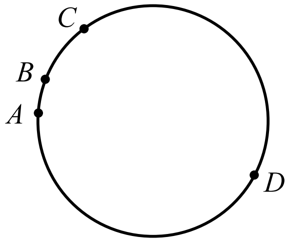
$$C = 2\pi r$$

For example, if a circle has a radius of 5.2, then its circumference is

$$(2)(\pi)(5.2) = (10.4)(\pi) \approx (10.4)(3.14)$$

which is approximately 32.7.

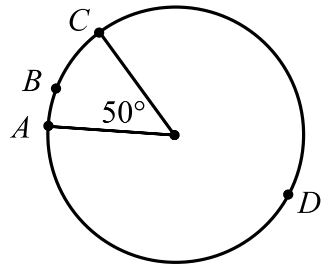
Given any two points on a circle, an **arc** is the part of the circle containing the two points and all the points between them. Two points on a circle are always the endpoints of two arcs. An arc is frequently identified by three points to avoid ambiguity. In Geometry Figure 22 below, there are four points on a circle. Going clockwise around the circle, the four points are A, B, C, and D. There are two different arcs between points A and C: arc ABC is the shorter arc between A and C, and arc ADC is the longer arc between A and C.



Geometry Figure 22

A **central angle** of a circle is an angle with its vertex at the center of the circle. The **measure of an arc** is the measure of its central angle, which is the angle formed by two radii that connect the center of the circle to the two endpoints of the arc. An entire circle is considered to be an arc with measure 360°.

In Geometry Figure 23 below, there are four points on a circle: points *A*, *B*, *C*, and *D*. It is given that the radius of the circle is 5.



Geometry Figure 23

In Geometry Figure 23, the measure of the shorter arc between points *A* and *C*, that is arc *ABC*, is 50°; and the measure of the longer arc, between points *A* and *C*, that is arc *ADC*, is 310°.

To find the **length of an arc** of a circle, note that the ratio of the length of an arc to the circumference is equal to the ratio of the degree measure of the arc to 360°. For example, since the radius of the circle in Geometry Figure 23 is 5, the circumference of the circle is 10π. Therefore,

$$\frac{\text{length of arc } ABC}{10\pi} = \frac{50}{360}$$

Multiplying both sides by 10π gives

$$\text{length of arc } ABC = \frac{50}{360}(10\pi)$$

Then, since

$$\frac{50}{360}(10\pi) = \frac{25\pi}{18} \approx \frac{(25)(3.14)}{18} \approx 4.4$$

it follows that the length of arc *ABC* is approximately 4.4.

The **area** of a circle with radius r is equal to πr^2. For example, since the radius of the circle in Geometry Figure 23 above is 5, the area of the circle is $\pi(5^2) = 25\pi$.

A **sector** of a circle is a region bounded by an arc of the circle and two radii. To find the **area of a sector**, note that the ratio of the area of a sector of a circle to the area of the entire circle is equal to the ratio of the degree measure of its arc to 360°. For example, in the circle in Geometry Figure 23 the region bounded by arc *ABC* and the two radii is a sector with central angle 50°, and the radius of the circle is 5. Therefore, if S represents the area of the sector with central angle 50°, then

$$\frac{S}{25\pi} = \frac{50}{360}$$

Multiplying both sides by 25π gives

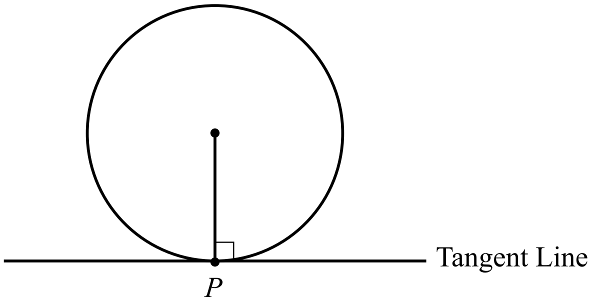
$$S = \left(\frac{50}{360}\right)(25\pi)$$

Then, since

$$\left(\frac{50}{360}\right)(25\pi) = \frac{125\pi}{36} \approx \frac{(125)(3.14)}{36} \approx 10.9$$

it follows that the area of the sector with central angle 50° is approximately 10.9.

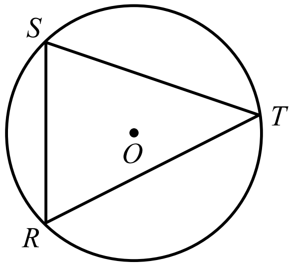
A **tangent** to a circle is a line that lies in the same plane as the circle and intersects the circle at exactly one point, called the **point of tangency**. If a line is tangent to a circle, then a radius drawn to the point of tangency is perpendicular to the tangent line. The converse is also true; that is, if a radius and a line intersect at a point on the circle and the line is perpendicular to the radius, then the line is a tangent to the circle at the point of intersection. Geometry Figure 24 below shows a circle, a line tangent to the circle at point P, and a radius drawn to point P.



Geometry Figure 24

A polygon is **inscribed** in a circle if all its vertices lie on the circle, or equivalently, the circle is **circumscribed** about the polygon.

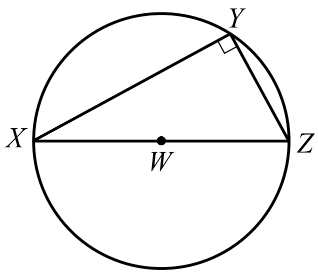
Geometry Figure 25 below shows triangle RST inscribed in a circle with center O. The center of the circle is inside the triangle.



Geometry Figure 25

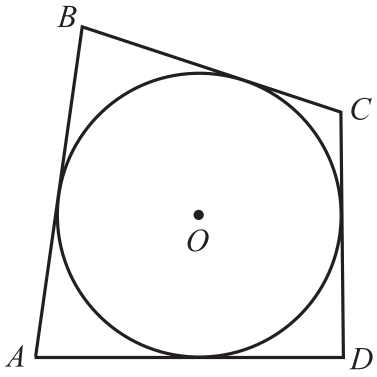
It is not always the case that if a triangle is inscribed in a circle, the center of the circle is inside the inscribed triangle. It is also possible for the center of the circle to be outside the inscribed triangle, or on one of the sides of the inscribed triangle. Note that if the center of the circle is on one of the sides of the inscribed triangle, that side is a diameter of the circle.

If one side of an inscribed triangle is a diameter of the circle, then the triangle is a right triangle. Conversely, if an inscribed triangle is a right triangle, then one of its sides is a diameter of the circle. Geometry Figure 26 below shows right triangle XYZ inscribed in a circle with center W. In triangle XYZ, side XZ is a diameter of the circle and angle Y is a right angle.



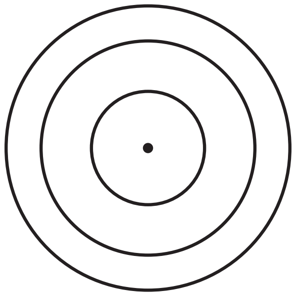
Geometry Figure 26

A polygon is circumscribed about a circle if each side of the polygon is tangent to the circle, or equivalently, the circle is inscribed in the polygon. Geometry Figure 27 below shows quadrilateral $ABCD$ circumscribed about a circle with center O.



Geometry Figure 27

Two or more circles with the same center are called **concentric circles**, as shown in Geometry Figure 28 below.

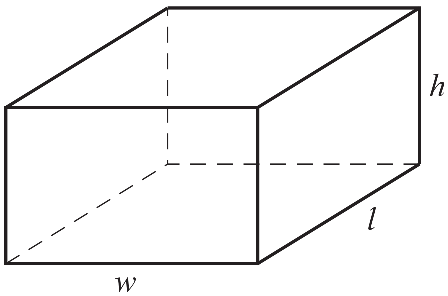


Geometry Figure 28

3.6 Three-Dimensional Figures

Basic three-dimensional figures include rectangular solids, cubes, cylinders, spheres, pyramids, and cones. In this section, we look at some properties of rectangular solids and right circular cylinders.

A **rectangular solid**, or **right rectangular prism**, has 6 rectangular surfaces called **faces**, as shown in Geometry Figure 29 below. Adjacent faces are perpendicular to each other. Each line segment that is the intersection of two faces is called an **edge**, and each point at which the edges intersect is called a **vertex**. There are 12 edges and 8 vertices. The dimensions of a rectangular solid are the length l, the width w, and the height h.



Geometry Figure 29

A rectangular solid with six square faces is called a **cube**, in which case $l = w = h$.

The **volume** V of a rectangular solid is the product of its three dimensions, or

$$V = lwh$$

The **surface area** A of a rectangular solid is the sum of the areas of the six faces, or

$$A = 2(lw + lh + wh)$$

For example, if a rectangular solid has length 8.5, width 5, and height 10, then its volume is

$$V = (8.5)(5)(10) = 425$$

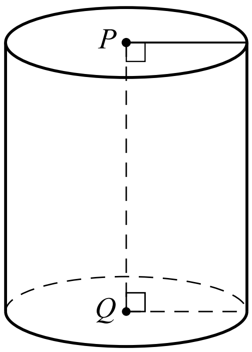
and its surface area is

$$A = 2((8.5)(5) + (8.5)(10) + (5)(10)) = 355$$

A **circular cylinder** consists of two bases that are congruent circles lying in parallel planes and a **lateral surface** made of all line segments that join points on the two circles and that are parallel to the line segment joining the centers of the two circles. The latter line segment is called the **axis** of the cylinder.

A **right circular cylinder** is a circular cylinder whose axis is perpendicular to its bases. The **height** of a right circular cylinder is the perpendicular distance between the two bases. Because the axis of a right circular cylinder is perpendicular to both bases, the height of a right circular cylinder is equal to the length of the axis.

The right circular cylinder shown in Geometry Figure 30 below has circular bases with centers P and Q. Line segment PQ is the axis of the cylinder and is perpendicular to both bases. The height of the cylinder is equal to the length of PQ.



Geometry Figure 30

The **volume** V of a right circular cylinder that has height h and a base with radius r is the product of the height and the area of the base, or

$$V = \pi r^2 h$$

The **surface area** A of a right circular cylinder is the sum of the areas of the two bases and the area of its lateral surface, or

$$A = 2(\pi r^2) + 2\pi rh$$

For example, if a right circular cylinder has height 6.5 and a base with radius 3, then its volume is

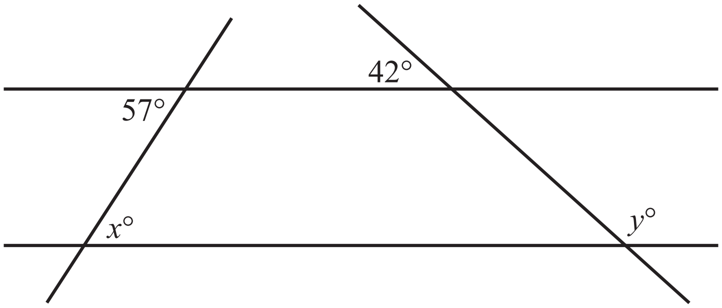
$$V = \pi(3^2)(6.5) = 58.5\pi$$

and its surface area is

$$A = 2\pi(3^2) + 2\pi(3)(6.5) = 57\pi$$

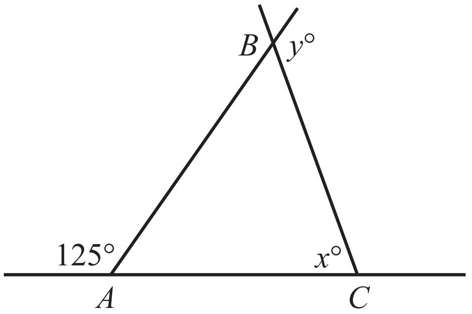
GEOMETRY EXERCISES

Exercise 1. In Geometry Figure 31 below, the two horizontal lines are parallel. Find the values of x and y.



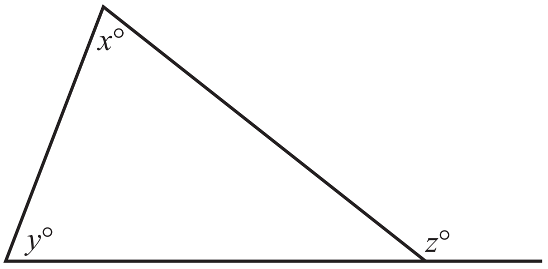
Geometry Figure 31

Exercise 2. In Geometry Figure 32 below, $AC = BC$. Find the values of x and y.



Geometry Figure 32

Exercise 3. In Geometry Figure 33 below, what is the relationship between x, y, and z ?



Geometry Figure 33

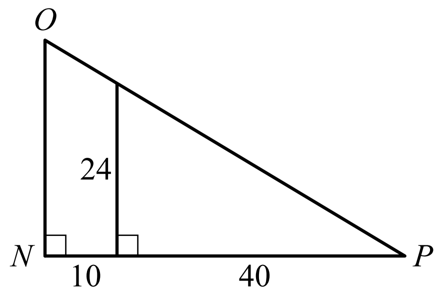
Exercise 4. What is the sum of the measures of the interior angles of a decagon (10-sided polygon) ?

Exercise 5. If the polygon in exercise 4 is regular, what is the measure of each interior angle?

Exercise 6. The lengths of two sides of an isosceles triangle are 15 and 22. What are the possible values of the perimeter?

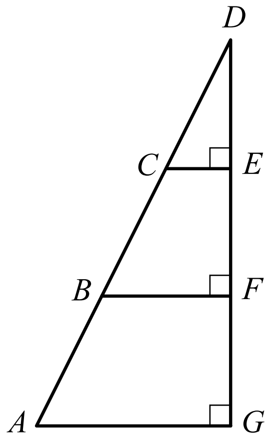
Exercise 7. Triangles PQR and XYZ are similar. If $PQ = 6$, $PR = 4$, and $XY = 9$, what is the length of side XZ?

Exercise 8. What are the lengths of sides *NO* and *OP* of triangle *NOP* in Geometry Figure 34 below?



Geometry Figure 34

Exercise 9. In Geometry Figure 35 below, $AB = BC = CD$. If the area of triangle *CDE* is 42, what is the area of triangle *ADG*?



Geometry Figure 35

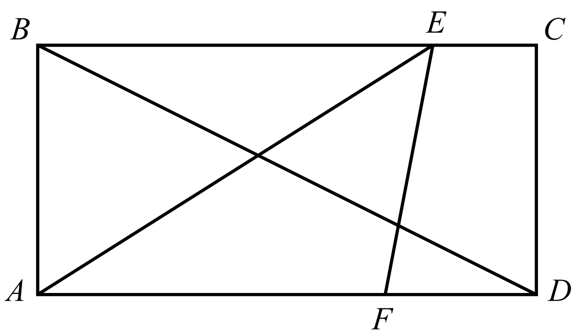
Exercise 10. In Geometry Figure 36 below, *ABCD* is a rectangle, $AB = 5$, $AF = 7$, and $FD = 3$. Find the following.

(a) The area of rectangle *ABCD*

(b) The area of triangle *AEF*

(c) The length of diagonal *BD*

(d) The perimeter of rectangle *ABCD*



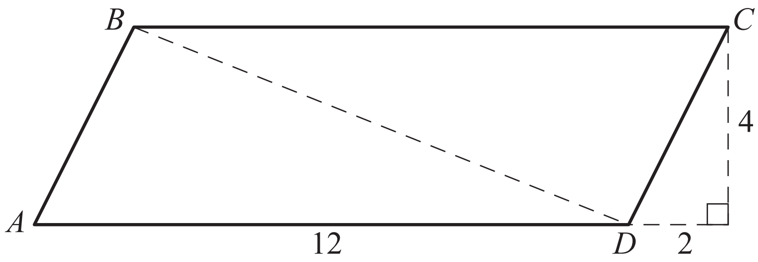
Geometry Figure 36

Exercise 11. In Geometry Figure 37 below, *ABCD* is a parallelogram. Find the following.

(a) The area of *ABCD*

(b) The perimeter of *ABCD*

(c) The length of diagonal *BD*



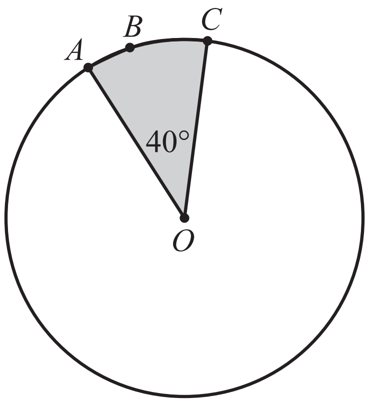
Geometry Figure 37

Exercise 12. In Geometry Figure 38 below, the circle with center *O* has radius 4. Find the following.

(a) The circumference of the circle

(b) The length of arc *ABC*

(c) The area of the shaded region



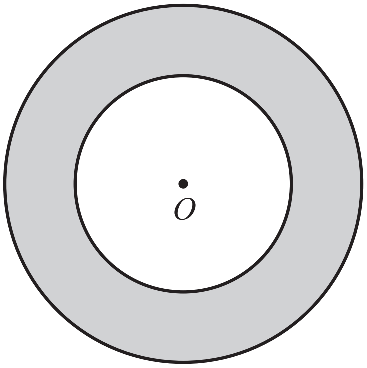
Geometry Figure 38

Exercise 13. Geometry Figure 39 below shows two concentric circles, each with center O. Given that the larger circle has radius 12 and the smaller circle has radius 7, find the following.

(a) The circumference of the larger circle

(b) The area of the smaller circle

(c) The area of the shaded region

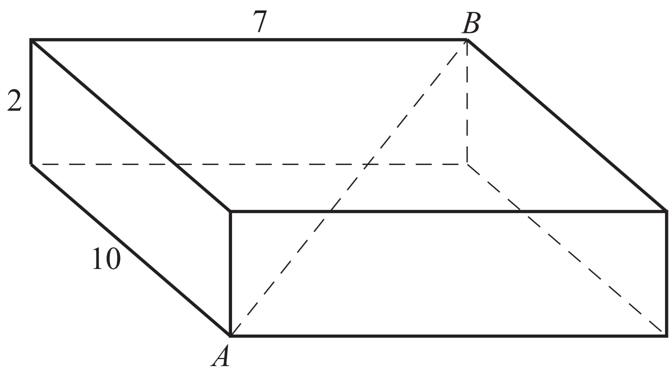


Geometry Figure 39

Exercise 14. For the rectangular solid in Geometry Figure 40 below, find the following.

(a) The surface area of the solid

(b) The length of diagonal AB



Geometry Figure 40

ANSWERS TO GEOMETRY EXERCISES

Exercise 1. $x = 57$ and $y = 138$

Exercise 2. $x = 70$ and $y = 125$

Exercise 3. $z = x + y$

Exercise 4. 1,440°

Exercise 5. 144°

Exercise 6. 52 and 59

Exercise 7. 6

Exercise 8. The length of side NO is 30 and the length of side OP is $10\sqrt{34}$.

Exercise 9. 378

Exercise 10. (a) 50

(b) 17.5

(c) $5\sqrt{5}$

(d) 30

Exercise 11. (a) 48

(b) $24 + 4\sqrt{5}$

(c) $2\sqrt{29}$

Exercise 12. (a) 8π

(b) $\frac{8\pi}{9}$

(c) $\frac{16\pi}{9}$

Exercise 13. (a) 24π

(b) 49π

(c) 95π

Exercise 14. (a) 208

(b) $3\sqrt{17}$

PART 4. DATA ANALYSIS

The review of data analysis begins with methods for presenting data, followed by counting methods and probability, and then progresses to distributions of data, random variables, and probability distributions. The review of data analysis ends with examples of data interpretation.

4.1 Methods for Presenting Data

Data can be organized and presented using a variety of methods. Tables are commonly used, and there are many graphical and numerical methods as well. In this section, we review tables and some common graphical methods for presenting and summarizing data.

In data analysis, a variable is any characteristic that can vary for a population of individuals or objects. Variables can be **quantitative**, or **numerical**, such as the age of individuals. Variables can also be **categorical**, or **nonnumerical**, such as the eye color of individuals.

Data are collected from a population by observing one or more variables. The **distribution of a variable**, or **distribution of data**, indicates how frequently different categorical or numerical data values are observed in the data.

Example 4.1.1: In a population of students in a sixth-grade classroom, a variable that can be observed is the height of each student. Note that the variable in this example is numerical.

Example 4.1.2: In a population of voters in a city's mayoral election, a variable that can be observed is the candidate that each voter voted for. Note that the variable in this example is nonnumerical.

The **frequency**, or **count**, of a particular category or numerical value is the number of times that the category or numerical value appears in the data. A **frequency distribution** is a table or graph that presents the categories or numerical values along with their corresponding frequencies. The **relative frequency** of a category or a numerical value is the corresponding frequency divided by the total number of data. Relative frequencies may be expressed in terms of percents, fractions, or decimals. A **relative frequency distribution** is a table or graph that presents the relative frequencies of the categories or numerical values.

Tables

Tables are used to present a wide variety of data, including frequency distributions and relative frequency distributions. The rows and columns provide clear associations between categories and data. A frequency distribution is often presented as a 2-column table in which the categories or numerical values of the data are listed in the first column and the corresponding frequencies are listed in the second column. A relative frequency distribution table has the same layout but with relative frequencies instead of frequencies. When data include a large number of categories or numerical values, the categories or values are often grouped together in a smaller number of groups and the corresponding frequencies are given.

Example 4.1.3: A survey was taken to find the number of children in each of 25 families. A list of the 25 values collected in the survey follows.

1 2 0 4 1 3 3 1 2 0 4 5 2 3 2 3 2 4 1 2 3 0 2 3 1

Here are tables that present the resulting frequency distribution and relative frequency distribution of the data.

Frequency Distribution

Number of Children	Frequency
0	3
1	5
2	7
3	6
4	3
5	1
Total	25

Relative Frequency Distribution

Number of Children	Relative Frequency
0	12%
1	20%
2	28%
3	24%
4	12%
5	4%
Total	100%

Note that in the relative frequency distribution table the relative frequencies are expressed as percents and that the total for the relative frequencies is 100%. If the relative frequencies were expressed as decimals or fractions instead of percents, the total would be 1.

Example 4.1.4: Thirty students took a history test. Here is a list of the 30 scores on the test, from least to greatest.

62 63 68 70 72 72 72 75 76 76 76 76 78 78 82
82 85 85 85 85 85 86 87 88 91 91 92 95 97 100

The 30 students achieved 18 different scores on the test. Displaying the frequency distribution of this many different scores would make the frequency distribution table very large, so instead we group the scores into four groups: the scores from 61 to 70, the scores from 71 to 80, the scores from 81 to 90, and the scores from 91 to 100. Here is the frequency distribution of the scores with these groups.

Score	Frequency
61 to 70	4
71 to 80	10
81 to 90	10
91 to 100	6

In addition to being used to present frequency and relative frequency distributions, tables are used to display a wide variety of other data. Here are two examples.

Example 4.1.5: The following table shows the annual per capita income in a certain state, from 1930 to 1980.

Year	Annual Per Capita Income
1930	\$656
1940	\$680
1950	\$1,717
1960	\$2,437
1970	\$4,198
1980	\$10,291

Example 4.1.6: The following table shows the closest and farthest distance of the eight planets from the Sun, in millions of kilometers.

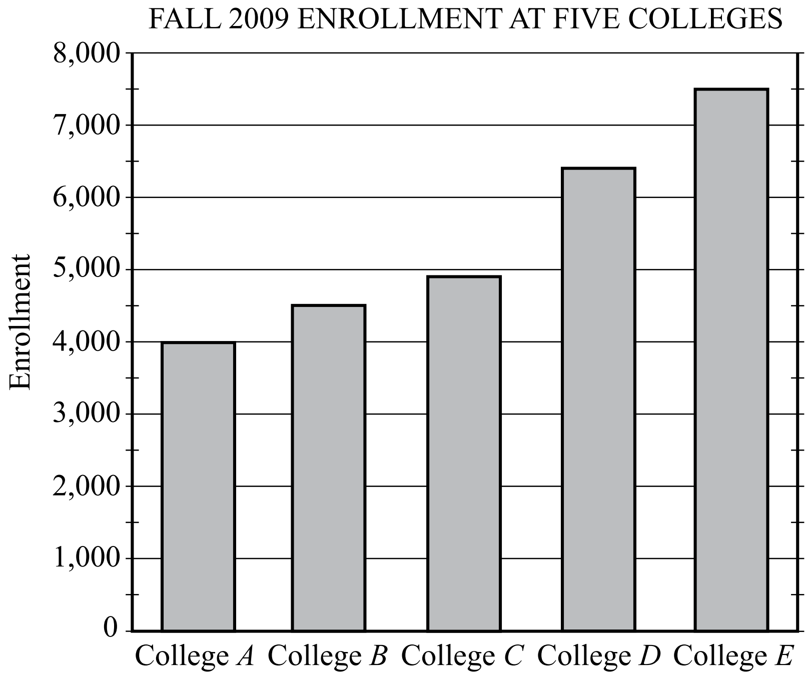
Planet	Closest Distance from the Sun (in millions of kilometers)	Farthest Distance from the Sun (in millions of kilometers)
Mercury	46	70
Venus	107	109
Earth	147	152
Mars	205	249
Jupiter	741	817
Saturn	1,350	1,510
Uranus	2,750	3,000
Neptune	4,450	4,550

Bar Graphs

A frequency distribution or relative frequency distribution of data collected from a population by observing one or more variables can be presented using a **bar graph**, or **bar chart**. In a bar graph, each of the data categories or numerical values is represented by a rectangular bar, and the height of each bar is proportional to the corresponding frequency or relative frequency. All of the bars are drawn with the same width, and the bars can be presented either vertically or horizontally. When data include a large number of different categories of numerical values, the categories or values are often grouped together in several groups and the corresponding frequencies or relative frequencies are given. Bar graphs enable comparisons across several categories more easily than tables do. For example, in a bar graph it is easy to identify the category with the greatest frequency by looking for the bar with the greatest height.

Here are two examples of frequency distributions presented as bar graphs.

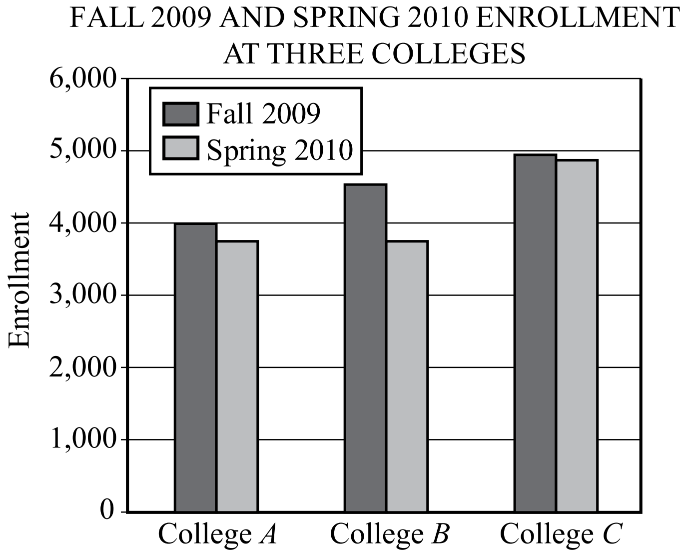
Example 4.1.7: Data Analysis Figure 1 is a bar graph with vertical bars. It shows the frequency distribution of one variable, fall 2009 enrollment. The variable is observed for five data categories, Colleges A, B, C, D, and E.



Data Analysis Figure 1

From the graph, we can conclude that the college with the greatest fall 2009 enrollment was College E and the college with the least enrollment was College A. Also, we can estimate that the enrollment for College D was about 6,400.

Example 4.1.8: Data Analysis Figure 2 is a bar graph with vertical bars. It shows the frequency distributions of two variables, fall 2009 enrollment and spring 2010 enrollment. Both variables are observed for three data categories, Colleges A, B, and C.



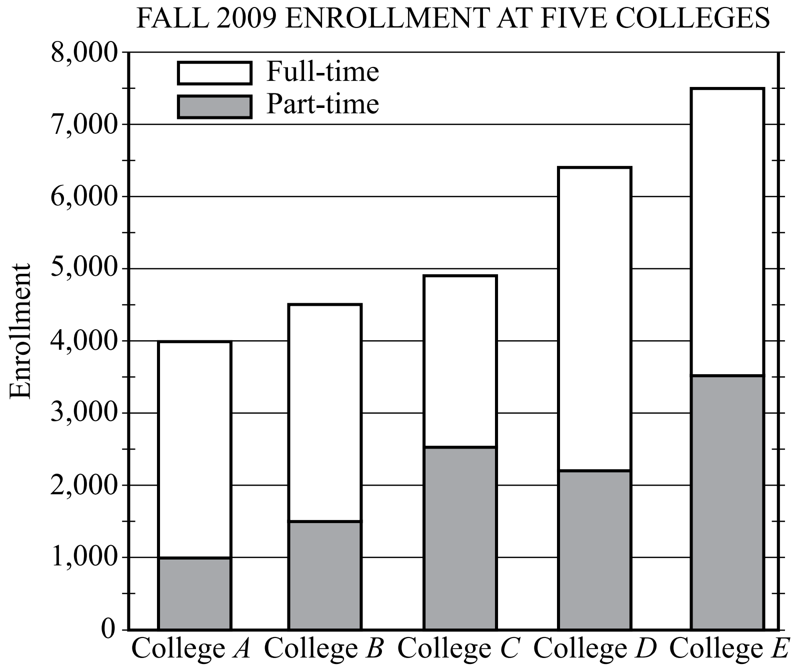
Data Analysis Figure 2

Observe that for all three colleges, the fall 2009 enrollment was greater than the spring 2010 enrollment. Also, the greatest decrease in the enrollment from fall 2009 to spring 2010 occurred at College B.

Segmented Bar Graphs

A **segmented bar graph**, or **stacked bar graph**, is similar to a regular bar graph except that in a segmented bar graph, each rectangular bar is divided, or segmented, into smaller rectangles that show how the variable is "separated" into other related variables. For example, rectangular bars representing enrollment can be divided into two smaller rectangles, one representing full-time enrollment and the other representing part-time enrollment, as shown in the following example.

Example 4.1.9: In Data Analysis Figure 3, the fall 2009 enrollment at the five colleges shown in Data Analysis Figure 1 is presented again, but this time each bar has been divided into two segments: one representing full-time enrollment and one representing part-time enrollment.



Data Analysis Figure 3

The total enrollment, the full-time enrollment, and the part-time enrollment at the five colleges can be estimated from the segmented bar graph in Data Analysis Figure 3. For example, for College D, the total enrollment was a little below 6,500, or approximately 6,400 students; the part-time enrollment was approximately 2,200; and the full-time enrollment was approximately 6,400 – 2,200, or 4,200 students.

Although bar graphs are commonly used to compare frequencies, as in the examples above, they are sometimes used to compare numerical data that could be displayed in a table, such as temperatures, dollar amounts, percents, heights, and weights. Also, the categories sometimes are numerical in nature, such as years or other time intervals.

Histograms

When a list of data is large and contains many different values of a numerical variable, it is useful to organize the data by grouping the values into intervals, often called classes. To do this, divide the entire interval of values into smaller intervals of equal length and then count the values that fall into each interval. In this way, each interval has a frequency and a relative frequency. The intervals and their frequencies (or relative frequencies) are often displayed in a **histogram**. Histograms are graphs of frequency distributions that are similar to bar graphs, but they must have a number line for the horizontal axis, which represents the numerical variable. Also, in a histogram, there are no regular spaces between the bars. Any spaces between bars in a histogram indicate that there are no data in the intervals represented by the spaces.

Example 4.5.1 in Section 4.5 illustrates a histogram with 50 bars.

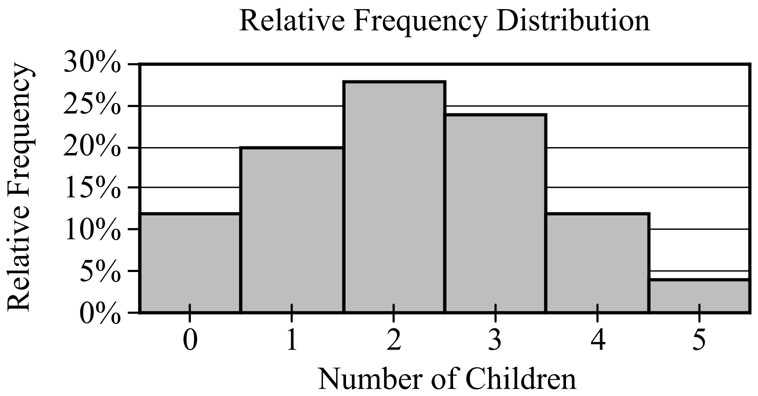
Numerical variables with just a few values can also be displayed using histograms, where the frequency or relative frequency of each value is represented by a bar centered over the value.

Example 4.1.10: In Example 4.1.3, the relative frequency distribution of the number of children of each of 25 families was presented as a 2-column table. For convenience, the table is repeated below.

Relative Frequency Distribution

Number of Children	Relative Frequency
0	12%
1	20%
2	28%
3	24%
4	12%
5	4%
Total	100%

This relative frequency distribution can also be displayed as a histogram, as shown in the following figure.



Data Analysis Figure 4

Histograms are useful for identifying the general shape of a distribution of data. Also evident are the "center" and degree of "spread" of the distribution, as well as high-frequency and low-frequency intervals. From the histogram in Data Analysis Figure 4 above, you can see that the distribution is shaped like a mound with one peak; that is, the data are frequent in the middle and sparse at both ends. The central values are 2 and 3, and the distribution is close to being symmetric about those values. Because the bars all have the same width, the area of each bar is proportional to the amount of data that the bar represents. Thus, the areas of the bars indicate where the data are concentrated and where they are not.

Finally, note that because each bar has a width of 1, the sum of the areas of the bars equals the sum of the relative frequencies, which is 100% or 1, depending on whether percents or decimals are used. This fact is central to the discussion of probability distributions in Section 4.5.

Circle Graphs

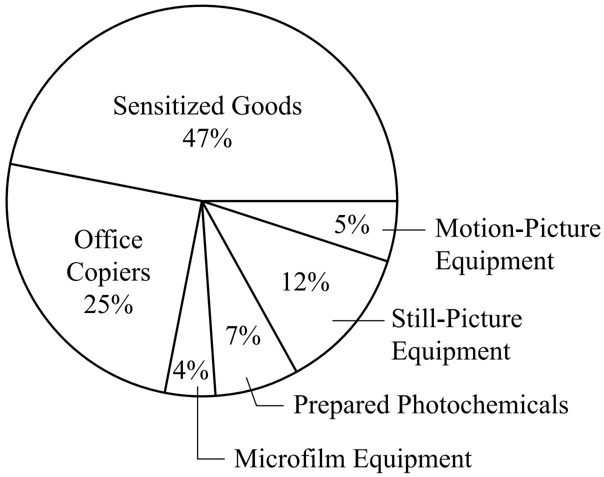
Circle graphs, often called **pie charts**, are used to represent data that have been separated into a small number of categories. They illustrate how a whole is separated into parts. The data are presented in a circle such that the area of the circle representing each category is proportional to the part of the whole that the category represents.

A circle graph may be used to represent a frequency distribution or a relative frequency distribution. More generally, a circle graph may represent any total amount that is distributed into a small number of categories, as in the following example.

Example 4.1.11:

UNITED STATES PRODUCTION OF PHOTOGRAPHIC EQUIPMENT AND SUPPLIES IN 1971

Total: $3,980 million



Data Analysis Figure 5

From the graph you can see that Sensitized Goods is the category with the greatest dollar value.

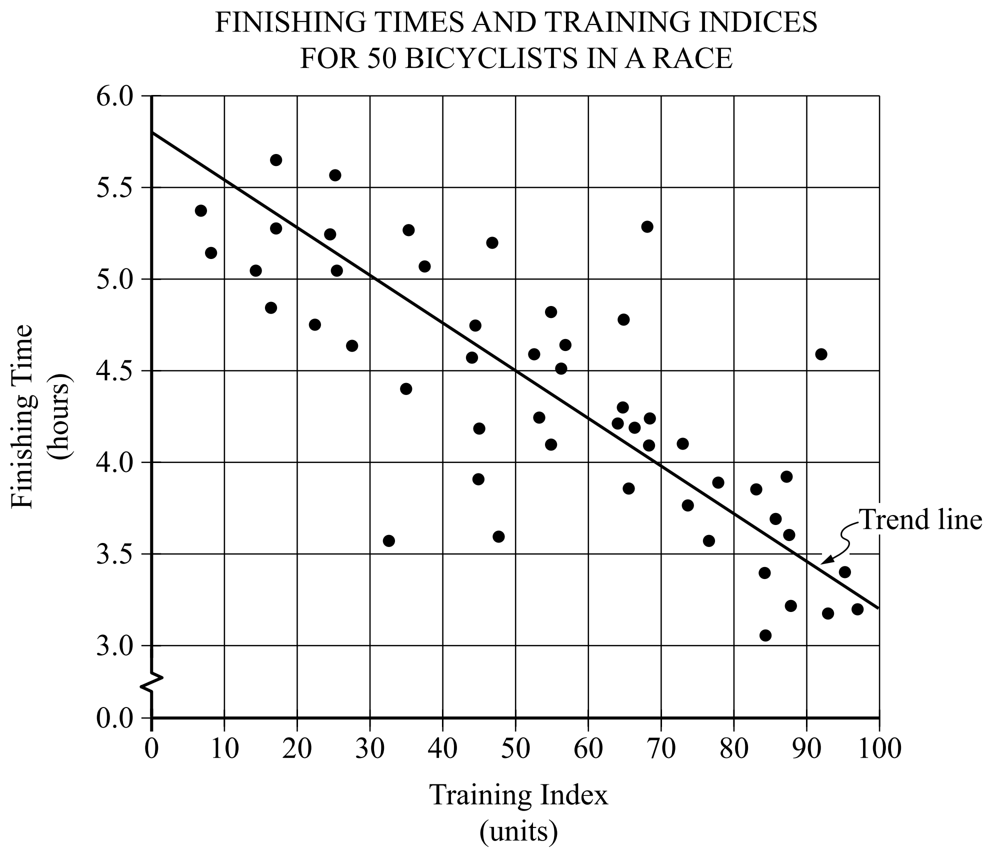
Each part of a circle graph is called a **sector**. Because the area of each sector is proportional to the percent of the whole that the sector represents, the measure of the central angle of a sector is proportional to the percent of 360 degrees that the sector represents. For example, the measure of the central angle of the sector representing the category Prepared Photochemicals is 7 percent of 360 degrees, or 25.2 degrees.

Scatterplots

A **scatterplot** is a type of graph that is useful for showing the relationship between two numerical variables whose values can be observed in a single population of individuals or objects. In a scatterplot, the values of one variable appear on the horizontal axis of a rectangular coordinate system and the values of the other variable appear on the vertical axis. For each individual or object in the data, an ordered pair of numbers is collected, one number for each variable, and the pair is represented by a point in the coordinate system.

A scatterplot makes it possible to observe an overall pattern, or **trend**, in the relationship between the two variables. Also, the strength of the trend as well as striking deviations from the trend are evident. In many cases, a line or a curve that best represents the trend is also displayed in the graph and is used to make predictions about the population.

Example 4.1.12: A bicycle trainer studied 50 bicyclists to examine how the finishing time for a certain bicycle race was related to the amount of physical training each bicyclist did in the three months before the race. To measure the amount of training, the trainer developed a training index, measured in "units" and based on the intensity of each bicyclist's training. The data and the trend of the data, represented by a line, are displayed in the scatterplot in Data Analysis Figure 6 below.



Data Analysis Figure 6

When a trend line is included in the presentation of a scatterplot, you can observe how scattered or close the data are to the trend line, or to put it another way, how well the trend line fits the data. In the scatterplot in Data Analysis Figure 6 above, almost all of the data points are relatively close to the trend line. The scatterplot also shows that the finishing times generally decrease as the training indices increase.

The trend line can be used to make predictions. For example, it can be predicted, based on the trend line, that a bicyclist with a training index of 70 units will finish the race in approximately 4 hours. This value is obtained by noting that the vertical line at the training index of 70 units intersects the trend line very close to 4 hours.

Another prediction that can be made, based on the trend line, is the approximate number of minutes by which a bicyclist will lower his or her finishing time for each increase of 10 training index units. This prediction is derived from the ratio of the change in finishing time to the change in training index, or the slope of the trend line. Note that the slope is negative. To estimate the slope, estimate the coordinates of any two points on the line, for instance, the points at the extreme left and right ends of the line: (0, 5.8) and (100, 3.2). The slope, which is measured in hours per unit, can be computed as follows:

$$\frac{3.2-5.8}{100-0}=\frac{-2.6}{100}=-0.026$$

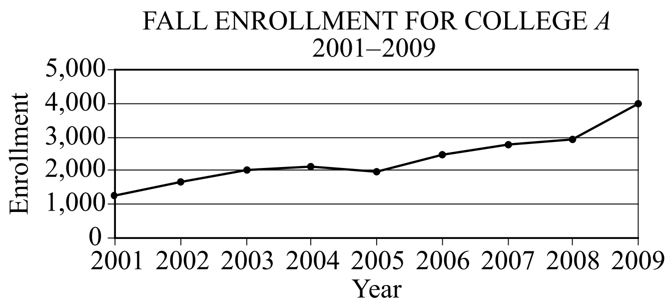
The slope can be interpreted as follows: The finishing time is predicted to decrease 0.026 hour for every unit by which the training index increases. Since we want to know how much the finishing time decreases for an increase of *10 units*, we multiply the rate by 10 to get 0.26 hour per 10 units. To compute the decrease in *minutes* per 10 units, we multiply 0.26 by 60 to get approximately 16 minutes. Based on the trend line, it can be predicted that the bicyclist will decrease his or her finishing time by approximately 16 minutes for every increase of 10 training index units.

Line Graphs

A **line graph** is another type of graph that is useful for showing the relationship between two numerical variables, especially if one of the variables is time. A line graph uses a coordinate plane, where each data point represents a pair of values observed for the two numerical variables. There is at most one data point for each value on the horizontal axis, similar to a function. The data points are in order from left to right, and consecutive data points are connected by a line segment.

When one of the variables is time, it is associated with the horizontal axis, which is labeled with regular time intervals. The data points may represent an interval of time, such as an entire day or year, or just an instant of time. Such a line graph is often called a **time series**.

Example 4.1.13:



Data Analysis Figure 7

The line graph shows that the greatest increase in fall enrollment between consecutive years was the change from 2008 to 2009. One way to determine this is by noting that the slope of the line segment joining the values for 2008 and 2009 is greater than the slopes of the line segments joining all other consecutive years. Another way to determine this is by noting that the increase in enrollment from 2008 to 2009 was greater than 1,000, but all other increases in enrollment were less than 1,000.

Although line graphs are commonly used to compare frequencies, as in Example 4.1.13 above, they can be used to compare any numerical data as the data change over time, such as temperatures, dollar amounts, percents, heights, and weights.

4.2 Numerical Methods for Describing Data

Data can be described numerically by various **statistics**, or **statistical measures**. These statistical measures are often grouped in three categories: measures of central tendency, measures of position, and measures of dispersion.

Measures of Central Tendency

Measures of **central tendency** indicate the “center” of the data along the number line and are usually reported as values that represent the data. There are three common measures of central tendency:

1. the **arithmetic mean**—usually called the **average** or simply the **mean**
2. the **median**
3. the **mode**

To calculate the **mean** of n numbers, take the sum of the n numbers and divide it by n.

Example 4.2.1: For the five numbers 6, 4, 7, 10, and 4, the mean is

$$\frac{6+4+7+10+4}{5}=\frac{31}{5}=6.2$$

When several values are repeated in a list, it is helpful to think of the mean of the numbers as a **weighted mean** of only those values in the list that are *different*.

Example 4.2.2: Consider the following list of 16 numbers.

2, 4, 4, 5, 7, 7, 7, 7, 7, 7, 8, 8, 9, 9, 9, 9

There are only 6 different values in the list: 2, 4, 5, 7, 8, and 9. The mean of the numbers in the list can be computed as

$$\frac{1(2)+2(4)+1(5)+6(7)+2(8)+4(9)}{1+2+1+6+2+4}=\frac{109}{16}=6.8125$$

The number of times a value appears in the list, or the frequency, is called the **weight** of that value. So the mean of the 16 numbers is the weighted mean of the values 2, 4, 5, 7, 8, and 9, where the respective weights are 1, 2, 1, 6, 2, and 4. Note that the sum of the weights is the number of numbers in the list, 16.

The mean can be affected by just a few values that lie far above or below the rest of the data, because these values contribute directly to the sum of the data and therefore to the mean. By contrast, the **median** is a measure of central tendency that is fairly unaffected by unusually high or low values relative to the rest of the data.

To calculate the median of n numbers, first order the numbers from least to greatest. If n is odd, then the median is the middle number in the ordered list of numbers. If n is even, then there are *two* middle numbers, and the median is the average of these two numbers.

Example 4.2.3: The five numbers 6, 4, 7, 10, and 4 listed in increasing order are 4, 4, 6, 7, 10, so the median is 6, the middle number. Note that if the number 10 in the list is replaced by the number 24, the mean increases from 6.2 to

$$\frac{4+4+6+7+24}{5}=\frac{45}{5}=9$$

but the median remains equal to 6. This example shows how the median is relatively unaffected by an unusually large value.

The median, as the "middle value" of an ordered list of numbers, divides the list into roughly two equal parts. However, if the median is equal to one of the data values and it is repeated in the list, then the numbers of data above and below the median may be rather different. For example, the median of the 16 numbers 2, 4, 4, 5, 7, 7, 7, 7, 7, 7, 8, 8, 9, 9, 9, 9 is 7, but four of the data are less than 7 and six of the data are greater than 7.

The **mode** of a list of numbers is the number that occurs most frequently in the list.

Example 4.2.4: The mode of the 6 numbers in the list 1, 3, 6, 4, 3, 5 is 3. A list of numbers may have more than one mode. For example, the list of 11 numbers 1, 2, 3, 3, 3, 5, 7, 10, 10, 10, 20 has two modes, 3 and 10.

Measures of Position

The three most basic **positions**, or locations, in a list of numerical data ordered from least to greatest are the beginning, the end, and the middle. It is useful here to label these as L for the least, G for the greatest, and M for the median. Aside from these, the most common measures of position are **quartiles** and **percentiles**. Like the median M, quartiles and percentiles are numbers that divide the data into roughly equal groups after the data have been ordered from the least value L to the greatest value G. There are three quartile numbers, called the **first quartile,** the **second quartile**, and the **third quartile**, that divide the data into four roughly equal groups; and there are 99 percentile numbers that divide the data into 100 roughly equal groups. As with the mean and median, the quartiles and percentiles may or may not themselves be values in the data.

In the following discussion of quartiles, the symbol Q_1 will be used to denote the first quartile, Q_2 will be used to denote the second quartile, and Q_3 will be used to denote the third quartile.

The numbers Q_1, Q_2, and Q_3 divide the data into 4 roughly equal groups as follows. After the data are listed in increasing order, the first group consists of the data from L to Q_1, the second group is from Q_1 to Q_2, the third group is from Q_2 to Q_3, and the fourth group is from Q_3 to G. Because the number of data may not be divisible by 4, there are various rules to determine the exact values of Q_1 and Q_3, and some statisticians use different rules, but in all cases Q_2 is equal to the median M. We use perhaps the most common rule for determining the values of Q_1 and $Q_{3.}$ According to this rule, after the data are listed in increasing order, Q_1 is the median of the first half of the data in the ordered list and Q_3 is the median of the second half of the data in the ordered list, as illustrated in the following example.

Example 4.2.5: To find the quartiles for the list of 16 numbers 2, 4, 4, 5, 7, 7, 7, 7, 7, 7, 8, 8, 9, 9, 9, 9 (which are already listed in increasing order), first divide the numbers in the list into two groups of 8 numbers each. The first group of 8 numbers is 2, 4, 4, 5, 7, 7, 7, 7, and the second group of 8 numbers is 7, 7, 8, 8, 9, 9, 9, 9, so that the second quartile, or median, is 7. To find the other quartiles, you can take each of the two smaller groups and find its median: the first quartile, Q_1, is 6 (the average of 5 and 7) and the third quartile, Q_3, is 8.5 (the average of 8 and 9).

In this example, note that the number 4 is in the lowest 25 percent of the distribution of data. There are different ways to describe this. We can say that 4 is below the first quartile, that is, below Q_1. We can also say that 4 is *in* the first quartile. The phrase "in a quartile" refers to being in one of the four groups determined by Q_1, Q_2, and Q_3.

Percentiles are mostly used for very large lists of numerical data ordered from least to greatest. Instead of dividing the data into four groups, the 99 percentiles P_1, P_2, P_3, …, P_{99} divide the data into 100 groups. Consequently, $Q_1 = P_{25}$, $M = Q_2 = P_{50}$, and $Q_3 = P_{75}$. Because the number of data in a list may not be divisible by 100, statisticians apply various rules to determine values of percentiles.

Measures of Dispersion

Measures of **dispersion** indicate the degree of spread of the data. The most common statistics used as measures of dispersion are the range, the interquartile range, and the standard deviation. These statistics measure the spread of the data in different ways.

The **range** of the numbers in a group of data is the difference between the greatest number G in the data and the least number L in the data; that is, $G - L$. For example, the range of the five numbers 11, 10, 5, 13, 21 is $21 - 5 = 16$.

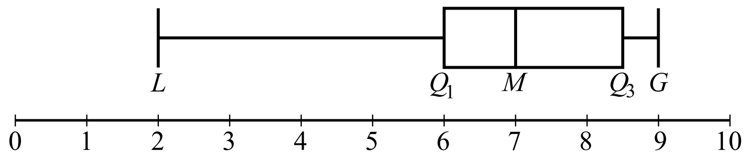
The simplicity of the range is useful in that it reflects the maximum spread of the data. Sometimes a data value is unusually small or unusually large in comparison with the rest of the data. Such data are called **outliers** because they lie so far out from the rest of the data. The range is directly affected by outliers.

A measure of dispersion that is not usually affected by outliers is the **interquartile range**. It is defined as the difference between the third quartile and the first quartile; that is, $Q_3 - Q_1$. Thus, the interquartile range measures the spread of the middle half of the data.

One way to summarize a group of numerical data and to illustrate its center and spread is to use the five numbers L, Q_1, Q_2, Q_3 and G. These five numbers can be plotted along a number line to show where the four quartile groups lie. Such plots are called **boxplots** or **box-and-whisker plots**, because a box is used to identify each of the two middle quartile groups of data, and "whiskers" extend outward from the boxes to the least and greatest values.

Example 4.2.6: In the list of 16 numbers 2, 4, 4, 5, 7, 7, 7, 7, 7, 7, 8, 8, 9, 9, 9, 9, the range is $9 - 2 = 7$, the first quartile, $Q_1 = 6$, and the third quartile, $Q_3 = 8.5$. So the interquartile range for the numbers in this list is $8.5 - 6 = 2.5$.

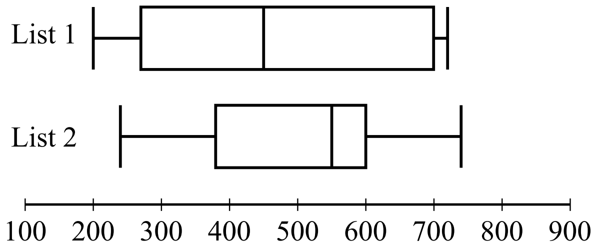
A boxplot for this list of 16 numbers is shown in the following figure.



Data Analysis Figure 8

There are a few variations in the way boxplots are drawn—the position of the ends of the boxes can vary slightly, and some boxplots identify outliers with certain symbols—but all boxplots show the center of the data at the median and illustrate the spread of the data in each of the four quartile groups. As such, boxplots are useful for comparing sets of data side by side.

Example 4.2.7: Two large lists of numerical data, list 1 and list 2, are summarized by the boxplots in the following figure.



Data Analysis Figure 9

Based on the boxplots, several different comparisons of the two lists can be made. First, the median of list 2, which is approximately 550, is greater than the median of list 1, which is approximately 450. Second, the two measures of spread, range and interquartile range, are greater for list 1 than for list 2. For list 1, these measures are approximately 520 and 430, respectively, and for list 2, they are approximately 500 and 220, respectively.

Unlike the range and the interquartile range, the **standard deviation** is a measure of spread that depends on each number in the list. Using the mean as the center of the data, the standard deviation takes into account how much each value differs from the mean and then takes a type of average of these differences. As a result, the more the data are spread away from the mean, the greater the standard deviation; and the more the data are clustered around the mean, the lesser the standard deviation.

The standard deviation of a group of numerical data is computed by

1. calculating the mean of the values,
2. finding the difference between the mean and each of the values,
3. squaring each of the differences,
4. finding the average of the squared differences, and
5. taking the nonnegative square root of the average of the squared differences.

Example 4.2.8: For the five data 0, 7, 8, 10, and 10, the standard deviation can be computed as follows. First, the mean of the data is 7, and the squared differences from the mean are

$$(7-0)^2, (7-7)^2, (7-8)^2, (7-10)^2, (7-10)^2$$

or 49, 0, 1, 9, 9. The average of the five squared differences is $\frac{68}{5}$, or 13.6, and the positive square root of 13.6 is approximately 3.7.

Note on terminology: The term "standard deviation" defined above is slightly different from another measure of dispersion, the **sample standard deviation**. The latter term is qualified with the word "sample" and is computed by dividing the sum of the squared differences by $n - 1$ instead of n. The sample standard deviation is only slightly different from the standard deviation but is preferred for technical reasons for a sample of data that is taken from a larger population of data. Sometimes the standard deviation is called the **population standard deviation** to help distinguish it from the sample standard deviation.

Example 4.2.9: Six hundred applicants for several post office jobs were rated on a scale from 1 to 50 points. The ratings had a mean of 32.5 points and a standard deviation of 7.1 points. How many standard deviations above or below the mean is a rating of 48 points? A rating of 30 points? A rating of 20 points?

Solution: Let d be the standard deviation, so $d = 7.1$ points. Note that 1 standard deviation above the mean is

$$32.5 + d = 32.5 + 7.1 = 39.6$$

and 2 standard deviations above the mean is

$$32.5 + 2d = 32.5 + 2(7.1) = 46.7$$

Using the same reasoning, if 48 is r standard deviations above the mean, then

$$32.5 + rd = 32.5 + r(7.1) = 48$$

Solving the equation $32.5 + r(7.1) = 48$ for r gives

$$r = \frac{48 - 32.5}{7.1} = \frac{15.5}{7.1} \approx 2.2$$

Similarly, any rating of p points is $\frac{p - 32.5}{7.1}$ standard deviations from the mean.

The number of standard deviations that a rating of 30 is away from the mean is

$$\frac{30 - 32.5}{7.1} = \frac{-2.5}{7.1} \approx -0.4$$

where the negative sign indicates that the rating is 0.4 standard deviation *below* the mean.

The number of standard deviations that a rating of 20 is away from the mean is

$$\frac{20 - 32.5}{7.1} = \frac{-12.5}{7.1} \approx -1.8$$

where the negative sign indicates that the rating is 1.8 standard deviations *below* the mean.

To summarize:

1. 48 points is 15.5 points above the mean, or approximately 2.2 standard deviations above the mean.
2. 30 points is 2.5 points below the mean, or approximately 0.4 standard deviation below the mean.
3. 20 points is 12.5 points below the mean, or approximately 1.8 standard deviations below the mean.

One more instance, which may seem trivial, is important to note:

32.5 points is 0 points from the mean, or 0 standard deviations from the mean.

Example 4.2.9 above shows that for a group of data, each value can be located with respect to the mean by using the standard deviation as a ruler. The process of subtracting the mean from each value and then dividing the result by the standard deviation is called **standardization**. Standardization is a useful tool because for each data value, it provides a measure of position relative to the rest of the data independently of the variable for which the data was collected and the units of the variable.

Note that the standardized values 2.2, –0.4, and –1.8 from the last example are all between –3 and 3; that is, the corresponding ratings 48, 30, and 20 are all within 3 standard deviations of the mean. This is not surprising, based on the following fact about the standard deviation.

Fact: In *any group of data*, most of the data are within 3 standard deviations of the mean.

Thus, when *any group of data* are standardized, most of the data are transformed to an interval on the number line centered about 0 and extending from –3 to 3. The mean is always transformed to 0.

4.3 Counting Methods

When a set contains a small number of objects, it is easy to list the objects and count them one by one. When the set is too large to count that way, and when the objects are related in a patterned or systematic way, there are some useful techniques for counting the objects without actually listing them.

Sets and Lists

The term **set** has been used informally in this review to mean a collection of objects that have some property, whether it is the collection of all positive integers, all points in a circular region, or all students in a school that have studied French. The objects of a set are called **members** or **elements**. Some sets are **finite**, which means that their members can be completely counted. Finite sets can, in principle, have all of their members listed, using curly brackets, such as the set of even digits $\{0, 2, 4, 6, 8\}$. Sets that are not finite are called **infinite** sets, such as the set of all integers. A set that has no members is called the **empty set** and is denoted by the symbol $\varnothing$. A set with one or more members is called **nonempty**. If A and B are sets and all of the members of A are also members of B, then A is a **subset** of B. For example, $\{2, 8\}$ is a subset of $\{0, 2, 4, 6, 8\}$. Also, by convention, $\varnothing$ is a subset of every set.

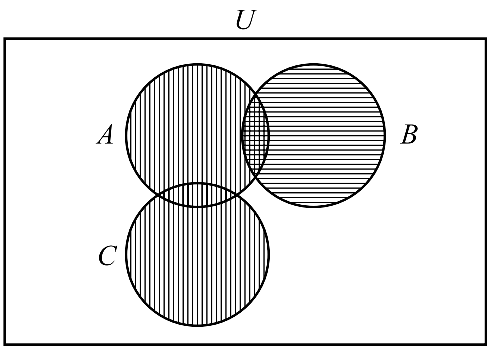
A **list** is like a finite set, having members that can all be listed, but with two differences. In a list, the members are ordered—that is, rearranging the members of a list makes it a different list. Thus, the terms "first element," "second element," and so on, make sense in a list. Also, elements can be repeated in a list and the repetitions matter. For example, the list 1, 2, 3, 2 and the list 1, 2, 2, 3 are different lists, each with four elements, and they are both different from the list 1, 2, 3, which has three elements.

In contrast to a list, when the elements of a set are given, repetitions are not counted as additional elements and the order of the elements does not matter. For example, the set $\{1, 2, 3, 2\}$ and the set $\{3, 1, 2\}$ are the same set, which has three elements. For any finite set S, the number of elements of S is denoted by $|S|$. Thus, if $S = \{6.2, -9, \pi, 0.01, 0\}$, then $|S| = 5$. Also, $|\varnothing| = 0$.

Sets can be formed from other sets. If S and T are sets, then the **intersection** of S and T is the set of all elements that are in both S and T and is denoted by $S \cap T$. The **union** of S and T is the set of all elements that are in either S or T or both and is denoted by $S \cup T$. If sets S and T have no elements in common, they are called **disjoint** or **mutually exclusive**.

A useful way to represent two or three sets and their possible intersections and unions is a **Venn diagram**. In a Venn diagram, sets are represented by circular regions that overlap if they have elements in common but do not overlap if they are disjoint. Sometimes the circular regions are drawn inside a rectangular region, which represents a **universal set**, of which all other sets involved are subsets.

Example 4.3.1: Data Analysis Figure 10 below is a Venn diagram using circular regions to represent the three sets A, B, and C. In the Venn diagram, the three circular regions are drawn in a rectangular region representing a universal set U.



Data Analysis Figure 10

The regions with vertical stripes represent the set $A \cup C$. The regions with horizontal stripes represent the set B. The region with both kinds of stripes represents the set $A \cap B$. The sets B and C are mutually exclusive, often written as $B \cap C = \varnothing$.

The last example can be used to illustrate an elementary counting principle involving intersecting sets, called the **inclusion-exclusion principle**. This principle relates the numbers of elements in the union and intersection of two finite sets. The number of elements in the union of two sets equals the sum of their individual numbers of elements minus the number of elements in their intersection. If the sets in the example are finite, then we have for the union of A and B,

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Because $A \cap B$ is a subset of both A and B, the subtraction is necessary to avoid counting the elements in $A \cap B$ twice. For the union of B and C, we have

$$|B \cup C| = |B| + |C|$$

because $B \cap C = \varnothing$.

Multiplication Principle

Suppose there are two choices to be made sequentially and that the second choice is independent of the first choice. Suppose also that there are k different possibilities for the first choice and m different possibilities for the second choice. The **multiplication principle** states that under those conditions, there are km different possibilities for the pair of choices.

For example, suppose that a meal is to be ordered from a restaurant menu and that the meal consists of one entrée and one dessert. If there are 5 entrées and 3 desserts on the menu, then there are (5)(3) = 15 different meals that can be ordered from the menu.

The multiplication principle applies in more complicated situations as well. If there are more than two independent choices to be made, then the number of different possible outcomes of all of the choices is the product of the numbers of possibilities for each choice.

Example 4.3.2: Suppose that a computer password consists of four characters such that the first character is one of the 10 digits from 0 to 9 and each of the next 3 characters is any one of the uppercase letters from the 26 letters of the English alphabet. How many different passwords are possible?

Solution: The description of the password allows repetitions of letters. Thus, there are 10 possible choices for the first character in the password and 26 possible choices for each of the next 3 characters in the password. Therefore, applying the multiplication principle, the number of possible passwords is $(10)(26)(26)(26) = 175{,}760$.

Note that if repetitions of letters are not allowed in the password, then the choices are not all independent, but a modification of the multiplication principle can still be applied. There are 10 possible choices for the first character in the password, 26 possible choices for the second character, 25 for the third character because the first letter cannot be repeated, and 24 for the fourth character because the first two letters cannot be repeated. Therefore, the number of possible passwords is $(10)(26)(25)(24) = 156{,}000$.

Example 4.3.3: Each time a coin is tossed, there are 2 possible outcomes—either it lands heads up or it lands tails up. Using this fact and the multiplication principle, you can conclude that if a coin is tossed 8 times, there are $(2)(2)(2)(2)(2)(2)(2)(2) = 2^8 = 256$ possible outcomes.

Permutations and Factorials

Suppose you want to determine the number of different ways the 3 letters A, B, and C can be placed in order from 1st to 3rd. The following is a list of all the possible orders in which the letters can be placed.

ABC ACB BAC BCA CAB CBA

There are 6 possible orders for the 3 letters.

Now suppose you want to determine the number of different ways the 4 letters A, B, C, and D can be placed in order from 1st to 4th. Listing all of the orders for 4 letters is time-consuming, so it would be useful to be able to count the possible orders without listing them.

To order the 4 letters, one of the 4 letters must be placed first, one of the remaining 3 letters must be placed second, one of the remaining 2 letters must be placed third, and the last remaining letter must be placed fourth. Therefore, applying the multiplication principle, there are (4)(3)(2)(1), or 24, ways to order the 4 letters.

More generally, suppose n objects are to be ordered from 1st to nth, and we want to count the number of ways the objects can be ordered. There are n choices for the first object, $n - 1$ choices for the second object, $n - 2$ choices for the third object, and so on, until there is only 1 choice for the nth object. Thus, applying the multiplication principle, the number of ways to order the n objects is equal to the product

$$n(n-1)(n-2)\cdots(3)(2)(1)$$

Each order is called a **permutation**, and the product above is called the number of permutations of n objects.

Because products of the form $n(n-1)(n-2)\cdots(3)(2)(1)$ occur frequently when counting objects, a special symbol $n!$, called ***n*-factorial**, is used to denote this product.

For example,

$$1! = 1$$
$$2! = (2)(1)$$
$$3! = (3)(2)(1)$$
$$4! = (4)(3)(2)(1)$$

As a special definition, $0! = 1$.

Note that $n! = n(n-1)! = n(n-1)(n-2)! = n(n-1)(n-2)(n-3)!$ and so on.

Example 4.3.4: Suppose that 10 students are going on a bus trip, and each of the students will be assigned to one of the 10 available seats. Then the number of possible different seating arrangements of the students on the bus is

$$10! = (10)(9)(8)(7)(6)(5)(4)(3)(2)(1) = 3{,}628{,}800$$

Now suppose you want to determine the number of ways in which you can select 3 of the 5 letters A, B, C, D, and E and place them in order from 1st to 3rd. Reasoning as in the preceding examples, you find that there are (5)(4)(3), or 60, ways to select and order them.

More generally, suppose that k objects will be selected from a set of n objects, where $k \le n$, and the k objects will be placed in order from 1st to kth. Then there are n choices for the first object, $n - 1$ choices for the second object, $n - 2$ choices for the third object, and so on, until there are $n - k + 1$ choices for the kth object. Thus, applying the multiplication principle, the number of ways to select and order k objects from a set of n objects is $n(n-1)(n-2)\dots(n-k+1)$. It is useful to note that

$$n(n-1)(n-2)\cdots(n-k+1) = n(n-1)(n-2)\cdots(n-k+1)\frac{(n-k)!}{(n-k)!} = \frac{n!}{(n-k)!}$$

This expression represents the number of **permutations of n objects taken k at a time**—that is, the number of ways to select and order k objects out of n objects. This number is commonly denoted by the notation ${}_nP_k$.

Example 4.3.5: How many different 5-digit positive integers can be formed using the digits 1, 2, 3, 4, 5, 6, and 7 if none of the digits can occur more than once in the integer?

Solution: This example asks how many ways there are to order 5 integers chosen from a set of 7 integers. According to the counting principle above, there are $(7)(6)(5)(4)(3) = 2{,}520$ ways to do this. Note that this is equal to $\frac{7!}{(7-5)!} = \frac{(7)(6)(5)(4)(3)(2!)}{2!} = (7)(6)(5)(4)(3)$.

Combinations

Given the 5 letters A, B, C, D, and E, suppose that you want to determine the number of ways in which you can select 3 of the 5 letters, but unlike before, you do not want to count different orders for the 3 letters. The following is a list of all of the ways in which 3 of the 5 letters can be selected without regard to the order of the letters.

ABC ABD ABE ACD ACE
ADE BCD BCE BDE CDE

There are 10 ways of selecting the 3 letters without order. There is a relationship between selecting with order and selecting without order.

The number of ways to select 3 of the 5 letters without order, which is 10, *multiplied by* the number of ways to order the 3 letters, which is 3!, or 6, *is equal to* the number of ways to select 3 of the 5 letters and order them, which is $\frac{5!}{2!}=60$. In short,

$$(\text{number of ways to select without order}) \times (\text{number of ways to order}) = (\text{number of ways to select with order})$$

This relationship can also be rewritten as follows.

$$(\text{number of ways to select without order}) = \frac{(\text{number of ways to select with order})}{(\text{number of ways to order})}$$

For the example above, the number of ways to select without order is $\frac{\frac{5!}{2!}}{3!}=\frac{5!}{3!2!}=10.$

More generally, suppose that k objects will be chosen from a set of n objects, where $k \le n$, but that the k objects will not be put in order. The number of ways in which this can be done is called the number of **combinations of *n* objects taken *k* at a time** and is given by the formula $\frac{n!}{k!(n-k)!}$.

Another way to refer to the number of combinations of n objects taken k at a time is ***n* choose *k***, and two notations commonly used to denote this number are ${}_nC_k$ and $\binom{n}{k}$.

Example 4.3.6: Suppose you want to select a 3-person committee from a group of 9 students. How many ways are there to do this?

Solution: Since the 3 students on the committee are not ordered, you can use the formula for the combination of 9 objects taken 3 at a time, or "9 choose 3":

$$\frac{9!}{3!(9-3)!}=\frac{9!}{3!6!}=\frac{(9)(8)(7)}{(3)(2)(1)}=84$$

Using the terminology of sets, given a set S consisting of n elements, n choose k is simply the number of subsets of S that consist of k elements.

The formula for n choose k, which is $\frac{n!}{k!(n-k)!}$ also holds when $k = 0$ and $k = n$. Therefore

1. n choose 0 is $\frac{n!}{0!n!}=1$.
   (This reflects the fact that there is only one subset of S with 0 elements, namely the empty set).
2. n choose n is $\frac{n!}{n!0!}=1$.
   (This reflects the fact that there is only one subset of S with n elements, namely the set S itself).

Finally, note that n choose k is always equal to n choose $n-k$, because

$$\frac{n!}{(n-k)!\,(n-(n-k))!}=\frac{n!}{(n-k)!\,k!}=\frac{n!}{k!(n-k)!}$$

4.4 Probability

Probability is a way of describing uncertainty in numerical terms. In this section, we review some of the terminology used in elementary probability theory.

A **probability experiment**, also called a **random experiment**, is an experiment for which the result, or **outcome**, is uncertain. We assume that all of the possible outcomes of an experiment are known before the experiment is performed, but which outcome will actually occur is unknown. The set of all possible outcomes of a random experiment is called the **sample space**, and any particular set of outcomes is called an **event**. For example, consider a cube with faces numbered 1 to 6, called a 6-sided die. Rolling the die once is an experiment in which there are 6 possible outcomes: either 1, 2, 3, 4, 5, or 6 will appear on the top face. The sample space for this experiment is $\{1, 2, 3, 4, 5, 6\}$. Here are two examples of events for this experiment.

Event 1: Rolling the number 4. This event has only one outcome.

Event 2: Rolling an odd number. This event has three outcomes.

The **probability** of an event is a number from 0 to 1, inclusive, that indicates the likelihood that the event occurs when the experiment is performed. The greater the number, the more likely the event.

Example 4.4.1: Consider the following experiment. A box contains 15 pieces of paper, each of which has the name of one of the 15 students in a high school class consisting of 7 juniors and 8 seniors, all with different names. The instructor will shake the box for a while and then choose a piece of paper at random and read the name. Here the sample space is the set of 15 names. The assumption of **random selection** means that each of the names is **equally likely** to be selected. If this assumption is made, then the probability that any one particular name will be selected is equal to $\frac{1}{15}$.

For any event E, the probability that E occurs is often written as $P(E)$.

For the sample space in this example, $P(E)$, that is, the probability that event E occurs, is equal to

$$\frac{\text{the number of names in the event } E}{15}$$

If J is the event that the student selected is a junior, then $P(J)=\frac{7}{15}$.

In general, for a random experiment with a finite number of possible outcomes, if each outcome is equally likely to occur, then the probability that an event E occurs is defined by

$$P(E)=\frac{\text{the number of outcomes in the event } E}{\text{the number of possible outcomes in the experiment}}$$

In the case of rolling a 6-sided die, if the die is "fair," then the 6 outcomes are equally likely. So the probability of rolling a 4 is $\frac{1}{6}$, and the probability of rolling an odd number (that is, rolling a 1, 3, or 5) can be calculated as $\frac{3}{6}=\frac{1}{2}$.

The following are six general facts about probability.

Fact 1: If an event E is certain to occur, then $P(E) = 1$.

Fact 2: If an event E is certain *not* to occur, then $P(E) = 0$.

Fact 3: If an event E is possible but not certain to occur, then $0 < P(E) < 1$.

Fact 4: The probability that an event E will not occur is equal to $1 - P(E)$.

Fact 5: If E is an event, then the probability of E is the sum of the probabilities of the outcomes in E.

Fact 6: The sum of the probabilities of all possible outcomes of an experiment is 1.

If E and F are two events of an experiment, we consider two other events related to E and F.

Event 1: The event that both E and F occur, that is, outcomes in the set $E \cap F$

Event 2: The event that E or F, or both, occur, that is, outcomes in the set $E \cup F$

Events that cannot occur at the same time are said to be **mutually exclusive**. For example, if a 6-sided die is rolled once, the event of rolling an odd number and the event of rolling an even number are mutually exclusive. But rolling a 4 and rolling an even number are not mutually exclusive, since 4 is an outcome that is common to both events.

For events E and F, we have the following three rules.

Rule 1: P(either E or F, or both, occur) = $P(E) + P(F) - P$(both E and F occur), which is the inclusion-exclusion principle applied to probability.

Rule 2: If E and F are mutually exclusive, then P(both E and F occur) = 0, and therefore, P(either E or F, or both, occur) = $P(E) + P(F)$.

Rule 3: E and F are said to be **independent** if the occurrence of either event does not affect the occurrence of the other. If two events E and F are independent, then P(both E and F occur) = $P(E)P(F)$. For example, if a fair 6-sided die is rolled twice, the event E of rolling a 3 on the first roll and the event F of rolling a 3 on the second roll are independent, and the probability of rolling a 3 on both rolls is $P(E)P(F) = \left(\frac{1}{6}\right)\left(\frac{1}{6}\right) = \frac{1}{36}$. In this example, the experiment is actually "rolling the die twice," and each outcome is an ordered pair of results like "4 on the first roll and 1 on the second roll." But event E restricts only the first roll—to a 3—having no effect on the second roll; similarly, event F restricts only the second roll—to a 3—having no effect on the first roll.

Note that if $P(E) \neq 0$ and $P(F) \neq 0$, then events E and F cannot be both mutually exclusive and independent. For if E and F are independent, then P(both E and F occur) $= P(E)P(F) \neq 0$, but if E and F are mutually exclusive, then P(both E and F occur) = 0.

It is common to use the shorter notation "E and F" instead of "both E and F occur" and use "E or F" instead of "E or F or both occur." With this notation, we can restate the previous three rules as follows.

Rule 1: $P(E \text{ or } F) = P(E) + P(F) - P(E \text{ and } F)$

Rule 2: $P(E \text{ or } F) = P(E) + P(F)$ if E and F are mutually exclusive.

Rule 3: $P(E \text{ and } F) = P(E)P(F)$ if E and F are independent.

Example 4.4.2: If a fair 6-sided die is rolled once, let E be the event of rolling a 3 and let F be the event of rolling an odd number. These events are *not* independent. This is because rolling a 3 makes certain that the event of rolling an odd number occurs. Note that $P(E \text{ and } F) \neq P(E)P(F)$, since

$$P(E \text{ and } F) = P(E) = \frac{1}{6} \quad \text{and} \quad P(E)P(F) = \left(\frac{1}{6}\right)\left(\frac{1}{2}\right) = \frac{1}{12}$$

Example 4.4.3: A 12-sided die, with faces numbered 1 to 12, is to be rolled once, and each of the 12 possible outcomes is equally likely to occur. The probability of rolling a 4 is $\frac{1}{12}$, so the probability of rolling a number that is *not* a 4 is $1 - \frac{1}{12} = \frac{11}{12}$.
The probability of rolling a number that is either a multiple of 5 (that is, rolling a 5 or a 10) or an odd number (that is, rolling a 1, 3, 5, 7, 9, or 11) is equal to

$$P(\text{multiple of 5}) + P(\text{odd}) - P(\text{multiple of 5 and odd}) = \frac{2}{12} + \frac{6}{12} - \frac{1}{12} = \frac{7}{12}$$

Another way to calculate this probability is to notice that rolling a number that is either a multiple of 5 or an odd number is the same as rolling one of the seven numbers 1, 3, 5, 7, 9, 10, and 11, which are equally likely outcomes. So by using the ratio formula to calculate the probability, the required probability is $\frac{7}{12}$.

Example 4.4.4: Consider an experiment with events A, B, and C for which $P(A) = 0.23$, $P(B) = 0.40$, and $P(C) = 0.85$.

Suppose that events A and B are mutually exclusive and events B and C are independent. What is $P(A \text{ or } B)$ and $P(B \text{ or } C)$?

Solution: Since A and B are mutually exclusive,

$$P(A \text{ or } B) = P(A) + P(B) = 0.23 + 0.40 = 0.63$$

Since B and C are independent, $P(B \text{ and } C) = P(B)P(C)$. So

$$P(B \text{ or } C) = P(B) + P(C) - P(B \text{ and } C) = P(B) + P(C) - P(B)P(C)$$

Therefore,

$$P(B \text{ or } C) = 0.40 + 0.85 - (0.40)(0.85) = 1.25 - 0.34 = 0.91$$

Example 4.4.5: Suppose that there is a 6-sided die that is weighted in such a way that each time the die is rolled, the probabilities of rolling any of the numbers from 1 to 5 are all equal, but the probability of rolling a 6 is twice the probability of rolling a 1. When you roll the die once, the 6 outcomes are *not equally likely*. What are the probabilities of the 6 outcomes?

Solution: Let p equal the probability of rolling a 1. Then each of the probabilities of rolling a 2, 3, 4, or 5 is equal to p, and the probability of rolling a 6 is equal to $2p$. Therefore, since the sum of the probabilities of all possible outcomes is 1, it follows that

$$1 = P(1) + P(2) + P(3) + P(4) + P(5) + P(6) = p + p + p + p + p + 2p = 7p$$

So the probability of rolling each of the numbers from 1 to 5 is $\frac{1}{7}$, and the probability of rolling a 6 is $\frac{2}{7}$.

Example 4.4.6: Suppose that you roll the weighted 6-sided die from Example 4.4.5 twice. What is the probability that the first roll will be an odd number and the second roll will be an even number?

Solution: To calculate the probability that the first roll will be odd and the second roll will be even, note that these two events are independent. To calculate the probability that both occur, you must multiply the probabilities of the two independent events. First compute the individual probabilities.

$$P(\text{odd}) = P(1) + P(3) + P(5) = \frac{3}{7}$$

$$P(\text{even}) = P(2) + P(4) + P(6) = \frac{4}{7}$$

Then $P(\text{first roll is odd and second roll is even}) = P(\text{odd})P(\text{even}) = \left(\frac{3}{7}\right)\left(\frac{4}{7}\right) = \frac{12}{49}$.

Two events that happen sequentially are not always independent. The occurrence of the first event may affect the occurrence of the second event. In this case, the probability that *both* events happen is equal to the probability that the first event happens multiplied by the probability that, *given that the first event has already happened*, the second event will happen as well.

Example 4.4.7: A box contains 5 orange disks, 4 red disks, and 1 blue disk. You are to select two disks at random and without replacement from the box. What is the probability that the first disk you select will be red and the second disk you select will be orange?

Solution: To solve, you need to calculate the following two probabilities and then multiply them.

1. The probability that the first disk selected from the box will be red
2. The probability that the second disk selected from the box will be orange, given that the first disk selected from the box is red

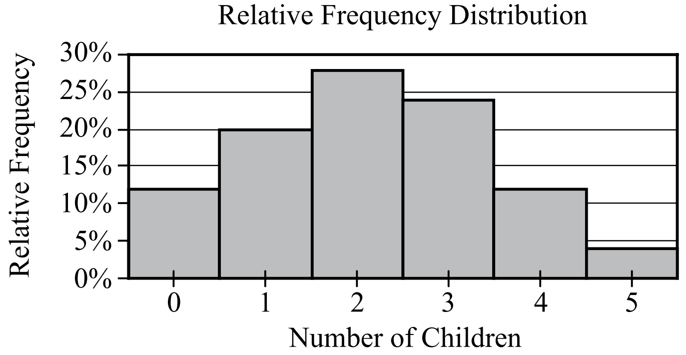
The probability that the first disk you select will be red is $\frac{4}{10} = \frac{2}{5}$. If the first disk you select is red, there will be 5 orange disks, 3 red disks, and 1 blue disk left in the box, for a total of 9 disks. Therefore, the probability that the second disk you select will be orange, given that the first disk you selected is red, is $\frac{5}{9}$. Multiply the two probabilities to get $\left(\frac{2}{5}\right)\left(\frac{5}{9}\right) = \frac{2}{9}$.

4.5 Distributions of Data, Random Variables, and Probability Distributions

In data analysis, variables whose values depend on chance play an important role in linking distributions of data to probability distributions. Such variables are called random variables. We begin with a review of distributions of data.

Distributions of Data

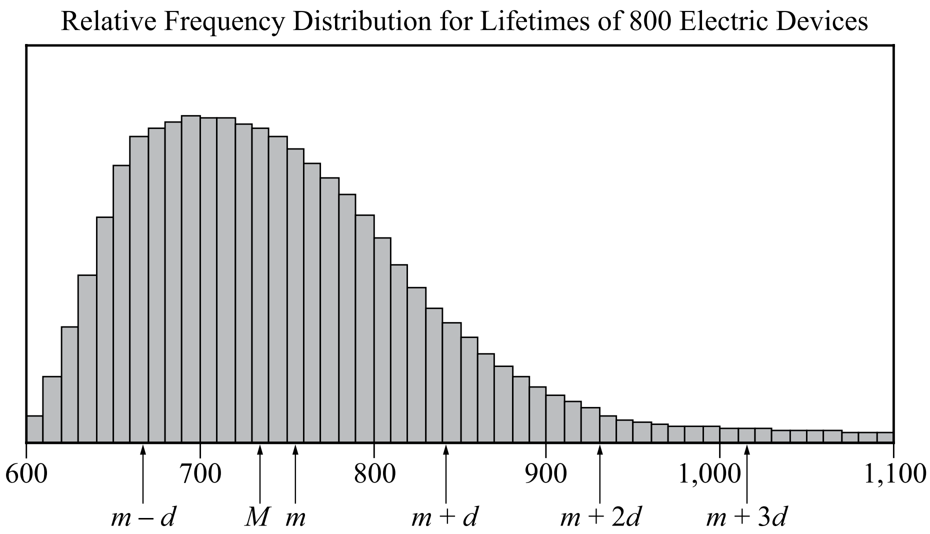
Recall that relative frequency distributions given in a table or histogram are a common way to show how numerical data are distributed. In a histogram, the areas of the bars indicate where the data are concentrated. The histogram of the relative frequency distribution of the number of children in each of 25 families in Data Analysis Figure 4 below illustrates a small group of data, with only 6 distinct data values and 25 data values altogether. (Note: This is the second occurrence of Data Analysis Figure 4, it was first encountered in Example 4.1.10.)



Data Analysis Figure 4 (repeated)

Many groups of data are much larger than 25 and have many more than 6 possible values, which are often measurements of quantities like length, money, or time.

Example 4.5.1: The lifetimes of 800 electric devices were measured. Because the lifetimes had many different values, the measurements were grouped into 50 intervals, or **classes**, of 10 hours each: 601 to 610 hours, 611 to 620 hours, and so on, up to 1,091 to 1,100 hours. The resulting relative frequency distribution, as a histogram, has 50 thin bars and many different bar heights, as shown in Data Analysis Figure 11 below.



Data Analysis Figure 11

In the histogram, the median is represented by M, the mean is represented by m, and the standard deviation is represented by d.

According to the graph:

- A data value 1 standard deviation below the mean, represented by $m - d$, is between 660 and 670.
- The median, represented by M, is between 730 and 740.
- The mean, represented by m, is between 750 and 760.

- A data value 1 standard deviation above the mean, represented by $m + d$, is between 840 and 850.
- A data value 2 standard deviations above the mean, represented by $m + 2d$, is approximately 930.
- A data value 3 standard deviations above the mean, represented by $m + 3d$, is between 1,010 and 1,020.

The standard deviation marks show how most of the data are within 3 standard deviations of the mean, that is, between the numbers $m - 3d$ and $m + 3d$. Note that $m + 3d$ is shown in the figure, but $m - 3d$ is not.

The tops of the bars of the relative frequency distribution in the histogram in Data Analysis Figure 11 have a relatively smooth appearance and begin to look like a curve. In general, histograms that represent very large data sets grouped into many classes have a relatively smooth appearance. Consequently, the distribution can be modeled by a smooth curve that is close to the tops of the bars. Such a model retains the shape of the distribution but is independent of classes.

Recall from Example 4.1.10 that the sum of the areas of the bars of a relative frequency histogram is 1. Although the units on the horizontal axis of a histogram vary from one data set to another, the vertical scale can be adjusted (stretched or shrunk) so that the sum of the areas of the bars is 1. With this vertical scale adjustment, the area under the curve that models the distribution is also 1. This model curve is called a **distribution curve**, but it has other names as well, including **density curve** and **frequency curve**.

The purpose of the distribution curve is to give a good illustration of a large distribution of numerical data that does not depend on specific classes. To achieve this, the main property of a distribution curve is that the area under the curve in any vertical slice, just like a histogram bar, represents the proportion of the data that lies in the corresponding interval on the horizontal axis, which is at the base of the slice.

Random Variables

When analyzing data, it is common to choose a value of the data at random and consider that choice as a random experiment, as introduced in Section 4.4. Then, the probabilities of events involving the randomly chosen value may be determined. Given a distribution of data, a variable, say X, may be used to represent a randomly chosen value from the distribution. Such a variable X is an example of a **random variable**, which is a variable whose value is a numerical outcome of a random experiment.

Example 4.5.2: In Example 4.1.3, data consisting of numbers of children in each of 25 families was summarized in a frequency distribution table. For convenience, the frequency distribution table is repeated below.

Frequency Distribution

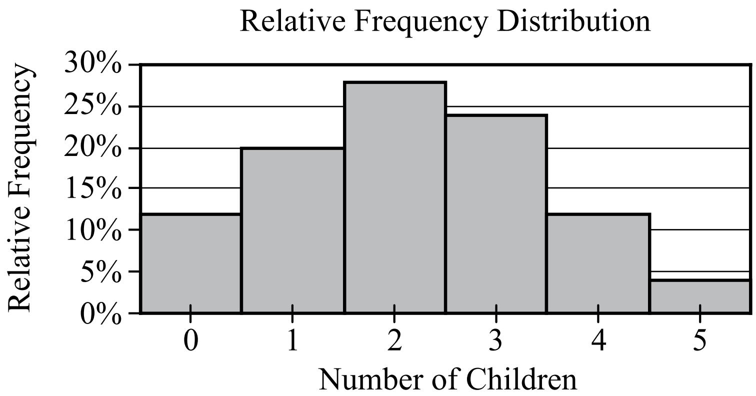
Number of Children	Frequency
0	3
1	5
2	7
3	6
4	3
5	1
Total	25

Let X be the random variable representing the number of children in a randomly chosen family among the 25 families. What is the probability that $X = 3$? That $X > 3$? That X is less than the mean of the distribution?

Solution: To determine the probability that $X = 3$, realize that this is the same as determining the probability that a family with 3 children will be chosen.

Since there are 6 families with 3 children and each of the 25 families is equally likely to be chosen, the probability that a family with 3 children will be chosen is $\frac{6}{25}$. That is, $X = 3$ is an event, and its probability is $P(X = 3) = \frac{6}{25}$, or 0.24. It is common to use the shorter notation $P(3)$ instead of $P(X = 3)$, so you could write $P(3) = 0.24$.

Note that in the histogram shown in Data Analysis Figure 4 below, the area of the bar corresponding to $X = 3$ as a proportion of the combined areas of all of the bars is equal to this probability. This indicates how probability is related to area in a histogram for a relative frequency distribution. (Note: This is the third occurrence of Data Analysis Figure 4, it was previously encountered in Example 4.1.10 and at the beginning of this section.)



Data Analysis Figure 4 (repeated)

To determine the probability that $X > 3$, notice that the event $X > 3$ is the same as the event "$X = 4$ or $X = 5$". Because $X = 4$ and $X = 5$ are mutually exclusive events, we can use the rules of probability from Section 4.4.

$$P(X > 3) = P(4) + P(5) = \frac{3}{25} + \frac{1}{25} = 0.12 + 0.04 = 0.16$$

To determine the probability that X is less than the mean of the distribution, first compute the mean of the distribution as follows.

$$\frac{0(3) + 1(5) + 2(7) + 3(6) + 4(3) + 5(1)}{25} = \frac{54}{25} = 2.16$$

Then, calculate the probability that X is less than the mean of the distribution (that is, the probability that X is less than 2.16).

$$P(X < 2.16) = P(0) + P(1) + P(2) = \frac{3}{25} + \frac{5}{25} + \frac{7}{25} = \frac{15}{25} = 0.6$$

The following table shows all 6 possible values of X and their probabilities. This table is called the **probability distribution** of the random variable X.

X	$P(X)$
0	0.12
1	0.20
2	0.28
3	0.24
4	0.12
5	0.04

Note that the probabilities are simply the relative frequencies of the 6 possible values expressed as decimals instead of percents. The following statement indicates a fundamental link between data distributions and probability distributions.

Statement: For a random variable that represents a randomly chosen value from a distribution of data, the probability distribution of the random variable is the same as the relative frequency distribution of the data.

Because the probability distribution and the relative frequency distribution are essentially the same, the probability distribution can be represented by a histogram. Also, all of the descriptive statistics—such as mean, median, and standard deviation—that apply to the distribution of data also apply to the probability distribution. For example, we say that the probability distribution above has a mean of 2.16, a median of 2, and a standard deviation of about 1.3, since the 25 data values have these statistics, as you can check.

These statistics are similarly defined for the random variable X above. Thus, we would say that the **mean of the random variable *X*** is 2.16. Another name for the mean of a random variable is **expected value**. So we would also say that the expected value of X is 2.16.

Note that the mean of X is equal to

$$\frac{0(3)+1(5)+2(7)+3(6)+4(3)+5(1)}{25}$$

which can also be expressed as

$$0\left(\frac{3}{25}\right)+1\left(\frac{5}{25}\right)+2\left(\frac{7}{25}\right)+3\left(\frac{6}{25}\right)+4\left(\frac{3}{25}\right)+5\left(\frac{1}{25}\right)$$

which is the same as

$$0P(0)+1P(1)+2P(2)+3P(3)+4P(4)+5P(5)$$

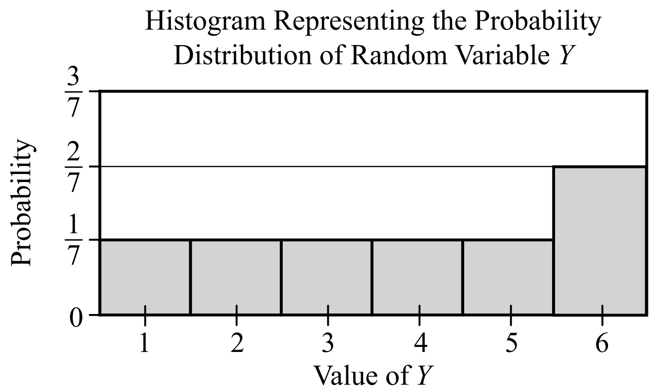
Therefore, the mean of the random variable X is the sum of the products $XP(X)$ for all values of X, that is, the sum of each value of X multiplied by its corresponding probability $P(X)$.

The preceding example involves a common type of random variable—one that represents a randomly chosen value from a distribution of data. However, the concept of a random variable is more general. A random variable can be any quantity whose value is the result of a random experiment. The possible values of the random variable are the same as the outcomes of the experiment. So any random experiment with numerical outcomes naturally has a random variable associated with it, as in the following example.

Example 4.5.3: Let Y represent the outcome of the experiment of rolling the weighted 6-sided die in Example 4.4.5. (In that example, the probabilities of rolling any of the numbers from 1 to 5 are all equal, but the probability of rolling a 6 is twice the probability of rolling a 1.) Then Y is a random variable with 6 possible values, the numbers 1 through 6. Each of the six values of Y has a probability. The probability distribution of the random variable Y is shown below, first in a table, then as a histogram.

Table Representing the Probability Distribution of Random Variable Y

Y	$P(Y)$
1	$\frac{1}{7}$
2	$\frac{1}{7}$
3	$\frac{1}{7}$
4	$\frac{1}{7}$
5	$\frac{1}{7}$
6	$\frac{2}{7}$



Data Analysis Figure 12

The mean, or expected value, of Y can be computed as

$$P(1) + 2P(2) + 3P(3) + 4P(4) + 5P(5) + 6P(6)$$

which is equal to

$$\left(\frac{1}{7}\right) + 2\left(\frac{1}{7}\right) + 3\left(\frac{1}{7}\right) + 4\left(\frac{1}{7}\right) + 5\left(\frac{1}{7}\right) + 6\left(\frac{2}{7}\right)$$

This sum simplifies to $\frac{1}{7} + \frac{2}{7} + \frac{3}{7} + \frac{4}{7} + \frac{5}{7} + \frac{12}{7}$, or $\frac{27}{7}$, which is approximately 3.86.

Both of the random variables X and Y above are examples of **discrete random variables** because their values consist of discrete points on a number line.

A basic fact about probability from Section 4.4 is that the sum of the probabilities of all possible outcomes of an experiment is 1, which can be confirmed by adding all of the probabilities in each of the probability distributions for the random variables X and Y above. Also, the sum of the areas of the bars in a histogram for the probability distribution of a random variable is 1. This fact is related to the following fundamental link between the areas of the bars of a histogram and the probabilities of a discrete random variable.

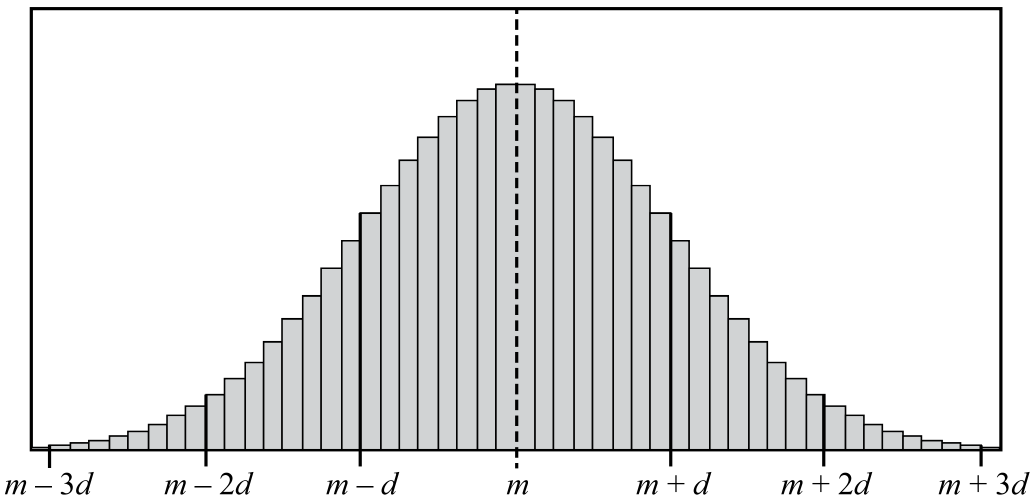
Fundamental Link: In a histogram representing the probability distribution of a random variable, the area of each bar is proportional to the probability represented by the bar.

If the die in Example 4.4.5 were a fair die instead of weighted, then the probability of each of the outcomes would be $\frac{1}{6}$, and consequently, each of the bars in the histogram of the probability distribution would have the same height. Such a flat histogram indicates a **uniform distribution**, since the probability is distributed uniformly over all possible outcomes.

The Normal Distribution

Many natural processes yield data that have a relative frequency distribution shaped somewhat like a bell, as in the distribution with mean m and standard deviation d in Data Analysis Figure 13 below.

Approximately Normal Relative Frequency Distribution



Data Analysis Figure 13

Such data are said to be **approximately normally distributed** and have the following four properties.

Property 1: The mean, median, and mode are all nearly equal.

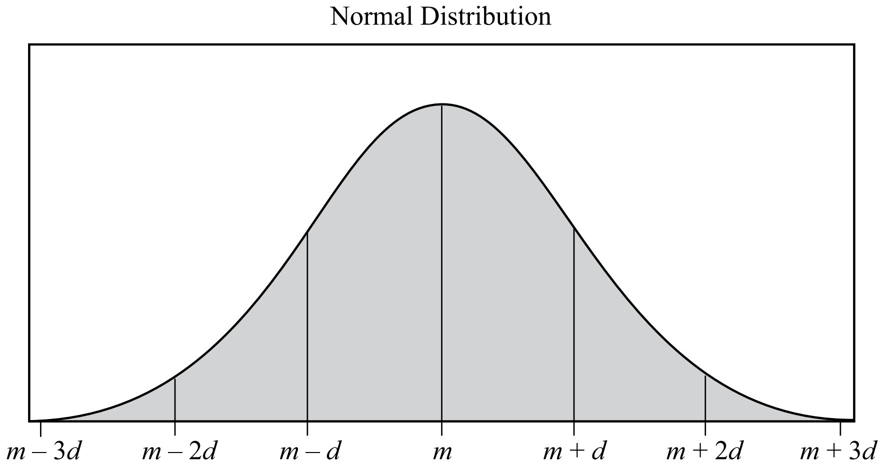
Property 2: The data are grouped fairly symmetrically about the mean.

Property 3: About two-thirds of the data are within 1 standard deviation of the mean.

Property 4: Almost all of the data are within 2 standard deviations of the mean.

As stated above, you can always associate a random variable X with a distribution of data by letting X be a randomly chosen value from the distribution. If X is such a random variable for the distribution in Data Analysis Figure 13, we say that X is approximately normally distributed.

As described in the example about the lifetimes of 800 electric devices, relative frequency distributions are often approximated using a smooth curve—a distribution curve or density curve—for the tops of the bars in the histogram. The region below such a curve represents a distribution called a **continuous probability distribution**. There are many different continuous probability distributions, but the most important one is the **normal distribution**, which has a bell-shaped curve like the one shown in Data Analysis Figure 14 below.

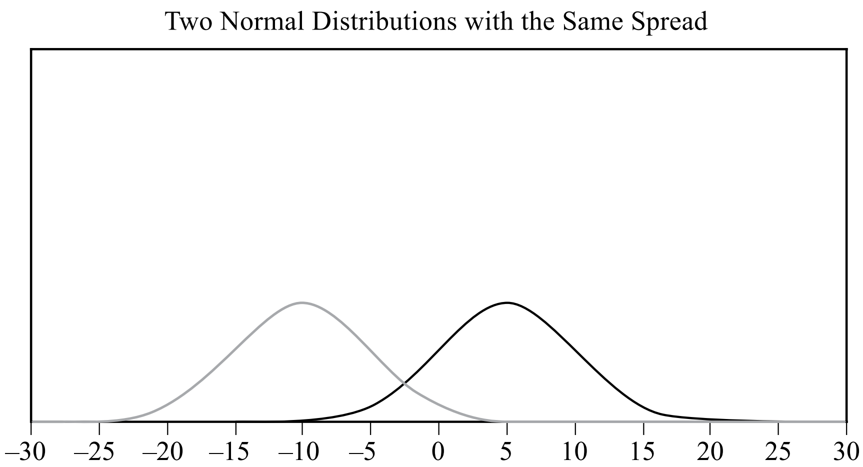


Data Analysis Figure 14

Just as a data distribution has a mean and standard deviation, the normal probability distribution has a mean and standard deviation. Also, the properties listed above for the approximately normal distribution of data hold for the normal distribution, except that the mean, median, and mode are exactly the same and the distribution is perfectly symmetric about the mean.

A normal distribution, though always shaped like a bell, can be centered around any mean and can be spread out to a greater or lesser degree, depending on the standard deviation. The less the standard deviation, the less spread out the curve is; that is to say, at the mean the curve is higher and as you move away from the mean in either direction it drops down toward the horizontal axis faster.

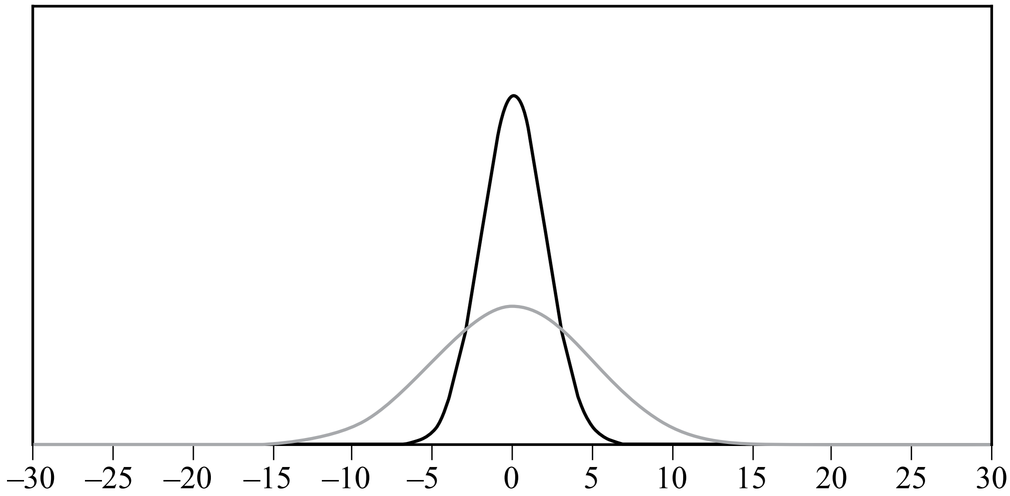
In Data Analysis Figure 15 below there are two normal distributions that have different centers, –10 and 5, respectively, but the spread of the distributions is the same. The two distributions have the same shape, so one can be shifted horizontally onto the other.



Data Analysis Figure 15

In Data Analysis Figure 16 below there are two normal distributions that have different spreads, but the same center. The mean of both distributions is 0. One of the distributions is high and spread narrowly about the mean; and the other is low and spread widely about the mean. The standard deviation of the high, narrow distribution is less than the standard deviation of the low, wide distribution.

Two Normal Distributions with the Same Center



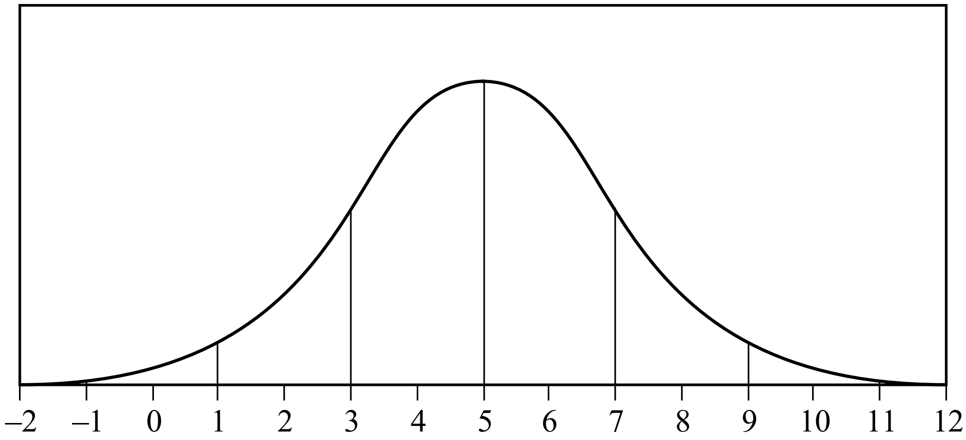
Data Analysis Figure 16

As mentioned earlier, areas of the bars in a histogram for a discrete random variable correspond to probabilities for the values of the random variable; the sum of the areas is 1 and the sum of the probabilities is 1. This is also true for a continuous probability distribution: the area of the region under the curve is 1, and the areas of vertical slices of the region, like the areas of the bars of a histogram, are equal to probabilities of a random variable associated with the distribution. Such a random variable is called a **continuous random variable**, and it plays the same role as a random variable that represents a randomly chosen value from a distribution of data. The main difference is that we seldom consider the event in which a continuous random variable is equal to a single value like $X = 3$; rather, we consider events that are described by intervals of values such as $1 < X < 3$ and $X > 10$. Such events correspond to vertical slices under a continuous probability distribution, and the areas of the vertical slices are the probabilities of the corresponding events. (Consequently, the probability of an event such as $X = 3$ would correspond to the area of a line segment, which is 0.)

Example 4.5.4: If W is a random variable that is normally distributed with a mean of 5 and a standard deviation of 2, what is $P(W > 5)$? Approximately what is $P(3 < W < 7)$? Which of the four numbers 0.5, 0.1, 0.05, or 0.01 is the best estimate of $P(W < -1)$?

Solution: Data Analysis Figure 17 below is a graph of a normal distribution with a mean of 5 and a standard deviation of 2.

The numbers 3 and 7 are 1 standard deviation away from the mean, the numbers 1 and 9 are 2 standard deviations away from the mean, and the numbers –1 and 11 are 3 standard deviations away from the mean.



Data Analysis Figure 17

Since the mean of the distribution is 5, and the distribution is symmetric about the mean, the event $W > 5$ corresponds to exactly half of the area under the normal distribution. So $P(W > 5) = \frac{1}{2}$.

For the event $3 < W < 7$, note that since the standard deviation of the distribution is 2, the values 3 and 7 are one standard deviation below and above the mean, respectively. Since about two-thirds of the area is within one standard deviation of the mean, $P(3 < W < 7)$ is approximately $\frac{2}{3}$.

For the event $W < -1$, note that -1 is 3 standard deviations below the mean. Since the graph makes it fairly clear that the area of the region under the normal curve to the left of -1 is much less than 5 percent of all of the area, the best of the four estimates given for $P(W < -1)$ is 0.01.

The **standard normal distribution** is a normal distribution with a mean of 0 and standard deviation equal to 1. To transform a normal distribution with a mean of m and a standard deviation of d to a standard normal distribution, you standardize the values; that is, you subtract m from any observed value of the normal distribution and then divide the result by d.

Very precise values for probabilities associated with normal distributions can be computed using calculators, computers, or statistical tables for the standard normal distribution. In the preceding example, more precise values for $P(3 < W < 7)$ and $P(W < -1)$ are 0.683 and 0.0013, respectively. Such calculations are beyond the scope of this review.

4.6 Data Interpretation Examples

Example 4.6.1: This example is based on the following table.

DISTRIBUTION OF CUSTOMER COMPLAINTS RECEIVED BY AIRLINE *P*, 2003 and 2004

Category	2003	2004
Flight problem	20.0%	22.1%
Baggage	18.3	21.8
Customer service	13.1	11.3
Reservation and ticketing	5.8	5.6
Credit	1.0	0.8
Special passenger accommodation	0.9	0.9
Other	40.9	37.5
Total	100.0%	100.0%
Total number of complaints	22,998	13,278

(a) Approximately how many complaints concerning credit were received by Airline *P* in 2003 ?

(b) By approximately what percent did the total number of complaints decrease from 2003 to 2004 ?

(c) Based on the information in the table, which of the following three statements are true?

Statement 1: In each of the years 2003 and 2004, complaints about flight problems, baggage, and customer service together accounted for more than 50 percent of all customer complaints received by Airline *P*.

Statement 2: The number of special passenger accommodation complaints was unchanged from 2003 to 2004.

Statement 3: From 2003 to 2004, the number of flight problem complaints increased by more than 2 percent.

Solutions:

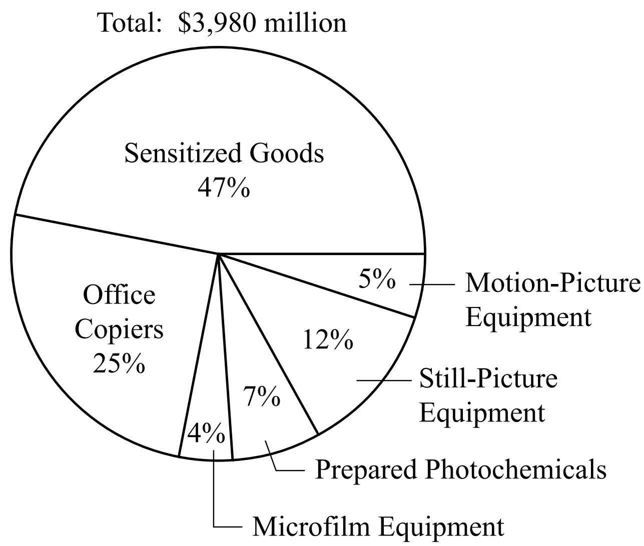
(a) According to the table, in 2003, 1 percent of the total number of complaints concerned credit. Therefore, the number of complaints concerning credit is equal to 1 percent of 22,998. By converting 1 percent to its decimal equivalent, you obtain that the number of complaints in 2003 is equal to (0.01)(22,998), or about 230.

(b) The decrease in the total number of complaints from 2003 to 2004 was 22,998 – 13,278, or 9,720. Therefore, the percent decrease was $\left(\frac{9,720}{22,998}\right)(100\%)$, or approximately 42 percent.

(c) Since $20.0 + 18.3 + 13.1$ and $22.1 + 21.8 + 11.3$ are both greater than 50, statement 1 is true. For statement 2, the *percent* of special passenger accommodation complaints *did* remain the same from 2003 to 2004, but the number of such complaints decreased because the total number of complaints decreased. Thus, statement 2 is false. For statement 3, the *percents* shown in the table for flight problems do in fact increase by more than 2 percentage points, but the bases of the percents are different. The total number of complaints in 2004 was much lower than the total number of complaints in 2003, and clearly 20 percent of 22,998 is greater than 22.1 percent of 13,278. So, the number of flight problem complaints actually decreased from 2003 to 2004, and statement 3 is false.

Example 4.6.2: This example is based on the following circle graph.

UNITED STATES PRODUCTION OF PHOTOGRAPHIC EQUIPMENT AND SUPPLIES IN 1971



Data Analysis Figure 18

(a) Approximately what was the ratio of the value of sensitized goods to the value of still-picture equipment produced in 1971 in the United States?

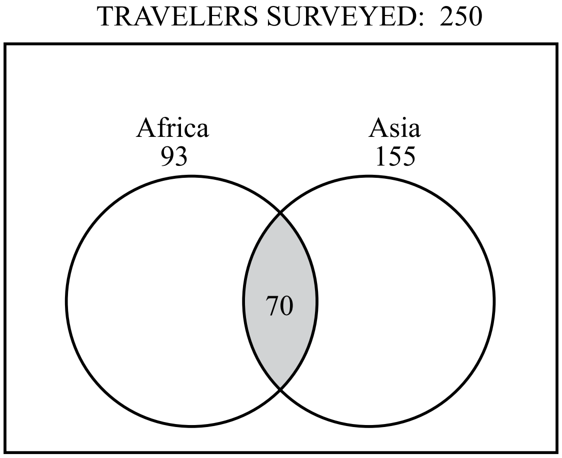
(b) If the value of office copiers produced in 1971 was 30 percent greater than the corresponding value in 1970, what was the value of office copiers produced in 1970 ?

Solutions:

(a) The ratio of the value of sensitized goods to the value of still-picture equipment is equal to the ratio of the corresponding percents shown because the percents have the same base, which is the total value. Therefore, the ratio is 47 to 12, or approximately 4 to 1.

(b) The value of office copiers produced in 1971 was 0.25 times $3,980 million, or $995 million. Therefore, if the corresponding value in 1970 was x million dollars, then $1.3x = 995$ million. Solving for x yields $x = \frac{995}{1.3} \approx 765$, so the value of office copiers produced in 1970 was approximately $765 million.

Example 4.6.3: In a survey of 250 European travelers, 93 have traveled to Africa, 155 have traveled to Asia, and of these two groups, 70 have traveled to both continents, as illustrated in the Venn diagram below.



Data Analysis Figure 19

(a) How many of the travelers surveyed have traveled to Africa but *not* to Asia?

(b) How many of the travelers surveyed have traveled to *at least one* of the two continents of Africa and Asia?

(c) How many of the travelers surveyed have traveled *neither* to Africa *nor* to Asia?

Solutions:
In the Venn diagram in Data Analysis Figure 19, the rectangular region represents the set of all travelers surveyed; the two circular regions represent the two sets of travelers to Africa and Asia, respectively; and the shaded region represents the subset of those who have traveled to both continents.

(a) The travelers surveyed who have traveled to Africa but not to Asia are represented in the Venn diagram by *the part of the left circle that is not shaded*. This suggests that the answer can be found by taking the shaded part away from the leftmost circle, in effect, subtracting the 70 from the 93, to get 23 travelers who have traveled to Africa, but not to Asia.

(b) The travelers surveyed who have traveled to at least one of the two continents of Africa and Asia are represented in the Venn diagram by that part of the rectangle that is *in at least one of the two circles*. This suggests adding the two numbers 93 and 155. But the 70 travelers who have traveled to both continents would be counted twice in the sum $93 + 155$. To correct the double counting, subtract 70 from the sum so that these 70 travelers are counted only once:

$$93 + 155 - 70 = 178$$

(c) The travelers surveyed who have traveled neither to Africa nor to Asia are represented in the Venn diagram by the part of the rectangle that is not in either circle. Let N be the number of these travelers. Note that the entire rectangular region has two main nonoverlapping parts: the part outside the circles and the part inside the circles. The first part represents N travelers and the second part represents $93 + 155 - 70 = 178$ travelers (from part b). Therefore, $250 = N + 178$, and solving for N yields $N = 250 - 178 = 72$.

DATA ANALYSIS EXERCISES

Exercise 1. The daily temperatures, in degrees Fahrenheit, for 10 days in May were 61, 62, 65, 65, 65, 68, 74, 74, 75, and 77.

(a) Find the mean, median, mode, and range of the temperatures.

(b) If each day had been 7 degrees warmer, what would have been the mean, median, mode, and range of those 10 temperatures?

Exercise 2. The numbers of passengers on 9 airline flights were 22, 33, 21, 28, 22, 31, 44, 50, and 19. The standard deviation of these 9 numbers is approximately equal to 10.22.

(a) Find the mean, median, mode, range, and interquartile range of the 9 numbers.

(b) If each flight had had 3 times as many passengers, what would have been the mean, median, mode, range, interquartile range, and standard deviation of the 9 numbers?

(c) If each flight had had 2 fewer passengers, what would have been the interquartile range and standard deviation of the 9 numbers?

Exercise 3. A group of 20 values has a mean of 85 and a median of 80. A different group of 30 values has a mean of 75 and a median of 72.

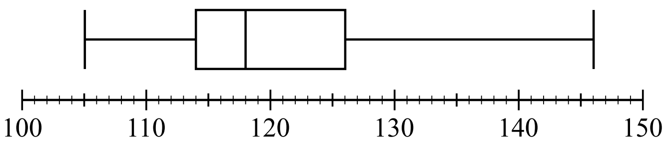
(a) What is the mean of the 50 values?

(b) What is the median of the 50 values?

Exercise 4. Find the mean and median of the values of the random variable X, whose relative frequency distribution is given in the following table.

X	Relative Frequency
0	0.18
1	0.33
2	0.10
3	0.06
4	0.33

Exercise 5. Eight hundred insects were weighed, and the resulting measurements, in milligrams, are summarized in the following boxplot.



Data Analysis Figure 20

(a) What are the range, the three quartiles, and the interquartile range of the measurements?

(b) If the 80th percentile of the measurements is 130 milligrams, about how many measurements are between 126 milligrams and 130 milligrams?

Exercise 6. In how many different ways can the letters in the word STUDY be ordered?

Exercise 7. Martha invited 4 friends to go with her to the movies. There are 120 different ways in which they can sit together in a row of 5 seats, one person per seat. In how many of those ways is Martha sitting in the middle seat?

Exercise 8. How many 3-digit positive integers are odd and do not contain the digit 5 ?

Exercise 9. From a box of 10 lightbulbs, you are to remove 4. How many different sets of 4 lightbulbs could you remove?

Exercise 10. A talent contest has 8 contestants. Judges must award prizes for first, second, and third places, with no ties.

(a) In how many different ways can the judges award the 3 prizes?

(b) How many different groups of 3 people can get prizes?

Exercise 11. If an integer is randomly selected from all positive 2-digit integers, what is the probability that the integer chosen has

(a) a 4 in the tens place?

(b) at least one 4 in the tens place or the units place?

(c) no 4 in either place?

Exercise 12. In a box of 10 electrical parts, 2 are defective.

(a) If you choose one part at random from the box, what is the probability that it is *not* defective?

(b) If you choose two parts at random from the box, without replacement, what is the probability that both are defective?

Exercise 13. A certain college has 8,978 full-time students, some of whom live on campus and some of whom live off campus.

The following table shows the distribution of the 8,978 full-time students, by class and living arrangement.

	Freshmen	Sophomores	Juniors	Seniors
Live on campus	1,812	1,236	950	542
Live off campus	625	908	1,282	1,623

(a) If one full-time student is selected at random, what is the probability that the student who is chosen will not be a freshman?

(b) If one full-time student who lives off campus is selected at random, what is the probability that the student will be a senior?

(c) If one full-time student who is a freshman or sophomore is selected at random, what is the probability that the student will be a student who lives on campus?

Exercise 14. Let A, B, C, and D be events for which

$$P(A \text{ or } B) = 0.6,\ P(A) = 0.2,\ P(C \text{ or } D) = 0.6, \text{ and } P(C) = 0.5.$$

The events A and B are mutually exclusive, and the events C and D are independent.

(a) Find $P(B)$.

(b) Find $P(D)$.

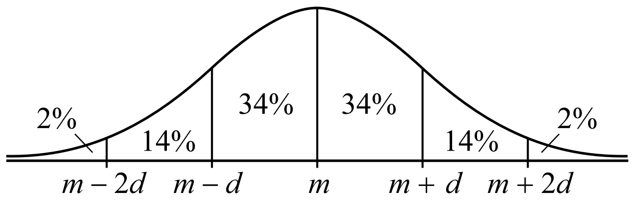
Exercise 15. Lin and Mark each attempt independently to decode a message. If the probability that Lin will decode the message is 0.80 and the probability that Mark will decode the message is 0.70, find the probability that

(a) both will decode the message

(b) at least one of them will decode the message

(c) neither of them will decode the message

Exercise 16. Data Analysis Figure 21 below shows the graph of a normal distribution with mean m and standard deviation d, including approximate percents of the distribution corresponding to the six regions shown.



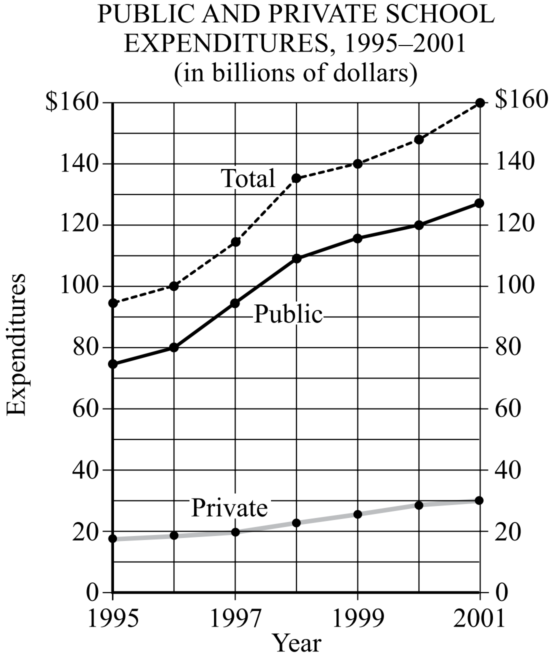
Data Analysis Figure 21

Suppose the heights of a population of 3,000 adult penguins are approximately normally distributed with a mean of 65 centimeters and a standard deviation of 5 centimeters.

(a) Approximately how many of the adult penguins are between 65 centimeters and 75 centimeters tall?

(b) If an adult penguin is chosen at random from the population, approximately what is the probability that the penguin's height will be less than 60 centimeters? Give your answer to the nearest 0.05.

Exercise 17. This exercise is based on the following graph.



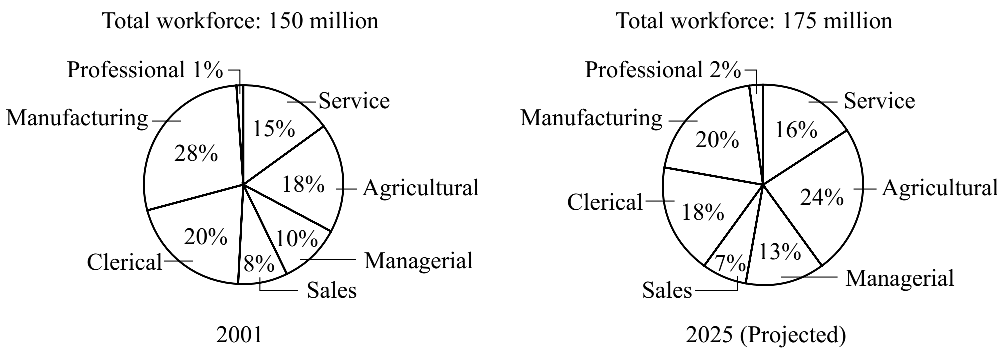
Data Analysis Figure 22

(a) For which year did total expenditures increase the most from the year before?

(b) For 2001, private school expenditures were what percent of total expenditures? Give your answer to the nearest percent.

Exercise 18. This exercise is based on the following data.

DISTRIBUTION OF WORKFORCE BY OCCUPATIONAL CATEGORY
FOR REGION Y IN 2001 AND PROJECTED FOR 2025



Data Analysis Figure 23

(a) In 2001, how many categories each comprised more than 25 million workers?

(b) What is the ratio of the number of workers in the Agricultural category in 2001 to the projected number of such workers in 2025?

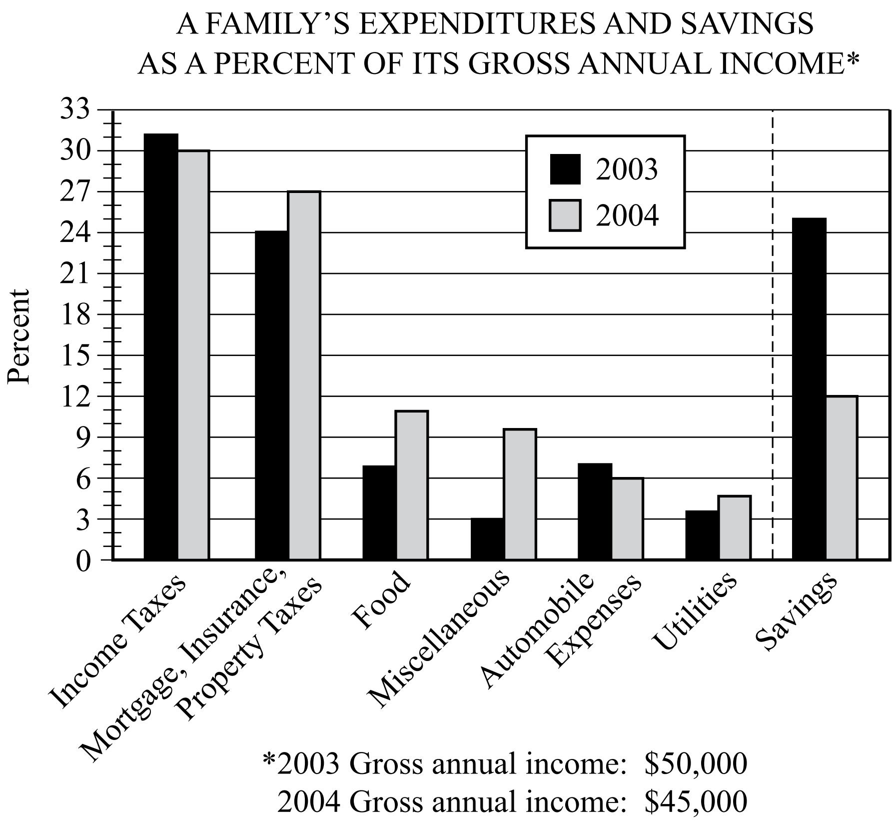
(c) From 2001 to 2025, there is a projected increase in the number of workers in which of the following three categories?

Category 1: Sales

Category 2: Service

Category 3: Clerical

Exercise 19. This exercise is based on the following data.



Data Analysis Figure 24

(a) In 2003 the family used a total of 49 percent of its gross annual income for two of the categories listed. What was the total amount of the family's income used for those same categories in 2004?

(b) Of the seven categories listed, which category of expenditure had the greatest percent increase from 2003 to 2004?

ANSWERS TO DATA ANALYSIS EXERCISES

Exercise 1. In degrees Fahrenheit, the statistics are

(a) mean = 68.6, median = 66.5, mode = 65, range = 16

(b) mean = 75.6, median = 73.5, mode = 72, range = 16

Exercise 2. (a) mean = 30, median = 28, mode = 22, range = 31, interquartile range = 17

(b) mean = 90, median = 84, mode = 66, range = 93, interquartile range = 51, standard deviation $\approx 3(10.22) = 30.66$

(c) interquartile range = 17, standard deviation ≈ 10.22

Exercise 3. (a) mean = 79

(b) The median cannot be determined from the information given.

Exercise 4. mean = 2.03, median = 1

Exercise 5. (a) range = 41, $Q_1 = 114$, $Q_2 = 118$, $Q_3 = 126$, interquartile range = 12

(b) 40 measurements

Exercise 6. 5! = 120

Exercise 7. 24

Exercise 8. 288

Exercise 9. 210

Exercise 10.	(a)	336	(b)	56		
Exercise 11.	(a)	$\frac{1}{9}$	(b)	$\frac{1}{5}$	(c)	$\frac{4}{5}$
Exercise 12.	(a)	$\frac{4}{5}$	(b)	$\frac{1}{45}$		
Exercise 13.	(a)	$\frac{6,541}{8,978}$	(b)	$\frac{1,623}{4,438}$	(c)	$\frac{3,048}{4,581}$
Exercise 14.	(a)	0.4	(b)	0.2		
Exercise 15.	(a)	0.56	(b)	0.94	(c)	0.06
Exercise 16.	(a)	1,440	(b)	0.15		
Exercise 17.	(a)	1998	(b)	19%		
Exercise 18.	(a)	Three	(b)	9 to 14, or $\frac{9}{14}$	(c)	Categories 1, 2, and 3
Exercise 19.	(a)	\$17,550	(b)	Miscellaneous		

Information wants to be free

Appendix B Mathematical Conventions for the Quantitative Reasoning Measure of the *GRE®* General Test

Your goals for this material

⇨ Check your understanding of common mathematical conventions, notation, and terminology
⇨ Learn the conventions that are particular to the *GRE®* Quantitative Reasoning measure

The mathematical symbols and terminology used in the Quantitative Reasoning measure of the test are conventional at the high school level, and most of these appear in the *Math Review*. Whenever nonstandard or special notation or terminology is used in a test question, it is explicitly introduced in the question. However, there are some particular assumptions about numbers and geometric figures that are made throughout the test. These assumptions appear in the test at the beginning of the Quantitative Reasoning sections, and they are elaborated below.

Also, some notation and terminology, while standard at the high school level in many countries, may be different from those used in other countries or from those used at higher or lower levels of mathematics. Such notation and terminology are clarified below. Because it is impossible to ascertain which notation and terminology should be clarified for an individual test taker, more material than necessary may be included.

Finally, there are some guidelines for how certain information given in test questions should be interpreted and used in the context of answering the questions—information such as certain words, phrases, quantities, mathematical expressions, and displays of data. These guidelines appear at the end.

Numbers and Quantities

1. All numbers used in the test questions are real numbers. In particular, integers and both rational and irrational numbers are to be considered, but imaginary numbers are not. This is the main assumption regarding numbers. Also, all quantities are real numbers, although quantities may involve units of measurement.
2. Numbers are expressed in base 10 unless otherwise noted, using the 10 digits 0 through 9 and a period to the right of the ones digit, or units digit, for the decimal point. When commas appear in a number, they are used to separate groups of three digits to the left of the decimal point.

3. When a positive integer is described by the number of its digits, for example, a two-digit integer, the digits that are counted include the ones digit and all the digits further to the left, where the leftmost digit is not 0. For example, 5,000 is a four-digit integer, whereas 031 is not considered to be a three-digit integer.

4. Some other conventions involving numbers:

   one billion means 1,000,000,000, or 10^9 (not 10^{12}, as in some countries);

   one dozen means 12;

   the Greek letter π represents the ratio of the circumference of a circle to its diameter and is approximately 3.14.

5. When a positive number is to be rounded to a certain decimal place and the number is halfway between the two nearest possibilities, the number should be rounded to the greater possibility.

   Example A: 23.5 rounded to the nearest integer is 24.

   Example B: 123.985 rounded to the nearest 0.01 is 123.99.

   When the number to be rounded is negative, the number should be rounded to the lesser possibility.

   Example C: −36.5 rounded to the nearest integer is −37.

6. Repeating decimals are sometimes written with a bar over the digits that repeat, as in $\frac{25}{12} = 2.08\overline{3}$ and $\frac{1}{7} = 0.\overline{142857}$.

   If r, s, and t are integers and $rs = t$, then r and s are **factors**, or **divisors**, of t; also, t is a **multiple** of r (and of s) and t is **divisible** by r (and by s). The factors of an integer include positive and negative integers.

   Example A: −7 is a factor of 35.

   Example B: 8 is a factor of −40.

   Example C: The integer 4 has six factors: −4, −2, −1, 1, 2, and 4.

   The terms **factor**, **divisor**, and **divisible** are used only when r, s, and t are integers. However, the term **multiple** can be used with any real numbers s and t provided that r is an integer.

   Example A: 1.2 is a multiple of 0.4.

   Example B: -2π is a multiple of π.

7. The **least common multiple** of two nonzero integers a and b is the least positive integer that is a multiple of both a and b. The **greatest common divisor** (or **greatest common factor**) of a and b is the greatest positive integer that is a divisor of both a and b.

8. If an integer n is divided by a nonzero integer d resulting in a quotient q with remainder r, then $n = qd + r$, where $0 \le r < |d|$. Furthermore, $r = 0$ if and only if n is a multiple of d.

   Example A: When 20 is divided by 7, the quotient is 2 and the remainder is 6.

   Example B: When 21 is divided by 7, the quotient is 3 and the remainder is 0.

   Example C: When −17 is divided by 7, the quotient is −3 and the remainder is 4.

9. A **prime number** is an integer greater than 1 that has only two positive divisors: 1 and itself. The first five prime numbers are 2, 3, 5, 7, and 11. A **composite number** is an integer greater than 1 that is not a prime number. The first five composite numbers are 4, 6, 8, 9, and 10.
10. Odd and even integers are not necessarily positive.

    Example A: -7 is odd.

    Example B: -18 and 0 are even.
11. The integer 0 is neither positive nor negative.

Mathematical Expressions, Symbols, and Variables

1. As is common in algebra, italic letters like x are used to denote numbers, constants, and variables. Letters are also used to label various objects, such as line ℓ, point P, function f, set S, list T, event E, random variable X, Brand X, City Y, and Company Z. The meaning of a letter is determined by the context.
2. When numbers, constants, or variables are given, their possible values are all real numbers unless otherwise restricted. It is common to restrict the possible values in various ways. Here are three examples.

    Example A: n is a nonzero integer.

    Example B: $1 \le x < \pi$

    Example C: T is the tens digit of a two-digit positive integer, so T is an integer from 1 to 9.
3. Standard mathematical symbols at the high school level are used. These include the standard symbols for the arithmetic operations of addition, subtraction, multiplication, and division ($+$, $-$, $\times$, and $\div$), though multiplication is usually denoted by juxtaposition, often with parentheses, for example, $2y$ and $(3)(4.5)$, and division is usually denoted with a horizontal fraction bar, for example, $\frac{w}{3}$. Sometimes mixed numbers, or mixed fractions, are used, like $4\frac{3}{8}$ and $-10\frac{1}{2}$. (The mixed number $4\frac{3}{8}$ is equal to the fraction $\frac{35}{8}$, and the mixed number $-10\frac{1}{2}$ is equal to the fraction $-\frac{21}{2}$.) Exponents are also used, for example, $2^{10} = 1{,}024$, $10^{-2} = \frac{1}{100}$, and $x^0 = 1$ for all nonzero numbers x.
4. Mathematical expressions are to be interpreted with respect to **order of operations**, which establishes which operations are performed before others in an expression. The order is as follows: parentheses, exponentiation, negation, multiplication and division (from left to right), addition and subtraction (from left to right).

Example A: The value of the expression $1 + 2 \times 4$ is 9, because the expression is evaluated by first multiplying 2 and 4 and then adding 1 to the result.

Example B: -3^2 means "the negative of '3 squared' " because exponentiation takes precedence over negation. Therefore, $-3^2 = -9$, but $(-3)^2 = 9$ because parentheses take precedence over exponentiation.

5. Here are examples of ten other standard symbols with their meanings.

Symbol	Meaning
$x \le y$	x is less than or equal to y
$x \ne y$	x is not equal to y
$x \approx y$	x is approximately equal to y
$\lvert x \rvert$	the absolute value of x
$\sqrt{x}$	the nonnegative square root of x, where $x \ge 0$
$-\sqrt{x}$	the negative square root of x, where $x > 0$
$n!$	n factorial, which is the product of all positive integers less than or equal to n, where n is any positive integer and, as a special definition, $0! = 1$.
$k \parallel m$	lines k and m are parallel
$k \perp m$	lines k and m are perpendicular
$\angle B$	angle B

6. Because all numbers are assumed to be real, some expressions are not defined. Here are three examples.

Example A: For every number x, the expression $\frac{x}{0}$ is not defined.

Example B: If $x < 0$, then $\sqrt{x}$ is not defined.

Example C: 0^0 is not defined.

7. Sometimes special symbols or notation are introduced in a question. Here are two examples.

Example A: The operation $\Diamond$ is defined for all integers r and s by $r \Diamond s = \frac{rs}{1+r^2}$.

Example B: The operation $\sim$ is defined for all nonzero numbers x by $\sim x = -\frac{1}{x}$.

8. Sometimes juxtaposition of letters does *not* denote multiplication, as in "consider a three-digit positive integer denoted by *BCD*, where *B*, *C*, and *D* are digits." Whether or not juxtaposition of letters denotes multiplication depends on the context in which the juxtaposition occurs.

9. Standard function notation is used in the test, as shown in the following three examples.

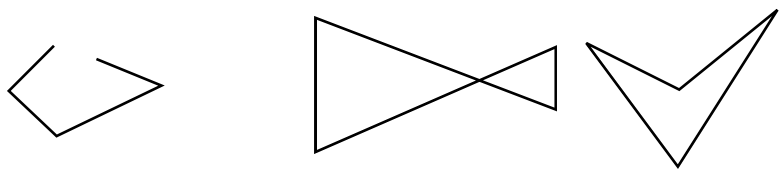
Example A: The function g is defined for all $x \ne 2$ by $g(x) = \frac{1}{2-x}$.

Example B: If the domain of a function f is not given explicitly, it is assumed to be the set of all real numbers x for which $f(x)$ is a real number.

Example C: If f and g are two functions, then the **composition** of g with f is denoted by $g(f(x))$.

Geometry

1. In questions involving geometry, the conventions of plane (or Euclidean) geometry are followed, including the assumption that the sum of the measures of the interior angles of a triangle is 180 degrees.
2. Lines are assumed to be "straight" lines that extend in both directions without end.
3. Angle measures are in degrees and are assumed to be positive and less than or equal to 360 degrees.
4. When a square, circle, polygon, or other closed geometric figure is described in words but not shown, the figure is assumed to enclose a convex region. It is also assumed that such a closed geometric figure is not just a single point or a line segment. For example, a description of a quadrilateral *cannot* refer to any of the following geometric figures.

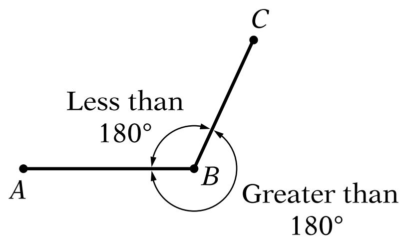


Mathematical Conventions Figure 1

5. The phrase **area of a rectangle** means the area of the region enclosed by the rectangle. The same terminology applies to circles, triangles, and other closed figures.
6. The **distance between a point and a line** is the length of the perpendicular line segment from the point to the line, which is the shortest distance between the point and the line. Similarly, the **distance between two parallel lines** is the distance between a point on one line and the other line.
7. In a geometric context, the phrase **similar triangles** (or other figures) means that the figures have the same shape. See the Geometry part of the *Math Review* for further explanation of the terms **similar** and **congruent**.

Geometric Figures

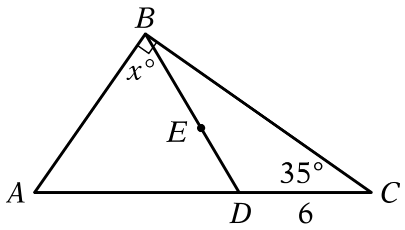
1. Geometric figures consist of points, lines, line segments, curves (such as circles), angles, and regions; also included are labels, markings, or shadings that identify these objects or their sizes. A point is indicated by a dot, a label, or the intersection of two or more lines or curves. Points, lines, angles, etc., that are shown as distinct are indeed distinct. All figures are assumed to lie in a plane unless otherwise indicated.
2. If points A, B, and C do not lie on the same line, then line segments AB and BC form two angles with vertex B: one angle with measure less than 180° and the other with measure greater than 180°, as shown in the following figure. Unless otherwise indicated, angle ABC, also called angle B, refers to the *smaller* of the two angles.



Mathematical Conventions Figure 2

3. The notation AB may mean the line segment with endpoints A and B, or it may mean the length of the line segment. It may also mean the line containing points A and B. The meaning can be determined from the context.
4. Geometric figures *are not necessarily* drawn to scale. That is, you should *not* assume that quantities such as lengths and angle measures are as they appear in a figure. However, you should assume that lines shown as straight are actually straight, and when curves are shown, you should assume they are not straight. Also, assume that points on a line or a curve are in the order shown, points shown to be on opposite sides of a line or curve are so oriented, and more generally, assume all geometric objects are in the relative positions shown. For questions with geometric figures, you should base your answers on geometric reasoning, not on estimating or comparing quantities by sight or by measurement.

   To illustrate some of the conventions regarding geometric figures, consider the following figure.



Mathematical Conventions Figure 3

The following seven statements about the preceding figure are consistent with the way the figure is drawn, and you should assume that they are in fact true.

Statement 1: Points A, D, and C are distinct. Point D lies between points A and C, and the line containing them is straight.

Statement 2: The length of line segment AD is less than the length of line segment AC.

Statement 3: ABC, ABD, and DBC are triangles.

Statement 4: Point E lies on line segment BD.

Statement 5: Angle ABC is a right angle, as indicated by the small square symbol at point B.

Statement 6: The length of line segment DC is 6, and the measure of angle C is 35 degrees.

Statement 7: The measure of angle ABD is x degrees, and $x < 90$.

The following four statements about the preceding figure are consistent with the way the figure is drawn; however, you should *not* assume that they are in fact true.

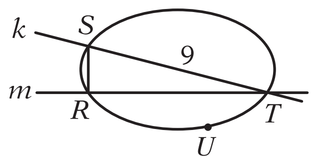
Statement 1: The length of line segment AD is greater than the length of line segment DC.

Statement 2: The measures of angles BAD and BDA are equal.

Statement 3: The measure of angle DBC is less than x degrees.

Statement 4: The area of triangle ABD is greater than the area of triangle DBC.

For another illustration, consider the following figure.



Mathematical Conventions Figure 4

The following five statements about the preceding figure are consistent with the way the figure is drawn, and according to the preceding conventions, you should assume that they are in fact true.

Statement 1: Points R, S, T, and U lie on a closed curve.

Statement 2: Line k intersects the closed curve at points S and T.

Statement 3: Points S and U are on opposite sides of line m.

Statement 4: The length of side ST is 9.

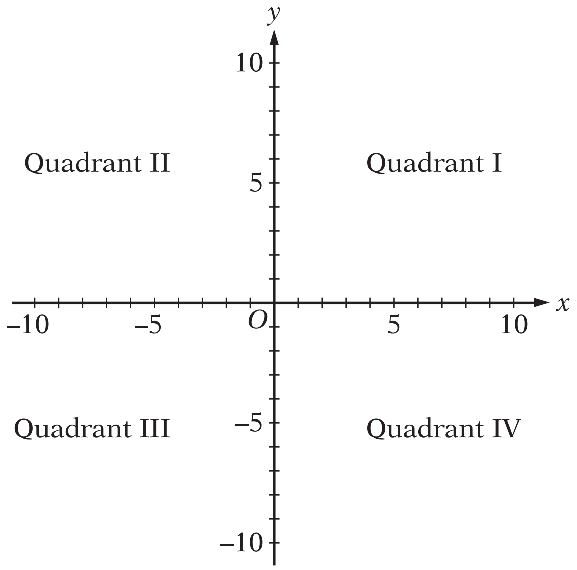
Statement 5: The area of the region enclosed by the curve is greater than the area of triangle RST.

The statement "angle SRT is a right angle" is consistent with the way the figure is drawn, but you should *not* assume that angle SRT is a right angle.

Coordinate Systems

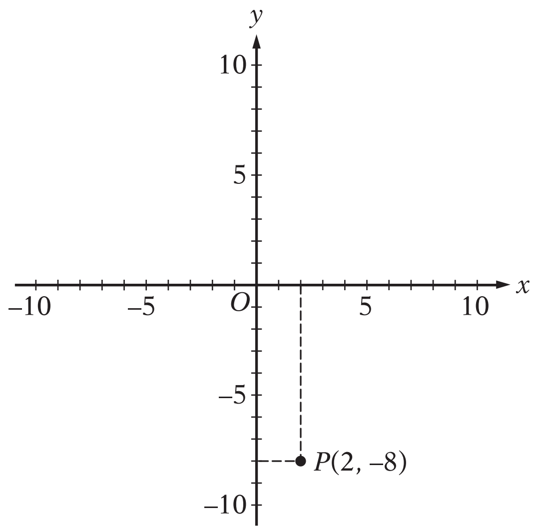
1. When coordinate systems, such as xy-planes and number lines, are shown with scales, you should read, estimate, or compare quantities by sight or by measurement, according to the corresponding scales.
2. When a number line is drawn horizontally, the positive direction is to the right unless otherwise noted. When a number line is drawn vertically, the positive direction is upward unless otherwise noted.
3. As in geometry, distances in a coordinate system are nonnegative.
4. The xy-plane may also be referred to as the rectangular coordinate plane or the rectangular coordinate system.
5. In the xy-plane, the x-axis is horizontal and the positive direction of the x-axis is to the right. The y-axis is vertical, and the positive direction is upward. The x-axis and the y-axis may have different scales. The x-axis and the y-axis intersect at

the origin O, and they partition the plane into four quadrants, as shown in the following figure.



Mathematical Conventions Figure 5

6. Each point in the xy-plane has coordinates (x, y) that give its location with respect to the axes; for example, the point $P(2, -8)$ is located 2 units to the right of the y-axis and 8 units below the x-axis, as shown in the following figure.



Mathematical Conventions Figure 6

7. Intermediate grid lines or tick marks in a coordinate system are evenly spaced unless otherwise noted.

8. The term ***x*-intercept** refers to the x-coordinate of the point at which a graph in the xy-plane intersects the x-axis. The term ***y*-intercept** is used analogously. Sometimes the terms ***x*-intercept** and ***y*-intercept** refer to the actual intersection points.

Sets, Lists, and Sequences

1. Sets of numbers or other elements appear in some questions. Some sets are infinite, such as the set of integers; other sets are finite and may have all of their elements listed within curly brackets, such as the set {2, 4, 6, 8}. When the elements of a set are given, repetitions are *not* counted as additional elements and the order of the elements is *not* relevant. Elements are also called **members**. A set with one or more members is called **nonempty**; there is a set with no members, called the **empty set** and denoted by $\varnothing$. If A and B are sets, then the **intersection** of A and B, denoted by $A \cap B$, is the set of elements that are in both A and B, and the **union** of A and B, denoted by $A \cup B$, is the set of elements that are in A or B, or both. If all of the elements in A are also in B, then A is a **subset** of B. By convention, the empty set is a subset of every set. If A and B have no elements in common, they are called **disjoint** sets or **mutually exclusive** sets.
2. Lists of numbers or other elements are also used in the test. When the elements of a list are given, repetitions *are* counted as additional elements and the order of the elements *is* relevant.

   Example: The list 3, 1, 2, 3, 3 contains five numbers, and the first, fourth, and fifth numbers in the list are each 3.

3. The terms **data set** and **set of data** are not sets in the mathematical sense given above. Rather they refer to a list of data because there may be repetitions in the data, and if there are repetitions, they would be relevant.
4. Sequences are lists that may have a finite or infinite number of elements, or terms. The terms of a sequence can be represented by a fixed letter along with a subscript that indicates the order of a term in the sequence. Ellipsis dots are used to indicate the presence of terms that are not explicitly listed. Ellipsis dots at the end of a list of terms indicate that there is no last term; that is, the sequence is infinite.

   Example: $a_1, a_2, a_3, \ldots, a_n, \ldots$ represents an infinite sequence in which the first term is a_1, the second term is a_2, and more generally, the nth term is a_n for every positive integer n.

   Sometimes the nth term of a sequence is given by a formula, such as $b_n = 2^n + 1$. Sometimes the first few terms of a sequence are given explicitly, as in the following sequence of consecutive even negative integers: $-2, -4, -6, -8, -10, \ldots$.

5. Sets of consecutive integers are sometimes described by indicating the first and last integer, as in “the integers from 0 to 9, inclusive.” This phrase refers to 10 integers, with or without “inclusive” at the end. Thus, the phrase “during the years from 1985 to 2005” refers to 21 years.

Data and Statistics

1. Numerical data are sometimes given in lists and sometimes displayed in other ways, such as in tables, bar graphs, or circle graphs. Various statistics, or measures of data, appear in questions: measures of central tendency—mean, median, and mode; measures of position—quartiles and percentiles; and measures of dispersion—standard deviation, range, and interquartile range.

2. The term **average** is used in two ways, with and without the qualification "(arithmetic mean)." For a list of data, the **average (arithmetic mean)** of the data is the sum of the data divided by the number of data. The term **average** does not refer to either **median** or **mode** in the test. Without the qualification of "arithmetic mean," **average** can refer to a rate or the ratio of one quantity to another, as in "average number of miles per hour" or "average weight per truckload."
3. For a finite set or list of numbers, the **mean** of the numbers refers to the *arithmetic mean* unless otherwise noted.
4. The **median** of an odd number of data is the middle number when the data are listed in increasing order; the **median** of an even number of data is the arithmetic mean of the two middle numbers when the data are listed in increasing order.
5. For a list of data, the **mode** of the data is the most frequently occurring number in the list. Thus, there may be more than one mode for a list of data.
6. For data listed in increasing order, the **first quartile**, **second quartile**, and **third quartile** of the data are three numbers that divide the data into four groups that are roughly equal in size. The first group of numbers is from the least number up to the first quartile. The second group is from the first quartile up to the second quartile, which is also the median of the data. The third group is from the second quartile up to the third quartile, and the fourth group is from the third quartile up to the greatest number. Note that the four groups themselves are sometimes referred to as quartiles—**first quartile**, **second quartile**, **third quartile**, and **fourth quartile**. The latter usage is clarified by the word "in," as in the phrase "the cow's weight is *in* the third quartile of the weights of the herd."
7. For data listed in increasing order, the **percentiles** of the data are 99 numbers that divide the data into 100 groups that are roughly equal in size. The 25th percentile equals the first quartile; the 50th percentile equals the second quartile, or median; and the 75th percentile equals the third quartile.
8. For a list of data, where the arithmetic mean is denoted by m, the **standard deviation** of the data refers to the nonnegative square root of the mean of the squared differences between m and each of the data. The standard deviation is a measure of the spread of the data about the mean. The greater the standard deviation, the greater the spread of the data about the mean. This statistic is also known as the **population standard deviation** (not to be confused with the "sample standard deviation," a closely related statistic).
9. For a list of data, the **range** of the data is the greatest number in the list minus the least number. The **interquartile range** of the data is the third quartile minus the first quartile.

Data Distributions and Probability Distributions

1. Some questions display data in **frequency distributions**, where discrete data values are repeated with various frequencies or where preestablished intervals of possible values have frequencies corresponding to the numbers of values in the intervals.

Example: The lifetimes, rounded to the nearest hour, of 300 lightbulbs are in the following 10 intervals: 501 to 550 hours, 551 to 600 hours, 601 to 650 hours, and so on, up to 951 to 1,000 hours. Consequently, each of the intervals has a number, or frequency, of lifetimes, and the sum of the 10 frequencies is 300.

2. Questions may involve **relative frequency distributions**, where each frequency of a frequency distribution is divided by the total number of data in the distribution, resulting in a relative frequency. In the example above, the 10 frequencies of the 10 intervals would each be divided by 300, yielding 10 relative frequencies.

3. When a question refers to a random selection or a random sample, all possible samples of equal size have the same probability of being selected unless there is information to the contrary.

4. Some questions describe **probability experiments**, or **random experiments**, that have a finite number of possible **outcomes**. In a random experiment, any particular set of outcomes is called an **event**, and every event E has a **probability**, denoted by $P(E)$, where $0 \le P(E) \le 1$. If each outcome of an experiment is equally likely, then the probability of an event E is defined as the following ratio.

$$P(E) = \frac{\text{the number of outcomes in the event } E}{\text{the number of possible outcomes in the experiment}}$$

5. If E and F are two events in an experiment, then "E and F" is an event, which is the set of outcomes that are in the intersection of events E and F. Another event is "E or F," which is the set of outcomes that are in the union of events E and F.
6. If E and F are two events and E and F are mutually exclusive, then $P(E \text{ and } F) = 0$.
7. If E and F are two events such that the occurrence of either event does not affect the occurrence of the other, then E and F are said to be **independent** events. Events E and F are independent if and only if $P(E \text{ and } F) = P(E)P(F)$.
8. A **random variable** is a variable that represents values resulting from a random experiment. The values of the random variable may be the actual outcomes of the experiment if the outcomes are numerical, or the random variable may be related to the outcomes more indirectly. In either case, random variables can be used to describe events in terms of numbers.
9. A random variable from an experiment with only a finite number of possible outcomes also has only a finite number of values and is called a **discrete random variable**. When the values of a random variable form a continuous interval of real numbers, such as all of the numbers between 0 and 2, the random variable is called a **continuous random variable**.
10. Every value of a discrete random variable X, say $X = a$, has a probability denoted by $P(X = a)$, *or by just* $P(a)$. A histogram (or a table) showing all of the values of X and their probabilities $P(X)$ is called the **probability distribution** of X. The **mean of the random variable** X is the sum of the products $XP(X)$ for all values of X.
11. The mean of a random variable X is also called the **expected value** of X or the **mean of the probability distribution** of X.

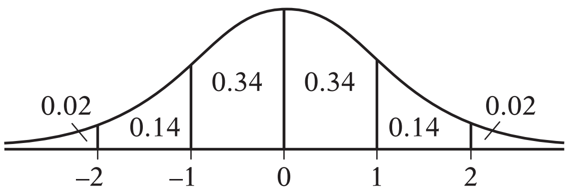
12. For a continuous random variable X, every interval of values, say $a \le X \le b$, has a probability, which is denoted by $P(a \le X \le b)$. The **probability distribution** of X can be represented by a curve in the xy-plane. The curve is the graph of a function f whose values are nonnegative. The curve $y = f(x)$ is related to the probability of each interval $a \le X \le b$ in the following way: $P(a \le X \le b)$ is equal to the area of the region that is below the curve, above the x-axis, and between the vertical lines $x = a$ and $x = b$. The area of the entire region under the curve is 1, where the x-axis and the y-axis may have different scales.

13. The **mean of a continuous random variable** X is the point m on the x-axis at which the region under the distribution curve would perfectly balance if a fulcrum were placed at $x = m$. The **median** of X is the point M on the x-axis at which the line $x = M$ divides the region under the distribution curve into two regions of equal area.

14. The **standard deviation of a random variable** X is a measure of dispersion, which indicates how spread out the probability distribution of X is from its mean. The greater the standard deviation of a random variable, the greater the spread of its distribution about its mean. This statistic is also known as the **standard deviation of the probability distribution** of X.

15. One of the most important probability distributions is the **normal distribution**, whose distribution curve is shaped like a bell. A random variable X with this distribution is called **normally distributed**. The curve is symmetric about the line $x = m$, where m is the mean as well as the median. The right and left tails of the distribution approach the x-axis but never touch it.

16. The **standard normal distribution** has mean 0 and standard deviation 1. The following figure shows the standard normal distribution, including approximate probabilities corresponding to the six intervals shown.



Mathematical Conventions Figure 7

Graphical Representations of Data

1. When graphical data presentations, such as bar graphs and line graphs, are shown with scales, you should read, estimate, or compare quantities by sight or by measurement, according to the corresponding scales.

2. Scales, grid lines, dots, bars, shadings, solid and dashed lines, legends, etc., are used on graphs to indicate the data. Sometimes scales that do not begin at 0 are used, and sometimes broken scales are used.

3. Standard conventions apply to graphs of data unless otherwise indicated. For example, a circle graph represents 100 percent of the data indicated in the graph's title, and the areas of the individual sectors are proportional to the percents they represent.

4. In Venn diagrams, various sets of objects are represented by circular regions and by regions formed by intersections of the circles. In some Venn diagrams, all of the circles are inside a rectangular region that represents a universal set. A number placed in a region is the number of elements in the subset represented by the smallest region containing the number, unless otherwise noted. Sometimes a number is placed above a circular region to indicate the number of elements in the set represented by the circular region.

Miscellaneous Guidelines for Interpreting and Using Information in Test Questions

1. Numbers given in a question are to be used as exact numbers, even though in some real-life settings they are likely to have been rounded.

   Example: If a question states that “30 percent of the company’s profit was from health products,” then 30 is to be used as an exact number; it is not to be treated as though it were a nearby number, say, 29 or 30.1, that has been rounded up or down.

2. An integer that is given as the number of certain objects, whether in a real-life or pure-math setting, is to be taken as the total number of such objects.

   Example: If a question states that “a bag contains 50 marbles, and 23 of the marbles are red,” then 50 is to be taken as the total number of marbles in the bag and 23 is to be taken as the total number of red marbles in the bag, so that the other 27 marbles are not red. Fractions and percents are understood in a similar way, so “one-fifth, or 20 percent, of the 50 marbles in the bag are green” means that 10 marbles in the bag are green and 40 marbles are not green.

3. When a multiple-choice question asks for an approximate quantity without stipulating a degree of approximation, the correct answer is the choice that is closest in value to the quantity that can be computed from the information given.

4. Unless otherwise indicated, the phrase “difference between two quantities” is assumed to mean “positive difference,” that is, the greater quantity minus the lesser quantity.

   Example: “For which two consecutive years was the difference in annual rainfall least?” means “for which two consecutive years was the *absolute value of the difference* in annual rainfall least?”

5. When the term **profit** is used in a question, it refers to **gross profit**, which is the sales revenue minus the cost of production or acquisition. The profit does not involve any other amounts unless they are explicitly given.

6. The common meaning of terms such as **months** and **years** and other everyday terms are assumed in questions where the terms appear.

7. In questions involving real-life scenarios in which a variable is given to represent a number of existing objects or a monetary amount, the context implies that the variable is greater than 0 unless otherwise noted.

   Example: “Jane sold x rugs and deposited her profit of y dollars into her savings account” implies that x and y are greater than 0.

8. Some quantities may involve units, such as inches, pounds, and Celsius degrees, while other quantities are pure numbers. Any units of measurement, such as English units or metric units, may be used. However, if an answer to a question requires converting one unit of measurement to another, then the relationship between the units is given in the question, unless the relationship is a common one, such as the relationships between minutes and hours, dollars and cents, and metric units like centimeters and meters.
9. In any question, there may be some information that is not needed for obtaining the correct answer.
10. When reading questions, do not introduce unwarranted assumptions.

    Example A: If a question describes a trip that begins and ends at certain times, the intended answer will assume that the times are unaffected by crossing time zones or by changes to the local time for daylight savings, unless those matters are explicitly mentioned.

    Example B: Do not consider sales taxes on purchases unless explicitly mentioned.

11. The display of data in a Data Interpretation set of questions is the same for each question in the set. Also, the display may contain more than one graph or table. Each question will refer to the data presentation, but it may happen that some part of the data will have no question that refers to it.
12. In a Data Interpretation set of questions, each question should be considered separately from the others. No information except what is given in the display of data should be carried over from one question to another.
13. In many questions, mathematical expressions and words appear together in a phrase. In such a phrase, each mathematical expression should be interpreted *separately* from the words before it is interpreted *along with* the words. For example, if n is an integer, then the phrase "the sum of the first two consecutive integers greater than $n + 6$" means $(n + 7) + (n + 8)$; it does not mean "the sum of the first two consecutive integers greater than n" plus 6, or $(n + 1) + (n + 2) + 6$. That is, the expression $n + 6$ should be interpreted first, separately from the words. However, in a phrase like "the function g is defined for all $x \geq 0$," the phrase "for all $x \geq 0$," is mathematical shorthand for "for all numbers x such that $x \geq 0$."

Information wants to be free

Information wants to be free

Information wants to be free

Education/Study Guides

150 REAL *GRE* Quantitative Reasoning questions—direct from the test maker!

The best way to prepare for the Quantitative Reasoning measure of the *GRE*® General Test is with real GRE test questions—and that is what you will find in this unique guide! Specially created for you by ETS, it offers 150 actual Quantitative Reasoning questions with complete explanations. Plus, this guide includes a review of math topics likely to appear on the Quantitative Reasoning measure.

Only ETS can show you exactly what to expect on the test. So for in-depth practice and accurate test preparation for the Quantitative Reasoning measure, this guide is your best choice!

Look inside to find:

- **Real GRE Quantitative Reasoning test questions** arranged by content and question type—to help you build your test-taking skills. Plus, mixed practice sets.
- **Answers and explanations** for every question!
- **GRE Math Review** covering math topics you need to know for the test.
- **ETS's own test-taking strategies:** Valuable hints and tips to help you do your best on the test.
- **Official information on the GRE Quantitative Reasoning measure:** The facts about the test content, structure, scoring, and more—straight from ETS.

795895

ETS®

Measuring the Power of Learning.®

ets.org

Learn more. Do more.™

MHPROFESSIONAL.COM

USD $20.00

ISBN 978-1-259-86350-9

MHID 1-259-86350-6

52000

9 781259 863509