# JAM 2006

## MATHEMATICAL STATISTICS TEST PAPER

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## Special Instructions / Useful Data

1. For an event $A$, $P(A)$ denotes the probability of the event $A$.
2. The complement of an event is denoted by putting a superscript "$c$" on the event, e.g. $A^c$ denotes the complement of the event $A$.
3. For a random variable $X$, $E(X)$ denotes the expectation of $X$ and $V(X)$ denotes its variance.
4. $N(\mu, \sigma^2)$ denotes a normal distribution with mean $\mu$ and variance $\sigma^2$.
5. Standard normal random variable is a random variable having a normal distribution with mean 0 and variance 1.
6. $P(Z > 1.96) = 0.025$, $P(Z > 1.65) = 0.050$, $P(Z > 0.675) = 0.250$ and $P(Z > 2.33) = 0.010$, where $Z$ is a standard normal random variable.
7. $P(\chi_2^2 \geq 9.21) = 0.01$, $P(\chi_2^2 \geq 0.02) = 0.99$, $P(\chi_3^2 \geq 11.34) = 0.01$, $P(\chi_4^2 \geq 9.49) = 0.05$, $P(\chi_4^2 \geq 0.71) = 0.95$, $P(\chi_5^2 \geq 11.07) = 0.05$ and $P(\chi_5^2 \geq 1.15) = 0.95$, where $P(\chi_n^2 \geq c) = \alpha$, where $\chi_n^2$ has a Chi-square distribution with $n$ degrees of freedom.
8. $n!$ denotes the factorial of $n$.
9. The determinant of a square matrix $A$ is denoted by $|A|$.
10. R: The set of all real numbers.
11. R": $n$-dimensional Euclidean space.
12. $y'$ and $y''$ denote the first and second derivatives respectively of the function $y(x)$ with respect to $x$.

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**NOTE: This Question-cum-Answer book contains THREE sections, the Compulsory Section A, and the Optional Sections B and C.**

- **Attempt ALL questions in the compulsory section A. It has 15 objective type questions of *six* marks each and also *nine* subjective type questions of *fifteen* marks each.**
- **Optional Sections B and C have *five* subjective type questions of *fifteen* marks each.**
- **Candidates seeking admission to either of the two programmes, M.Sc. in Applied Statistics & Informatics at IIT Bombay and M.Sc. in Statistics & Informatics at IIT Kharagpur, are required to attempt ONLY Section B (Mathematics) from the Optional Sections.**
- **Candidates seeking admission to the programme, M.Sc. in Statistics at IIT Kanpur, are required to attempt ONLY Section C (Statistics) from the Optional Sections.**

**You must therefore attempt either Optional Section B or Optional Section C depending upon the programme(s) you are seeking admission to, and accordingly tick one of the boxes given below.**

| | | |
|---|---|---|
| **Optional Section Attempted** | **B** | |
| | **C** | |

- ***The negative marks for the Objective type questions will be carried over to the total marks.***
- **Write the answers to the objective questions in the *Answer Table for Objective Questions* provided on page MS 11/63 only.**



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## Compulsory Section A

1. If $a_n > 0$ for $n \geq 1$ and $\lim_{n \to \infty} (a_n)^{1/n} = L < 1$, then which of the following series is not convergent?

   (A) $\sum_{n=1}^{\infty} \sqrt{a_n a_{n+1}}$

   (B) $\sum_{n=1}^{\infty} a_n^2$

   (C) $\sum_{n=1}^{\infty} \sqrt{a_n}$

   (D) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{a_n}}$

2. Let $E$ and $F$ be two mutually disjoint events. Further, let $E$ and $F$ be independent of $G$. If $p = P(E) + P(F)$ and $q = P(G)$, then $P(E \cup F \cup G)$ is

   (A) $1 - pq$

   (B) $q + p^2$

   (C) $p + q^2$

   (D) $p + q - pq$

3. Let $X$ be a continuous random variable with the probability density function symmetric about 0. If $V(X) < \infty$, then which of the following statements is true?

   (A) $E(|X|) = E(X)$

   (B) $V(|X|) = V(X)$

   (C) $V(|X|) < V(X)$

   (D) $V(|X|) > V(X)$

4. Let

$$f(x) = x|x| + |x-1|, \quad -\infty < x < \infty.$$

   Which of the following statements is true?

   (A) $f$ is not differentiable at $x = 0$ and $x = 1$.

   (B) $f$ is differentiable at $x = 0$ but not differentiable at $x = 1$.

   (C) $f$ is not differentiable at $x = 0$ but differentiable at $x = 1$.

   (D) $f$ is differentiable at $x = 0$ and $x = 1$.

5. Let $A\underset{\sim}{x} = \underset{\sim}{b}$ be a non-homogeneous system of linear equations. The augmented matrix $[A : \underset{\sim}{b}]$ is given by

$$\left[\begin{array}{cccc|c} 1 & 1 & -2 & 1 & 1 \\ -1 & 2 & 3 & -1 & 0 \\ 0 & 3 & 1 & 0 & -1 \end{array}\right].$$



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Which of the following statements is true?
(A) Rank of $A$ is 3.
(B) The system has no solution.
(C) The system has unique solution.
(D) The system has infinite number of solutions.

6. An archer makes 10 independent attempts at a target and his probability of hitting the target at each attempt is $\frac{5}{6}$. Then the conditional probability that his last two attempts are successful given that he has a total of 7 successful attempts is
(A) $\frac{1}{5^5}$
(B) $\frac{7}{15}$
(C) $\frac{25}{36}$
(D) $\frac{8!}{3!\ 5!}\left(\frac{5}{6}\right)^7\left(\frac{1}{6}\right)^3$

7. Let
$$f(x)=(x-1)(x-2)(x-3)(x-4)(x-5),\ -\infty<x<\infty.$$
The number of distinct real roots of the equation $\frac{d}{dx}f(x)=0$ is exactly
(A) 2 (B) 3 (C) 4 (D) 5

8. Let
$$f(x)=\frac{k\,|x|}{(1+|x|)^4},\quad -\infty<x<\infty.$$
Then the value of $k$ for which $f(x)$ is a probability density function is
(A) $\frac{1}{6}$
(B) $\frac{1}{2}$
(C) 3
(D) 6

9. If $M_X(t)=e^{3t+8t^2}$ is the moment generating function of a random variable $X$, then $P(-4.84<X\le 9.60)$ is
(A) equal to 0.700
(B) equal to 0.925
(C) equal to 0.975
(D) greater than 0.999



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10. Let $X$ be a binomial random variable with parameters $n$ and $p$, where $n$ is a positive integer and $0 \le p \le 1$. If $\alpha = P\left(|X - np| \ge \sqrt{n}\right)$, then which of the following statements holds true for all $n$ and $p$?

(A) $0 \le \alpha \le \frac{1}{4}$

(B) $\frac{1}{4} < \alpha \le \frac{1}{2}$

(C) $\frac{1}{2} < \alpha < \frac{3}{4}$

(D) $\frac{3}{4} \le \alpha \le 1$

11. Let $X_1, X_2, ..., X_n$ be a random sample from a Bernoulli distribution with parameter $p$; $0 \le p \le 1$. The bias of the estimator $\dfrac{\sqrt{n} + 2\sum_{i=1}^{n} X_i}{2\left(n + \sqrt{n}\right)}$ for estimating $p$ is equal to

(A) $\frac{1}{\sqrt{n}+1}\left(p - \frac{1}{2}\right)$

(B) $\frac{1}{n+\sqrt{n}}\left(\frac{1}{2} - p\right)$

(C) $\frac{1}{\sqrt{n}+1}\left(\frac{1}{2} + \frac{p}{\sqrt{n}}\right) - p$

(D) $\frac{1}{\sqrt{n}+1}\left(\frac{1}{2} - p\right)$

12. Let the joint probability density function of $X$ and $Y$ be

$$f(x, y) = \begin{cases} e^{-x}, & \text{if } 0 \le y \le x < \infty, \\ 0, & \text{otherwise.} \end{cases}$$

Then $E(X)$ is

(A) 0.5

(B) 1

(C) 2

(D) 6



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13. Let $f : \square \to \square$ be defined as

$$f(t) = \begin{cases} \dfrac{\tan t}{t}, & t \neq 0, \\ 1, & t = 0. \end{cases}$$

Then the value of $\lim\limits_{x \to 0} \dfrac{1}{x^2} \displaystyle\int_{x^2}^{x^3} f(t)\, dt$

(A) is equal to $-1$
(B) is equal to 0
(C) is equal to 1
(D) does not exist

14. Let $X$ and $Y$ have the joint probability mass function;

$$P(X = x, Y = y) = \frac{1}{2^{y+2}(y+1)} \left( \frac{2y+1}{2y+2} \right)^x, \quad x, y = 0, 1, 2, \ldots.$$

Then the marginal distribution of $Y$ is

(A) Poisson with parameter $\lambda = \dfrac{1}{4}$

(B) Poisson with parameter $\lambda = \dfrac{1}{2}$

(C) Geometric with parameter $p = \dfrac{1}{4}$

(D) Geometric with parameter $p = \dfrac{1}{2}$

15. Let $X_1$, $X_2$ and $X_3$ be a random sample from a $N(3, 12)$ distribution. If $\bar{X} = \dfrac{1}{3} \displaystyle\sum_{i=1}^{3} X_i$ and $S^2 = \dfrac{1}{2} \displaystyle\sum_{i=1}^{3} (X_i - \bar{X})^2$ denote the sample mean and the sample variance respectively, then $P(1.65 < \bar{X} \leq 4.35,\ 0.12 < S^2 \leq 55.26)$ is

(A) 0.49
(B) 0.50
(C) 0.98
(D) none of the above



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16. (a) Let $X_1, X_2, \ldots, X_n$ be a random sample from an exponential distribution with the probability density function;

$$f(x;\theta) = \begin{cases} \theta\, e^{-\theta x}, & \text{if } x > 0, \\ 0, & \text{otherwise,} \end{cases}$$

where $\theta > 0$. Obtain the maximum likelihood estimator of $P(X > 10)$. **9 Marks**

(b) Let $X_1, X_2, \ldots, X_n$ be a random sample from a discrete distribution with the probability mass function given by

$$P(X = 0) = \frac{1-\theta}{2}; \quad P(X = 1) = \frac{1}{2}; \quad P(X = 2) = \frac{\theta}{2}, \quad 0 \le \theta \le 1.$$

Find the method of moments estimator for $\theta$. **6 Marks**

17. (a) Let $A$ be a non-singular matrix of order $n$ $(n > 1)$, with $|A| = k$. If $adj(A)$ denotes the adjoint of the matrix $A$, find the value of $|adj(A)|$. **6 Marks**

(b) Determine the values of $a$, $b$ and $c$ so that $(1, 0, -1)$ and $(0, 1, -1)$ are eigenvectors of the matrix,

$$\begin{bmatrix} 2 & 1 & 1 \\ a & 3 & 2 \\ 3 & b & c \end{bmatrix}.$$

**9 Marks**

18. (a) Using Lagrange's mean value theorem, prove that

$$\frac{b-a}{1+b^2} < \tan^{-1} b - \tan^{-1} a < \frac{b-a}{1+a^2},$$

where $0 < \tan^{-1} a < \tan^{-1} b < \frac{\pi}{2}$. **6 Marks**

(b) Find the area of the region in the first quadrant that is bounded by $y = \sqrt{x}$, $y = x - 2$ and the $x-$axis. **9 Marks**

19. Let $X$ and $Y$ have the joint probability density function;

$$f(x, y) = \begin{cases} c\, x\, y\, e^{-(x^2 + 2y^2)}, & \text{if } x > 0,\ y > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Evaluate the constant $c$ and $P(X^2 > Y^2)$.

20. Let $PQ$ be a line segment of length $\beta$ and midpoint $R$. A point $S$ is chosen at random on $PQ$. Let $X$, the distance from $S$ to $P$, be a random variable having the uniform distribution on the interval $(0, \beta)$. Find the probability that $PS$, $QS$ and $PR$ form the sides of a triangle.

21. Let $X_1, X_2, \ldots, X_n$ be a random sample from a $N(\mu, 1)$ distribution. For testing $H_0 : \mu = 10$ against $H_1 : \mu = 11$, the most powerful critical region is $\bar{X} \ge k$, where $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$. Find $k$ in terms of $n$ such that the size of this test is 0.05. Further determine the minimum sample size $n$ so that the power of this test is at least 0.95.



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22. Consider the sequence $\{s_n\}$, $n \geq 1$, of positive real numbers satisfying the recurrence relation

$$s_{n-1} + s_n = 2\, s_{n+1} \quad \text{for all} \quad n \geq 2.$$

(a) Show that $|s_{n+1} - s_n| = \dfrac{1}{2^{n-1}} |s_2 - s_1|$ for all $n \geq 1$.

(b) Prove that $\{s_n\}$ is a convergent sequence.

23. The cumulative distribution function of a random variable $X$ is given by

$$F(x) = \begin{cases} 0, & \text{if } x < 0, \\ \dfrac{1}{5}\left(1 + x^3\right), & \text{if } 0 \leq x < 1, \\ \dfrac{1}{5}\left[3 + (x-1)^2\right], & \text{if } 1 \leq x < 2, \\ 1, & \text{if } x \geq 2. \end{cases}$$

Find $P(0 < X < 2)$, $P(0 \leq X \leq 1)$ and $P\left(\dfrac{1}{2} \leq X \leq \dfrac{3}{2}\right)$.

24. Let $A$ and $B$ be two events with $P(A \mid B) = 0.3$ and $P(A \mid B^c) = 0.4$. Find $P(B \mid A)$ and $P(B^c \mid A^c)$ in terms of $P(B)$. If $\dfrac{1}{4} \leq P(B \mid A) \leq \dfrac{1}{3}$ and $\dfrac{1}{4} \leq P(B^c \mid A^c) \leq \dfrac{9}{16}$, then determine the value of $P(B)$.

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## Optional Section B

25. Solve the initial value problem

$$y' - y + y^2\left(x^2 + 2x + 1\right) = 0, \quad y(0) = 1.$$

26. Let $y_1(x)$ and $y_2(x)$ be the linearly independent solutions of

$$x\, y'' + 2\, y' + x\, e^x\, y = 0.$$

If $W(x) = y_1(x)\, y_2'(x) - y_2(x)\, y_1'(x)$ with $W(1) = 2$, find $W(5)$.

27. (a) Evaluate $\displaystyle\int_0^1 \int_y^1 x^2 e^{xy}\, dx\, dy$. **9 Marks**

(b) Evaluate $\displaystyle\iiint_W z\, dx\, dy\, dz$, where $W$ is the region bounded by the planes $x = 0$, $y = 0$, $z = 0$, $z = 1$ and the cylinder $x^2 + y^2 = 1$ with $x \geq 0$, $y \geq 0$.

**6 Marks**

28. A linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ is given by

$$T(x, y, z) = (3x + 11y + 5z,\ x + 8y + 3z).$$

Determine the matrix representation of this transformation relative to the ordered bases $\{(1,0,1), (0,1,1), (1,0,0)\}$, $\{(1,1), (1,0)\}$. Also find the dimension of the null space of this transformation.



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29. (a) Let $f(x, y) = \begin{cases} \dfrac{x^2 + y^2}{x + y}, & \text{if } x + y \neq 0, \\ 0, & \text{if } x + y = 0. \end{cases}$

Determine if $f$ is continuous at the point $(0, 0)$. **6 Marks**

(b) Find the minimum distance from the point $(1, 2, 0)$ to the cone $z^2 = x^2 + y^2$. **9 Marks**

## Optional Section C

30. Let $X_1, X_2, \ldots, X_n$ be a random sample from an exponential distribution with the probability density function;

$$f(x; \theta) = \begin{cases} \dfrac{1}{\theta} e^{-\frac{x}{\theta}}, & \text{if } x > 0, \\ 0, & \text{otherwise,} \end{cases}$$

where $\theta > 0$. Derive the Cramér-Rao lower bound for the variance of any unbiased estimator of $\theta$. Hence, prove that $T = \dfrac{1}{n} \sum_{i=1}^{n} X_i$ is the uniformly minimum variance unbiased estimator of $\theta$.

31. Let $X_1, X_2, \ldots$ be a sequence of independently and identically distributed random variables with the probability density function;

$$f(x) = \begin{cases} \dfrac{1}{2} x^2 e^{-x}, & \text{if } x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Show that $\lim_{n \to \infty} P\left(X_1 + \ldots + X_n \geq 3\left(n - \sqrt{n}\right)\right) \geq \dfrac{1}{2}$.

32. Let $X_1, X_2, \ldots, X_n$ be a random sample from a $N(\mu, \sigma^2)$ distribution, where both $\mu$ and $\sigma^2$ are unknown. Find the value of $b$ that minimizes the mean squared error of the estimator $T_b = \dfrac{b}{n-1} \sum_{i=1}^{n} \left(X_i - \bar{X}\right)^2$ for estimating $\sigma^2$, where $\bar{X} = \dfrac{1}{n} \sum_{i=1}^{n} X_i$.

33. Let $X_1, X_2, \ldots, X_5$ be a random sample from a $N(2, \sigma^2)$ distribution, where $\sigma^2$ is unknown. Derive the most powerful test of size $\alpha = 0.05$ for testing $H_0 : \sigma^2 = 4$ against $H_1 : \sigma^2 = 1$.

34. Let $X_1, X_2, \ldots, X_n$ be a random sample from a continuous distribution with the probability density function;

$$f(x; \lambda) = \begin{cases} \dfrac{2x}{\lambda} e^{-\frac{x^2}{\lambda}}, & \text{if } x > 0, \\ 0, & \text{otherwise,} \end{cases}$$

where $\lambda > 0$. Find the maximum likelihood estimator of $\lambda$ and show that it is sufficient and an unbiased estimator of $\lambda$.



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