MATHEMATICS

Q.1 Given: \( f(x) = \begin{cases} x & ; 0 \leq x < \frac{1}{2} \\ \frac{1}{2} & ; x = \frac{1}{2} \\ 1 - x & ; \frac{1}{2} < x \leq 1 \end{cases} \) and \( g(x) = \left(x - \frac{1}{2}\right)^2 \), \( x \in \mathbb{R} \). Then the area (in sq. units) of the region bounded by the curves \( y = f(x) \) and \( y = g(x) \) between the lines, \( 2x = 1 \) and \( 2x = \sqrt{3} \), is -

(1) \( \frac{1}{3} + \frac{\sqrt{3}}{4} \)  
(2) \( \frac{1}{2} + \frac{\sqrt{3}}{4} \)  
(3) \( \frac{1}{2} - \frac{\sqrt{3}}{4} \)  
(4) \( \frac{\sqrt{3}}{4} - \frac{1}{3} \)

Ans. [4]

Sol. Co-ordinates of \( P \left( \frac{1}{2}, 0 \right) \), \( Q \left( \frac{1}{2}, \frac{1}{2} \right) \), \( R \left( \frac{\sqrt{3}}{2}, 1 - \frac{\sqrt{3}}{2} \right) \) and \( S \left( \frac{\sqrt{3}}{2}, 0 \right) \)

\[ \text{Required area} = \text{Area of trapezium PQRS} - \int_{1/2}^{\sqrt{3}/2} \left(x - \frac{1}{2}\right)^2 \, dx \]

\[ = \frac{1}{2} \left( \frac{\sqrt{3}}{2} - \frac{1}{2} \right) \left( \frac{1}{2} + 1 - \frac{\sqrt{3}}{2} \right) - \frac{1}{3} \left[ \left(x - \frac{1}{2}\right)^3 \right]_{1/2}^{\sqrt{3}/2} \]

\[ = \frac{1}{2} \left( \frac{\sqrt{3} - 1}{2} \right) \left( \frac{3 - \sqrt{3}}{2} \right) - \frac{1}{3} \left[ \left( \frac{\sqrt{3} - 1}{2} \right)^3 - 0 \right] \]

\[ = \frac{\sqrt{3}}{4} - \frac{1}{3} \]
Q.2 Let \( a, b \in \mathbb{R}, a \neq 0 \) be such that the equation, \( ax^2 - 2bx + 5 = 0 \) has a repeated root \( \alpha \), which is also a root of the equation, \( x^2 - 2bx - 10 = 0 \). If \( \beta \) is the other root of this equation, then \( \alpha^2 + \beta^2 \) is equal to -

(1) 26  
(2) 24  
(3) 25  
(4) 28

Ans. [3]

Sol. Roots of equation \( ax^2 - 2bx + 5 = 0 \) are \( \alpha, \alpha \)

\[
\therefore \alpha + \alpha = \frac{2b}{a} \Rightarrow \alpha = \frac{b}{a} \quad \text{...(i)}
\]

and \( \alpha^2 = \frac{5}{a} \) \quad \text{...(ii)}

from eq. (i) & (ii)

\[
\frac{b^2}{a^2} = \frac{5}{a} \Rightarrow b^2 = 5a \quad \text{...(iii)} \quad (a \neq 0)
\]

Roots of equation \( x^2 - 2bx - 10 = 0 \) are \( \alpha, \beta \)

\[
\therefore \alpha + \beta = 2b
\]

and \( \alpha \beta = -10 \)

Now, \( \alpha = \frac{b}{a} \) is root of equation \( x^2 - 2bx - 10 = 0 \)

\[
\therefore \frac{b^2}{a^2} - \frac{2b^2}{a} - 10 = 0
\]

\[
\Rightarrow \frac{5}{a} - 10 - 10 = 0 \quad (\because \: b^2 = 5a)
\]

\[
\Rightarrow a = \frac{1}{4} \text{ and } b^2 = \frac{5}{4}
\]

So, \( \alpha^2 = \frac{b^2}{a^2} = 20 \) and \( \beta^2 = 5 \)

\[
\Rightarrow \alpha^2 + \beta^2 = 25
\]

Q.3 If \( A = \{x \in \mathbb{R} : |x| < 2\} \) and \( B = \{x \in \mathbb{R} : |x - 2| \geq 3\}; \) then -

(1) \( A \cup B = R - (2, 5) \)  
(2) \( A \cap B = (-2, -1) \)  
(3) \( B - A = R - (-2, 5) \)  
(4) \( A - B = [-1, 2) \)

Ans. [3]

Sol. \( A = \{x \in \mathbb{R} : |x| < 2\} \)

\( A \in (-2, 2) \)

and \( B = \{x \in \mathbb{R} : |x - 2| \geq 3\} \)

\( B \in (-\infty, -1] \cup [5, \infty) \)

\( A \cup B \in (-\infty, 2) \cup [5, \infty) \)

\( A \cap B \in (-2, -1] \)

\( B - A \in (-\infty, -2] \cup [5, \infty) \text{ or } R - (-2, 5) \)

\( A - B \in (-1, 2) \)
Q.4 Let $a_n$ be the $n^{th}$ term of a G.P. of positive terms. If $\sum_{n=1}^{100} a_{2n+1} = 200$ and $\sum_{n=1}^{100} a_{2n} = 100$, then $\sum_{n=1}^{200} a_n$ is equal to

(1) 175  (2) 225  (3) 300  (4) 150

Ans. [4]

Sol. Let the G.P. is $a, ar, ar^2$ ....

$$\sum_{n=1}^{100} a_{2n+1} = 200 \Rightarrow a_3 + a_5 + a_7 + .... + a_{201} = 200$$

$$\Rightarrow \frac{a^2 (r^{200} - 1)}{r^2 - 1} = 200 \quad \text{...(1)}$$

$$\sum_{n=1}^{100} a_{2n} = 100 \Rightarrow a_2 + a_4 + .... + a_{200} = 100$$

$$\Rightarrow \frac{ar(r^{200} - 1)}{r^2 - 1} = 100 \quad \text{...(2)}$$

Dividing (1) by (2)

we get, $r = 2$

Adding both eq. (1) & (2)

$\Rightarrow a_2 + a_3 + a_4 + a_5 + .... + a_{201} = 300$

$\Rightarrow r(a_1 + a_2 + .... + a_{200}) = 300$

$\Rightarrow a_1 + 2a_2 + .... + 200a_2 = \frac{300}{r}$

$\Rightarrow \sum_{n=1}^{200} a_1 = \frac{300}{2} = 150$

Q.5 If $z$ be a complex number satisfying $|\text{Re}(z)| + |\text{Im}(z)| = 4$, then $|z|$ cannot be -

(1) $\sqrt{10}$  (2) $\sqrt{8}$  (3) $\sqrt{7}$  (4) $\sqrt{\frac{17}{2}}$

Ans. [3]

Sol. Let $z = x + iy$

given that $|\text{Re}(z)| + |\text{Im}(z)| = 4$

$\therefore |x| + |y| = 4$

|B(0, 4)|

(-4,0)C

O

A(4, 0)

D(0, -4)
Maximum value of $|z| = 4$
Minimum value of $|z| =$ perpendicular distance of line AB from (0, 0)

$= 2\sqrt{2}$

So, $|z| \in [2\sqrt{2}, 4]$
So, $|z|$ can not be $\sqrt{7}$

Q.6 If $x = \sum_{n=0}^{\infty} (-1)^n \tan^2 \theta$ and $y = \sum_{n=0}^{\infty} \cos^2 n \theta$, for $0 < \theta < \frac{\pi}{4}$, then -

(1) $y(1-x) = 1$  
(2) $x(1-y) = 1$  
(3) $y(1+x) = 1$  
(4) $x(1+y) = 1$

Ans. [1]

Sol.
\[
x = \sum_{n=0}^{\infty} (-1)^n \tan^2 \theta \\
\Rightarrow x = 1 - \tan^2 \theta + \tan^4 \theta - \tan^6 \theta ....
\]
\[
\Rightarrow x = \frac{1}{1 + \tan^2 \theta} = \cos^2 \theta ....(1)
\]

and $y = \sum_{n=0}^{\infty} \cos^2 n \theta$

\[
y = 1 + \cos^2 \theta + \cos^4 \theta + \cos^6 \theta + ....
\]
\[
y = \frac{1}{1 - \cos^2 \theta} = \frac{1}{\sin^2 \theta}
\]
\[
\Rightarrow \sin^2 \theta = \frac{1}{y} ....(2)
\]

Adding (1) & (2), we get,
\[
x + \frac{1}{y} = 1
\]
\[
\Rightarrow y(1-x) = 1
\]

Q.7 Let $[t]$ denote the greatest integer $\leq t$ and $\lim_{x\to 0} \left[\frac{4}{x}\right] = A$. Then the function, $f(x) = [x^2] \sin(\pi x)$ is discontinuous, when $x$ is equal to -

(1) $\sqrt{A+1}$  
(2) $\sqrt{A}$  
(3) $\sqrt{A+21}$  
(4) $\sqrt{A+5}$

Ans. [1]

Sol.
\[
\lim_{x\to 0} \left[\frac{4}{x}\right] = A
\]
\[
\Rightarrow \lim_{x\to 0} \left(\frac{4}{x} - \left[\frac{4}{x}\right]\right) = A
\]
\[
\Rightarrow \lim_{x\to 0} \left(4 - x \left[\frac{4}{x}\right]\right) = A
\]
\[
\Rightarrow A = 4
\]

Now, $f(x) = [x^2] \sin(\pi x)$ is continuous at every integer point but discontinuous at non integer points then by options, $\sqrt{A+1}$ is correct answer.
Q.8 If \( p \rightarrow (p \land \neg q) \) is false, then the truth values of \( p \) and \( q \) are respectively -

1. F, F  
2. T, F  
3. T, T  
4. F, T

Ans. [3]

Sol. \( p \rightarrow (p \land \neg q) \) will be false only when \( p \) is true and \( (p \land \neg q) \) is false.
So, \( p = T, q = T \)

Q.9 Let a function \( f : [0, 5] \rightarrow \mathbb{R} \) be continuous, \( f(1) = 3 \) and \( F \) be defined as \( F(x) = \int_{1}^{x} t^2 g(t) \, dt \), where

\( g(t) = \int_{1}^{t} f(u) \, du \). Then for the function \( F \), the point \( x = 1 \) is

1. not a critical point  
2. a point of local maxima  
3. a point of local minima  
4. a point of inflection

Ans. [3]

Sol. \( F(x) = \int_{1}^{x} t^2 g(t) \, dt \)

\( \Rightarrow F'(x) = x^2 g(x) \)

at \( F'(1) = (1) g(1) = 0 \) \{\because g(1) = 0\}

Now, \( F''(x) = 2xg(x) + x^2 g'(x) \)

\( F''(1) = 2g(1) + g'(1) \)

\( F''(1) = 0 + g'(1) \) \{\( g'(t) = f(t) \); \( g'(1) = f'(1) = 3\} \)

\( F''(1) = 3 \)

So, at \( x = 1, F'(1) = 0 \) but \( F''(1) > 0 \)

\( \therefore \) For the function \( f(x) \), \( x = 1 \) is a point of local minima.

Q.10 A random variable \( X \) has the following probability distribution:

| \( X \) | 1 \( \quad \) 2 \( \quad \) 3 \( \quad \) 4 \( \quad \) 5 \n|---|---|---|---|---|
| \( P(X) \) | \( K^2 \) \( \) 2K \( \) K \( \) 2K \( \) 5K^2 |

Then \( P(X > 2) \) is equal to -

1. \( \frac{1}{36} \)  
2. \( \frac{1}{6} \)  
3. \( \frac{7}{12} \)  
4. \( \frac{23}{36} \)

Ans. [4]

Sol. \( \sum_{i=1}^{5} P(X) = 1 \)

\( \Rightarrow K^2 + 2K + K + 2K + 5K^2 = 1 \)

\( \Rightarrow 6K^2 + 5K - 1 = 0 \)

\( \Rightarrow K = \frac{1}{6}, K = -1 \) (rejected)
So, \( K = \frac{1}{6} \)

\[
P(X > 2) = K + 2K + 5K^2 = \frac{1}{6} + \frac{2}{6} + \frac{5}{36} = \frac{23}{36}\]

**Q.11** If

\[
\int \frac{d\theta}{\cos^2 \theta (\tan 2\theta + \sec 2\theta)} = \lambda \tan \theta + 2 \log_e |f(\theta)| + C \text{ where } C \text{ is a constant of integration, then the ordered pair } (\lambda, f(\theta)) \text{ is equal to -}
\]

\begin{align*}
\text{(1)} & \quad (1, 1 - \tan \theta) \\
\text{(2)} & \quad (-1, 1 - \tan \theta) \\
\text{(3)} & \quad (-1, 1 + \tan \theta) \\
\text{(4)} & \quad (1, 1 + \tan \theta)
\end{align*}

**Ans. [3]**

**Sol.**

\[
\int \frac{d\theta}{\cos^2 \theta (\tan 2\theta + \sec 2\theta)}
\]

\[
= \int \frac{d\theta}{\cos^2 \theta \left( \frac{2 \tan \theta}{1 - \tan^2 \theta} + \frac{1 + \tan^2 \theta}{1 - \tan^2 \theta} \right)}
\]

\[
= \int \frac{\sec^2 \theta (1 - \tan^2 \theta)}{(1 + \tan \theta)^2} d\theta
\]

Put \( \tan \theta = t \)

\[\Rightarrow \sec^2 \theta d\theta = dt\]

\[
= \int \frac{(1 - t^2)}{(1 + t)^2} dt = \int \left( \frac{1 - t}{1 + t} \right) dt
\]

\[
= \int \left( -1 + \frac{2}{1 + t} \right) dt
\]

\[= -t + 2 \log_e |1 + t| + C\]

\[= - \tan \theta + 2 \log_e |1 + \tan \theta| + C\]

\[\therefore \lambda = -1 \text{ and } f(\theta) = 1 + \tan \theta\]

**Q.12** The length of the minor axis (along y-axis) of an ellipse in the standard form is \( \frac{4}{\sqrt{3}} \). If this ellipse touches the line, \( x + 6y = 8 \), then its eccentricity is -

\begin{align*}
\text{(1)} & \quad \frac{1}{2} \sqrt{\frac{11}{3}} \\
\text{(2)} & \quad \frac{1}{3} \sqrt{\frac{11}{3}} \\
\text{(3)} & \quad \frac{1}{2} \sqrt{\frac{5}{3}} \\
\text{(4)} & \quad \sqrt{\frac{5}{6}}
\end{align*}

**Ans. [1]**

**Sol.**

Let the equation of ellipse \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad (a > b) \)

Given that \( 2b = \frac{4}{\sqrt{3}} \Rightarrow b = \frac{2}{\sqrt{3}} \)
Equation of tangent $y = mx \pm \sqrt{a^2 m^2 + b^2}$ ...(1)

Given tangent is $x + 6y = 8$

$\Rightarrow y = -\frac{1}{6}x + \frac{8}{6}$ ...(2)

from eq. (1) & (2)

$m = -\frac{1}{6}$ and $a^2 m^2 + b^2 = \frac{16}{9}$

$\Rightarrow a^2 \left(\frac{1}{36}\right) + \frac{4}{3} = \frac{16}{9}$

$\Rightarrow a^2 = 16$

Now, $e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{4/3}{16}} = \frac{11}{12} = \frac{1}{2} \frac{11}{3}$

Q.13 In the expansion of $\left(\frac{x}{\cos \theta} + \frac{1}{x \sin \theta}\right)^{16}$, if $\ell_1$ is the least value of the term independent of $x$ when $\frac{\pi}{8} \leq \theta \leq \frac{\pi}{4}$

and $\ell_2$ is the least value of the term independent of $x$ when $\frac{\pi}{16} \leq \theta \leq \frac{\pi}{8}$, then the ratio $\ell_2 : \ell_1$ is equal to -

(1) 16 : 1 (2) 1 : 8 (3) 8 : 1 (4) 1 : 16

Ans. [1]

Sol. General term $T_{r+1} = 16 C_r \left(\frac{x}{\cos \theta}\right)^{16-r} \left(\frac{1}{x \sin \theta}\right)^r$

$T_{r+1} = 16 C_r \left(\frac{1}{\cos \theta}\right)^{16-r} \left(\frac{1}{\sin \theta}\right)^r x^{16-2r}$

term is independent of $x$

$\therefore 16 - 2r = 0$

$\Rightarrow r = 8$

$T_9 = 16 C_8 \left(\frac{1}{\cos \theta}\right)^8 \left(\frac{1}{\sin \theta}\right)^8$

$T_9 = 16 C_8 \frac{2^8}{(\sin 2\theta)^8}$

$\theta \in \left[\frac{\pi}{8}, \frac{\pi}{4}\right] \Rightarrow 2\theta \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right]$

For least value $\Rightarrow \sin 2\theta$ should be maximum

$\therefore 2\theta = \frac{\pi}{2}$

So, $\ell_1 = 16 C_8 (2^8)$ ...(1)

Again, $\theta \in \left[\frac{\pi}{16}, \frac{\pi}{8}\right] \Rightarrow 2\theta \in \left[\frac{\pi}{8}, \frac{\pi}{4}\right]$
Minimum value, \( \ell_2 = 16C_8 \cdot \frac{2^8}{(1/\sqrt{2})^8} = 16C_8 2^{12} \) ...(2)

\[
\frac{\ell_2}{\ell_1} = \frac{2^4}{1} = \frac{16}{1}
\]

Q.14 If \( x = 2 \sin \theta - \sin 2\theta \) and \( y = 2 \cos \theta - \cos 2\theta \), \( \theta \in [0, 2\pi] \), then \( \frac{d^2 y}{dx^2} \) at \( \theta = \pi \) is -

(1) \( \frac{3}{2} \)

(2) \( \frac{3}{4} \)

(3) \( \frac{3}{4} \)

(4) \( \frac{3}{8} \)

Ans. [Bonus]

Sol. \( x = 2 \sin \theta - \sin 2\theta \)

\[ \Rightarrow \frac{dx}{d\theta} = 2 \cos \theta - 2 \cos 2\theta \]

and \( y = 2 \cos \theta - \cos 2\theta \)

\[ \Rightarrow \frac{dy}{d\theta} = -2 \sin \theta + 2 \sin 2\theta \]

Now, \( \frac{dy}{dx} = \frac{2(\sin 2\theta - \sin \theta)}{(\cos \theta - \cos 2\theta)} = \frac{2 \cos \frac{3\theta}{2} \sin \frac{\theta}{2}}{2 \sin \frac{3\theta}{2} \sin \frac{\theta}{2}} \]

\[ \Rightarrow \frac{dy}{dx} = \cot \frac{3\theta}{2} \]

\[ \Rightarrow \frac{d^2 y}{dx^2} = -\sec^2 \frac{3\theta}{2} \left( \frac{3}{2} \right) \frac{d\theta}{dx} \]

\[ \Rightarrow \frac{d^2 y}{dx^2} = -\frac{3}{2} \left( \sec^2 \frac{3\theta}{2} \right) \left( \frac{1}{2 \cos \theta - 2 \cos 2\theta} \right) \]

at \( \theta = \pi \)

\[ \frac{d^2 y}{dx^2} = \left( \frac{3}{2} \right) \left( 1 \right) \left( \frac{1}{-2 - 2} \right) = \frac{3}{8} \]

Q.15 If \( \frac{dy}{dx} = \frac{xy}{x^2 + y^2} \); \( y(1) = 1 \); then a value of \( x \) satisfying \( y(x) = e \) is -

(1) \( \frac{1}{2} \sqrt{3} e \)

(2) \( \sqrt{2} e \)

(3) \( \frac{e}{\sqrt{2}} \)

(4) \( \sqrt{3} e \)

Ans. [4]

Sol. \( \frac{dy}{dx} = \frac{xy}{x^2 + y^2} \)

Put \( y = vx \)

then \( \frac{dy}{dx} = v + x \frac{dv}{dx} \)
\[ v + x \frac{dv}{dx} = \frac{x(vx)}{x^2 + v^2 x^2} \]
\[ x \frac{dv}{dx} = \frac{v}{1 + v^2} - v \]
\[ \int \frac{1 + v^2}{v^2} \ dv = - \int \frac{dx}{x} \]
\[ \Rightarrow - \frac{1}{2v^2} + \log v = - \log x + C \]
\[ \Rightarrow - \frac{1}{2} x^2 + \log \left( \frac{y}{x} \right) = - \log x + C \]
\[ \text{put } x = 1, \ y = 1 \]
\[ \Rightarrow C = - \frac{1}{2} \]
from eq. (1)
\[ - \frac{1}{2} x^2 + \log \left( \frac{y}{x} \right) = - \log x - \frac{1}{2} \]
Put \( y = e \)
\[ \Rightarrow x^2 = 3e^2 \]
\[ \Rightarrow x = \pm \sqrt{3} \ e \]
\[ \Rightarrow x = \sqrt{3} \ e \]

Q.16 Let \( a - 2b + c = 1 \). If \( f(x) = \begin{vmatrix} x + a & x + 2 & x + 1 \\ x + b & x + 3 & x + 2 \\ x + c & x + 4 & x + 3 \end{vmatrix} \), then -
(1) \( f(-50) = -1 \)
(2) \( f(50) = 1 \)
(3) \( f(50) = -501 \)
(4) \( f(-50) = 501 \)

Ans. [2]

Sol. \( f(x) = \begin{vmatrix} x + a & x + 2 & x + 1 \\ x + b & x + 3 & x + 2 \\ x + c & x + 4 & x + 3 \end{vmatrix} \)
\( R_1 \rightarrow R_1 - 2R_2 + R_3 \)
\[ f(x) = \begin{vmatrix} 1 & 0 & 0 \\ x + b & x + 3 & x + 2 \\ x + c & x + 4 & x + 3 \end{vmatrix} \]
\[ f(x) = 1 \]
\[ \therefore f(50) = 1 \]

Q.17 If 10 different balls are to be placed in 4 distinct boxes at random, then the probability that two of these boxes contain exactly 2 and 3 balls is -
(1) \( \frac{945}{2^{10}} \)
(2) \( \frac{965}{2^{11}} \)
(3) \( \frac{945}{2^{11}} \)
(4) \( \frac{965}{2^{10}} \)
Q.18 The following system of linear equations
7x + 6y − 2z = 0  
3x + 4y + 2z = 0  
x − 2y − 6z = 0, has
(1) no solution  
(2) infinitely many solutions, (x, y, z) satisfying y = 2z  
(3) only the trivial solution  
(4) infinitely many solutions, (x, y, z) satisfying x = 2z

Ans. [4]

Sol. 7x + 6y − 2z = 0  ....(1)  
3x + 4y + 2z = 0  ....(2)  
x − 2y − 6z = 0  ....(3)

\[ \Delta = \begin{vmatrix} 7 & 6 & -2 \\ 3 & 4 & 2 \\ 1 & -2 & -6 \end{vmatrix} \]

\[ \Delta = 7(-24 + 4) - 6(-18 - 2) - 2(-6 - 4) = 0 \]

\[ \Delta = 0 \]

∴ infinite non-trivial solution exist

to eliminate y we operate eq. (1) − (2) + (3)

\[ 5x = 10z \]

\[ x = 2z \]

Q.19 If one end of a focal chord AB of the parabola \( y^2 = 8x \) is at A\( \left( \frac{1}{2}, -2 \right) \), then the equation of the tangent to it at B is -

(1) 2x + y − 24 = 0  
(2) x + 2y + 8 = 0  
(3) x − 2y + 8 = 0  
(4) 2x − y − 24 = 0

Ans. [3]

Sol. \( y^2 = 8x \)  \quad (a = 2)

Let one end of focal chord is \( (at^2, 2at) = \left( \frac{1}{2}, -2 \right) \)

\[ 2at = -2 \]

\[ t = -1/2 \]
other end of focal chord will be \[ \left( \frac{a}{t^2}, -\frac{2a}{t} \right) \equiv (8, 8) \]

Now, tangent at B(8, 8)
\[ \Rightarrow y(8) = 8\left(\frac{x + 8}{2}\right) \]
\[ \Rightarrow x - 2y + 8 = 0 \]

Q.20 Let \( f \) and \( g \) be differentiable functions on \( \mathbb{R} \) such that \( fog \) is the identity function. If for some \( a, b \in \mathbb{R} \), \( g'(a) = 5 \) and \( g(a) = b \), then \( f'(b) \) is equal to -

(1) \( \frac{1}{5} \)  
(2) \( \frac{2}{5} \)  
(3) 5  
(4) 1

Ans. [1]
Sol. \( fog \) is an identity function
\[ \therefore \; fog(x) = x \]
\[ \Rightarrow f'(g(x)) \cdot g'(x) = 1 \]
put \( x = a \)
\[ \Rightarrow f'(g(a)) \cdot g'(a) = 1 \]
\[ \Rightarrow f'(b) (5) = 1 \]
\[ \Rightarrow f'(b) = \frac{1}{5} \]

Q.21 If the distance between the plane, \( 23x - 10y - 2z + 48 = 0 \) and the plane containing the lines
\[ \frac{x + 1}{2} = \frac{y - 3}{4} = \frac{z + 1}{3} \quad \text{and} \quad \frac{x + 3}{2} = \frac{y + 2}{6} = \frac{z - 1}{\lambda} \; (\lambda \in \mathbb{R}) \] is equal to \( \frac{k}{\sqrt{633}} \), then \( k \) is equal to ____________.

Ans. [3]
Sol. Required distance = perpendicular distance of plane \( 23x - 10y - 2z + 48 = 0 \) either from (-1, 3, -1) or (-3, -2, 1)
\[ \Rightarrow \left| \frac{-23 -30 + 2 + 48}{\sqrt{(23)^2 + (10)^2 + (2)^2}} \right| = \frac{k}{\sqrt{633}} \]
\[ \Rightarrow \frac{3}{\sqrt{633}} = \frac{k}{\sqrt{633}} \]
\[ \therefore \; k = 3 \]

Q.22 If \( C_r = 25C_r \) and \( C_0 + 5 \cdot C_1 + 9 \cdot C_2 + \ldots + (101) \cdot C_{25} = 2^{25} \cdot k \), then \( k \) is equal to ____________.

Ans. [51]
Sol. \( C_0 + 5 \cdot C_1 + 9 \cdot C_2 + \ldots + (101) \cdot C_{25} \)
\[ = \sum_{r=0}^{25} (4r + 1) \cdot 25C_r = 4 \sum_{r=0}^{25} r \cdot 25C_r + \sum_{r=0}^{25} 25C_r \)
\[
= 4 \sum_{r=0}^{25} r \frac{25}{r} \binom{24}{r-1} + 2^{25}
= 100 \cdot 2^{24} + 2^{25}
= 2^{25}(50 + 1)
\]
\[\therefore k = 51\]

Q.23 The number of terms common to the two A.P.’s 3, 7, 11, ..... 407 and 2, 9, 16, ..... 709 is ____________.

Ans. [14]

Sol. First A.P. is 3, 7, 11, 15, 19, 23, ..... 407
Second A.P. is 2, 9, 16, 23, ..... 709
First common term = 23
Common difference d = L.C.M. (4, 7) = 28
Last term \(\leq 407\)
\[
\Rightarrow 23 + (n - 1) (28) \leq 407
\]
\[
\Rightarrow n \leq 14.7
\]
So, \(n = 14\)

Q.24 If the curves, \(x^2 - 6x + y^2 + 8 = 0\) and \(x^2 - 8y + y^2 + 16 - k = 0\), \((k > 0)\) touch each other at a point, then the largest value of \(k\) is ____________.

Ans. [36]

Sol. \(C_1 = (3, 0), \ C_2(0, 4)\)
\[
r_1 = \sqrt{9 + 0 - 8} = 1; \ r_2 = \sqrt{16 - 16 + k} = \sqrt{k}
\]
Two circles touch each other
\[
\therefore \ C_1C_2 = |r_1 \pm r_2|
\]
\[
= |1 \pm \sqrt{k}|
\]
\[
\Rightarrow 1 + \sqrt{k} = 5 \text{ or } \sqrt{k} - 1 = 5
\]
\[
\Rightarrow k = 16 \text{ or } k = 36
\]
Maximum value of \(k = 36\)

Q.25 Let \(\vec{a}, \vec{b}\) and \(\vec{c}\) be three vectors such that \(|\vec{a}| = \sqrt{5}, |\vec{b}| = 5, \vec{b} \cdot \vec{c} = 10\) and the angle between \(\vec{b}\) and \(\vec{c}\) is \(\frac{\pi}{3}\). If \(\vec{a}\) is perpendicular to the vector \(\vec{b} \times \vec{c}\), then \(|\vec{a} \times (\vec{b} \times \vec{c})|\) is equal to ________.

Ans. [30]

Sol.
\[
|\vec{a} \times (\vec{b} \times \vec{c})| = |\vec{a}||\vec{b}||\vec{c}| \sin \frac{\pi}{2}
\]
\[
= |\vec{a}||\vec{b}||\vec{c}| \sin \frac{\pi}{3}
\]
\[ (\sqrt{3}) (5) | \vec{c} | \cdot \frac{\sqrt{3}}{2} \]

\[ = \frac{15}{2} | \vec{c} | \quad \text{...(1)} \]

\[ \Rightarrow \cos \frac{\pi}{3} = \frac{\vec{b} \cdot \vec{c}}{|\vec{b}| |\vec{c}|} \]

\[ \Rightarrow \frac{1}{2} = \frac{10}{5 | \vec{c} |} \quad \Rightarrow |\vec{c}| = 4 \]

from eq. (1)

\[ |\vec{a} \times (\vec{b} \times \vec{c})| = \frac{15}{2} \times 4 = 30 \]